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On Some Numerical Schemes for Transonic
Flow Problems

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ON SOME NUMERICAL SCHEMES FOR TRANSONIC FLOW PROBLEMS

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ABSTRACT

New second order accurate finite difference approximations for a class of nonlinear pde's of mixed type which includes the 2D Low Frequency Transonic Small Disturbance equation (TSD) and the 2D Full Potential equation (FP) are presented.

For the TSD equation, the scheme is implemented via a time splitting algorithm; the inclusion of flux limiters keeps the total variation non-increasing and eliminates spurious oscillations near shocks. Global Linear Stability, Total Variation Diminishing and Entropy Stability results are proven. Numerical results for the flow over a thin airfoil are presented. Current techniques used to solve the TSD equation may easily be extended to second order accuracy by this method.

For the FP equation, the new scheme requires no subsonic/supersonic switching and no numerical flux biasing. Global Linear Stability for all values of the Mach number is proven.

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INTRODUCTION

Recently a number of new shock capturing finite difference approximations for solving scalar conservation law nonlinear partial differential equations in several space dimensions have been constructed and applied to solve numerically the equations of inviscid compressible flows of aerodynamics. Those partial differential equations are, in the time-independent (steady) case, of mixed-type, i.e. their type changes from elliptic to hyperbolic as the flow regime changes from subsonic to supersonic and vice-versa.

In this paper, we present some new shock capturing finite difference approximations for solving scalar conservation laws. Our new schemes have the following properties:

(i)   second order accuracy throughout the computational domain;

(ii) global linear stability in all elliptic and all hyperbolic regions;

(iii) sharp steady discrete shock solutions;

(iv) total variation non-increasing property of the approximate solutions;

(v)   entropy stability, at least in some cases, i.e. the approximate solutions satisfy a discrete entropy condition consistent with the differential entropy condition of the p.d.e.; this property ensures that the approximate solutions are admissible on physical grounds.

A model 2D conservation law equation is constructed and a finite difference approximation scheme is proposed for this model equation. The above properties are proven for this scheme. This model can serve to represent and solve numerically two commonly used equations for simulating inviscid, isentropic potential flow problems at transonic speeds: the Transonic Small Disturbance (TSD) equation and the Full Potential (FP) equation.

The new schemes are studied first in their semi-discrete (method of lines) version. A new Alternate Direction Implicit (ADI)-like time discretization is also presented for the particular application to the low-frequency, unsteady, two-dimensional TSD equation; the results of the numerical implementation of this implicit scheme on a variable computational mesh proved satisfactory.

In [2], [18], [17], [13] and [3], a number of shock capturing finite difference approximations for solving the TSD and the FP equations have been proposed. These schemes satisfy properties (iii) & (v), and with the inclusion of flux limiters, property (iv) as well. Properties (i) & (ii) are usually satisfied in all elliptic regions; in hyperbolic regions, only first order accuracy is attained and the linear stability of the method is typically limited to values of the Mach number in $[0, M_c]$ where M_c is large enough to include the transonic regime. These schemes have a four-point bandwith and are type-dependent, i.e. they use *different* formulas for the difference approximations in the elliptic and the hyperbolic regions. They use *central* differencing in the elliptic regions

and *upwind* differencing in the hyperbolic regions. The upwinding is designed to take into account the correct region of influence and to keep the shock front sharper. For the TSD equation, since the flow is quasi-unidirectional, the upwinding is performed in that direction ([3]). For the FP equation, the upwinding can be performed separately for the x-dependent term and the y-dependent term. This approach was labeled directional flux biasing in [13]. Recently this approach was refined by introducing the method of streamwise flux biasing (see [16]) in which the upwinding is performed in a direction close to that of the actual flow. Unfortunately, the method hence obtained is only first order accurate (see [11], Section 8, for a review of the schemes based on this method).

Our new method does not use flux biasing but a special kind of upwinding *uniformly* in all regions. The resulting stencil, the same in all regions, is of 7-point bandwidth, with 4 points upwind and 2 points downwind.

The format of this paper is as follows.

In Section 1, we introduce our new second order accurate numerical schemes for a class of 2D conservation law nonlinear p.d.e.'s which includes the TSD equation and the FP equation. We prove a convergence result à la Lax-Wendroff.

In Section 2, we prove the linear stability of these schemes for the most commonly used numerical fluxes for the TSD equation and the FP equation.

In Section 3, we present an extended version of these schemes which makes use of flux limiters to keep the total variation non-increasing.

In Section 4, we prove a discrete entropy inequality satisfied by our finite difference approximation in the case of the low-frequency, unsteady TSD equation and we show that this inequality is consistent with the differential entropy inequality of the problem.

In Section 5, we describe a time-splitting algorithm for solving the unsteady TSD equation.

In Section 6, we present the numerical results obtained by implementing our finite difference method for the TSD equation for the flow over a thin airfoil.

1. NEW SECOND ORDER ACCURATE SCHEMES FOR 2D SCALAR CONSERVATION LAWS

In this section, we introduce our model equation and our new schemes for two-dimensional, nonlinear, scalar conservation law p.d.e.'s. and we present a convergence theorem à la Lax-Wendroff.

1.1. Model Equation

We consider the following 2D model scalar conservation law:

$$\partial_t F_0(\vec{\nabla}\Phi) + \vec{\nabla}\cdot\vec{F}(\vec{\nabla}\Phi) = 0 \qquad \text{(MODEL)}$$

where F_0 and $\vec{F}$ are smooth functions of their arguments and $\vec{\nabla} = [\partial_x, \partial_y]^T$, to be solved for $(x,y)\in\Omega$, where Ω a region of R^2, and for $t > 0$ together with:

(i) some initial conditions: $\Phi(x,y,0)=\Phi_0(x,y)$ for $(x,y)\in\Omega$, where Φ_0 is assumed to be such that $\vec{\nabla}\Phi_0 \in L^1(\Omega)\cap L^\infty(\Omega)\cap BV(\Omega)$, and

(ii) some boundary conditions on $\partial\Omega$.

This model initial boundary value problem includes the following two problems for the isentropic flow over an airfoil at transonic speeds (for the derivation of these two problems from the basic conservation equations of fluid dynamics, see [11], Section 1).

1.1.1. Example 1: Transonic Small Disturbance Equation

Choosing: $F_0(\vec{\nabla}\Phi) = 2k\,\partial_x\Phi$, $\vec{F}(\vec{\nabla}\Phi) = [f(\partial_x\Phi), -\partial_y\Phi]^T$, $f(u) = \dfrac{\gamma+1}{2}u^2 - Ku$ where[1] $K = \dfrac{1-M_\infty^2}{[M_\infty^2\delta]^{2/3}}$,

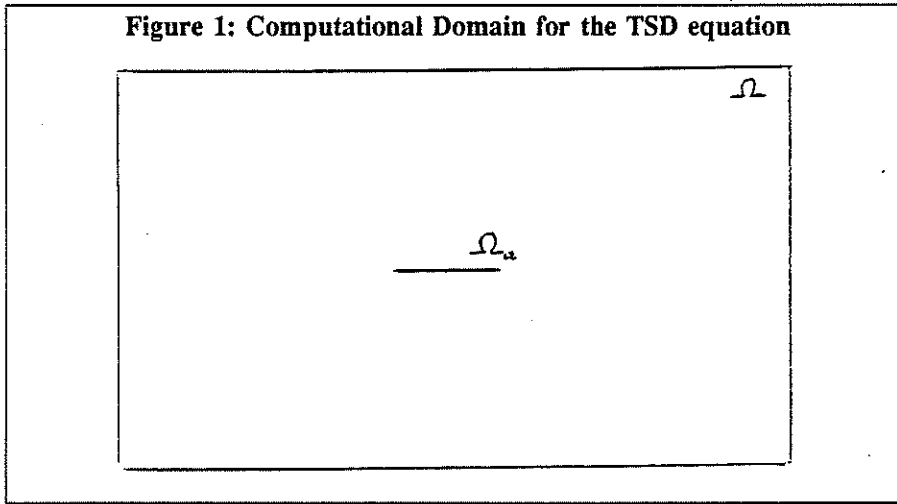and Ω to be the domain shown on Figure 1 yields the TSD equation:

$$2k\,\partial_t\partial_x\Phi + \partial_x f(\partial_x\Phi) - \partial_y\partial_y\Phi = 0 \tag{TSD}$$

The boundary conditions are:

(i) the tangential velocity condition on the airfoil: $\partial_y\Phi_{|\partial\Omega_a} = \partial_x h + \delta^{2/3}M_\infty^{-2/3}\partial_t h$ where Ω_a is the slit[2]

$$\left\{(x,y)\in\Omega \mid y=0, 0\le x\le 1\right\} \text{ and}$$

(ii) some *computational* farfield boundary conditions on $\partial\Omega\backslash\Omega_a$, typically a combination of Dirichlet and Neuman boundary conditions.

Figure 1: Computational Domain for the TSD equation

1.1.2. Example 2: Full Potential Equation

Choosing: $F_0(\vec{\nabla}\Phi) = \rho(|\vec{\nabla}\Phi|)$, $\vec{F}(\vec{\nabla}\Phi) = \rho(|\vec{\nabla}\Phi|)\vec{\nabla}\Phi$

where $\rho(q) = \left[1 - \dfrac{\gamma-1}{2}M_\infty^2(q^2-1)\right]^{\frac{1}{\gamma-1}}$ and Ω to be the domain shown on Figure 2 yields the FP equation:

$$\partial_t\rho(|\vec{\nabla}\Phi|) + \vec{\nabla}\cdot[\rho(|\vec{\nabla}\Phi|)\vec{\nabla}\Phi] = 0 \tag{FP}$$

The boundary conditions are:

(i) the tangential velocity condition on the airfoil Ω_a (no boundary layer effects): $\partial_y\Phi/\partial_x\Phi_{|\partial\Omega_a} = \delta\dfrac{dH}{dx}$

where $\partial\Omega_a = \left\{(x,y)\in\Omega \mid y=\pm\delta\max[0,H(x,t)]\right\}$ and

(ii) some *computational* farfield boundary conditions on $\partial\Omega\backslash\Omega_a$, typically a combination of Dirichlet and Neuman boundary conditions.

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[1] K is the transonic similarity parameter, M_∞ is the freestream Mach number and $\delta\ll 1$ is the airfoil thickness ratio.

[2] $y=\pm\delta^{4/3}M_\infty^{2/3}\max[0,h(x,t)]$, $0\le x\le 1$, describes the shape of the airfoil in similarity variables.

