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SINGULAR PERTURBATIONS OF SINGULAR POINTS WITH APPLICATION TO TUBULAR CHEMICAL REACTORS

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ABSTRACT

Singular perturbation techniques are used to study solutions of certain boundary value problems defined on annular domains with outer radius normalized to one and inner radius ε, in the limit $\varepsilon \to 0$. Asymptotic expansions are constructed to describe the behavior of solutions at singular points such as bifurcation and limit points. First, a detailed analysis is carried out for a generalization of the classical eigenvalue problem for an elastic membrane. Second, the behavior of axisymmetric solutions at a limit point for a class of nonlinear equations is investigated. The results are applied to a model problem arising in chemical reactor theory. The asymptotic analysis predicts a surprisingly large sensitivity of singular points to the ε-domain perturbation considered here.

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1. Introduction

Let $u(\underline{x},\lambda)$ be the solution of a boundary value problem (BVP), where $\underline{x}$ is a point in a plane bounded domain $D \subseteq \mathbb{R}^2$, depending on a real parameter λ. Of particular interest are certain critical values of λ where a solution branch becomes singular, such as limit or turning points λ_L or bifurcation points λ_B. Let D be modified by piercing a small circular hole of radius ε centered at $\underline{x}_o$, that is, the domain D_ε is obtained by subtracting from D a ball $\{\underline{x} \in D, |\underline{x} - \underline{x}_o| < \varepsilon\}$. This process will affect the singular points $\lambda_L(\varepsilon)$, $\lambda_B(\varepsilon)$. We are interested in asymptotic expressions for $\lambda_L(\varepsilon) - \lambda_L$ and $\lambda_B(\varepsilon) - \lambda_B$ for small ε.

This problem is not only of intrinsic mathematical interest, also for $D \subseteq \mathbb{R}^n$, $n \geq 2$, but there are applications in a number of different areas. One is the buckling of plates and shells with a small hole. Both snapping (limit points) and buckling (bifurcation points) are known to occur in such structures, where it is usually simpler to calculate critical points for the structure without hole. Therefore, it is of interest to assess analytically, for small ε, the effect of the singular domain variation on the critical load. Another area of potential application is chemical reactor problems, in particular tubular reactors with both external and internal cooling.

In these applications, the domain D in question is usually two-dimensional.

We shall not address the problem in any generality in this paper. Rather, we shall show for specific classes of problems how singular perturbation techniques can be applied to yield asymptotic expressions for $\lambda_L(\varepsilon)$ and $\lambda_B(\varepsilon)$. We restrict ourselves to circular domains with a centrally located hole, that is, an annular domain, where $\varepsilon := a/b$ is small (a = inner radius, b= outer radius). In that case, rotationally symmetric solutions are of particular interest in the limit point case, but we may also have symmetry breaking bifurcation points, where nonsymmetric solutions branch from a symmetric solution at λ_B. When primary bifurcation from a trivial solution occurs, the bifurcation points are simply the eigenvalues of the linearized problem. Hence we shall study (1) the dependence of eigenvalues on ε for linear problems and (2) the variation of limit points with ε for nonlinear problems. In both cases we shall confine attention to certain ordinary differential equations of second order.

The linear eigenvalue problem with an ε-domain perturbation has previously been considered by Ozawa [1], who derived an asymptotic expression for the eigenvalues of the Laplacian for bounded two- and three-dimensional domains D_ε, when ε tends to zero. Assuming Dirichlet boundary conditions at the outer boundary and at the hole $|\underline{x} - \underline{x}_o| = \varepsilon$, he showed that for <u>simple</u> eigenvalues the relation

$$\lambda_j(\varepsilon) - \lambda_j = 2\pi(\log\varepsilon)^{-1}\varphi_j(\underline{x}_o)^2 + O((\log\varepsilon)^{-2}), \quad \varepsilon \to 0 \tag{1.1}$$

holds for a plane domain, where λ_j is the corresponding eigenvalue of the unperturbed domain D and φ_j is the normalized eigenfunction of the Laplacian associated with λ_j. For the circular domain, we obtain the above result by singular perturbation methods, which also gives us coefficients of higher order terms in an asymptotic expansion of $\lambda_B(\varepsilon)$. The above estimate holds for the radially symmetric eigenfunction. We show that for nonsymmetric eigenfunctions of the Laplacian for the circular domain

$$\lambda_j(\varepsilon) - \lambda_j = 2\pi\varepsilon^2 \varphi_j(\underline{x}_o)^2 + O(\varepsilon^4 \log\varepsilon), \qquad \varepsilon \to 0. \tag{1.2}$$

The method of Ozawa is quite different from ours, it is valid for general domains D, whereas our calculations are for circular domains only. In principle, the singular perturbation method should also be applicable to general domains, but we have not attempted to carry out such an analysis. On the other hand, we get the detailed behavior of the solution in the boundary layer near $|\underline{x} - \underline{x}_o| = \varepsilon$ by our method, in contrast to the method of Ozawa.

We are not aware of any previous treatment of the limit point problem. Considering a nonlinear problem of the form $\Delta u + \lambda f(u) = 0$, and assuming Dirichlet boundary conditions at both the inner and outer boundary of D_ε, we obtain the same type of asymptotic relation for simple (quadratic) limit points as given in (1.1), provided the solutions for D and D_ε are axially symmetric. For the special case $f = e^u$, which occurs in the modeling of thermal ignition problems, the first two terms of the asymptotic expansion for $\lambda_L(\varepsilon) - \lambda_L$ are worked out in detail. The results show that piercing a small hole can have a remarkably large influence on the location of the perturbed limit point.

The problems to be analyzed in this paper bear some resemblance to the following model problem in the asymptotic theory of incompressible flow at low Reynolds numbers, originally introduced by Kaplun and Lagerstrom in 1957. Let $y(r;\varepsilon)$ be defined by [2]

$$\frac{d^2y}{dr^2} + \frac{n-1}{r}\frac{dy}{dr} + y\frac{dy}{dr} = 0,\ y = 0 \text{ at } r = \varepsilon,\ y = 1 \text{ at } r = \infty \qquad (1.3)$$

The first two terms represent the Laplacian of an axisymmetric function in n dimensions, say the temperature, r being the radial variable (yy' may be considered as a heat loss). In the absence of the hole the equilibrium temperature is $y = 1$ everywhere. For $n = 2$, the introduction of a cylindrical cooling rod of radius ε constitutes a perturbation, which we expect to be small if $\varepsilon \ll 1$, except near the surface of the rod, as $y = 0$ at $r = \varepsilon$. Hence, the convergence of $y(r;\varepsilon)$ as ε tends to zero is nonuniform and we have a singular perturbation problem of the layer-type, where the boundary layer occurs at $r = \varepsilon$. An asymptotic solution of (1.3) was given in [2], it is described in more detail in [3], where it is also shown that the BVP (1.3) has a unique solution.

In contrast to this situation, we here treat two cases where the solution is not unique. In Section 2, we consider primary bifurcation from the trivial solution of a nonlinear BVP, that is, we consider the perturbation of the eigenvalues of the linearized problem. In Section 2.1, we begin with a simple example: a circular membrane, which admits an analytic solution showing explicitly the effect of a small hole on the eigenvalues for both axisymmetric and nonsymmetric eigenfunctions. The singular perturbation treatment of a more general case is contained in subsections 2.2 and 2.3, in analogy to axisymmetric and nonsymmetric eigenfunctions, respectively. Numerical calculations confirm the range of validity of the asymptotic solutions. In Section 3, the limit point problem is treated for a general class of nonlinear BVPs. The formulation of the problem is given in Section 3.1, asymptotic expansions of the solution are constructed at and near the limit point in Section 3.2. The results of Section 3 are applied in Section 4 to the example of a tubular chemical reactor involving exothermic reaction. The asymptotic expansion is worked out in more detail for this particular case and the results are compared with accurate numerical solutions that confirm the validity of our perturbation solutions.

2. Singular Perturbation of Bifurcation Points

Consider a nonlinear BVP on the unit disk, which admits axisymmetric solutions in some range of a real parameter λ. At certain critical values of λ^*, new branches of axisymmetric and/or nonsymmetric solutions may bifurcate from a 'primary' solution branch. A typical situation is

$$\begin{aligned} Lu + N(r,u;\lambda) &= 0 \qquad 0 \leq r < 1, \qquad 0 \leq \theta < 2\pi \\ u &= 0 \qquad \text{on} \quad r = 1 \end{aligned} \tag{2.1}$$

where L is a second order elliptic differential operator and N is a nonlinear operator, r being the radial variable. Examples where such BVPs occur are fluid flow problems, elastic plates and shells, chemical reactor problems, etc. We are interested in the effect of a small circular hole of radius ε on the behavior of the solutions of (2.1). Some problems of this type have been treated in [2] - [5], using singular perturbation techniques.

For axisymmetric solutions $u = U(r)$ of (2.1) one has $U'(0) = 0$. This situation is now perturbed by the deletion of a small disk of radius ε, centered at the origin. Imposing the condition $u(\varepsilon) = 0$, a condition which cannot be satisfied in the limit case $\varepsilon = 0$, the solution will exhibit boundary layer behavior near $x = \varepsilon$ in much the same way as in the Lagerstrom model problem (1.3); hence we have a singular perturbation problem.

Asymptotic solutions corresponding to regular points (λ, u) on a solution branch $u = u(r;\lambda)$ of (2.1), with $N(r,0,\lambda) \neq 0$ were constructed in [4] and [5] for some elastic membrane and shell problems. Here we are interested in the solution behavior at singular points (λ_o, u_o) such as bifurcation and limit points. In this section we consider the situation where $N(r,0,\lambda)=0$ in (2.1), so that $u \equiv 0$ is the primary solution branch. It is known that bifurcation points are given by the eigenvalues λ_n of the linearized problem of (2.1). In order to exhibit the asymptotic behavior of the eigenvalues $\lambda_n(\varepsilon)$ and the eigenfunctions $u_n(r;\varepsilon)$ of the perturbed problem for small ε, we first consider a simple example: the change in the characteristic frequencies of a circular drum due to a small hole at the center.

2.1 The Circular Membrane with a Small Hole

The BVP for the determination of the characteristic frequencies of an annular membrane fixed at the inner and outer edge is given by $\Delta u + \lambda^2 u = 0$, with $u = 0$ on the circles $r = \varepsilon$ and $r = 1$. The radially symmetric eigenfunctions must satisfy

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \lambda^2 u = 0 \qquad u(\varepsilon) = u(1) = 0 \tag{2.2}$$

They are given by $u_n(r;\varepsilon) = J_o(\lambda_n(\varepsilon)r) + c_n Y_o(\lambda_n(\varepsilon)r)$, where the standard notation for the Bessel functions J_k and Y_k has been used. The eigenvalues $\lambda_n(\varepsilon)$, $n=1,2,\ldots$, are determined from the boundary conditions in (2.2), yielding the transcendental equation

$$J_o(\lambda\varepsilon)Y_o(\lambda) - J_o(\lambda)Y_o(\lambda\varepsilon) = 0 \tag{2.3}$$

For $\varepsilon > 0$, all zeros of (2.3) are real and simple, asymptotic formulas for large ε are known [6]. Here we obtain asymptotic expressions for small positive ε. Let λ_n be the n^{th} eigenvalue for the circular membrane, given by $J_o(\lambda_n) = 0$, that is, $\lambda_n = j_{o,n}$ [6]. Setting $\lambda = \lambda_n + \lambda'$ in (2.3), assuming $\lambda' \ll \lambda_n$, and expanding (2.3) for small ε, we obtain

$$(1 - \frac{1}{4}\varepsilon^2(\lambda_n + \lambda')^2 + \dots)(Y_o(\lambda_n) + \lambda' Y_o'(\lambda_n) + \frac{1}{2}(\lambda')^2 Y_o''(\lambda_n) + \dots)$$
$$- \frac{2}{\pi}\{[\ell n(\lambda_n + \lambda')\varepsilon + \gamma_o](1 - \frac{1}{4}(\lambda_n + \lambda')^2\varepsilon^2 + \dots) + \frac{1}{4}(\lambda_n + \lambda')^2\varepsilon^2 - \dots\}$$
$$\cdot [-\lambda' J_1(\lambda_n) + \frac{1}{2\lambda_n}(\lambda')^2 J_1(\lambda_n) + \dots] = 0 \qquad (2.4)$$

where $\gamma_o = \gamma - \ell n2$ and γ is the Euler constant 0.577216. Collecting the leading terms we find

$$Y_o(\lambda_n) - \lambda' Y_1(\lambda_n) + \frac{2}{\pi}\lambda' J_1(\lambda_n)(\ell n\, \lambda_n + \gamma_o) + \frac{2\lambda'}{\pi}(\ell n\varepsilon) J_1(\lambda_n) + \dots = 0 \qquad (2.5)$$