Figure 2: Computational Domain for the FP equation
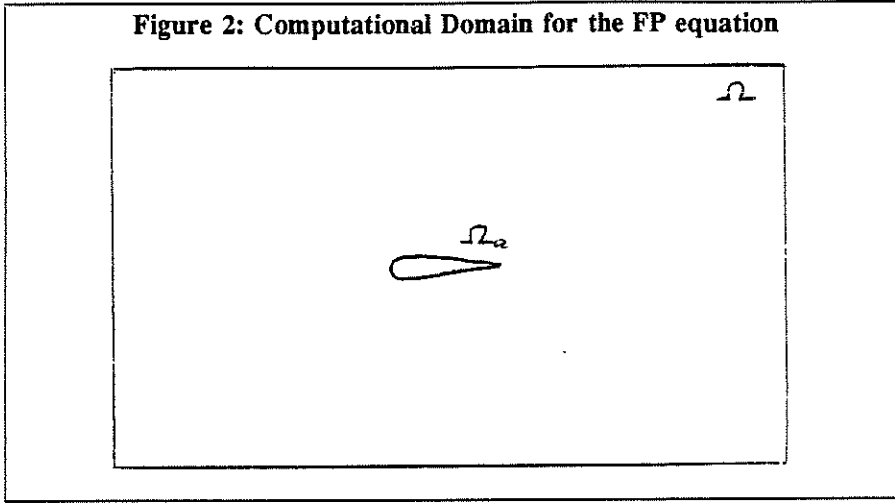
1.2. Semi-Discrete Finite Difference Approximations

We now consider a semi-discrete, method of lines approximation to the model equation (MODEL). We subdivide the region Ω (assumed to be the rectangle $[a,b] \times [c,d]$ for simplicity) into cells of the form

$$\omega_{jk} = \left\{ (x,y) \in \Omega \mid x_{j-1/2} \leq x < x_{j+1/2} , \; y_{k-1/2} \leq y < y_{k+1/2} \right\},$$

where $a \leq \cdots < x_{j-1/2} < x_{j+1/2} < \cdots \leq b$, $c \leq \cdots < y_{k-1/2} < y_{k+1/2} < \cdots \leq d$.

Let (x_j, y_k) denote the center of the cell ω_{jk}.

Set $\Delta x_j = x_{j+1/2} - x_{j-1/2}$, $\Delta y_k = y_{k+1/2} - y_{k-1/2}$ and $\Delta x = \max_j \Delta x_j$, $\Delta y = \max_k \Delta y_k$.

For each $t > 0$, define the step function $\Phi^\Delta(x,y,t) \equiv \Phi_{jk}(t)$, for $(x,y) \in \omega_{jk}$.

The initial data is discretized via: $\Phi_{jk}(0) \equiv T^\Delta \Phi_0(x,y)$ for $(x,y) \in \omega_{jk}$

where the space averaging operator T^Δ is defined by:

$$T^\Delta \Psi(x,y, \;) = \frac{1}{meas(\omega_{jk})} \int_{\omega_{jk}} \Psi(x,y, \;) d\omega_{jk} \equiv \Psi_{jk}(\;) \quad \text{for } (x,y) \in \omega_{jk}$$

At grid point (x_j, y_k), we approximate:
$$\begin{cases} \Phi & \sim \; \Phi_{jk} \\ F_0(\vec{\nabla}\Phi) & \sim \; F_0(\vec{L}_-\Phi_{jk}) \\ \vec{\nabla}\cdot\vec{F}(\vec{\nabla}\Phi) & \sim \; \vec{L}_- \cdot \vec{H}(\vec{L}_-\Phi_{jk}, \vec{L}_+\Phi_{jk}) \end{cases}$$

A method of lines, conservation law discretization of the initial-boundary value problem (MODEL) + initial conditions is:

$$\begin{cases} \partial_t F_0(\vec{L}_-\Phi_{jk}) + \vec{L}_- \cdot \vec{H}(\vec{L}_-\Phi_{jk}, \vec{L}_+\Phi_{jk}) = 0 \\ \Phi_{jk}(0) = T^\Delta \Phi_0(x,y) \end{cases}$$
(MODEL-scheme)

In the above, $\vec{L}_\pm$ are difference operators which approximate $\vec{\nabla}$ with second order accuracy (examples of such difference operators follow), i.e., for any step function Ψ_{jk}, we have:

$$|\vec{L}_\pm \Psi_{jk} - \vec{\nabla}\Psi(x_j, y_k)| = O(\Delta x^2) + O(\Delta y^2) \text{ for all } j,k$$

and the *numerical flux* $\vec{H}$ is assumed to satisfy the following *consistency* condition:

$$\vec{H}(\vec{q},\vec{q}) = \vec{F}(\vec{q})$$

For any step function Ψ_{jk}, we define the following discrete operators:

$$S_{\pm}^x \Psi_{jk} = \Psi_{j\pm1,k} \ , \ \ S_{\pm}^y \Psi_{jk} = \Psi_{j,k\pm1} \ ,$$

$$\Delta_{\pm}^x = \mp [I - S_{\pm}^x] \ , \ \ \Delta_{\pm}^y = \mp [I - S_{\pm}^y],$$

$$D_{\pm}^x = \Delta_{\pm}^x / \Delta_{\pm}^x x_j \ , \ \ D_{\pm}^y = \Delta_{\pm}^y / \Delta_{\pm}^y y_k,$$

$$D_0^x = (S_+^x - S_-^x) / (S_+^x - S_-^x)x_j \ , \ \ D_0^y = (S_+^y - S_-^y) / (S_+^y - S_-^y)y_k \ ,$$

$$L_{\pm}^x = D_{\pm}^x + S_{\pm}^x(D_0^x - D_{\pm}^x) \ , \ \ L_{\pm}^y = D_{\pm}^y + S_{\pm}^y(D_0^y - D_{\pm}^y).$$

In the case of a uniform mesh ($\Delta x_j \equiv \Delta x$, $\Delta y_k \equiv \Delta y$), the difference operators $L_{\pm}^x$ and $L_{\pm}^y$ take the simpler form:

$$L_{\pm}^x = D_{\pm}^x (1 \mp \frac{1}{2}\Delta_{\pm}^x) \text{ and } L_{\pm}^y = D_{\pm}^y (1 \mp \frac{1}{2}\Delta_{\pm}^y).$$

Theorem 1.1: Let Ψ be a smooth function defined on Ω and let Ψ_{jk} be the step function $\Psi^\Delta(x,y)$.
Then the above difference operators satisfy the following accuracy properties:

(i) $L_{\pm}^x \Psi_{jk} = \partial_x \Psi(x_j,y_k) + O(\Delta x^2)$, $L_{\pm}^y \Psi_{jk} = \partial_y \Psi(x_j,y_k) + O(\Delta y^2)$;

(ii) $D_-^y D_+^x \Psi_{jk} = \partial_y \partial_y \Psi(x_j,y_k) + O(\Delta y^2)$

Proof: The proof of this theorem follows directly from the definition of $L_{\pm}^x$, $D_{\pm}^x$ and a Taylor series expansion.

Lemma 1.2: Let $D_\pm$ denote either $D_{\pm}^x$ or $D_{\pm}^y$, $S_\pm$ denote either $S_{\pm}^x$ or $S_{\pm}^y$ and $L_\pm$ denote either $L_{\pm}^x$ or $L_{\pm}^y$.
For any step functions Ψ_{jk} and Ψ'_{jk}, the following "product rule" identities hold:

(i) $\Psi_{jk}D_\pm\Psi'_{jk} + (S_\pm\Psi'_{jk})(D_\pm\Psi_{jk}) = D_\pm(\Psi_{jk}\Psi'_{jk})$

(ii) $(S_\pm\Psi_{jk})(D_0\Psi'_{jk}) + (S_\mp\Psi'_{jk})(D_0\Psi_{jk}) = D_0(\Psi_{jk}\Psi'_{jk})$

(iii) $\Psi_{jk}D_+D_-\Psi'_{jk} - \Psi'_{jk}D_+D_-\Psi_{jk} = D_+[\Psi_{jk}D_-\Psi'_{jk} - \Psi'_{jk}D_-\Psi_{jk}]$

(iv) $(S_+S_+\Psi_{jk})L_-\Psi'_{jk} + (S_-S_-\Psi'_{jk})L_+\Psi_{jk} = D_0[(S_+\Psi_{jk})(S_-\Psi'_{jk})] + (S_+S_+\Psi_{jk})\Delta_-D_-\Psi'_{jk} - (S_-S_-\Psi'_{jk})\Delta_+D_+\Psi_{jk}$

Proof: The above identities can be verified by inspection.

1.2.1. Example 1: Transonic Small Disturbance Equation

A second order accurate method of lines finite difference approximation to the TSD equation is obtained by the choice of:

$$\vec{L}_\pm = [L_{\pm}^x, D_{\pm}^y]^T \ , \ \ \vec{H}(\vec{q}_-,\vec{q}_+) = [h(u_-,u_+), -v_+]^T \ \text{ where } \ \vec{q}_\pm = [u_\pm, v_\pm]^T$$

Theorem 1.3: Suppose that $\Phi_{jk}(t)$ is determined by:

$$\begin{cases} 2k\partial_t L_-^x \Phi_{jk} + L^x h(L_-^x \Phi_{jk}, L_+^x \Phi_{jk}) - D_-^y D_+^y \Phi_{jk} = 0 \\ \Phi_{jk}(0) = T^\Delta \Phi_0(x,y) \end{cases}$$ (TSD-scheme)

where h is a numerical flux satisfying the consistency condition: $h(u,u) = f(u)$.
Suppose that Φ_{jk}, $L_{\pm}^x \Phi_{jk}$ and $D_{\pm}^y \Phi_{jk}$ converge boundedly a.e. as Δx, $\Delta y \to 0^+$ to Φ, $\partial_x \Phi$ and $\partial_y \Phi$ respectively.
Then Φ is a weak solution[3] of:

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[3] For a review of the theory of weak solutions, see appendix A in [11].

$$\begin{cases} 2k\,\partial_t\partial_x\Phi + \partial_x f(\partial_x\Phi) - \partial_y\partial_y\Phi = 0 \\ \Phi(x,y,0) = \Phi_0(x,y) \end{cases} \tag{TSD}$$

Proof (à la Lax-Wendroff [9] *):* Let $\Psi \in C_0^\infty(\Omega\times[0,T])$ be a test function. Set $\Psi_{jk}(.) = T^\Delta\Psi(x,y,.)$.
Since Ψ has compact support, we have:

$$\sum_j \Delta x_j D_0 \left[(S_-^x\Psi_{jk}).S_-^x\left(\partial_t\Phi_{jk} + h(L_-^x\Phi_{jk}, L_+^x\Phi_{jk})\right) \right] = 0$$

For any Q_{jk}, in particular for $Q_{jk} = \partial_t\Phi_{jk}$ and $Q_{jk} = h(L_-^x\Phi_{jk}, L_+^x\Phi_{jk})$, we have:

$$\sum_j (\Delta x_j)(S_\pm^x S_\pm^x Q_{jk})(\Delta_\mp^x D_\mp^x \Psi_{jk}) = O(\Delta x)$$

Since Ψ has compact support, we have:

$$\sum_k (\Delta y_k) D_+^y \left[\Phi_{jk} D_-^y \Psi_{jk} - \Psi_{jk} D_-^y \Phi_{jk} \right] = 0$$

Multiply (TSD-scheme) by $S_+^x S_+^x \Psi_{jk}(t)\Delta x_j\Delta y_k$, sum over j,k and integrate over $[0,\infty)$ w.r. to t.

Using integration by parts (w.r. to t), summation by parts (via the properties of Lemma 1.2) and the above observations, we obtain:

$$\sum_{j,k}\Delta x_j\Delta y_k\left(2k S_-^x S_-^x\Phi_{jk}(0).L_+^x\Psi_{jk}(0) + \int_0^\infty dt \left[2k\left(S_-^x S_-^x\Phi_{jk}\right)\left(L_+^x\partial_t\Psi_{jk}\right) - \right. \right.$$

$$\left.\left. \left(S_-^x S_-^x h(L_-^x\Phi_{jk}, L_+^x\Phi_{jk})\right)\left(L_+^x S_-^x S_-^x\Psi_{jk}\right) - \Phi_{jk} D_+^y D_-^y S_-^x S_-^x\Psi_{jk} \right] \right) = 0$$

Taking the limit as $\Delta x,\Delta y \to 0^+$, using the consistency condition and Theorem 1.1, and applying Lebesgue Dominated Convergence Theorem, we obtain:

$$\iint_\Omega d\omega \left(2k\,\Phi_0\partial_x\Psi(0) + \int_0^\infty dt \left[2k\,\Phi\partial_t\partial_x\Psi - f(\partial_x\Phi)\partial_x\Psi - \Phi\partial_y\partial_y\Psi \right] \right) = 0$$

i.e. Φ is a weak solution of (TSD).

1.2.2. Example 2: Full Potential Equation

A second order accurate method of lines finite difference approximation to the TSD equation is obtained by the choice of:

$$\vec{L}_\pm(\ ) = \left[\{L_\pm^x(\ )\}^+ - \{L_\mp^x(\ )\}^-, \{L_\pm^y(\ )\}^+ - \{L_\mp^y(\ )\}^- \right]^T \text{ and } \vec{H}(\vec{q}_-,\vec{q}_+) = \rho(|\vec{q}_-|)\,\vec{q}_+$$

where, for any real number x, we set $x^\pm = \frac{1}{2}(x\pm|x|)$.

Theorem 1.4: Suppose that $\Phi_{jk}(t)$ is determined by:

$$\begin{cases} \partial_t\,\rho(|\vec{L}_-\Phi_{jk}|) + \vec{L}_-\cdot\left[\rho(|\vec{L}_-\Phi_{jk}|)\vec{L}_+\Phi_{jk}\right] = 0 \\ \Phi_{jk}(0) = T^\Delta\Phi_0(x,y) \end{cases} \tag{FP-scheme}$$

Suppose that Φ_{jk}, $L_\pm^x\Phi_{jk}$ and $L_\pm^y\Phi_{jk}$ converge boundedly a.e. as $\Delta x,\Delta y \to 0^+$ to Φ, $\partial_x\Phi$ and $\partial_y\Phi$ respectively.[4]

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[4] i.e. $\vec{L}_\pm\Phi_{jk} \to \vec{\nabla}\Phi$ boundedly a.e. as $\Delta x,\Delta y \to 0^+$.

Then Φ is a weak solution of:

$$\begin{cases} \partial_t \rho(|\vec{\nabla}\Phi|) + \vec{\nabla}\cdot[\rho(|\vec{\nabla}\Phi|)\vec{\nabla}\Phi] = 0 \\ \Phi(x,y,0) = \Phi_0(x,y) \end{cases} \tag{FP}$$

Proof (à la Lax-Wendroff): We look only at the case $\partial_x\Phi$, $\partial_y\Phi \geq 0$; the other cases are treated in a similar fashion.

Let $\Psi \in C_0^\infty(\Omega \times [0,T])$ be a test function. Set $\Psi_{jk}(.) = T^\Delta \Psi(x,y,.)$ and $\rho_{jk} = \rho(|\vec{L}_-\Phi_{jk}|)$.
Since Ψ has compact support, we have:

$$\sum_j (\Delta x_j) D_0^x \left[\left(S_+^x(\rho_{jk} L_+^x \Phi_{jk}) \right) \left(S_-^x S_-^x S_-^x \Psi_{jk} \right) \right] = 0$$

$$\sum_k (\Delta y_k) D_0^y \left[\left(S_+^y(\rho_{jk} L_+^y \Phi_{jk}) \right) \left(S_-^y S_-^y S_-^y \Psi_{jk} \right) \right] = 0$$

Multiply (FP-scheme) by $\Psi_{jk}(t)\Delta x_j \Delta y_k$, sum over j,k and integrate over $[0,\infty)$ w.r. to t.

Using integration by parts (w.r. to t), summation by parts (via the properties of Lemma 1.2) and the above observations, we obtain:

$$\sum_{j,k} \Delta x_j \Delta y_k \left(\rho_{jk}(0)\Psi_{jk}(0) + \int_0^\infty dt \left[\rho_{jk} \partial_t \Psi_{jk} + \right. \right.$$

$$\left. \left. \left(\left[S_-^x S_-^x (\rho_{jk} L_+^x \Phi_{jk}) \right] L_+^x S_-^x S_-^x \Psi_{jk} \right) + \left(\left[S_-^y S_-^y (\rho_{jk} L_+^y \Phi_{jk}) \right] L_+^y S_-^y S_-^y \Psi_{jk} \right) \right] \right) = 0$$

Taking the limit as $\Delta x, \Delta y \to 0^+$, using the consistency condition and Theorem 1.1, and applying Lebesgue Dominated Convergence Theorem, we obtain:

$$\iint_\Omega d\omega \left(\rho(|\vec{\nabla}\Phi_0|)\Psi(0) + \int_0^\infty dt \left[\rho(|\vec{\nabla}\Phi|)\vec{\nabla}\Phi \cdot \vec{\nabla}\Psi \right] \right) = 0$$

i.e. Φ is a weak solution of (FP).

2. LINEAR STABILITY ANALYSIS

In this section, we derive a necessary and sufficient condition for the finite difference method (MODEL-scheme) of Section 1 to be linearly stable.

For the sake of simplicity, a uniform mesh of cell size $\Delta x \times \Delta y$ is assumed. All the results of this section also hold if a variable mesh is assumed.

2.1. Necessary and sufficient condition for linear stability

Lemma 2.1: Assume that the (consistent) numerical flux $\vec{H}$ in (MODEL-scheme) is differentiable. Apply the method (MODEL-scheme) to: $\Phi_{jk}(t) = (j\Delta x, k\Delta y)\cdot\vec{q} + \varepsilon\phi_{jk}(t)$, ε small, where $\vec{q} = [u,v]^T$ is a constant state. Set $q = |\vec{q}|$.

Then the *linearized* method of lines corresponding to (MODEL-scheme) is given by:

$$\frac{\partial F_0}{\partial \vec{q}}(\vec{q})\cdot\vec{L}_-\frac{d}{dt}\phi_{jk} + \vec{L}_-\cdot\left[\frac{\partial \vec{H}}{\partial \vec{q}_-}(\vec{q},\vec{q})\bullet\vec{L}_-\phi_{jk} + \frac{\partial \vec{H}}{\partial \vec{q}_+}(\vec{q},\vec{q})\bullet\vec{L}_+\phi_{jk} \right] = 0 \quad \text{(linearized-MODEL-scheme)}$$

where $\cdot$ denotes the vector dot product and $\bullet$ denotes the matrix-vector multiplication.

Proof: The linearized finite difference method is obtained using the following equations:

$$\vec{L}_-\Phi_{jk} \quad = \quad \vec{q} + \varepsilon\vec{L}_-\phi_{jk}$$

$$F_0(\vec{L}_-\Phi_{jk}) \quad = \quad F_0(\vec{q}) + \varepsilon\frac{\partial F_0}{\partial\vec{q}}(\vec{q})\cdot\vec{L}_-\phi_{jk} + O(\varepsilon^2)$$

$$\frac{d}{dt}F_0(\vec{L}_-\Phi_{jk}) \quad = \quad \varepsilon\frac{\partial F_0}{\partial\vec{q}}(\vec{q})\cdot\vec{L}_-\phi_{jk} + O(\varepsilon^2)$$

$$\vec{H}_{jk} \quad = \quad \vec{H}(\vec{q},\vec{q}) + \varepsilon\Big[\frac{\partial\vec{H}}{\partial\vec{q}_-}(\vec{q},\vec{q})\bullet\vec{L}_-\phi_{jk} + \frac{\partial\vec{H}}{\partial\vec{q}_+}(\vec{q},\vec{q})\bullet\vec{L}_+\phi_{jk}\Big] + O(\varepsilon^2)$$

$$\vec{L}_-\vec{H}_{jk} \quad = \quad \varepsilon\vec{L}_-\Big[\frac{\partial\vec{H}}{\partial\vec{q}_-}(\vec{q},\vec{q})\bullet\vec{L}_-\phi_{jk} + \frac{\partial\vec{H}}{\partial\vec{q}_+}(\vec{q},\vec{q})\bullet\vec{L}_+\phi_{jk}\Big] + O(\varepsilon^2)$$

Theorem 2.2: A necessary and sufficient condition for the method (linearized-MODEL-scheme) to be stable is that the operator $\frac{d}{dt}$ satisfies the inequality: $\text{Re}\frac{\hat{d}}{dt} \leq 0$, where $\hat{\ }$ denotes the Fourier transform and $\frac{\hat{d}}{dt}$ is given by:

$$\frac{\partial F_0}{\partial\vec{q}}(\vec{q})\cdot\hat{\vec{L}}_-\frac{\hat{d}}{dt} + \hat{\vec{L}}_-\Big[\frac{\partial\vec{H}}{\partial\vec{q}_-}(\vec{q},\vec{q})\bullet\hat{\vec{L}}_- + \frac{\partial\vec{H}}{\partial\vec{q}_+}(\vec{q},\vec{q})\bullet\hat{\vec{L}}_+\Big] = 0 \tag{2.1}$$

For our two generic examples, we prove below that the linearized methods corresponding to (TSD-scheme) and (FP-scheme) satisfy the condition of Theorem 2.2. We need first the following lemma:

Lemma 2.3: The first order accurate difference operators $D^y_\pm$ and the second order accurate difference operators $L^x_\pm$ and $L^y_\pm$ defined in Section 1 satisfy the following equations:

(i) $\quad \hat{D}^y_+\hat{D}^y_- = -\Big(\frac{2\sigma}{\Delta y}\Big)^2$

(ii) $\quad \hat{L}^x_\pm = \frac{2s}{\Delta x}[\mp 2s^3 + ic\,(1+2s^2)]$; $\quad \hat{L}^y_\pm = \frac{2\sigma}{\Delta y}[\mp 2\sigma^3 + i\gamma(1+2\sigma^2)]$

(iii) $\quad \hat{L}^x_+\hat{L}^x_- = -\Big(\frac{2s}{\Delta x}\Big)^2(1+3s^2)$; $\quad \hat{L}^y_+\hat{L}^y_- = -\Big(\frac{2\sigma}{\Delta y}\Big)^2(1+3\sigma^2)$

where $\frac{\alpha}{\Delta}x$ and $\frac{\beta}{\Delta}y$ are the Fourier variables and where we set: $s = \sin\frac{\alpha}{2}$, $\sigma = \sin\frac{\beta}{2}$, $c = \cos\frac{\alpha}{2}$ and $\gamma = \cos\frac{\beta}{2}$.