In order to balance the term $Y_o(\lambda_n)$, we must have $\lambda' \ell n\varepsilon = 0(1)$, which yields

$$\lambda' = \frac{\pi}{2}\frac{Y_o(\lambda_n)}{J_1(\lambda_n)}\frac{1}{\ell n(1/\varepsilon)} \qquad (2.6)$$

The next term is obtained by carrying the expansion (2.4) one step further, replacing $\lambda_n + \lambda'$ by $\lambda_n + \lambda' + \lambda''$. Inspection of the terms following the leading terms (2.5) shows that $\lambda'' \ell n^2\varepsilon =: \alpha_2 = 0(1)$, and a straightforward calculation yields $\alpha_2 = \alpha_{2,n}$ as given below. At this point it is apparent that the general form of the expansion for $\lambda_n(\varepsilon)$ should be

$$\lambda_n(\varepsilon) \sim \lambda_n + \sum_{j=1}^{\infty} \alpha_{j,n}\, \delta(\varepsilon)^j \qquad \text{as} \quad \varepsilon \to 0+ \qquad (2.7)$$

where

$$\delta(\varepsilon) = \frac{1}{\ln(1/\varepsilon)} \qquad (2.8)$$

$$\alpha_{1,n} = \frac{\pi Y_o(\lambda_n)}{2 J_1(\lambda_n)}, \quad \alpha_{2,n} = \frac{\pi^2 Y_o(\lambda_n)}{4 J_1^2(\lambda_n)}[\frac{2}{\pi} J_1(\lambda_n)(\ln\lambda_n + \gamma_o) - Y_1(\lambda_n) + \frac{Y_o(\lambda_n)}{2\lambda_n}] \qquad (2.9)$$

Note that the terms $O(\varepsilon^2)$ in (2.4) are transcendentally small compared to $\delta(\varepsilon)$ and therefore do not contribute to the expansion (2.7). Further coefficients $\alpha_{k,n}$ $k > 2$ can be determined by substituting (2.7) into (2.3) and expanding as indicated above.

Substituting the $\alpha_{k,n}$ into the expansions for $J_o(\lambda)$ and $Y_o(\lambda)$, we also get the coefficient $c_n = -J_o(\lambda)/Y_o(\lambda)$ in the eigenfunctions $u_n(r;\varepsilon) = J_o(\lambda_n(\varepsilon)r) + c_n(\varepsilon)Y_o(\lambda_n(\varepsilon)r)$. To the order computed in (2.9), we find

$$c_n(\varepsilon) = \frac{\pi}{2}\,\delta(\varepsilon) + \frac{\pi}{2}(\ln\lambda_n + \gamma_o)\delta^2(\varepsilon) + 0(\delta^3(\varepsilon)) \quad \text{as } \varepsilon\to 0+ \tag{2.10}$$

Expanding $J_o(\lambda_n(\varepsilon)r)$ and $Y_o(\lambda_n(\varepsilon)r)$ in the same way as $J_o(\lambda_n(\varepsilon))$ and $Y_o(\lambda_n(\varepsilon))$, respectively, and using (2.11), we obtain the following approximation for the eigenfunction $u_n(r;\varepsilon)$

$$u_n(r;\varepsilon)=J_o(\lambda_n r)+\delta(\varepsilon)[-\alpha_{1,n}rJ_1(\lambda_n r)+\frac{\pi}{2}Y_o(\lambda_n r)]+\delta(\varepsilon)^2[-\alpha_{2,n}rJ_1(\lambda_n r)$$
$$+\frac{1}{2}\alpha_{1,n}^2\left(\frac{r}{\lambda_n}J_1(\lambda_n r) - r^2J_o(\lambda_n r)\right)+\frac{\pi}{2}(\gamma_o+\ln\lambda_n)Y_o(\lambda_n r)-\frac{\pi}{2}\alpha_{1,n}rY_1(\lambda_n r)]+0(\delta^3) \tag{2.11}$$

Clearly, we have

$$\lim_{\substack{\varepsilon\to 0\\ r\text{ fixed}}} u_n(r;\varepsilon) = J_o(\lambda_n r)$$

for each $0<r\le 1$. In fact, on any fixed interval $r_o\le r\le 1$, $r_o>0$, $u_n(r;\varepsilon)$ converges uniformly to the unperturbed solution $J_o(\lambda_n r)$, where $J_o(0)=1$. The expression in (2.11) can be viewed as an outer expansion for $u_n(r;\varepsilon)$. An inner approximation valid for r near ε can be obtained directly by appropriately rearranging terms in (2.11), that is, the outer expansion here contains the inner expansion. Setting $s=r/\varepsilon$ and carrying out a limit process for $\varepsilon\to 0$, with s fixed, (2.11) yields the leading order inner approximation

$$u_n(r;\varepsilon) = \frac{\ln s}{\ln 1/\varepsilon} + 0((\ln 1/\varepsilon)^{-2}), \qquad \text{as } \varepsilon\to 0+ \tag{2.12}$$

For the lowest eigenvalue $\lambda_1 = 2.404826$ we find

$$\alpha_{1,1} = 1.542890, \qquad \alpha_{2,1} = 1.190264$$
$$c_1(\varepsilon) = \frac{\pi}{2}\,[\,\delta(\varepsilon) + 1.965402\,\delta(\varepsilon)^2 + 0(\delta(\varepsilon)^3)] \qquad \varepsilon \to 0 \tag{2.13}$$

At first sight, it may seem surprising that piercing a small hole and fixing the membrane at the edge of the hole should change the lowest frequency of the drum 'only' by an amount of order $1/\ln(1/\varepsilon)$; for the lowest characteristic mode of vibration is maximal at r=0 for the circular membrane, while that mode for the annular membrane has the circle $r=\varepsilon$ as a nodal curve. However, due to the boundary layer character of $u_n(r;\varepsilon)$, the modes of vibration of the pierced drum deviate sharply from those of the circular drum only in a small area adjacent to the hole. Thus the energy balance, which essentially determines the frequencies, is but slightly perturbed. Quantitively, the eigenvalues are remarkably sensitive to the ε-perturbation because of the relatively slow convergence of $\delta(\varepsilon)$ to zero as ε tend to zero, e.g., for $\varepsilon=0.01$ we have $\lambda_1(\varepsilon) - \lambda_1 \doteq 0.5912$, which is a 25% increase in λ_1 (more numerical results are given at the end of this section).

Next we consider the nonsymmetric eigenfunctions

$$v_n(r,\theta) = y(r)(A_n\cos n\theta + B_n\sin n\theta), \qquad n=1,\ 2,\ \ldots$$

where y(r) is a solution of the BVP

$$y'' + \frac{1}{r}y' + (\lambda^2 - \frac{n^2}{r^2})y = 0, \qquad y(\varepsilon) = y(1) = 0 \tag{2.14}$$

The eigenfunctions are given by $y_{n,m}=J_n(\lambda r)+c_{m,n}Y_n(\lambda r)$, where the eigenvalues $\lambda=\lambda_{m,n}(\varepsilon)$ are determined by the zeros of the equation

$$J_n(\lambda\varepsilon)\,Y_n(\lambda) - J_n(\lambda)\,Y_n(\lambda\varepsilon) = 0 \tag{2.15}$$

The nonsymmetric eigenvalues $\lambda_{m,n}$ for the circular membrane are given by

$J_n(\lambda_{m,n})=0$, $m=1,2,\ldots$ As in the symmetric case, setting $\lambda=\lambda_{n,m}+\lambda'$ in (2.15) and using the expansions for J_n and Y_n yields a first correction to $\lambda_{m,n}$ for small ε. For $n=1$, the leading term of $Y_n(\varepsilon\lambda)$ is $-4/(\varepsilon\lambda_{m,1}+\varepsilon\lambda')\pi$, while $J_n(\lambda\varepsilon)$ begins with the term $(\lambda_{m,1}+\lambda')\varepsilon/2$. Thus we obtain the dominant balance

$$\frac{1}{2}\lambda_{m,1}\varepsilon Y_1(\lambda_{m,1})+\frac{4\lambda'}{\varepsilon\pi\lambda_{m,1}}J_1'(\lambda_{m,1})=0,$$

and therefore

$$\lambda'=-\frac{\pi}{8}\lambda_{m,1}^2\frac{Y_1(\lambda_{m,1})}{J_1'(\lambda_{m,1})}\varepsilon^2$$

The $\ln\varepsilon$ -terms of $Y_1(\lambda\varepsilon)$ appear in the next correction, which is of order $\varepsilon^4\ln\varepsilon$. Hence the asymptotic expansion for $\lambda_{m,1}(\varepsilon)$ is of the form

$$\lambda_{m,1}(\varepsilon)=\lambda_{m,1}+\alpha_{1,m,1}\varepsilon^2+\alpha_{2,m,1}\varepsilon^4\ln\varepsilon+\alpha_{3,m,1}\varepsilon^4+\ldots \text{ as } \varepsilon\to 0 \tag{2.16}$$

where

$$\alpha_{1,m,1}=-\frac{\pi}{8}\lambda_{m,1}^2\frac{Y_1(\lambda_{m,1})}{J_1'(\lambda_{m,1})},\qquad \alpha_{2,m,1}=\frac{1}{2}\lambda_{m,1}^2\alpha_{1,m,1}. \tag{2.17}$$

The constant $c_{m,n}=-J_n(\lambda_{m,n}(\varepsilon))/Y_n(\lambda_{m,n}(\varepsilon))$ in the eigenfunction $y_{n,m}(r)$ is obtained in the same way as before. For $n=1$ we find

$$c_{m,1}=\frac{\pi}{8}\lambda_{m,1}^2\varepsilon^2+\frac{\pi}{16}\lambda_{m,1}^4\varepsilon^4\ln\varepsilon+O(\varepsilon^4)\qquad \text{as } \varepsilon\to 0 \tag{2.18}$$

It can be seen from the series expansions for J_n and Y_n that for $n\geqslant 1$ the asymptotic expansion (2.16) generalizes to

$$\lambda_{m,n}(\varepsilon)-\lambda_{m,n}=\alpha_{1,m,n}\varepsilon^{2n}+\alpha_{2,m,n}\varepsilon^{2n+2}\ln\varepsilon+O(\varepsilon^{2n+2}),\quad \varepsilon\to 0 \tag{2.19}$$

We conclude from (2.16) and (2.19) that the effect of the ε-perturbation on the eigenvalues is much less significant than in the symmetric case. This result is a consequence of $J_n(\lambda_{m,n}r)=0$ for r=0, $n \geq 1$, which is also responsible for a weaker boundary layer effect. The radial part $y_{n,m}(r)$ of the nonsymmetric eigenfunctions simply has a corner layer near $r=\varepsilon$.

Summarizing the results for the annular membrane, we have the asymptotic formula (2.7) for simple eigenvalues, in agreement with the estimate (1.1) of Ozawa, but we have the considerably sharper asymptotic estimate (2.19), $n \geq 1$, for double eigenvalues.

2.2 Perturbation of Eigenvalues for a more General Case

In this section we first consider a class of eigenvalue problems generalizing the symmetric eigenfunction problem (2.2) defined by

$$\begin{gathered} L_s u := u'' + \frac{1}{x} a(x)\, u' + \lambda^2 \rho(x)\, u = 0, \qquad \varepsilon < x < 1 \\ u(\varepsilon) = 0, \quad Bu(1) = 0, \quad Bu := \beta_0 u + \beta_1 u' \end{gathered} \tag{2.20}$$

assuming a(0)=1, a(x) and $\rho(x)$ being analytic functions for $|x| \leq 1$. We construct an approximate asymptotic solution to (2.20), using the previously established results for the special case (2.2) as a guide. In the absence of the ε-perurbation, we have the eigenvalue problem

$$L_s U = 0, \qquad 0 < x < 1, \qquad U'(0) = 0, \qquad BU(1) = 0 \tag{2.21}$$

The basic assumption is that this problem has a discrete set of eigenvalues λ_k with associated eigenfunctions $U_k(x)$, k=1, 2,..., with the normalization $U_k(0)=1$. Denote the eigenvalues and eigenfunctions of the perturbed problem (2.20) by $\lambda_k(\varepsilon)$ and $u_k(x;\varepsilon)$, respectively. For small ε we expect the deviations $\lambda_k(\varepsilon) - \lambda_k$ and $u_k(x;\varepsilon) - U_k(x)$ to be small, except for x near ε. It is convenient to approximate $u_k(x;\varepsilon)$ by an outer expansion which satisfies the boundary condition at x=1 and matches with an inner expansion valid near $x=\varepsilon$.