Proof: The proof of this lemma is purely algebraic and is left to the reader.

2.2. Example 1: Transonic Small Disturbance Equation

Lemma 2.4: Assume that $h(u_-, u_+)$ is a differentiable numerical flux.

A necessary and sufficient condition for the method (TSD-scheme) to be linearly stable is that the operator $\frac{d}{dt}$ in:

$$2k\,L^x_-\frac{d}{dt}\phi_{jk} + L^x_-\left[\frac{\partial h}{\partial u_-}(u,u)L^x_-\phi_{jk} + \frac{\partial h}{\partial u_+}(u,u)L^x_+\phi_{jk}\right] - D^y_-D^y_+\phi_{jk} = 0 \qquad \text{(linearized-TSD-scheme)}$$

satisfies the inequality: $\text{Re}\dfrac{\hat{d}}{dt} \le 0$.

Proof: The result of this Lemma is an immediate application of Theorem 2.2 since:

$$\frac{\partial F_0}{\partial \vec{q}} \equiv [2k,0]^T \; , \quad \frac{\partial \vec{H}}{\partial \vec{q}_-}(\vec{q},\vec{q}) = \begin{bmatrix} \dfrac{\partial h}{\partial u_-}(u,u) & 0 \\ 0 & 0 \end{bmatrix} , \quad \frac{\partial \vec{H}}{\partial \vec{q}_+}(\vec{q},\vec{q}) = \begin{bmatrix} \dfrac{\partial h}{\partial u_+}(u,u) & 0 \\ 0 & -1 \end{bmatrix}$$

Theorem 2.5: A sufficient condition for the method (TSD-scheme) to be linearly stable is:

$$\frac{\partial h}{\partial u_-}(u,u) - \frac{\partial h}{\partial u_+}(u,u) \ge 0 \tag{2.2}$$

Proof: Assume that (2.2) holds. By part (ii) of Lemma 2.3, we have: $\text{Re}\hat{L}^x_- = -\text{Re}\hat{L}^x_+ = \dfrac{4s^4}{\Delta x} \ge 0$. By part (i) of Lemma 2.3, we have: $\hat{D}^y_+\hat{D}^y_- = -\dfrac{4\sigma^2}{\Delta y^2} \le 0$. Moreover $[\hat{L}^x_-]^{-1} = \dfrac{\hat{L}^x_+}{\hat{L}^x_-\hat{L}^x_+}$ thus, by parts (ii) & (iii) of Lemma 2.3, $\text{Re}[\hat{L}^x_-]^{-1} = \dfrac{s^2}{1+3s^2}\Delta x \ge 0$. Substitution into (linearized-TSD-scheme) multiplied by $[\hat{L}^x_-]^{-1}$ then implies $\text{Re}\dfrac{\hat{d}}{dt} \le 0$, thus, by Lemma 2.4, the method (TSD-scheme) is linearly stable.

Remark: The sufficient condition (2.2) is satisfied for most commonly used differentiable numerical fluxes, in particular differentiable monotone fluxes. In particular, one can use any of the following fluxes:

(Engquist-Osher)

$$h^{EO}(u_-,u_+) = \int_{\bar{u}}^{u_-} [f'(u)]^+du + \int_{\bar{u}}^{u_+} [f'(u)]^- du + f(\bar{u})$$

(Lax-Friedrichs-type)

$$h^{LF}(u_-,u_+) = \frac{1}{2}[f(u_-) + \alpha u_-] + \frac{1}{2}[f(u_+) - \alpha u_+] \quad \text{where } \alpha > |f'|$$

(No-Switch-type)[5]

$$h^{NS}(u_-,u_+) = \begin{cases} u_+\dfrac{f(u_-)}{u_-} & \text{if } M_\infty \le 1 \\[3mm] u_-\dfrac{f(u_+)}{u_+} & \text{if } M_\infty \ge 1 \end{cases}$$

---

[5] For the TSD equation, this flux, introduced by the author in [11], has the advantage that no local switching is needed.

2.3. Example 2: Full Potential Equation

For the FP example, we further introduce the following streamwise and normal finite difference operators[6] :

$$L_{\pm}^s \equiv \frac{\vec{q}}{q} \cdot \vec{L}_{\pm} \qquad \text{(streamwise)}$$

$$L_{\pm}^n \equiv (\frac{\vec{q}}{q} \times \vec{L}_{\pm}) \cdot (\hat{i}_x \times \hat{i}_y) \qquad \text{(normal)}$$

Lemma 2.6: The streamwise and normal finite difference operators $L_{\pm}^s$ and $L_{\pm}^n$ satisfy the following equations:

(i) $\quad \hat{L}_{\pm}^s \quad = 2\left[\mp 2\left[\dfrac{|\xi|s^4}{\Delta x} + \dfrac{|\eta|\sigma^4}{\Delta y}\right] + i\left[\dfrac{\xi sc\,(1+2s^2)}{\Delta x} + \dfrac{\eta \sigma \gamma(1+2\sigma^2)}{\Delta y}\right]\right]$

(ii) $\quad \hat{L}_{\pm}^n \quad = 2\left[\mp 2\left[\dfrac{|\eta|s^4}{\Delta x} - \dfrac{|\xi|\sigma^4}{\Delta y}\right] + i\left[\dfrac{\eta sc\,(1+2s^2)}{\Delta x} - \dfrac{\xi \sigma \gamma(1+2\sigma^2)}{\Delta y}\right]\right]$

(iii) $\quad \vec{L}_+ \cdot \vec{L}_- = L_+^s L_-^s + L_+^n L_-^n = L_+^x L_-^x + L_+^y L_-^y$

(iv) $\quad \hat{L}_+^s \hat{L}_-^s \quad = -4\left[\dfrac{\xi^2 s^2(1+3s^2)}{(\Delta x)^2} + \dfrac{2\xi\eta s\sigma[4s^2\sigma^2(s\sigma\,\text{sign}[\xi\eta]+c\gamma)+c\gamma(1+2s^2+2\sigma^2)]}{\Delta x\,\Delta y} + \dfrac{\eta^2\sigma^2(1+3\sigma^2)}{(\Delta y)^2}\right]$

(v) $\quad \hat{L}_+^n \hat{L}_-^n \quad = -4\left[\dfrac{\eta^2 s^2(1+3s^2)}{(\Delta x)^2} - \dfrac{2\xi\eta s\sigma[4s^2\sigma^2(s\sigma\,\text{sign}[\xi\eta]+c\gamma)+c\gamma(1+2s^2+2\sigma^2)]}{\Delta x\,\Delta y} + \dfrac{\xi^2\sigma^2(1+3\sigma^2)}{(\Delta y)^2}\right]$

(vi) $\quad \hat{L}_+^s \hat{L}_-^s \quad \leq 0$

(vii) $\quad \hat{L}_+^n \hat{L}_-^n \quad \leq 0$

where we set $\xi = \dfrac{u}{q}$ and $\eta = \dfrac{v}{q}$.

Proof: The proof of the above equalities is purely algebraic and can be found in [11].

Lemma 2.7: The following relations hold:

$$\rho'(q) = -\rho(q)\frac{M^2(q)}{q} \quad \text{and} \quad \frac{d}{dq}[\rho(q)q] = \rho(q)[1 - M^2(q)]$$

where the *Mach number* $M(q)$ is defined by: $M(q) = \dfrac{q}{a(q)}$ and $a(q)$ is the local *speed of sound* related to the density $\rho(q)$ by the isentropic relation: $a^2 = \dfrac{\rho^{\gamma-1}}{M_\infty^2}$.

Proof: This is a simple consequence of the definition of the functions $\rho(q)$ and $M(q)$.

Theorem 2.8: The method (FP-scheme) is linearly stable.

Proof: Assume[7] $\partial_x\Phi, \partial_y\Phi \geq 0$ (thus $u, v \geq 0$). We have:

---

[6] If $\partial_x\Phi, \partial_y\Phi \geq 0$, then $L_{\pm}^s = \frac{u}{q}L_{\pm}^x + \frac{v}{q}L_{\pm}^y$ and $L_{\pm}^n = -\frac{v}{q}L_{\pm}^x + \frac{u}{q}L_{\pm}^y$ where $u, v \geq 0$. The other cases yield similar expressions.

[7] The proof for the other cases is similar.

$$\vec{H}(\vec{q}_- , \vec{q}_+) = \rho(q_-)\vec{q}_+ , \quad \frac{\partial F_0}{\partial \vec{q}}(\vec{q}) = \rho'(q)\frac{\vec{q}}{q} , \quad \frac{\partial \vec{H}}{\partial \vec{q}_-}(\vec{q},\vec{q}) = \rho'(q)\frac{\vec{q}\,\vec{q}^T}{q} , \quad \frac{\partial \vec{H}}{\partial \vec{q}_+}(\vec{q},\vec{q}) = \rho(q)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Set $\rho = \rho(q)$ and $M = M(q)$. Then, using Lemma 2.7, the linearized finite difference method corresponding to (FP-scheme) can be rewritten as:

$$-\frac{\rho M^2}{q}\frac{\vec{q}}{q}\cdot\vec{L}_-\frac{d}{dt}\phi_{jk} + \vec{L}_-\cdot[-\frac{\rho M^2}{q}\frac{\vec{q}\,\vec{q}^T}{q} \bullet \vec{L}_-\phi_{jk} + \rho\vec{L}_+\phi_{jk}] = 0 \qquad \text{(linearized-FP-scheme)}$$

Using $\vec{L}_-\cdot\left[\frac{\vec{q}}{q}\frac{\vec{q}^T}{q} \bullet \vec{L}_-\right] = \vec{L}_-\cdot\left[\frac{\vec{q}}{q}(\frac{\vec{q}}{q}\cdot\vec{L}_-)\right] = \left[\frac{\vec{q}}{q}\cdot\vec{L}_-\right]^2 \equiv (L^s_-)^2$, (linearized-FP-scheme) can be rewritten as:

$$\left(L^s_-\frac{d}{dt} + q\left[(L^s_-)^2 - M^{-2}\vec{L}_-\cdot\vec{L}_+\right]\right)\phi_{jk} = 0$$

and equation (2.1) reduces to:

$$\hat{L}^s_-\frac{\hat{d}}{dt} + q\left[(\hat{L}^s_-)^2 - M^{-2}\hat{\vec{L}}_-\cdot\hat{\vec{L}}_+\right] = 0 \qquad (2.3)$$

By part (i) of Lemma 2.6, we have: $\text{Re}\hat{L}^s_- \ge 0$. By part (iii) of Lemma 2.6, we have $\hat{\vec{L}}_-\cdot\hat{\vec{L}}_+ \le 0$. Moreover $[\hat{L}^s_-]^{-1} = \dfrac{\hat{L}^s_+}{\hat{L}^s_-\hat{L}^s_+}$, thus by parts (i) & (vi) of Lemma 2.6, we have: $\text{Re}[\hat{L}^s_-]^{-1} \ge 0$. Substitution into (2.3) multiplied by $[\hat{L}^s_-]^{-1}$ then implies $\text{Re}\dfrac{\hat{d}}{dt} \le 0$, hence, by Corollary 2.2, the method (FP-scheme) is linearly stable.

3. TOTAL VARIATION STABILITY

A desirable property for a numerical scheme is a bound on the total variation of its approximate solutions. This ensures that overshoots and undershoots near shocks do not appear in the approximate solutions.

In this section, we propose a technique to render the method (MODEL-scheme) introduced in Section 1 total variation stable by applying flux limiters to the nonlinear terms of the difference equations dimension by dimension. For the sake of simplicity, a uniform mesh is assumed. The generalization to a non-uniform mesh is straightforward.

It turns out that the new method obtained using this technique specializes, in the one-dimensional case, to an **extension** of the *generalized MUSCL*[8] *schemes* which have been used to solve the well-known inviscid Burgers' equation (see [12]).

We observe that, when restricted to the 1D case, both the TSD equation and the FP equation have the form of the inviscid Burgers' equation.

In the case of the TSD equation, the 1D restriction of (MODEL-limited-scheme) can easily be written as a generalized MUSCL scheme.

In the case of the FP equation, the 1D restriction of (MODEL-limited-scheme) can be written as an extension of a generalized MUSCL scheme which we will introduce.

3.1. Total Variation Non-Increasing version of our Model Scheme

We extend the second order accurate method (MODEL-scheme) introduced in Section 1 to:

$$\partial_t F_0(\vec{L}_-\Phi_{jk}) + \vec{L}_-\cdot\vec{H}(\underline{L}_-\Phi_{jk} , \underline{L}_+\Phi_{jk}) = 0 \qquad \text{(MODEL-limited-scheme)}$$

---

[8] The notion of MUSCL (Monotone Upstream Centered Schemes for Conservation Laws) schemes is due to van Leer (see [19]).

where $\vec{L}_{\pm}$ are difference operators which approximate $\vec{\nabla}$ with second order accuracy.

We define some *limited* second order difference operators by:

$$\underline{L}_{\pm}^{x} = D_{\pm}^{x} + m\left(S_{\pm}^{x}(D_{0}^{x}-D_{\pm}^{x}),D_{\mp}^{x}-D_{0}^{x}\right) \text{ and } \underline{L}_{\pm}^{y} = D_{\pm}^{y} + m\left(S_{\pm}^{y}(D_{0}^{y}-D_{\pm}^{y}),D_{\mp}^{y}-D_{0}^{y}\right)$$

In the above, the **minmod** function m is defined by:

$$m(u_{-},u_{+}) = \begin{cases} s\ \min(|u_{-}|,|u_{+}|) & \text{if } sgn(u_{-}) = sgn(u_{+}) = s \\ 0 & \text{else} \end{cases} = sgn(u_{-}) \cdot \max[0,\min(|u_{-}|,u_{+}sgn(u_{-})]$$

In the uniform mesh case, we have:

$$\underline{L}_{\pm}^{x} = D_{\pm}^{x} \mp \frac{1}{2}m(D_{+}^{x}\Delta_{\pm}^{x},\ D_{-}^{x}\Delta_{\pm}^{x}) \text{ and } \underline{L}_{\pm}^{y} = D_{\pm}^{y} \mp \frac{1}{2}m(D_{+}^{y}\Delta_{\pm}^{y},\ D_{-}^{y}\Delta_{\pm}^{y})$$

For the TSD case, we choose: $\vec{L}_{\pm} = [\underline{L}_{\pm}^{x}, D_{\pm}^{y}]^{T}$.

For the FP case, we choose: $\vec{L}_{\pm}(\ ) = \left[\{\underline{L}_{\pm}^{x}(\ )\}^{+} - \{\underline{L}_{\mp}^{x}(\ )\}^{-},\ \{\underline{L}_{\pm}^{y}(\ )\}^{+} - \{\underline{L}_{\mp}^{y}(\ )\}^{-}\right]^{T}$.

3.2. One-dimensional case/ Inviscid Burgers' Equation

We now look at the one-dimensional inviscid Burgers' equation:

$$\partial_{t}u + \partial_{x}f(u) = 0 \qquad\qquad\qquad \text{(Burgers)}$$

where f is a convex function.

Consider the following method of lines conservation form finite difference approximation to Burgers' equation:

$$\partial_{t}u_{j-1/2} + D_{-}^{x}h(u_{j-1/2},u_{j+1/2}) = 0 \qquad\qquad \text{(Burgers-scheme)}$$

where the numerical flux h is assumed to satisfy the consistency condition: $h(u,u)=f(u)$.

We recall some definitions and a theorem due to Osher.

Definition 3.1: The numerical flux h is said to be *monotone* if $h(u_{-},u_{+})$ is a non-decreasing function of $-u_{+}$ and of u_{-}.

The method (Burgers-scheme) is said to be a *monotone scheme* if its numerical flux is monotone.

In particular, if h is C^{1}, a necessary and sufficient condition for the method (Burgers-scheme) to be monotone is: $\dfrac{\partial h}{\partial u_{+}} \leq 0 \leq \dfrac{\partial h}{\partial u_{-}}$.

Definition 3.2: The numerical flux $h(u_{-},u_{+})$ is said to be an *E-flux* (see [14]) if the following condition holds:

$$sgn(u_{+}-u_{-})[h(u_{-},u_{+}) - f(u)] \leq 0 \ , \text{ for any } u \in I[u_{-},u_{+}]^{9}$$

E-fluxes can be characterized as the fluxes for which: $sgn(u_{+}-u_{-})[h(u_{-},u_{+}) - h^{G}(u_{-},u_{+})] \geq 0$

where $h^{G}(u_{-},u_{+}) = \begin{cases} \min\limits_{[u_{-},u_{+}]} f & \text{if } u_{-} < u_{+} \\ \max\limits_{[u_{+},u_{-}]} f & \text{if } u_{-} \geq u_{+} \end{cases}$ is the Godunov numerical flux ([5]).

---

9 $I[a,b]$ denotes the closed interval $[\min(a,b),\max(a,b)]$ and $sgn(\alpha) = \begin{cases} \dfrac{\alpha}{|\alpha|} & \text{if } \alpha \neq 0 \\ 0 & \text{else} \end{cases}$

The method (Burgers-scheme) is said to be an *E-scheme* if its numerical flux is an E-flux.

Definition 3.3: Define the *total variation* of the function u by: $TV(u) = \sum_j |\Delta_+^x u_{j-1/2}|$ where $u_{j-1/2} = T^\Delta u$.

The method (Burgers-scheme) is said to be *total variation non-increasing*[10] if the total variation of its solutions satisfies:

$$TV[u(t)] \le TV[u(0)] \quad \text{for all } t > 0$$

In [7], Harten showed that a sufficient condition for the $(2p+1)$-point scheme

$$\partial_t u_{j-1/2} + D_-^x h(u_{j-p+1/2}, \cdots, u_{j-1/2}, u_{j+1/2}, \cdots, u_{j+p-1/2}) = 0$$

to be TVD is that the flux h satisfies an **incremental** form:

$$D_- h(u_{j-p+1/2}, \cdots, u_{j-1/2}, u_{j+1/2}, \cdots, u_{j+p-1/2}) = -C_j^+ D_+ u_{j-1/2} + C_{j-1}^- D_- u_{j-1/2} \tag{3.1}$$

where $\left\{ \begin{array}{l} C_j^+ = C^+(u_{j-p+1/2}, \cdots, u_{j-1/2}, u_{j+1/2}, \cdots, u_{j+p-1/2}) \\ C_{j-1}^- = C^-(u_{j-p-1/2}, \cdots, u_{j-1/2}, u_{j+1/2}, \cdots, u_{j+p-3/2}) \end{array} \right\}$ for some nonnegative functions C^+ and C^-.

We now extend (Burgers-scheme) to a second order accurate TVD scheme.

Theorem 3.4 (Osher [12]): Consider the scheme:

$$\partial_t u_{j-1/2} + D_-^x h\left(u_{j-1/2} + \frac{\Delta x}{2}\sigma_{j-1/2}, u_{j+1/2} - \frac{\Delta x}{2}\sigma_{j+1/2}\right) = 0 \qquad \text{(MUSCL-scheme)}$$

If $h(u_-, u_+)$ is an E-flux (resp. a differentiable monotone flux), then the above scheme is TVD provided that the following condition holds:

$$0 \le \frac{\sigma_{j-1/2}}{D_\pm^x u_{j-1/2}} \le 1 \quad (\text{resp. } 1 \ge \pm\frac{1}{2}\frac{\Delta_+^x \sigma_{j-1/2}}{D_+^x u_{j-1/2}} \text{ near points where } \frac{\partial h}{\partial u_\pm} \ne 0).$$

3.3. Example 1: Transonic Small Disturbance Equation

We now propose a *limited* version of (TSD-scheme):

$$2k\partial_t L_-^x \Phi_{jk} + L_-^x h(\underline{L}_-^x \Phi_{jk}, \underline{L}_+^x \Phi_{jk}) - D_-^y D_+^y \Phi_{jk} = 0 \qquad \text{(TSD-limited-scheme)}$$

The 1D restriction of the TSD equation:

$$2k\,\partial_t\partial_x\Phi + \partial_x f(\partial_x\Phi) = 0 \quad \text{where } f(u) = \frac{\gamma+1}{2}u^2 - Ku$$

can be rewritten as Burgers' equation:

$$\partial_t u + \partial_x f(u) = 0 \quad \text{where } u \equiv \partial_x\Phi$$

We note that: $f''(u) = \gamma+1 > 0$ i.e. f is a convex function.

The y-independent restriction of (TSD-limited-scheme) is given by:

$$2k\partial_t L_-^x \Phi_{jk} + L_-^x h(\underline{L}_-^x \Phi_{jk}, \underline{L}_+^x \Phi_{jk}) = 0 \tag{3.2}$$

---

[10] This concept was first introduced in [7]. Traditionally the abbreviation TVD (total variation diminishing) is preferred to TVNI.

Theorem 3.5: If h is either a consistent E-flux or a consistent differentiable monotone flux, then the scheme (3.2) is TVD.

Proof: Set $u_{j-1/2} = D_-^x \Phi_j$ and $\sigma_{j-1/2} = m(D_-^x u_{j-1/2}, D_+^x u_{j-1/2})$ [11]. Equation (3.2) then takes the form:

$$\left(1 + \frac{1}{2}\Delta^x\right)\left[2k\,\partial_t u_{j-1/2} + D_-^x h\left(u_{j-1/2} + \frac{\Delta x}{2}\sigma_{j-1/2}, u_{j+1/2} - \frac{\Delta x}{2}\sigma_{j+1/2}\right)\right] = 0$$

which is of the form (MUSCL-scheme).

It is immediate from the definition of $\sigma_{j-1/2}$ and the definition of the minmod function that the first condition of Theorem 3.4 is always satisfied.