We assume that the outer expansion takes the form

$$u_k(x;\varepsilon) \sim U_k(x) + \sum_{n=1}^{\infty} \mu_n(\varepsilon) h_{n,k}(x), \quad \lambda_k(\varepsilon) \sim \lambda_k + \sum_{n=1}^{\infty} \delta_n(\varepsilon)\alpha_{n,k}, \ \varepsilon \to 0 \quad (2.22)$$

with $0 < x \leq 1$ fixed, where the asymptotic sequences δ_n and μ_n are to be determined. In what follows, the line of argument is very similar to that in the analysis of (1.3) (see [3], pp. 90-93), except that we have the additional expansion for the eigenvalue $\lambda_k(\varepsilon)$. Substituting (2.22) into (2.20) and requiring that the equations for the $h_{n,k}(x)$ have forcing terms, as in [3], we obtain in succession $\delta_1 = \mu_1$, $\delta_2 = \mu_2 = \mu_1^2$, etc. Dropping the subscripts k for simplicity, the equations to be satisfied by $h_1(x)$ and $h_2(x)$ are

$$L_s h_1 = -2\alpha_1 \lambda U, \qquad Bh_1(1) = 0, \qquad (2.23)$$

$$L_s h_2 = -2\alpha_2 \lambda U - (\alpha_1^2 U + 2\alpha\lambda h_1), \qquad Bh_2(1) = 0. \qquad (2.24)$$

Using a(0)=1, the general solution of $L_s h=0$ can be written in the form

$$h(x) = c_1 U(x) + c_2 V(x), \qquad V(x) = (\ln x)W(x) + Z(x),$$

U, W and Z being analytic functions for $|x| \leq 1$. For $\lambda = \lambda_k$ we can take $U=U_k$, then the solution of (2.23) can be written as

$$h_1(x) = c_1 U(x) + c_2 V(x) - \alpha_1 H(x), \quad U(x)=U_k(x), \quad c_2 BV(1) - \alpha_1 BH(1) = 0 \qquad (2.25)$$

where H(x) is a particular (regular) solution of $L_s H = 2\lambda U$. Once c_2 is determined, the coefficient $\alpha_{1,k}$ in (2.22) is determined by

$$\alpha_{1,k} = \alpha_1 = c_2 \frac{BV(1)}{BH(1)} \qquad (2.26)$$

Similarly we find the solution h_2 of (2.24)

$$h_2(x) = c_3U(x) + c_4V(x) - (\alpha_2 + \frac{\alpha_1^2}{\lambda})H(x) - \alpha_1 G(x) \tag{2.27}$$

where G(x) is a particular solution of $L_s G=2\lambda h_1$. The coefficient $\alpha_{2,k}$ is obtained from the boundary conditions for h_2,

$$\alpha_{2,k} = \alpha_2 = -\frac{1}{2\lambda}\alpha_1^2 + \frac{1}{BH(1)}[c_4 BV(1) - \alpha_1 BG(1)] \tag{2.28}$$

Next an inner expansion is constructed, in order to satisfy the boundary condition at $x=\varepsilon$. Introducing the inner (stretched) variable $s=x/\varepsilon$, the differential equation for $u(x) = \bar{u}(s)$ is

$$\ddot{\bar{u}} + \frac{1}{s}a(\varepsilon s)\dot{\bar{u}} + \varepsilon^2\lambda^2(\varepsilon)\varrho(\varepsilon s)\bar{u} = 0 \qquad \dot{\bar{u}} := \frac{d\bar{u}}{ds} \tag{2.29}$$

Assuming an inner expansion for $u_k(x;\varepsilon)$ in the form

$$u_k(x;\varepsilon) \sim \sum_{n=1}^{\infty} \nu_n(\varepsilon)\, g_{n,k}(s) \qquad \text{as } \varepsilon \to 0 \tag{2.30}$$

one finds for the leading term, again omitting subscripts k,

$$\ddot{g}_1 + \frac{1}{s}\dot{g}_1 = 0\,, \qquad g_1(1) = 0$$

and therefore

$$g_1(s) = d_1 \ln s\,.$$

The constant d_1 and $\nu_1(\varepsilon)$ must be determined by matching the inner with the outer solution. Obviously the term $\nu_1(\varepsilon)d_1(\ln x - \ln\varepsilon)$ should match the terms $U_k(x)$ and $\mu_1 c_2 W(x)\ln x$ of the outer solution for $x\to 0$. Hence we have $-\nu_1 d_1 \ln\varepsilon = U_k(0)=1$ and $\nu_1 d_1 = \mu_1 c_2 W(0)$. Together with (2.26) we obtain the results

$$\nu_1(\varepsilon) = -\frac{1}{\ln\varepsilon} = \delta(\varepsilon), \qquad d_1 = 1, \quad \mu_1 = \nu_1 = \delta(\varepsilon), \qquad c_2 = W(0)^{-1}$$
$$\alpha_1 = (W(0)BH(1))^{-1}[\beta_o Z(1) + \beta_1(W(1) + Z'(1))] \tag{2.31}$$

where $\delta(\varepsilon)$ and d_1 are the same as in (2.8) and (2.13), respectively.

It is clear from (2.29) that the forcing term in the differential equation for $g_2(s)$ will be of the order $\varepsilon\nu_1/\nu_2$ (or $\varepsilon^2\nu_1/\nu_2$ if $a(x)=a(-x)$), hence the constant terms in the Taylor expansions of U, W, and Z in $h_1(x)$ give rise to a term $C\mu_1$ for which there is no corresponding term in the inner expansion. The situation is exactly as in problem (1.3), that is, we must insert a term $g^*(s)\,\delta(\varepsilon)^2$ in the inner expansion , with $g^*=d^*\ln s$. Then $\delta^2 d^*(\ln x - \ln\varepsilon)$ serves to match the term $C\mu_1$ of the outer expansion. Thus the inner expansion should be sought in the form

$$u_k(x;\varepsilon) \sim (\ln s)\sum_{n=1}^{\infty} d_n\,\delta(\varepsilon)^n \qquad \text{as} \quad \varepsilon\to 0 \tag{2.32}$$

where d_n are constants, $d_1=1$. In effect, the inner expansion satisfies (2.29) with ε set equal to zero. From $d^*=d_2=C$ we find

$$d_2 = c_1 + \frac{Z(0)}{W(0)} - \alpha_1 H(0) \tag{2.33}$$

with α_1 given by (2.31). The constant c_1 is determined by a normalization of the eigenfunction $u_k(x;\varepsilon)$, it can be set to zero.

The matching procedure can evidently be continued. The term $c_4W(0)\ln x$ in (2.27) should match the term $d_2\ln x$ in (2.32), and $\delta^3 d_3(-\ln\varepsilon)$ should match $\mu_2 h_2(0) = \delta^2 h_2(0)$. Thus we get, with $c_3=0$,

$$c_4 = \frac{d_2}{W(0)}, \qquad d_3 = d_2 \frac{Z(0)}{W(0)} - (\alpha_2 + \frac{1}{2\lambda}\alpha_1^2)H(0) - \alpha_1 G(0) \tag{2.34}$$

The first equation, together with (2.32) and $c_1=0$ yields the final formula for the coefficient $\alpha_{2,k}$

$$\alpha_{2,k} = \alpha_2 = -\frac{1}{2\lambda}\alpha_1^2 + \frac{1}{BH(1)}\left[\frac{BV(1)}{W(0)}\left(\frac{Z(0)}{W(0)} - \alpha_1 H(0)\right) - \alpha_1 BG(1)\right] \tag{2.35}$$

At this point the first three terms of the outer and inner expansions (2.22) and (2.32) have been completely determined. It is obvious how to continue the process. We find that a three term asymptotic approximation is appropriate - unless ε is extremely small - because of the slow convergence of $\delta(\varepsilon)$ to zero as ε tends to zero. This observation is confirmed by the numerical results in Table 1.

As an example we retrieve the solution given in Section 2.1 from our general formulas. If $a(x) = \rho(x) = 1$, $\beta_o=1$ and $\beta_1=0$, we have $U(x) = J_o(\lambda_k x)$ and $V(x)=Y_o(\lambda_k x)$. It can be verified by direct substitution into (2.23) and (2.24) that the functions H and G are, with $\lambda = \lambda_k$,

$$H(x) = xJ_1(\lambda x), \qquad G(x) = \frac{\pi}{2} xY_1(\lambda x) + \frac{\alpha_1}{2}[x^2 J_o(\lambda x) - \frac{2}{\lambda} xJ_1(\lambda x)]$$

From the expansion of Y_o we find $W(0)=2/\pi$ and

$$Z(x) = \frac{2}{\pi}(\ln\lambda + \gamma_o)J_o(\lambda x) + O(x^2) \qquad x \to 0$$

Substituting these expressions into (2.26) and (2.28), the values for α_1 and α_2 given in (2.9) are obtained. Furthermore, we find $d_2 = \ln\lambda + \gamma_o =: \gamma_1$, $c_4 = \pi\gamma_1/2$ and $d_3 = \gamma_1^2 + Y_o(\lambda)/J_1(\lambda)$.

2.3 Perturbation of eigenvalues corresponding to the nonsymmetric case

A class of eigenvalue problems generalizing the case of nonsymmetric modes of vibration of an annular membrane is given by

$$L_a u := u'' + \frac{1}{x} a(x)\, u' - \frac{1}{x^2} b(x)\, u + \lambda^2 \rho(x)\, u = 0, \quad \varepsilon < x < 1$$
$$u(\varepsilon) = 0 = Bu(1), \quad a(x) = 1 + \sum_{j=1}^{\infty} a_j x^j, \quad b(x) = b_o + \sum_{j=1}^{\infty} b_j x^j, \quad \rho(x) = \rho_o + \sum_{j=1}^{\infty} \rho_j x^j, \qquad (2.36)$$

where Bu is the same boundary operator as defined in (2.20). From the results of Section 2.1 we expect the form of the asymptotic solution to depend on the value of b(0). We consider here only the case b(0)=1, which corresponds to n=1 in (2.12). The modifications to be made for $n \neq 1$, $b(0) \neq 0$ will be evident from the procedure given in what follows. Without the ε-perturbation we have the eigenvalue problem

$$L_a U = 0, \qquad 0 < x < 1, \qquad U'(0) = BU(1) = 0 \qquad (2.37)$$

The outer expansion for the solution to problem (2.36) is formally the same as (2.22), resulting in equations for h_1 and h_2 given by (2.23) and (2.24), respectively.

The general solution of $L_a h = 0$ is of the form

$$h(x) = c_1 U(x) + c_2 V(x), \quad U(x) = xZ(x), \quad V(x) = AxZ(x)\ln x + \frac{1}{x} W(x) \qquad (2.38)$$

as the roots of the indicial equation are +1 and -1. The functions Z(x) and W(x) are analytic for $|x| \leq 1$, A is a constant. For $\lambda = \lambda_k$, U is the eigenfunction $U_k(x)$. With these definitions of U and V, (2.25) and (2.26) remain valid, provided H(x) is a particular solution of $L_a H = 2\lambda U$.

The inner expansion is again taken in the general form (2.30), yielding the following equations for $g_1(s)$ and $g_2(s)$

$$\ddot{g}_1 + \frac{1}{s}\dot{g}_1 - \frac{1}{s^2} g_1 = 0 \qquad g_1(1) = 0 \tag{2.39}$$

$$\ddot{g}_2 + \frac{1}{s}\dot{g}_2 - \frac{1}{s^2} g_2 = \begin{cases} -a_1\dot{g}_1 + b_1 g_1 s^{-1} & \text{if } a_1^2 + b_1^2 \neq 0, \ \nu_2 = \varepsilon\nu_1 \\ -a_2 s\dot{g}_1 + (b_2 - \lambda^2\rho_o)g_1 & \text{if } a_1^2 + b_1^2 = 0 \ \ \nu_2 = \varepsilon^2\nu_1 \end{cases} \tag{2.40}$$

The solution of (2.39) is

$$g_1(s) = d_1(s - \frac{1}{s})$$

Although both cases of (2.40) can be analyzed by the present method, we shall restrict the discussion to the somewhat simpler case $a_1 = b_1 = 0$ (example: a and b even functions of x). The solution for g_2 is then given by

$$g_2(s) = d_2(s - \frac{1}{s}) + d_1A_1 s\ln s + d_1A_2 s^3, \quad 2A_1 := \lambda^2\rho_o - a_2 - b_2, \quad 10A_2 := b_2 - a_2 - \lambda^2\rho_o \tag{2.41}$$

To this point we have for $\nu_1 g_1 + \nu_2 g_2$, in terms of the variable x,

$$\nu_1 d_1(\frac{x}{\varepsilon} - \frac{\varepsilon}{x}) + \varepsilon^2\nu_1\left\{d_2(\frac{x}{\varepsilon} - \frac{\varepsilon}{x}) + d_1A_1\frac{x}{\varepsilon}(\ln x - \ln\varepsilon) + d_1A_2(\frac{x}{\varepsilon})^3\right\} \tag{2.42}$$

which must be matched with the outer solution $xZ + \mu_1 h_1(x)$ + higher order terms. From the first term in (2.42) it follows that $\nu_1 = \varepsilon$ and $d_1 = Z(0)$. The next term, $-\nu_1 d_1\varepsilon/x$, can be matched only by the $O(1/x)$ term in h_1, which is $\mu_1 c_2 W(0)/x$.

Hence, $\mu_1 = \varepsilon^2$ and $d_1 = -c_2 W(0)$, so that c_2 is determined, which in turn yields α_1 via (2.27). To sum up, we have

$$\nu_1 = \varepsilon, \quad \nu_2 = \varepsilon^3, \quad \mu_1 = \delta_1 = \varepsilon^2, \quad d_1 = Z(0)$$

$$c_2 = -\frac{Z(0)}{W(0)}, \qquad \alpha_1 = -\frac{Z(0)}{W(0)}\frac{BV(1)}{BH(1)} \tag{2.43}$$

As the term $-d_1 A_1 x\,\varepsilon^2 \ln \varepsilon$ in (2.42) has no counterpart in the outer solution, we must insert a (switchback) term $(\varepsilon^3 \ln \varepsilon) g^*(s)$ in the inner expansion, where $g^* = d^* g_1/d_1$, with $d^* = d_1 A_1$. Writing this term in the form

$$(\varepsilon^3 \ln \varepsilon)\, g^*(s) = d_1 A_1 (x\, \varepsilon^2 \ln \varepsilon - \frac{1}{x}\, \varepsilon^4 \ln \varepsilon)$$

shows that the outer solution should also contain a term $(\varepsilon^4 \ln \varepsilon)/x$, implying that $\mu_2 = \delta_2 = \varepsilon^4 \ln \varepsilon$, in agreement with the special case of section 2.1. The term $d_1 A_1\, \varepsilon^2 x \ln x$ in (2.42) should match with the term $\mu_1 c_2 A x \ln x$ in the outer solution. Since c_2 and d_1 have already been determined by (2.43), c_2 and d_1 should satisfy the equation $d_1 A_1 = c_2 A$, that is,

$$-2A = W(0)(\lambda^2 \rho_o - a_2 - b_2) \tag{2.44}$$

Similarly, all other terms in the inner and outer expansions can be matched. The truth of (2.44) can be verified from the series expansions of W(x) and Z(x).

The above reasoning indicates how to proceed to obtain higher order terms. For the perturbed eigenvalues (in the case $a_1 = b_1 = 0$) we find

$$\lambda_k(\varepsilon) = \lambda_k + \alpha_{1,k}\, \varepsilon^2 + \alpha_{2,k}\, \varepsilon^4 \ln \varepsilon + O(\varepsilon^4) \qquad \text{as } \varepsilon \to 0$$

with $\alpha_{1,k}$ given in (2.43) and

$$\alpha_{2,k} = - \frac{Z(0)\ BV(1)}{W(0)\ BH(1)} A_1 = \frac{1}{2} \alpha_{1,k}(\lambda_k^2 \rho_o - a_2 - b_2) \tag{2.45}$$

The inner and outer expansions of the perturbed eigenfunctions are of the form

$$u_k(x, \varepsilon) = \begin{cases} \varepsilon g_{1,k}(s) + \varepsilon^3 \ln \varepsilon\ g_k^*(s) + \varepsilon^3 g_{2,k}(s) + \ldots & \text{inner} \\ U_k(x) + \varepsilon^2 h_{1,k}(x) + \varepsilon^4 \ln\varepsilon\ h_k^*(x) + \varepsilon^4 h_{2,k}(x) + \ldots & \text{outer} \end{cases} \tag{2.46}$$

It is a simple exercise to apply the above results to the membrane problem (2.12), and to derive the formulas (2.16) from (2.43) and (2.45).

The asymptotic solutions of problem (2.2) are now compared with analytical solutions for the case of the annular membrane, where the eigenvalues $\lambda_n(\varepsilon)$ were computed numerically by solving equation (2.3). In Table 1, we compare numerical results with approximate values of $\lambda_n(\varepsilon)$, n=1, 2, for both two and three terms on the right hand side of formula (2.7). The accuracy clearly increases with decreasing values of ε.

ε	numerical solution for $\lambda_1(\varepsilon)$	2 - term asymptotic approximation	3 - term asymptotic approximation	numerical solution for $\lambda_2(\varepsilon)$	2 - term asymptotic approximat.	3 - term asymptotic approximat.
0.1	3.3139	3.0749	3.2994	6.8576	6.1996	6.6699
0.05	3.0644	2.9198	3.0525	6.4254	6.0424	6.3202
0.01	2.8009	2.7399	2.7960	6.0109	5.8598	5.9774
0.005	2.7419	2.6960	2.7384	5.9265	5.8154	5.9042
0.001	2.6548	2.6282	2.6531	5.8090	5.7466	5.7988
0.0001	2.5871	2.5723	2.5864	5.7236	5.6900	5.7193

Table 1, Comparison of numerical and asymptotic approximations to the two lowest eigenvalues $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$ of problem (2.2)

The two term asymptotic solutions for the eigenfunctions $u_1(x;\varepsilon)$ and $u_2(x;\varepsilon)$ are compared with accurate numerical solutions in figures 1 and 2, both for ε=0.01. In view of the relatively large value of the expansion parameter $\delta(0.01)$=0.2171, the agreement for ε=0.01 is remarkably good. A three term asymptotic solution is indistinguishable from the numerical solution within plotting accuracy.

3. Singular perturbations of limit points

The results of the preceding section amply demonstrate the sensitivity of bifurcation points to small changes in the domain for an important class of BVP's. Our purpose in this section is to carry out a parallel study of the sensitivity of limit points for certain related nonlinear model problems involving the Laplacian and cylindrically symmetric solutions. We begin by describing the basic equations and the conditions which define a limit point.

3.1 The basic equations

The class of problems that we wish to consider can be viewed as originating from the two-dimensional BVP in the unit disk

$$\begin{aligned} &\Delta U + \lambda f(r,U) = 0 , \quad 0 \leq r < 1 , \; 0 \leq \theta < 2\pi \\ &\quad U = 0 \quad \text{on} \quad r = 1 , \end{aligned} \tag{3.1}$$

with r being the radial variable, λ a parameter and f a smooth nonlinear function of r and U. Cylindrically symmetric solutions $U(r,\lambda)$ of (3.1) satisfy what we shall refer to as

$$\text{BVP(I)} \quad \begin{cases} U'' + \dfrac{U'}{r} + \lambda f(r,U) = 0 , & 0 < r < 1 , \\ U'(0,\lambda) = 0 = U(1,\lambda) , & \end{cases} \tag{3.2}$$

where primes denote differentiation with respect to r.

Associated with BVP(I) is the linear "variational" equation

$$V'' + \frac{V'}{r} + \lambda f_U(U,\lambda)V = - f(U,\lambda) \ , \qquad 0 < r < 1 \ . \tag{3.3}$$

for

$$V(r,\lambda) = \frac{\partial U}{\partial \lambda}(r,\lambda), \tag{3.4}$$

obtained by differentiating BVP(I) with respect to λ. It is well-known (e.g. see [7,8]) that a necessary condition for a solution branch of BVP(I) to have a simple (quadratic) limit point at , say, $\lambda = \lambda_o$ is that the homogeneous version of the variational equation has a nontrivial solution satisfying the boundary condition in (3.5). We assume that this is the case. When $\lambda = \lambda_o$, $U_o(r)=U(r,\lambda_o)$ is a solution of BVP(I) and $V_o(r)$ is a nontrivial (smooth) solution of

$$\text{Var(I)} \begin{cases} \mathcal{L}V_o \equiv V_o'' + \dfrac{V_o'}{r} + \lambda_o f_{U_o}(r,U_o)V_o = 0, \qquad 0 < r < 1 \\ V_o'(0) = 0 = V_o(1) \ . \end{cases} \tag{3.5}$$

Without loss of generality we may assume that $V_o(0) = 1$.

Later we shall make use of the fact that the homogeneous differential equation

$$\mathcal{L}V = 0 \tag{3.6}$$

has a second, linearly independent, solution

$$W_o(r) \equiv V_o(r) \int \frac{dr}{rV_o^2(r)} \ , \tag{3.7}$$

which, in view of the smoothness of $V_o(r)$, satisfies the asymptotic relation

$$W_o(r) = \ln r + O(1) \quad \text{as} \quad r \to 0+. \tag{3.8}$$

The modified problem consists of deleting a small circular hole of radius ε from the center of the unit disk and extending the Dirichlet condition to the boundary curve $r = \varepsilon$. Let $u(r,\lambda;\varepsilon)$ denote the cylindrically symmetric solution of this modified problem. Then, u satisfies

$$\text{BVP(II)} \begin{cases} u'' + \dfrac{u'}{r} + \lambda f(r,u) = 0 , & \varepsilon < r < 1 , \\ u(\varepsilon,\lambda;\varepsilon) = 0 = u(1,\lambda;\varepsilon) . \end{cases} \tag{3.9}$$

We want to study the behavior of the solution branches $u(r,\lambda;\varepsilon)$ as $\varepsilon \to 0$.

Let $U(r,\lambda)$ denote a solution branch of BVP(I) with limit point at $\lambda = \lambda_o$, $U = U_o$. The results of Section 2 suggest that under suitable conditions BVP(II) will have a nearby solution branch $u(r,\lambda;\varepsilon)$ satisfying

$$\lim_{\substack{\varepsilon\to 0 \\ r \text{ fixed}}} u(r,\lambda;\varepsilon) = U(r,\lambda) , \tag{3.10}$$

for each $0 < r \leq 1$. Of course we don't expect the convergence in (3.10) to be uniform near $r = 0$. Moreover, for sufficiently small $\varepsilon > 0$, we expect this solution branch $u(r,\lambda;\varepsilon)$ to have a family of limit points

$$\lambda = \lambda_o(\varepsilon) , \quad u_o(r;\varepsilon) \equiv u(r,\lambda_o(\varepsilon);\varepsilon) , \tag{3.11}$$

satisfying

$$\lim_{\varepsilon\to 0} \lambda_o(\varepsilon) = \lambda_o , \quad \lim_{\varepsilon\to 0} u_o(r;\varepsilon) = U_o(r) , \tag{3.12}$$

for each r, $0 < r \leq 1$.

It is apparent that the existence of this family of limit points requires the existence of a family of nontrivial solutions $v_o(r;\varepsilon)$ of the homogeneous variational problem

$$\text{Var(II)} \begin{cases} v_o'' + \dfrac{v_o'}{r} + \lambda_o(\varepsilon) f_u(r,u_o) v_o = 0 \;, \quad \varepsilon < r < 1, \\ v_o(\varepsilon;\varepsilon) = 0 = v_o(1;\varepsilon) \;. \end{cases} \tag{3.13}$$

Consistent with the limits in (3.12) we expect the function $v_o(r;\varepsilon)$ to satisfy

$$\lim_{\substack{\varepsilon \to 0 \\ r \text{ fixed}}} v_o(r;\varepsilon) = V_o(r) \;, \tag{3.14}$$

for each r, $0 < r \leq 1$. We now undertake the task of constructing asymptotic expansions for $\lambda_o(\varepsilon)$, u_o, and (necessarily) v_o.

3.2 Construction of the expansions

In constructing asymptotic approximations to the perturbed family of limit points $\lambda_o(\varepsilon)$ we essentially follow the formalism described in Section 2 for bifurcation points. The present problem is slightly more complicated because we must solve both parts of the extended system formed by BVP(II) and Var(II) simultaneously. Understanding the interplay between these two BVP's is crucial to obtaining the correct form of the inner and outer expansions for $u_o(r;\varepsilon)$ and $v_o(r;\varepsilon)$. To make this interplay more transparent we shall first consider simple approximations to λ_o, u_o and v_o. Later we describe general expansions.

Based on the limits in (3.12) and (3.14) we attempt simple outer approximations for u_o and v_o in the form

$$u_o(r,\varepsilon) \sim U_o(r) + \phi(\varepsilon)\,\tilde{u}(r) \quad ,$$
$$v_o(r;\varepsilon) \sim V_o(r) + \psi(\varepsilon)\,\tilde{v}(r) \quad , \tag{3.15}$$

for $\varepsilon \to 0$ with $0 < r < 1$ fixed. An analogous expression for $\lambda_o(\varepsilon)$ is given by

$$\lambda_o(\varepsilon) \sim \lambda_o + \chi(\varepsilon)\,\tilde{\lambda} \qquad \text{as } \varepsilon \to 0 \ . \tag{3.16}$$

Here ϕ, ψ and χ are unknown $o(1)$ order functions; $\tilde{u}$ and $\tilde{v}$ are unknown functions of r, and $\tilde{\lambda}$ is an unknown constant.