The second condition of Theorem 3.4 follows from [12] :

$$\frac{1}{2}\left|\frac{\sigma_{j+1/2} - \sigma_{j-1/2}}{D_+^x u_{j-1/2}}\right| = \frac{1}{2}\left|\frac{m(D_+^x u_{j-1/2}, D_+^x u_{j+1/2})}{D_+^x u_{j-1/2}} - \frac{m(D_-^x u_{j-1/2}, D_+^x u_{j-1/2})}{D_+^x u_{j-1/2}}\right| \le \frac{1}{2} < 1$$

due to the following property of the minmod function:

$$\frac{m(x,y)}{x} \in [0,1] \text{ if } x \ne 0, \quad \frac{m(x,y)}{y} \in [0,1] \text{ if } y \ne 0$$

Therefore (3.2) is TVD for any consistent numerical flux h which is either an E-flux or a differentiable monotone flux. This class of numerical fluxes contains the most commonly used ones.

3.4. Example 2: Full Potential Equation

We now propose a *limited* version of (FP-scheme).

$$\partial_t \rho(|\vec{L}_-\Phi_{jk}|) + \vec{L}_- \cdot \left[\rho(|\vec{L}_-\Phi_{jk}|)\vec{L}_+\Phi_{jk}\right] = 0 \qquad \text{(FP-limited-scheme)}$$

where we set $\vec{L}_\pm = [L_\pm^x, L_\pm^y]^T$.

The 1D restriction of the FP equation:

$$\partial_t \rho(q) + \partial_x[\rho(q)q] = 0 \quad \text{where} \quad \rho(q) = \left[1 - \frac{\gamma-1}{2}M_\infty^2(q^2-1)\right]^{\frac{1}{\gamma-1}}$$

can be rewritten as Burgers' equation:

$$\partial_t u + \partial_x f(u) = 0$$

where $u \equiv -\rho$ and $f(u) = u\,q(u)$, $q(u) \equiv \left[1 + \frac{1-(-u)^{\gamma-1}}{\frac{\gamma-1}{2}M_\infty^2}\right]^{1/2}$

We note that, by Lemma 2.7, we have:

$$f'(u) = q(u)\left[1 - M^{-2}(q(u))\right]$$

thus $f' > 0$ (resp. $f' < 0$) in supersonic (resp. subsonic) regions and:

$$f''(u) = \frac{q(u)}{-u}M^{-2}(q(u))\left[\gamma + M^{-2}(q(u))\right] > 0$$

i.e. f is a convex function.

---

[11] the k-subscripts are dropped for the sake of clarity.

[12] WLOG we can assume $D_+^x u_{j-1/2} \ne 0$.

The 1D restriction of (FP-limited-scheme) is given by:

$$\partial_t \rho(|L^x_- \Phi_{jk}|) + L^x_- \left[\rho(|\underline{L}^x_- \Phi_{jk}|)\underline{L}^x_+ \Phi_{jk}\right] = 0 \tag{3.3}$$

Theorem 3.6: The scheme (3.3) is TVD in the linear case.

Proof: Set[13] $q_{j-1/2} = D^x_- \Phi_j$ and $u_{j-1/2} = -\Lambda^{-1}\rho(\Lambda q_{j-1/2})$, i.e. $q_{j-1/2} = \Lambda^{-1}q(\Lambda u_{j-1/2})$, where $\Lambda \equiv 1 + \frac{1}{2}\Delta^x$ and

where we *formally* define: $\Lambda^{-1} = \sum_{n \in Z_+}(-\frac{1}{2}\Delta^x)^n$.

Set $\sigma_{j-1/2} = m(D^x_- q_{j-1/2}, D^x_+ q_{j-1/2})$ and $h(q_-, q_+) = -\rho(q_-)q_+$.

Equation (3.3) then[14] takes the form:

$$-\left(1 + \frac{1}{2}\Delta^x\right)\left[\partial_t u_{j-1/2} + D^x_- h\left(q_{j-1/2} + \frac{\Delta x}{2}\sigma_{j-1/2}, q_{j+1/2} - \frac{\Delta x}{2}\sigma_{j+1/2}\right)\right] = 0$$

i.e.

$$\partial_t u_{j-1/2} + D^x_- h\left(q_{j-1/2} + \frac{\Delta x}{2}\sigma_{j-1/2}, q_{j+1/2} - \frac{\Delta x}{2}\sigma_{j+1/2}\right) = 0 \qquad \text{(extended-MUSCL-scheme)}$$

Note that:

(i) $h(q(u_-), q(u_+))$ is consistent with f since $h(q(u), q(u)) = -\rho(q(u))q(u) = u\, q(u) = f(u)$.

(ii) h is an E-flux since for any $q \in I[q_-, q_+]$, we have:

$sgn(q_+ - q_-) \cdot [\rho(q_-) - \rho(q)] \geq 0$ since $\rho'(q) \leq 0$

Thus $sgn(q_+ - q_-) \cdot [h(q_-, q_+) - h(q, q)] = -sgn(q_+ - q_-) \cdot [\rho(q_-)q_+ - \rho(q)q]$

$\leq sgn[\rho(q)(q_+ - q_-)] \cdot (q - q_+) \leq 0$ since $\rho(q) \geq 0$ and $(q_+ - q_-)(q - q_+) \leq 0$

(iii) h is a differentiable monotone flux since:

$$\frac{\partial h}{\partial q_-}(q_-, q_+) = \frac{\rho(q_-)M^2(q_-)}{q_-}q_+ \geq 0 \quad \text{and} \quad \frac{\partial h}{\partial q_+}(q_-, q_+) = -\rho(q_-) \leq 0$$

For the **linear case**, let: $\Phi_j(t) = \bar{q}\, j\Delta x + \varepsilon\phi_j(t)$, ε small, where $\bar{q}$ is a constant state.

Set $v_{j-1/2} = D^x_- \phi_j$ and $s_{j-1/2} = m(D^x_- v_{j-1/2}, D^x_+ v_{j-1/2})$. Using:

$$q_{j-1/2} \qquad\qquad\qquad\qquad = \bar{q} + \varepsilon v_{j-1/2}$$

$$u_{j-1/2} = -\Lambda^{-1}\rho(\Lambda q_{j-1/2}) \qquad = -\rho(\bar{q}) + \varepsilon\frac{\rho(\bar{q})M^2(\bar{q})}{\bar{q}}v_{j-1/2} + O(\varepsilon^2)$$

$$\sigma_{j-1/2} = m(\varepsilon D^x_- v_{j-1/2}, \varepsilon D^x_+ v_{j-1/2}) = \varepsilon s_{j-1/2}$$

$$q_{j\pm1/2} \mp \frac{\Delta x}{2}\sigma_{j\pm1/2} \qquad\qquad = \bar{q} + \varepsilon\left(v_{j\pm1/2} \mp \frac{\Delta x}{2}s_{j\pm1/2}\right)$$

$$h\left(q_{j-1/2} + \frac{\Delta x}{2}\sigma_{j-1/2}, q_{j+1/2} - \frac{\Delta x}{2}\sigma_{j+1/2}\right)$$

$$= \varepsilon\frac{\rho(\bar{q})M^2(\bar{q})}{\bar{q}}\, l\left(v_{j-1/2} + \frac{\Delta x}{2}s_{j-1/2}, v_{j+1/2} - \frac{\Delta x}{2}s_{j+1/2}\right) + O(\varepsilon^2)$$

---

[13] the k-subscripts are dropped for the sake of clarity.

[14] in the case $\partial_x \Phi \geq 0$. The case $\partial_x \Phi \leq 0$ can be treated in a similar fashion.

where we set $l(v_-, v_+) = \overline{q}[v_- - M^{-2}(\overline{q})v_+]$, yields the linear version of scheme (3.3):

$$\partial_t v_{j-1/2} + D_-^x \; l \; (v_{j-1/2} + \frac{\Delta x}{2} s_{j-1/2}, v_{j+1/2} - \frac{\Delta x}{2} s_{j+1/2}) = 0 \tag{3.4}$$

Note that l is an E-flux since for any $v \in I[v_-, v_+]$ we have:

$$sgn\,(v_+ - v_-) \cdot \left[l(v_-, v_+) - l(v, v) \right] = \overline{q} \; sgn\,(v_+ - v_-) \cdot \left[l(v_- - v) - M^{-2}(\overline{q})(v_+ - v) \right] \le 0$$

The inequality $0 \le \dfrac{\sigma_{j-1/2}}{D_{\pm}^x u_{j-1/2}} \le 1$ is immediate from the definition of $s_{j-1/2}$ and the definition of the minmod function, therefore the first condition of Theorem 3.4 is satisfied thus the scheme (3.4) is TVD, i.e. the scheme (3.3) is TVD in the linear case.[15]

For the **nonlinear** case, it is not possible to apply Theorem 3.4 and it appears difficult to obtain an incremental form such as (3.1).

Numerical evidence supporting or contradicting total variation stability for the scheme (3.3) in the nonlinear case will presented in a subsequent paper.

4. ENTROPY CONDITION FOR THE TSD SCHEME

In this section, we prove that the approximate solutions of our new second order accurate, TVD, semi-discrete finite difference method (TSD-limited-scheme) for the low-frequency unsteady TSD equation satisfy a discrete entropy inequality which guarantees that they converge to the unique (physical) solution of (TSD).

We restrict ourselves to the case when an E-flux is used as a building block. For the sake of simplicity, a uniform mesh is assumed.

Part of the proof of the main theorem of this section relies on arguments similar to the ones found in [3] and [12] and makes use of Plancherel's theorem.

4.1. Differential Entropy Inequality

Up to some rescaling of t ($2k = 1$), the low-frequency, unsteady TSD equation (TSD) of Section 1 can be rewritten as the following system:

$$\partial_t [\Phi_x, 0] + \partial_x [f(\Phi_x), \Phi_y] + \partial_y [-\Phi_y, -\Phi_x]_y = [0, 0] \tag{4.1}$$

This system is symmetrizable[16] by multiplying it by $\vec{\nabla}\Phi = \dfrac{\partial}{\partial \vec{q}} [\dfrac{1}{2} |\vec{q}|^2] \Big|_{\vec{q} = \vec{\nabla}\Phi}$

The entropy function is: $V(\vec{q}) = \dfrac{1}{2} u^2$ and the corresponding entropy fluxes are:

$F^{(1)}(\vec{q}) = \int u \, f'(u) \, du + \dfrac{1}{2} v^2$ and $F^{(2)}(\vec{q}) = -uv$ where $\vec{q} = [u, v]^T$.

The resulting differential entropy inequality is:

$$\partial_t V(\vec{\nabla}\Phi) + \vec{\nabla} \cdot [F^{(1)}(\vec{\nabla}\Phi), F^{(2)}(\vec{\nabla}\Phi)]^T \le 0 \tag{4.2}$$

---

[15] l is also a differentiable monotone flux so we could also have used the second condition of Theorem 3.4.

[16] The reader is referred to [8], [4] and [10] for an interesting discussion on the derivation of additional systems of conservation laws, additional symmetric hyperbolic systems of conservation laws and an entropy inequality from a given system of nonlinear conservation laws. See also Appendix A in [11].