We don't expect the outer approximations in (3.15) to remain valid near $r = \varepsilon$. As in the bifurcation problem a suitable inner coordinate is

$$s = r/\varepsilon \ , \tag{3.17}$$

which transforms the differential equation in (3.9) to

$$\frac{d^2u_o}{ds^2} + \frac{1}{s}\,\frac{du_o}{ds} + \varepsilon^2\lambda_o(\varepsilon)f(\varepsilon s, u_o) = 0 \ , \qquad s > 1 \ . \tag{3.19}$$

Moreover, u_o must satisfy the boundary condition

$$u_o = 0 \quad \text{for} \quad s = 1 \ . \tag{3.20}$$

For sufficiently smooth f the derivative terms dominate in (3.19). We deduce that the inner expansion for u_o has the form

$$u_o(r;\varepsilon) = \delta(\varepsilon)(\ln s)[\, U_o(0) + o(1)\,] \tag{3.21}$$

as $\varepsilon \to 0$ with $s > 1$ fixed. The order function $\delta(\varepsilon)$ is defined as before

$$\delta(\varepsilon) \equiv 1/\ln(1/\varepsilon). \tag{3.22}$$

The constant $U_o(0)$ in (3.21) follows from a straightforward leading-order matching with the outer expansion in (3.15). By a similar argument based on the form of (3.13) we find that the inner expansion for v_o satisfies

$$v_o(r;\varepsilon) = \delta(\varepsilon)(\ln s)[\, 1 + o(1)\,] , \tag{3.23}$$

as $\varepsilon \to 0$ with $s > 1$ fixed. In deriving (3.23) we have made use of the fact that $V_o(0) = 1$.

Equations for the correction terms $\tilde{u}$ and $\tilde{v}$ in the outer approximations are obtained by substituting (3.15) and (3.16) into (3.9) and (3.13) and carrying out the standard limit process. We obtain

$$\begin{aligned} \mathcal{L}\tilde{u} &= -\,[\lim_{\varepsilon\to 0} \frac{\chi(\varepsilon)}{\phi(\varepsilon)}]\,\tilde{\lambda}\, f(r,U_o) , \\ \tilde{u}(1) &= 0 \end{aligned} \tag{3.24}$$

and

$$\mathcal{L}\tilde{v} = -\left[\lim_{\varepsilon\to 0} \frac{\chi(\varepsilon)}{\psi(\varepsilon)}\right] \tilde{\lambda} f_u(r,U_o)V_o(r)$$

$$-\left[\lim_{\varepsilon\to 0} \frac{\phi(\varepsilon)}{\psi(\varepsilon)}\right] \lambda_o f_{uu}(r,U_o)\tilde{u}(r)V_o(r) , \quad (3.25)$$

$$\tilde{v}(1) = 0 ,$$

where $\mathcal{L}$ is defined in (3.5).

First we focus on (3.24). Since $f(r,U_o) \not\equiv 0$, we can rule out the possibility that $\chi >> \phi$ as $\varepsilon \to 0$. Suppose $\chi = O(\phi)$, say

$$\lim_{\varepsilon\to 0} \frac{\chi(\varepsilon)}{\phi(\varepsilon)} = 1 , \quad (3.26)$$

Then (3.24) becomes

$$\mathcal{L}\tilde{u} = -\tilde{\lambda} f(r,U_o) , \quad \tilde{u}(1) = 0 \quad (3.27)$$

It proves convenient to express a particular solution of (3.27) in terms of the solution of the related initial-value problem

$$\mathcal{L} z = f(r,U_o) , \quad r > 0 , \quad z(0) = z'(0) = 0 . \quad (3.28)$$

It is not difficult to show that for smooth U_o and f (3.28) has a smooth solution z(r).

We can write the solution of (3.27) as

$$\tilde{u}(r) = c_1 V_o(r) + \tilde{\lambda}[\, z(1)\frac{W_o(r)}{W_o(1)} - z(r)\,] \,, \qquad (3.29)$$

where c_1 is an arbitrary constant. To gain further information we must carry out a matching with the inner expansion. For small r, the outer approximation (3.15) satisfies

$$u_o(r,\varepsilon) = [\, U_o(0)+o(1)\,] + \phi(\varepsilon)\,[\, \tilde{\lambda}\frac{z(1)}{W_o(1)}(\ln r) + O(1)\,] \,, \qquad (3.30)$$

as $\varepsilon \to 0$ with $\varepsilon < r << 1$. Matching of this expression with the inner approximation (3.21) to $O(\delta)$ occurs if and only if

$$\phi(\varepsilon) \sim \delta(\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \,, \qquad (3.31)$$

and

$$\tilde{\lambda} = \frac{W_o(1)}{z(1)} U_o(0) \,. \qquad (3.32)$$

(If $z(1) = 0$, we are dealing with a higher order limit point.)

An interesting feature of this partial result is that it provides a first correction to the value of the limit point $\lambda_o(\varepsilon)$ without any input from the variational problem (3.25). However, in order to determine the constant c_1 in (3.29) we would have to solve (3.25) for $\tilde{v}$.

There is a second, more subtle way in which the companion variational problem influences the form of the expansion for u_o. We have yet to consider the possibility that

$$\lim_{\varepsilon \to 0} \frac{\chi(\varepsilon)}{\phi(\varepsilon)} = 0 \tag{3.33}$$

In this event (3.24) becomes

$$\mathcal{L}\tilde{u} = 0 \ , \quad \tilde{u}(1) = 0 \ , \tag{3.34}$$

which has the solution

$$\tilde{u}(r) = d_1 \ V_o(r) \ , \tag{3.35}$$

with d_1 an arbitrary constant. The outer approximation (3.15) then becomes

$$u_o(r;\varepsilon) \sim U_o(r) + \phi(\varepsilon) \ d_1 \ V_o(r) \ , \tag{3.36}$$

as $\varepsilon \to 0$ with $0 < r < 1$ fixed. Since $V_o(r)$ is smooth for small r this expression is consistent with the inner approximation (3.21) so long as

$$\delta(\varepsilon) << \phi(\varepsilon) << 1 \quad \text{as} \quad \varepsilon \to 0 \ . \tag{3.37}$$

Without recourse to the variational problem we cannot say anything further about ϕ .

Since v_o satisfies a homogeneous BVP we may assume that $\tilde{v}$ in (3.15) is not simply a multiple of V_o. Thus, we assume that

$$\psi(\varepsilon) \sim \phi(\varepsilon) \quad \text{as} \quad \varepsilon \to 0 \ , \tag{3.38}$$

with the consequence that (3.25) becomes

$$\mathcal{L}\tilde{v} = -\lambda_o d_1 f_{uu}(r,U_o) V_o^2 , \qquad \tilde{v}(1) = 0 . \tag{3.39}$$

At this point our argument proceeds as before for $\tilde{u}$. The inhomogeneous term in (3.39) forces $\tilde{v}$ to have $\ln r$ behavior for small r. But with $\delta \ll \psi(\varepsilon) \ll 1$ as $\varepsilon \to 0$, we can't match the resulting outer expansion for v_o in (3.15) with the inner expansion (3.23). Thus we conclude that the limit (3.33), and consequently (3.37), is not possible. The only viable choice is (3.31) and (3.32). It then becomes apparent that we must take

$$\psi(\varepsilon) \sim \delta(\varepsilon) \qquad \text{as} \qquad \varepsilon \to 0 , \tag{3.40}$$

so that (3.25) becomes

$$\mathcal{L}\tilde{v} = -\tilde{\lambda} f_u(r,U_o)V_o - \lambda_o f_{uu}(r,U_o)\,\tilde{u}\,V_o , \qquad \tilde{v}(1) = 0 . \tag{3.41}$$

Upon substituting (3.29) and (3.32) into (3.41), solving and matching the resulting expression for v_o to $O(\delta)$ with the inner expansion (3.23) we obtain the value of the constant c_1. We defer this calculation to the next section for a special case. Thus, in principle at least, we have completed our determination of the leading-order corrections to the family of limit points.

Motivated by these preliminary results we assume that full expansions for the familiy of limit points have the form

$$\lambda_o(\varepsilon) \sim \lambda_o + \sum_{k=1}^{\infty} \lambda_k \delta^k(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \; ; \tag{3.42}$$

with inner expansions for u_o and v_o given by

$$\begin{aligned} u_o(r;\varepsilon) &\sim (\ln s) \sum_{k=1}^{\infty} a_k \delta^k(\varepsilon), \\ v_o(r;\varepsilon) &\sim (\ln s) \sum_{k=1}^{\infty} b_k \delta^k(\varepsilon), \end{aligned} \tag{3.43}$$

as $\varepsilon \to 0$ with $s = r/\varepsilon > 1$ fixed; and with outer expansions for u_o and v_o given by

$$\begin{aligned} u_o(r;\varepsilon) &\sim U_o(r) + \sum_{k=1}^{\infty} u_k(r)\delta^k(\varepsilon), \\ v_o(r;\varepsilon) &\sim V_o(r) + \sum_{k=1}^{\infty} v_k(r)\delta^k(\varepsilon), \end{aligned} \tag{3.44}$$

as $\varepsilon \to 0$ with $0 < r < 1$ fixed. The unknowns in (3.42) - (3.44) are the constants λ_k, a_k and b_k and the functions $u_k(r)$ and $v_k(r)$ for $k = 1,2,\ldots.$

A few remarks are in order regarding the form of these expansions. An induction argument based on our preceding analysis can be employed to verify that no intermediate terms, e.g. $O(\delta^{3/2})$, have been missed. On the other hand, our expansions fail to account for $O(\varepsilon)$ effects because such terms are transcendentally small compared to $O(\delta)$ terms. As a consequence, the effect of the nonlinearity never appears in the inner expansions. The constants a_k and b_k are determined by a straightforward matching. It can be shown that the outer expansions contain the inner expansions.

In order to determine equations for the unknowns λ_k, u_k and v_k one simply substitutes (3.42) and (3.44) into the system of differential equations (3.9) and (3.13), and equates coefficients of powers of δ to zero. Assuming the implied smoothness of f holds we find

$$O(\delta) \qquad \begin{aligned} \mathcal{L}u_1 &= -\lambda_1 f(r,U_o), \\ \mathcal{L}v_1 &= -\lambda_1 V_o f_u(r,U_o) - \lambda_o u_1 V_o f_{uu}(r,U_o), \end{aligned} \tag{3.45}$$

$$O(\delta^2) \qquad \begin{aligned} \mathcal{L}u_2 &= -\lambda_2 f(r,U_o) - \lambda_1 u_1 f_u(r,U_o) - \frac{1}{2}\lambda_o u_1^2 f_{uu}(r,U_o), \\ \mathcal{L}v_2 &= -\lambda_2 V_o f_u(r,U_o) - \lambda_1[v_1 f_u(r,U_o) + u_1 V_o f_{uu}(r,U_o)] \\ &\quad -\lambda_o\ (u_2 V_o + u_1 v_1) f_{uu}(r,U_o) - \frac{1}{2} u_1^2 V_o f_{uuu}(r,U_o)] , \end{aligned} \tag{3.46}$$

$$O(\delta^k) \qquad \begin{aligned} \mathcal{L}u_k &= g_k(r,\lambda_1,\ldots,\lambda_k,u_1,\ldots,u_{k-1}), \\ \mathcal{L}v_k &= h_k(r,\lambda_1,\ldots,\lambda_k,u_1,\ldots,u_k,v_1,\ldots,v_{k-1}), \end{aligned} \tag{3.47}$$

where g_k and h_k are functions of the indicated arguments. The appropriate outer boundary conditions are

$$u_k(1) = v_k(1) = 0, \qquad k = 1,2,\ldots \quad . \tag{3.48}$$

This system of problems must be solved in sequence. As our preliminary results indicate, in order to determine λ_k, $k = 1,2,\ldots$, one must first determine u_{k-1} <u>and</u> v_{k-1}. Then, solution of the equation for u_k, subject to (3.48) and subsequent matching with the inner expansion in (3.43) yields λ_k. The function u_k

will still involve an arbitrary constant c_k in the form

$$u_k(r) = c_k V_o(r) + \dots \quad . \tag{3.49}$$

To ascertain c_k one must solve the companion problem for v_k. Analogous arbitrary constants in v_k may be set to zero since $v_o(r;\varepsilon)$ equals $V_o(r)$ to leading order and v_o satisfies a homogeneous BVP (3.13).

The preceding analysis provides a means of constructing approximations to the limit point itself. Constructing approximations to $u(r,\lambda;\varepsilon)$ for other values of λ is also a relatively easy task. Provided $0 < \lambda < \lambda_o$ with λ not too near to λ_o the deviation $u(r,\lambda;\varepsilon) - U(r,\lambda)$, with U the solution of the unperturbed BVP(I) in (3.2), will be $O(\delta)$ except near $r = \varepsilon$. Due to the quadratic nature of the limit point the deviation will be larger when λ is near λ_o.

To analyze the solution near the limit point it proves convenient to set

$$\lambda = \lambda_o + \alpha\,\delta \tag{3.50}$$

and to carry out an asymptotic analysis for $\alpha = O(1)$ as $\varepsilon \to 0$. The appropriate outer expansion for u has the form

$$u(r,\lambda;\varepsilon) \sim U_o(r) + \sum_{k=1}^{\infty} w_k(r)\delta^{k/2}(\varepsilon) \ , \tag{3.51}$$

as $\varepsilon \to 0$ with $0 < r < 1$ fixed. The inner expansion also proceeds in powers of $\sqrt{\delta}$. The unknown w_k's in (3.51) are found by solving (3.9) in the obvious manner. The variational problem plays no role in this calculation. One easily finds, for example, that

$$w_1(r) = b_1 V_o(r) \ , \tag{3.52}$$

with the constant b_1 equal to $\pm$ a certain multiple of $\sqrt{|\lambda_1-\alpha|}$. The odd powers of $\sqrt{\delta}$ disappear in (3.51) as $\alpha \to \lambda_1$ making this expression consistent with the limit point expansion.

This completes our discussion of the perturbed limit point problem for general nonlinear functions f. In order to better clarify the mechanics and to give some indication of the range of validity of our perturbation procedure we discuss at length a special case in the next section.

4. Application to tubular chemical reactors

Limit points play a prominent role in the study of reactors involving a selfheating chemical whose reaction velocity follows the Arrhenius Law and which dissipates energy by conduction only. When the exothermicity of the reactant mixture, measured by a parameter λ, reaches a limit point λ_o a thermal explosion can occur.

We address the following problem. Suppose that a reacting material is confined to an "infinite" circular cylinder of radius normalized to one and that the temperature on the boundary is maintained at a constant value, say T_o. Such a situation would fix the value of λ_o. Now suppose one were to place a "cooling" rod of radius ε along the axis of the reactor and to maintain the temperature at the constant value T_o on this inner cylinder as well. Clearly the presence of such a small rod will have an effect on the limit point. The problem is to estimate this effect. We shall show that it is quite dramatic.

For simplicity we restrict our attention to the limiting situation of infinite activation energy. The resulting problem is usually called the Frank-Kamenetzkii approximation. The reader may refer to [9] for a full derivation of the basic equations. In nondimensional form they are given for the case of an infinite circular cylinder geometry by

$$\Delta U + \lambda e^{U} = 0 \quad \text{in} \quad 0 \leq r < 1, \quad 0 \leq \theta < 2\pi$$
$$U = 0 \quad \text{on} \quad r = 1, \tag{4.1}$$

where U is proportional to the deviation of the temperature from T_o. This BVP is of the form of (3.1) with

$$f(r,U) = e^{U} .$$

Cylindrically symmetric solutions $U(r,\lambda)$ satisfy

$$\text{BVP(I)} \quad \begin{aligned} & U'' + \frac{U'}{r} + \lambda e^{U} = 0 , \qquad 0 < r < 1, \\ & U'(0,\lambda) = 0 = U(1,\lambda) \end{aligned} \tag{4.2}$$

As is well-known BVP(I) has the simple exact solution

$$U(r,\lambda) = \ln \left[\frac{\gamma}{r^2 + \gamma - 1} \right]^2 \tag{4.3}$$

where

$$\gamma \equiv \frac{4}{\lambda} (1 \pm \sqrt{1 - \lambda/2}) \tag{4.4}$$

For $0 < \lambda < 2$ these expressions describe two solutions. Clearly

$$\lambda = \lambda_o = 2, \quad U(r,2) = U_o(r) = \ln \left(\frac{2}{r^2 + 1}\right)^2 \tag{4.5}$$

defines a limit point. For this case the homogeneous variational problem (3.5) becomes

$$\text{Var(I)} \begin{cases} \mathcal{L}V_o \equiv V_o'' + \dfrac{V_o'}{r} + \dfrac{8}{(r^2+1)^2} V_o = 0, & 0 < r < 1, \\ V_o'(0) = 0 = V_o(1). \end{cases} \tag{4.6}$$

A smooth solution of Var(I) normalized to one at $r = 0$ is given by

$$V_o(r) = \frac{1 - r^2}{1 - r^2} , \tag{4.7}$$

with a second, linearly independent solution having $\ln r$ behavior given by

$$W_o(r) = \frac{2}{1 + r^2} + \frac{1 - r^2}{1 + r^2} \ln r, \qquad 0 < r \leq 1. \tag{4.8}$$

Introducing the "cooling" rod of radius ε into the tubular reactor leads to the following perturbed problem for $u(r,\lambda;\varepsilon)$

$$\text{BVP(II)} \begin{cases} u'' + \dfrac{u'}{r} + \lambda e^u = 0 , & \varepsilon < r < 1, \\ u(\varepsilon,\lambda;\varepsilon) = 0 = u(1,\lambda;\varepsilon). \end{cases} \tag{4.9}$$

We seek approximations to the family of limit points defined by $\lambda_o(\varepsilon)$ and $u_o(r;\varepsilon)$ which reduces to (4.5) as $\varepsilon \to 0$. The associated homogeneous variational problem in (3.13) for $v_o(r;\varepsilon)$ takes the form

$$\text{Var(II)} \quad \begin{cases} v_o'' + \dfrac{v_o'}{r} + \lambda_o(\varepsilon)e^{u_o} v_o = 0, & \varepsilon < r < 1 , \\ v_o(\varepsilon;\varepsilon) = 0 = v_o(1;\varepsilon) . & \end{cases} \tag{4.10}$$

Construction of the expansions for $\lambda_o(\varepsilon)$, u_o and v_o is carried out as described in the preceding section. We set

$$\lambda_o(\varepsilon) \sim 2 + \lambda_1\delta + \lambda_2\delta^2 + \ldots \quad \text{as} \quad \varepsilon \to 0, \tag{4.11}$$

with outer expansions

$$\begin{aligned} u_o(r;\varepsilon) &\sim \ln\left(\frac{2}{r^2+1}\right)^2 + u_1(r)\delta + u_2(r)\delta^2 + \ldots , \\ v_o(r;\varepsilon) &\sim \frac{1-r^2}{1+r^2} + v_1(r)\delta + v_2(r)\delta^2 + \ldots, \end{aligned} \tag{4.12}$$

and inner expansions

$$\begin{aligned} u_o(r;\varepsilon) &\sim \delta(\ln s)\,[\ln 4 + a_1\delta + \ldots] , \\ v_o(r;\varepsilon) &\sim \delta(\ln s)\,[1 + b_1\delta + \ldots] . \end{aligned} \tag{4.13}$$

The coefficients of the leading-order terms in (4.13) follow from matching with the outer expansions.

From (3.45) we obtain the governing equation for $u_1(r)$

$$\mathcal{L}u_1 = -\lambda_1 e^{U_o} = -\frac{4\lambda_1}{(r^2+1)^2} , \qquad (4.14)$$

$$u_1(1) = 0$$

which has the solution (using (4.7) and (4.8))

$$u_1(r) = c_1 \frac{1-r^2}{1+r^2} + \frac{\lambda_1}{2} \cdot \frac{1-r^2}{1+r^2} \ln r , \qquad (4.15)$$

with c_1 an arbitrary constant. Matching the $\ln r$ terms to leading order in the inner and outer expansions for $u_o(r;\varepsilon)$ yields

$$\lambda_1 = 2 \ln 4 .$$

Thus we have an explicit value for the leading-order correction to the limit point.

To this point the constant c_1 in (4.15) is still arbitrary. To determine it we must solve for v_1. The governing problem for v_1 is obtained from (3.45) as

$$\mathcal{L}v_1 = -\lambda_1 V_o e^{U_o} - \lambda_o u_1 V_o e^{U_o} ,$$

$$= 8 \cdot \frac{r^2-1}{(r^2+1)^3} \left[\ln 4 + c_1 \frac{1-r^2}{1+r^2} + (\ln 4) \frac{1-r^2}{1+r^2} (\ln r) \right], \qquad (4.17)$$

$$v_1(1) = 0 .$$

This problem can be solved either analytically or numerically. We briefly describe a numerical approach that facilitates matching. Set

$$v_1(r) = d_1 W_o(r) + c_1 z_1(r) + z_2(r) + z_3(r) \ln r, \qquad (4.18)$$

where d_1 is an arbitrary constant and the $z_j(r)$ are arbitrary functions. (Since c_1 is unknown we must separate it out.) The point is that the "splitting" in (4.18) allows the $z_j(r)$ to be smooth functions on $0 \le r \le 1$. We simply substitute (4.18) into (4.17) and set the coefficients of the $\ln r$ terms and the non $\ln r$ terms to zero in the obvious manner. To ensure smoothness we impose the appropriate initial conditions on the z_j at $r = 0$. This algorithm leads to the following IVP's:

$$\begin{aligned} \mathcal{L} z_3 &= -8 \ln 4 \frac{(r^2 - 1)^2}{(r^2 + 1)^4}, && r > 0 \\ \mathcal{L} z_2 &= -2 \frac{z_3'}{r} + 8(\ln 4) \frac{r^2 - 1}{(r^2 + 1)^3}, && r > 0 \qquad (4.19) \\ \mathcal{L} z_1 &= -8 \frac{(r^2 - 1)^2}{(r^2 + 1)^4}, && r > 0 \end{aligned}$$

with

$$z_j(0) = z_j'(0) = 0, \qquad j = 1,2,3. \qquad (4.20)$$

The initial conditions in (4.20) ensure that the z_j are $O(r^2)$ as $r \to 0$. Thus, matching the $\ln r$ terms to leading order in the inner and outer epxansions for v_o yields

$$d_1 = 1 \qquad (4.21)$$

and, satisfaction of the boundary condition at $r = 1$ in (4.17) provides the value of c_1

$$c_1 = -\frac{1}{z_1(1)} (1 + z_2(1)) \tag{4.22}$$

We find, in fact, that

$$c_1 = 2 . \tag{4.23}$$

It is a straightforward matter to calculate further terms in the limit-point expansions. For example, consideration of the problem for u_2 in the outer expansion yields the next coefficient in (4.11) to be

$$\lambda_2 = 2.9609 \quad \text{(numerical)} \tag{4.24}$$

Summarizing our results to this point for the limit-point expansions, we have established that

$$\lambda_o(\varepsilon) = 2 + \frac{2 \ln 4}{\ln 1/\varepsilon} + \frac{2.9609}{(\ln 1/\varepsilon)^2} + 0\left(\frac{1}{(\ln 1/\varepsilon)^3}\right) , \tag{4.25}$$

as $\varepsilon \to 0$; with the inner expansion for u_o given by

$$u_o(r;\varepsilon) = (\ln s)\left[\frac{\ln 4}{\ln 1/\varepsilon} + \frac{2}{(\ln 1/\varepsilon)^2} + 0\left(\frac{1}{(\ln 1/\varepsilon)^3}\right)\right] , \tag{4.26}$$

as $\varepsilon \to 0$ with $s = r/\varepsilon > 1$ fixed; and with the outer expansion for u_o given by

$$u_o(r;\varepsilon) = \ln\left(\frac{2}{r^2+1}\right)^2 + \frac{1}{\ln 1/\varepsilon}\,\frac{1-r^2}{1+r^2}\,[2+(\ln 4)\ln r] + 0\left(\frac{1}{(\ln 1/\varepsilon)^2}\right), \quad (4.27)$$

as $\varepsilon \to 0$ with $0 < r \leq 1$ fixed. We have written these expressions in terms of $\ln 1/\varepsilon$ to emphasize the fundamental role of this expansion parameter.

The approximation to $\lambda_o(\varepsilon)$ provided by (4.25) represents the centerpiece of this section. The slow convergence to zero of $1/\ln(1/\varepsilon)$ as $\varepsilon \to 0$ implies that even for very thin cooling rods we can expect significant increases in the safe operating range of λ beyond the unperturbed value of $\lambda_o = 2$. For example, for $\varepsilon = .0001$ our formula predicts an increase in λ_o of more than fifteen percent.

In order to measure the region of validity of our formulas we have carried out extensive numerical computations. Efficient continuation methods which allow for the calculation of limit points are now quite standard [10,11]. In Table 2 we compare the predictions of both two and three terms of formula (4.25) with numerical calculations of $\lambda_o(\varepsilon)$ for several values of ε. In each instance three terms of formula (4.25) provide a better approximation than do just two terms. Even for $\varepsilon = .1$ the accuracy is impressive ($1/\ln 10 \doteq 0.4343$).