It is well known that weak solutions of (4.1) are not unique and that the differential entropy inequality (4.2) is satisfied (in the weak sense) only by the physical solution, i.e. the one which does not admit expansion shocks.[17]

In the next paragraph, we show that (TSD-limited-scheme) satisfies a discrete entropy inequality which implies (4.2), thus the approximate solutions converge to the unique (physical) solution of the problem.

4.2. Discrete Entropy Inequality

Consider the second order accurate TVD scheme for the TSD equation derived in Section 3:

$$\partial_t L_-^x \Phi_{jk} + L_-^x h(\underline{L}_-^x \Phi_{jk}, \underline{L}_+^x \Phi_{jk}) - D_+^y D_-^y \Phi_{jk} = 0 \qquad \text{(TSD-limited-scheme)}$$

where h is a Lipschitz continuous E-flux consistent with the *convex* function f.

Set $u_{j-1/2,k} = D_-^x \Phi_{jk}$, $v_{j,k-1/2} = D_-^y \Phi_{jk}$ and $\sigma_{j-1/2,k} = m(D_-^x u_{j-1/2,k}, D_+^x u_{j-1/2,k})$.

The scheme can be rewritten as:

$$\partial_t u_{j-1/2,k} + D_-^x h\left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}, u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right) - \left[1+\frac{1}{2}\Delta_-^x\right]^{-1} D_+^y v_{j,k-1/2} = 0 \qquad (4.3)$$

The following equations hold:

(i) $\quad u_{j-1/2,k} \partial_t u_{j-1/2,k} = \partial_t \left[\frac{1}{2} u_{j-1/2,k}^2\right]$

(ii) $\quad u_{j-1/2,k} D_-^x h\left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}, u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right)$

$$= D_+^x \int^{u_{j-1/2,k}} u f'(u)\, du - D_+^x \left[u_{j-1/2,k}\, f(u_{j-1/2,k})\right] + D_+^x \int^{u_{j-1/2,k}} f(u)\, du$$

$$+ D_+^x \left[u_{j-1/2,k}\, h\left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}, u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right)\right]$$

$$- \left[D_+^x u_{j-1/2,k}\right] h\left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}, u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right)$$

$$= D_+^x \left[\int^{u_{j-1/2,k}} u f'(u)\, du + u_{j-1/2,k}\left\{h\left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}, u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right) - f(u_{j-1/2,k})\right\}\right]$$

$$- \int_{u_{j-1/2,k}}^{u_{j+1/2,k}} \left[h\left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}, u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right) - f(u)\right] \frac{du}{\Delta x}$$

(iii) $\quad -u_{j-1/2,k}\left[1+\frac{1}{2}\Delta_-^x\right]^{-1} D_+^y v_{j,k-1/2} = -u_{j-1/2,k} D_+^y v_{j,k-1/2} - u_{j-1/2,k}\left[\left[1+\frac{1}{2}\Delta_-^x\right]^{-1} - 1\right] D_+^y v_{j,k-1/2}$

$$= -D_+^y \left[u_{j-1/2,k}\, v_{j,k-1/2}\right] + \left[D_+^y u_{j-1/2,k}\right] v_{j,k+1/2} - u_{j-1/2,k}\left[\left[1+\frac{1}{2}\Delta_-^x\right]^{-1} - 1\right] D_+^y v_{j,k-1/2}$$

$$= D_+^y F^{(2)}(u_{j-1/2,k}, v_{j,k-1/2}) + \left[D_-^x v_{j,k+1/2}\right] v_{j,k+1/2} - u_{j-1/2,k}\left[\left[1+\frac{1}{2}\Delta_-^x\right]^{-1} - 1\right] D_+^y v_{j,k-1/2}$$

---

[17] In [10], Mock showed that (4.2) is equivalent to the (distribution) inequality: $\partial_x \partial_x \Phi \leq 0$.

$$= D_+^y F^{(2)}(u_{j-1/2,k}, v_{j,k-1/2}) + D_+^x \left[\frac{1}{2} v_{j-1,k+1/2}^2 \right] + \frac{\Delta x}{2} \left[D_-^x v_{j,k+1/2} \right]^2 - u_{j-1/2,k} \cdot \left[\left[1 + \frac{1}{2} \Delta^x \right]^{-1} - 1 \right] D_+^y v_{j,k-1/2}$$

$$= D_+^y F^{(2)}(u_{j-1/2,k}, v_{j,k-1/2}) + D_+^x \left[\frac{1}{2} v_{j-1,k+1/2}^2 \right] + \frac{\Delta x}{2} \left[D_-^x D_+^y \Phi_{jk} \right]^2 - D_-^x \Phi_{jk} \left[\left[1 + \frac{1}{2} \Delta^x \right]^{-1} - 1 \right] D_+^y D_-^y \Phi_{jk}$$

Thus multiplying the scheme (4.3) by $u_{j-1/2,k}$ yields:

$$\partial_t V_{jk} + D_+^x F_{j-1/2,k}^{(1)} + D_+^y F_{j,k-1/2}^{(2)} - \boxed{\text{II}} - \boxed{\text{III}} = 0$$

where:

$$V_{jk} = V(D_-^x \Phi_{jk}, .)$$
$$F_{j-1/2,k}^{(1)} = F^{(1)}(D_-^x \Phi_{jk}, S^x D_+^y \Phi_{jk}) + \boxed{\text{IIII}}$$
$$F_{j,k-1/2}^{(2)} = F^{(2)}(D_-^x \Phi_{jk}, D_-^y \Phi_{jk})$$

$$\boxed{\text{II}} = \int_{u_{j-1/2,k}}^{u_{j+1/2,k}} \left[h\left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}, u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right) - f(u) \right] \frac{du}{\Delta x}$$

$$\boxed{\text{III}} = D_-^x \Phi_{jk} \left[\left[1 + \frac{1}{2}\Delta^x \right]^{-1} - 1 \right] D_+^y D_-^y \Phi_{jk} - \frac{\Delta x}{2} \left[D_-^x D_+^y \Phi_{jk} \right]^2$$

$$\boxed{\text{IIII}} = u_{j-1/2,k} \left[h\left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}, u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right) - f(u_{j-1/2,k}) \right]$$

Lemma 4.1: The quantity $\boxed{\text{II}}$ is ≤ 0.

Proof: We first note that, since $1 - \dfrac{\sigma_{j-1/2,k} + \sigma_{j+1/2,k}}{2D_+^x u_{j-1/2,k}} \in [0,1]$, we have:

$$sgn\left[\left(u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right) - \left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}\right) \right] = sgn\left\{ \Delta_+^x u_{j-1/2,k} \cdot \left[1 - \frac{\sigma_{j-1/2,k} + \sigma_{j+1/2,k}}{2D_+^x u_{j-1/2,k}} \right] \right\}$$

$$= sgn\left(\Delta_+^x u_{j-1/2,k}\right)$$

Therefore, for any $\tilde{u}_{jk}$, $\boxed{\text{II}}$ can be written as:

$$\boxed{\text{II}} = |D_+^x u_{j-1/2,k}| \ \boxed{\text{A}} + \frac{1}{\Delta x} \boxed{\text{B}}$$

where the quantities $\boxed{\text{A}}$ and $\boxed{\text{B}}$ have the following expressions:

$$\boxed{\text{A}} = sgn\left[\left(u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right) - \left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}\right) \right]$$

$$\left[h\left(u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}, u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}\right) - f(\tilde{u}_{jk}) \right]$$

$$\boxed{\text{B}} = sgn\left(\Delta_+^x u_{j-1/2,k}\right) \int_{\min(u_{j-1/2,k}, u_{j+1/2,k})}^{\max(u_{j-1/2,k}, u_{j+1/2,k})} \left[f(\tilde{u}_{jk}) - f(u) \right] du$$

Set $u_{jk} = \dfrac{u_{j-1/2,k} + u_{j+1/2,k}}{2}$

(i) In the case of a **rarefaction** $(u_{j-1/2,k} < u_{j+1/2,k})$, we have: $D_+^x u_{j-1/2,k} > 0$ and:

$$u_{j-1/2,k} \leq u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k} \leq u_{jk} \leq u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k} \leq u_{j+1/2,k} \tag{4.4}$$

Choose $\tilde{u}_{jk} = u_{jk}$. Due to inequalities (4.4) and the assumption that h is an E-flux, we have: $\boxed{A} \le 0$.

Since f is convex, we have:

$$f(u_{jk}) = f\left[\frac{u_{j-1/2,k} + u_{j+1/2,k}}{2}\right] = f\left[\int\limits_{u_{j-1/2,k}}^{u_{j+1/2,k}} u \, \frac{du}{\Delta^x_+ u_{j-1/2,k}}\right] \le^{18} \int\limits_{u_{j-1/2,k}}^{u_{j+1/2,k}} f(u) \, \frac{du}{\Delta^x_+ u_{j-1/2,k}} = \frac{1}{\Delta^x_+ u_{j-1/2,k}} \int\limits_{u_{j-1/2,k}}^{u_{j+1/2,k}} f(u) \, du$$

thus $\boxed{B} = \int\limits_{u_{j-1/2,k}}^{u_{j+1/2,k}} [f(u_{jk}) - f(u)] \, du \le 0$.

(ii) In the case of a **shock** $(u_{j+1/2,k} < u_{j-1/2,k})$, we have: $D^x_+ u_{j-1/2,k} < 0$ and:

$$u_{j+1/2,k} \le u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k} \le u_{jk} \le u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k} \le u_{j-1/2,k} \tag{4.5}$$

where $\sigma_{j\pm1/2,k} = -\min(-D^x_+ u_{j-1/2,k}, |D^x_\pm u_{j\pm1/2,k}|)$.

Due to inequalities (4.5) and the assumption that h is an E-flux, we have:

$$\boxed{A} \le 0 \text{ if } \tilde{u}_{jk} \in [u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}, u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}].$$

The quantity $\boxed{B}$ is given by:

$$\boxed{B} = \int\limits_{u_{j+1/2,k}}^{u_{j-1/2,k}} [f(u) - f(\tilde{u}_{jk})] \, du$$

Let $\overline{u}$ denote the sonic point $(f'(\overline{u}) = 0)$. Assume that the shock is **nonsonic**, i.e. $\overline{u} \notin [u_{j+1/2,k}, u_{j-1/2,k}]$.

Let $\tilde{u}_{jk} = \dfrac{\displaystyle\int\limits_{u_{j+1/2,k}}^{u_{j-1/2,k}} u f'(u) \, du}{f(u_{j-1/2,k}) - f(u_{j+1/2,k})}$ \hfill (4.5)

We then have:

$$\boxed{B} = \int\limits_{u_{j+1/2,k}}^{u_{j-1/2,k}} [f(u) - f(\tilde{u}_{jk})] \, du = \int\limits_{u_{j+1/2,k}}^{u_{j-1/2,k}} [f(u) - f(\tilde{u}_{jk}) - (u - \tilde{u}_{jk})f'(u)] \, du$$

Set $g(u) = f(u) - f(\tilde{u}_{jk}) - (u - \tilde{u}_{jk})f'(u)$. Since $g'(u) = -(u - \tilde{u}_{jk})f''(u)$ and since f is convex, g has an absolute maximum at $u = \tilde{u}_{jk}$ thus $g(u) \le g(\tilde{u}_{jk}) = 0$, which yields $\boxed{B} \le 0$.