ε	accurate numerical solution for $\lambda_o(\varepsilon)$	2 Term asymptotic approximation	3 Term asymptotic approximation
.1	3.8915	3.2041	3.7626
.04	3.2003	2.8613	3.1471
.01	2.7610	2.6020	2.7416
.001	2.4693	2.4013	2.4634
.0001	2.3384	2.3010	2.3359

Table 2. Comparison of numerical and asymptotic approximations (using formula (4.25)) to the limit point $\lambda_o(\varepsilon)$ for the perturbed BVP(II) in (4.9).

A more detailed picture of the effect of introducing a small cooling rod is provided in the bifurcation diagram in Fig. 3 , where

$$||u||_2 \equiv \left(\frac{1}{1-\varepsilon}\int_\varepsilon^1 u^2(r,\lambda;\varepsilon)dr\right)^{1/2} . \tag{4.28}$$

The $\varepsilon = 0$ curve was obtained directly from the exact solution in (4.3) - (4.4); and the curves for $\varepsilon > 0$ were generated numerically. Qualitatively these curves are very similar, but the location of the limit point (physically the value of λ at which a thermal explosion can occur) is very sensitive to the size of the cooling rod.

The inner and outer approximations to $u_o(r;\varepsilon)$ provided by (4.26) and (4.27), respectively, are also quite accurate. For example, in Fig. 4 we compare the exact (numerical) solution with both the two-term inner and the two-term outer approximations for $\varepsilon = .04$. The approximations are both very good in their assumed regions of validity. As ε is decreased the accuracy improves, in agreement with the order of magnitude error estimates in (4.26) and (4.27).

It is also possible to combine the inner and outer approximations into uniformly valid representations. Adding one term of (4.26) to one term of (4.27) and subtracting out the common (overlapping) part leads to the one term composite approximation

$$u(r;\varepsilon) = \ln\left(\frac{2}{r^2+1}\right)^2 + (\ln 4)\,\frac{\ln r}{\ln 1/\varepsilon} + o(1/\ln 1/\varepsilon) \ ,$$

as $\varepsilon \to 0$ uniformly on $\varepsilon \le r \le 1$. The same process, except taking two terms from both (4.26) and (4.27), yields the two term composite approximation

$$u(r;\varepsilon) = \ln\left(\frac{2}{r^2+1}\right)^2 + \frac{1-r^2}{1+r^2}\,\frac{1}{\ln 1/\varepsilon}\,(2 + (\ln 4)(\ln r)) + \frac{2\ln r}{(\ln 1/\varepsilon)^2} + o(1/\ln^2 1/\varepsilon) \ ,$$

as $\varepsilon \to 0$ uniformly on $\varepsilon \le r \le 1$.

In Fig. 5 and 6 we compare both the one term and the two term composite approximations with the exact (numerical) solutions of $u_o(r;\varepsilon)$ for $\varepsilon = .04$ and .001, respectively. The improvement in the accuracy of the two term approximation over the one term approximation is readily apparent, as is the improvement in each approximation for smaller values of ε. On the other hand, the two term composite approximation is not as accurate as either the corresponding inner or outer approximations in their regions of validity (compare Fig. 4 and 5). This situation is common in problems where the boundary layer terms do not decay exponentially.

We recognize that the example treated in this section has restricted applicability to chemical reactors in which rapid exothermic reactions take place. However, it should be stressed that recognition of circumstances causing instability and theoretical results on how to prevent it are generally important aspects of reactor design; they could save expensive experimentation.

Acknowledgment

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References

[1] S. Ozawa, Singular variation of domains and eigenvalues of the Laplacian, Duke Math. Journ. 48, 767-778 (1981).

[2] P.A. Lagerstrom and R.G. Casten, Basic concepts underlying singular perturbation techniques, SIAM Rev. 14, 63-120 (1972).

[3] J. Kevorkian and J.D. Cole, Perturbation methods in applied mathematics. Springer-Verlag, New York, Heidelberg, Berlin (1981).

[4] F.Y.M. Wan and H.J. Weinitschke, Boundary layer solutions for some nonlinear elastic membrane problems, ZAMP 38, 79-91 (1987).

[5] H.J. Weinitschke and C.G. Lange, Asymptotic solutions for finite deformation of thin shells of revolution with a small circular hole, Quart. Appl. Math. 45, (1987).

[6] M. Abramowitz and I.A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, Dover Publ., New York (1965).

[7] J.B. Keener and H.B. Keller, Perturbed bifurcation theory, Arch. Rat. Mech. Anal. 50, 159-175 (1973).

[8] G. Moore and A. Spence, The calculation of turning points of non-linear equations, SIAM J. Numer. Anal. 17, 567-576 (1980).

[9] D.A. Frank-Kamenetskii, Diffusion and Heat Exchange in Chemical Kinetics (translated by N. Thon) Princeton University Press, Princeton, N.J. (1955).

[10] E. Riks, The application of Newton's method to the problem of elastic stability, J. Appl. Mech. 39, 1060-1065 (1972).

[11] H.B. Keller, Numerical solution of bifurcation and nonlinear eigenvalue problems, in: Applications of bifurcation theory, P.H. Rabinowitz, ed., Academic Press, New York, 359-384 (1977).

Figure 1

Comparison of two term asymptotic solution (solid curve) with accurate numerical solution (closed circles) for $u_1(r;\varepsilon)$, $\varepsilon = 0.01$ and with leading term outer solution $J_o(\lambda_1 r)$ (open circles)

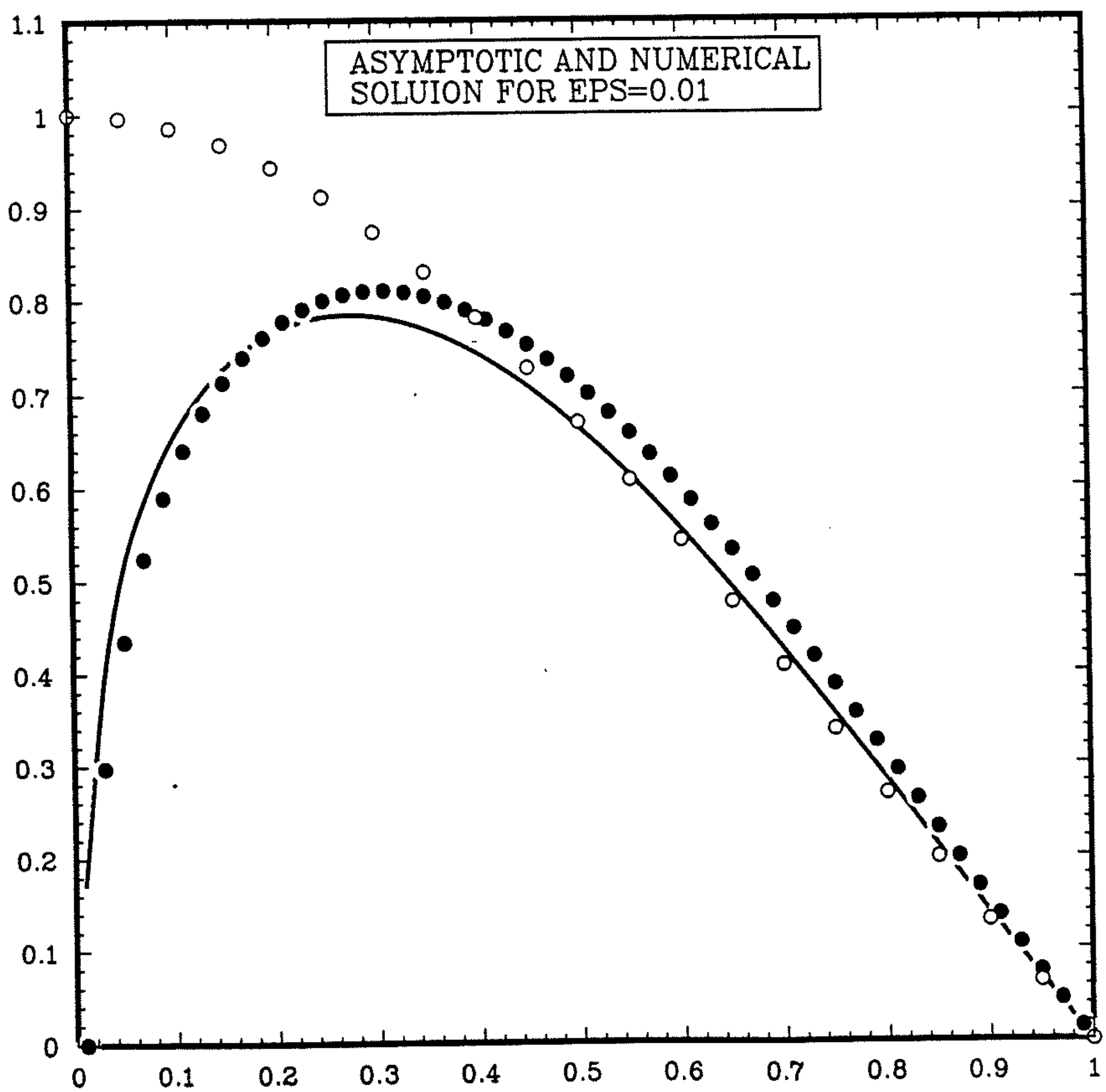

Figure 2

Comparison of two term asymptotic solution (solid curve) with accurate numerical solution (closed circles) for $u_2(r;\varepsilon)$, $\varepsilon=0.01$ and with leading term outer solution $J_o(\lambda_2 r)$ (open circles).

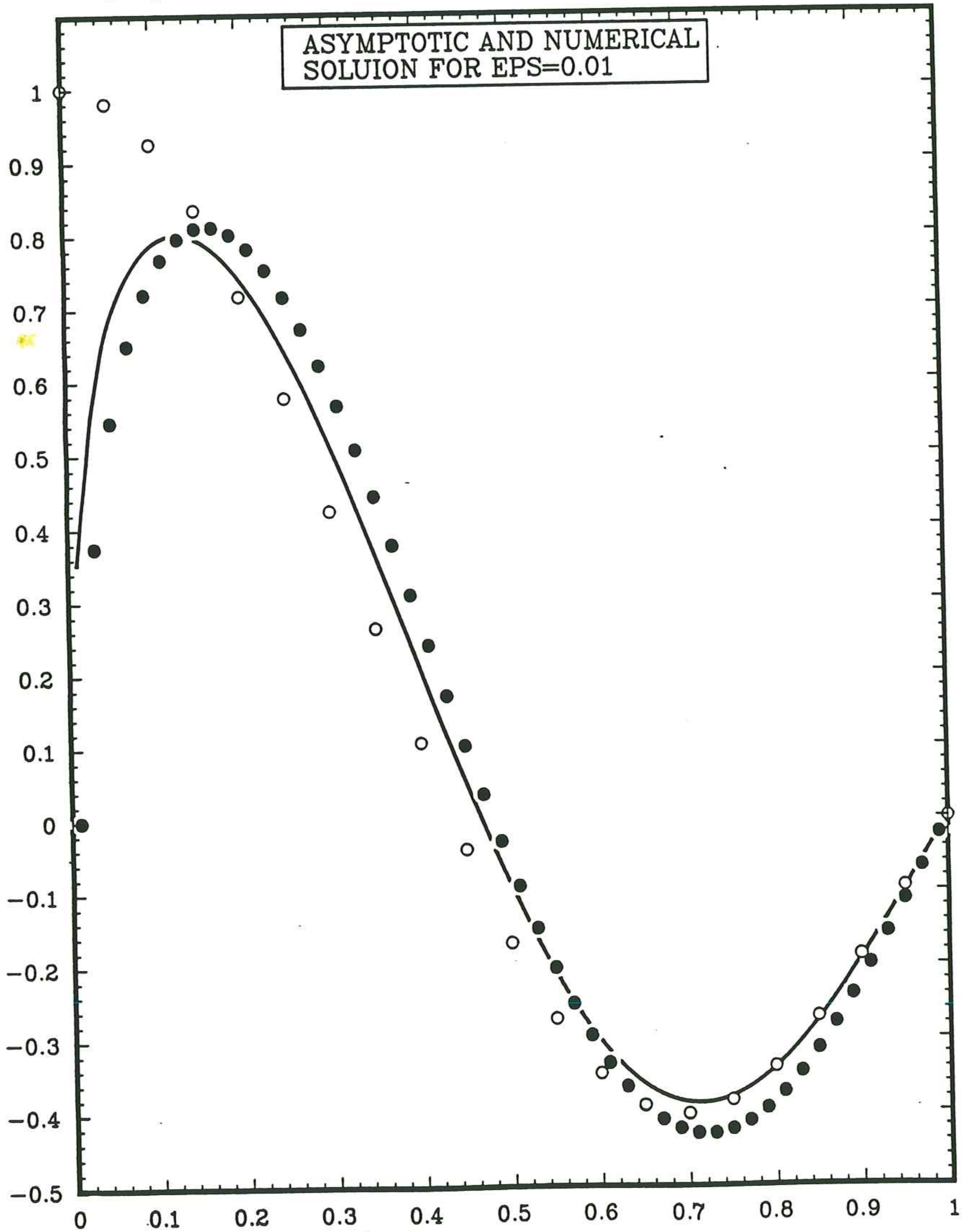

Figure 3

Bifurcation diagram $\|u\|_2$ vs. λ for BVP (II), equation (4.9), for $\varepsilon = 0$, .1, .01, and .001

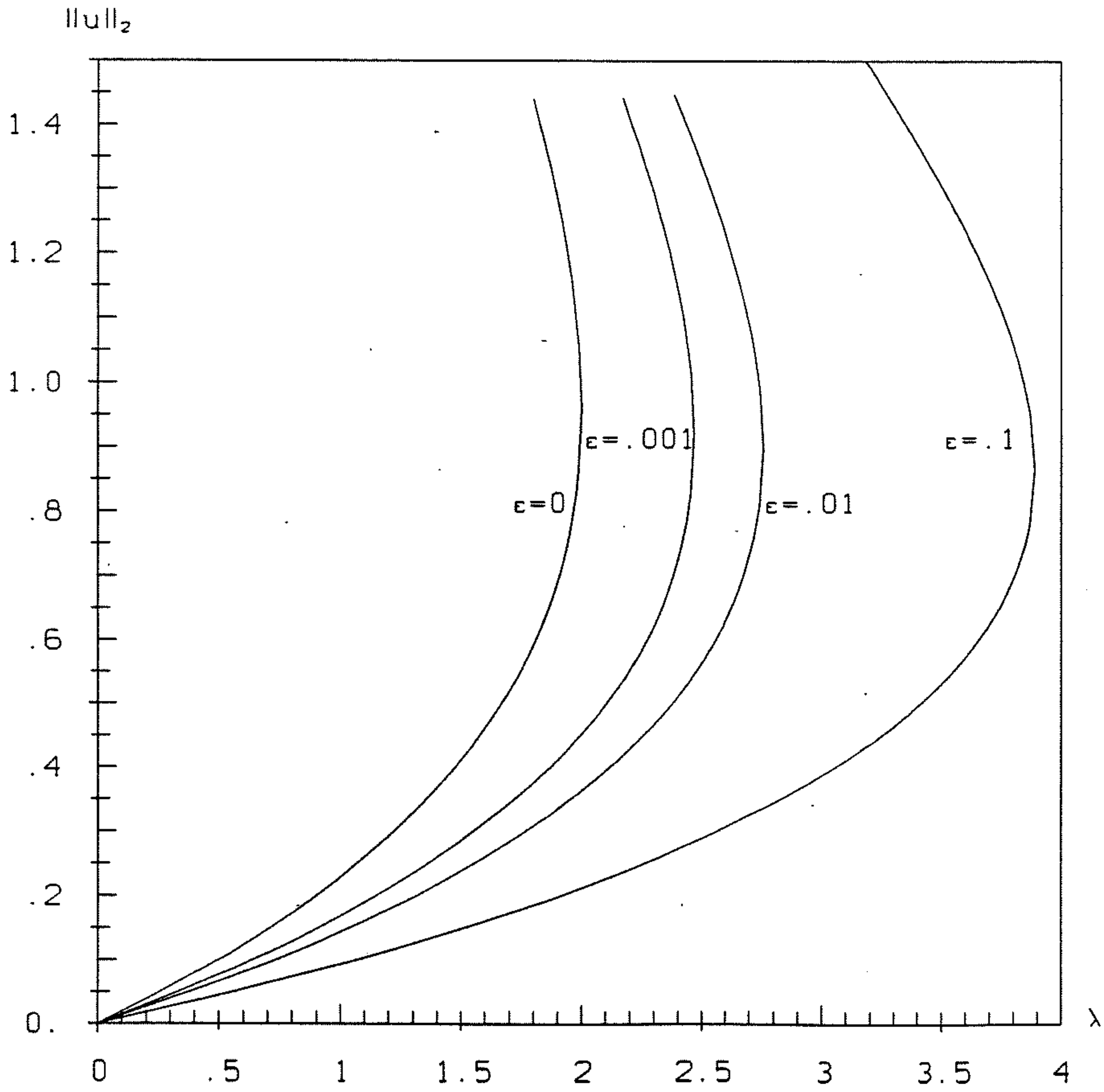

Figure 4

Comparison of two term inner and outer expansions of $u_o(r;\varepsilon)$ with accurate numerical solution for $\varepsilon=0.04$ at the limit point $\lambda_o(\varepsilon)$

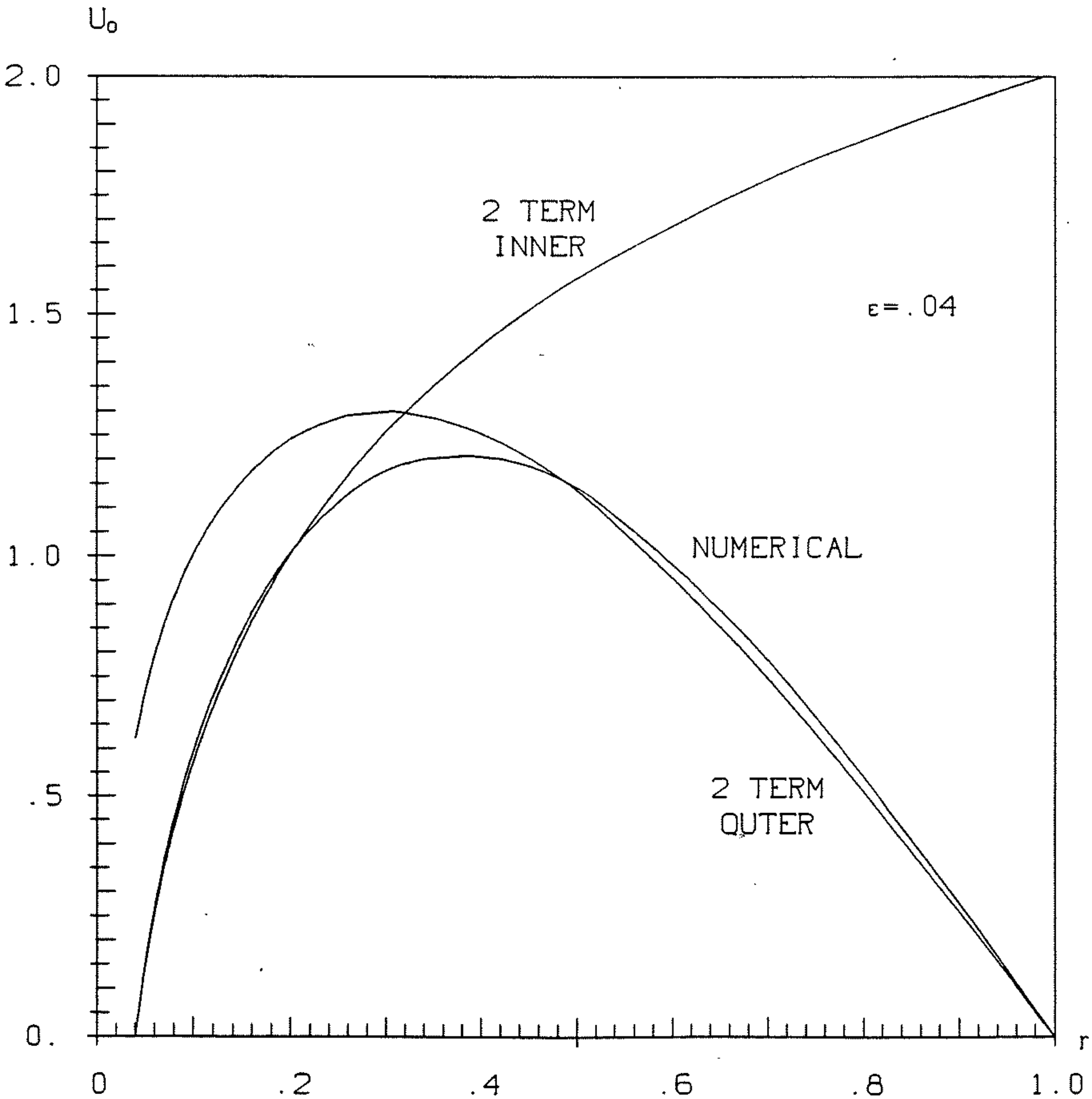

Figure 5

Comparison of one and two term composite approximations of $u(r;\varepsilon)$ with accurate numerical solution for $\varepsilon = 0.04$ at the limit point $\lambda_o(\varepsilon)$

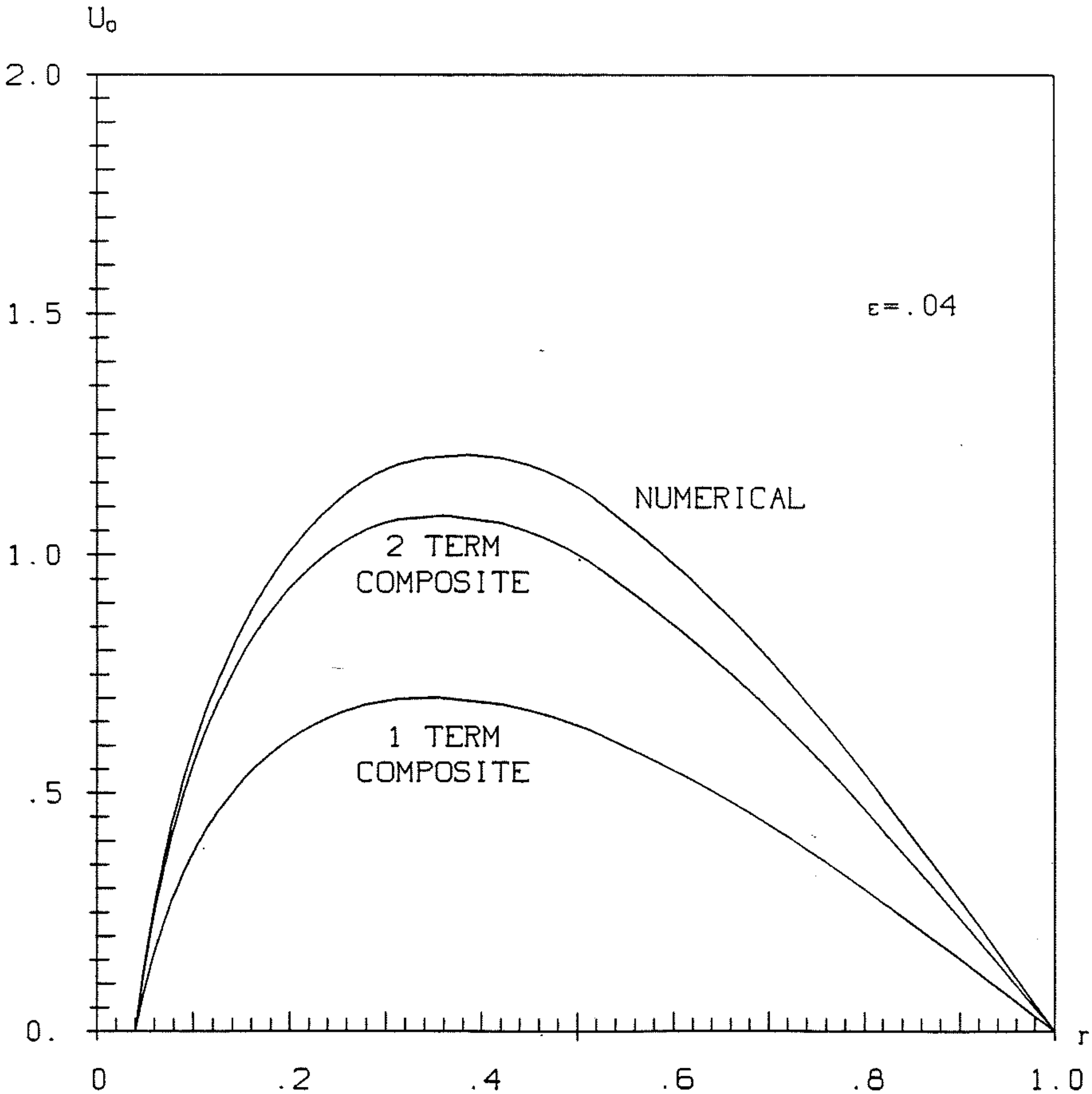

Figure 6

Comparison of one and two term composite expansions of $u(r;\varepsilon)$ with accurate numerical solution for $\varepsilon = 0.001$ at the limit point $\lambda_o(\varepsilon)$