For any $u \in [u_{j+1/2,k}, u_{j-1/2,k}]$, we have: $sgn \, f'(u) = sgn[f(u_{j-1/2,k}) - f(u_{j+1/2,k})] = sgn(u_{jk} - \overline{u})$.

Moreover:

$$\frac{\tilde{u}_{jk} - u_{jk}}{\dfrac{u_{j-1/2,k} - u_{j+1/2,k}}{2}} = \int\limits_{u_{j+1/2,k}}^{u_{j-1/2,k}} \frac{u - u_{jk}}{\dfrac{u_{j-1/2,k} - u_{j+1/2,k}}{2}} \cdot \frac{f'(u) \, du}{f(u_{j-1/2,k}) - f(u_{j+1/2,k})} \in [-1, +1]$$

thus $\tilde{u}_{jk} \in [u_{j+1/2,k}, u_{j-1/2,k}]$ and $2|\tilde{u}_{jk} - u_{jk}| - (u_{j-1/2,k} - u_{j+1/2,k}) \le 0$.

The condition $\tilde{u}_{jk} \in [u_{j+1/2,k} - \frac{\Delta x}{2}\sigma_{j+1/2,k}, u_{j-1/2,k} + \frac{\Delta x}{2}\sigma_{j-1/2,k}]$, i.e.

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[18] by Jensen's inequality

$$\frac{\bar{u}_{jk} - u_{jk}}{\dfrac{u_{j-1/2,k} - u_{j+1/2,k}}{2}} \in \left[-1 - \frac{\Delta x \, \sigma_{j+1/2,k}}{u_{j-1/2,k} - u_{j+1/2,k}} , +1 + \frac{\Delta x \, \sigma_{j-1/2,k}}{u_{j-1/2,k} - u_{j+1/2,k}} \right]$$

is satisfied if we change $\sigma_{j-1/2,k}$ and $\sigma_{j+1/2,k}$ to:

$$\Delta x \, \sigma_{j-1/2,k} = \max \left[2 \, | \, \bar{u}_{jk} - u_{jk} \, | - (u_{j-1/2,k} - u_{j+1/2,k}), - \min \left(u_{j-1/2,k} - u_{j-1/2,k} , [u_{j+1/2,k} - u_{j-3/2,k}]^- \right) \right]$$

$$\Delta x \, \sigma_{j+1/2,k} = \max \left[2 \, | \, \bar{u}_{jk} - u_{jk} \, | - (u_{j-1/2,k} - u_{j+1/2,k}), - \min \left(u_{j-1/2,k} - u_{j+1/2,k} , [u_{j+1/2,k} - u_{j+3/2,k}]^+ \right) \right]$$

These changes are necessary only if $u_{j-1/2,k} < u_{j-3/2,k}$ or $u_{j+1/2,k} > u_{j+3/2,k}$.

The second order accuracy is not affected since:

$$\int\limits_{u_{j+1/2,k}}^{u_{j-1/2,k}} (u - u_{jk}) \, du = 0 \quad \text{and} \quad \int\limits_{u_{j+1/2,k}}^{u_{j-1/2,k}} (u - u_{jk})^2 \, du = \frac{1}{6} (u_{j-1/2,k} - u_{j+1/2,k})^2 \, \frac{u_{j-1/2,k} - u_{j+1/2,k}}{2}$$

$$\text{and} \quad \bar{u}_{jk} - u_{jk} = \frac{\displaystyle\int\limits_{u_{j+1/2,k}}^{u_{j-1/2,k}} (u - u_{jk})^2 \, \frac{f'(u) - f'(u_{jk})}{u - u_{jk}} \, du}{f(u_{j-1/2,k}) - f(u_{j+1/2,k})}$$

thus $\quad \dfrac{| \, \bar{u}_{jk} - u_{jk} \, |}{\dfrac{u_{j-1/2,k} - u_{j+1/2,k}}{2}} < \dfrac{\dfrac{1}{6}(u_{j-1/2,k} - u_{j+1/2,k})}{\dfrac{| \, f(u_{j-1/2,k}) - f(u_{j+1/2,k}) \, |}{u_{j-1/2,k} - u_{j+1/2,k}}} \max\limits_{[u_{j+1/2,k} , u_{j-1/2,k}]} f''$

thus $\quad 2 \, | \, \bar{u}_{jk} - u_{jk} \, | - (u_{j-1/2,k} - u_{j+1/2,k}) = O \left(| \, u_{j-1/2,k} - u_{j+1/2,k} \, |^2 \right)$.

In the case: $f(u) = \dfrac{\gamma + 1}{2} (u^2 - 2 \bar{u} u)$, we obtain:

$$\bar{u}_{jk} = \frac{1/3(u_{j+1/2,k}^2 + u_{j+1/2,k} u_{j-1/2,k} + u_{j-1/2,k}^2) - \bar{u} u_{jk}}{u_{jk} - \bar{u}} \quad \text{and} \quad \bar{u}_{jk} - u_{jk} = \frac{1}{6} \frac{u_{j-1/2,k} - u_{j+1/2,k}}{2} \frac{u_{j-1/2,k} - u_{j+1/2,k}}{u_{jk} - \bar{u}}$$

thus $\quad 2 \, | \, \bar{u}_{jk} - u_{jk} \, | - (u_{j-1/2,k} - u_{j+1/2,k}) = (u_{j-1/2,k} - u_{j+1/2,k}) \left[\dfrac{1}{6} \dfrac{u_{j-1/2,k} - u_{j+1/2,k}}{| \, u_{jk} - \bar{u} \, |} - 1 \right]$.

Assume now that the shock is **sonic**, i.e. $\bar{u} \in [u_{j+1/2,k} , u_{j-1/2,k}]$. We restrict $\sigma_{j-1/2,k}$ and $\sigma_{j+1/2,k}$ to be 0 (at the loss of second order accuracy).

If $f(u_{j-1/2,k}) \neq f(u_{j+1/2,k})$, then choose $\bar{u}_{jk}$ as given by formula (4.5).

Else, choose $\bar{u}_{jk}$ such that $f(\bar{u}_{jk}) = \max\limits_{[u_{j+1/2,k} , u_{j-1/2,k}]} f$.

Lemma 4.2: Let $\Psi_{jk}(t) = T^\Delta \Psi(t)$ where Ψ is a nonnegative test function with compact support $\subset \Omega \times [0,T]$. Then we have: $\sum\limits_{j,k} \boxed{\text{III}} \, \Psi_{jk} \Delta x \Delta y \leq 0$.

Proof: We can write[19]

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[19] For $a = \{a_{jk}\}$, $b = \{b_{jk}\} \in l^2$, we define: $(a,b)_{l^2} = \sum\limits_{j,k} a_{jk} \bar{b}_{jk} \Delta x \Delta y$, $| a |_{l^2} = (a,a)_{l^2}$, $\hat{a} = \{\hat{a}_{J,K}\} = \left\{ \sum\limits_{j,k} a_{jk} e^{-iJJ\Delta x} e^{-ikK\Delta y} \Delta x \Delta y \right\}$, $(\hat{a}, \hat{b})_{L^2} = \sum\limits_{J,K} \hat{a}_{J,K} \bar{\hat{b}}_{J,K} \Delta x \Delta y$ and $| \hat{a} |_{L^2} = (\hat{a}, \hat{a})_{L^2}$. Plancherel's theorem states that: $| a |_{l^2} = | \hat{a} |_{L^2}$ and $(a,b)_{l^2} = (\hat{a}, \hat{b})_{L^2}$.

$$\sum_{j,k} \boxed{\text{III}}\, \Psi_{jk}\Delta x\,\Delta y = \sum_{j,k}\left[D^x_{-}\Phi_{jk}\Big[[1+\tfrac{1}{2}\Delta^x]^{-1}-1\Big] D^y_+D^y_-\Phi_{jk} - \frac{\Delta x}{2}\Big[D^x_-D^y_+\Phi_{jk}\Big]^2\right]\Psi_{jk}\Delta x\,\Delta y$$

$$= \mathrm{Re}\left(\sqrt{\Psi}D^x_-\Phi,\sqrt{\Psi}\Big[[1+\tfrac{1}{2}\Delta^x]^{-1}-1\Big]D^y_+D^y_-\Phi\right)_{l^2} - \frac{1}{2}\mid\sqrt{\Delta x}\sqrt{\Psi}D^y_+D^y_-\Phi\mid^2_{l^2}$$

$$= \mathrm{Re}\left(\sqrt{\hat\Psi}\hat D^x_-\hat\Phi,\sqrt{\hat\Psi}\Big[[1+\tfrac{1}{2}\hat\Delta^x]^{-1}-1\Big]\hat D^y_+\hat D^y_-\hat\Phi\right)_{L^2} - \frac{1}{2}\mid\sqrt{\Delta x}\sqrt{\hat\Psi}\hat D^y_+\hat D^y_-\hat\Phi\mid^2_{L^2}$$

$$= \sum_{J,K}\mathrm{Re}\left[\hat D^x_-\Big[[1+\tfrac{1}{2}\overline{\Delta^x}]^{-1}-1\Big]\hat D^y_+\hat D^y_- - \frac{\Delta x}{2}\mid\hat D^x_-\hat D^y_+\mid^2\right]\mid\sqrt{\hat\Psi_{J,K}}\hat\Phi_{J,K}\mid^2\Delta x\,\Delta y$$

$$= \sum_{J,K}\Delta x\,\mathrm{Re}\left[\frac{[[1+\tfrac{1}{2}\overline{\Delta^x}]^{-1}-1]}{\overline{\Delta^x}}-\frac{1}{2}\right]\mid\hat D^x_-\hat D^y_+\mid^2\mid\sqrt{\hat\Psi_{J,K}}\hat\Phi_{J,K}\mid^2\Delta x\,\Delta y$$

$$\le 0\quad\text{since }\mathrm{Re}\left[\frac{[[1+\tfrac{1}{2}\overline{\Delta^x}]^{-1}-1]}{\overline{\Delta^x}}-\frac{1}{2}\right] = -\frac{1+2\sin^2\frac{J\Delta x}{2}}{1+3\sin^2\frac{J\Delta x}{2}} \le 0$$

Lemma 4.3: a.e. $\lim\limits_{\Delta x\to 0}\boxed{\text{III}} = 0$.

Proof: This is due to the Lipschitz continuity of h, the continuity of f and of the minmod function, and the consistency condition:

$$\text{a.e. }\lim_{\Delta x\to 0}\left[h\Big(u_{j-1/2,k}+\frac{\Delta x}{2}\sigma_{j-1/2,k},u_{j+1/2,k}-\frac{\Delta x}{2}\sigma_{j+1/2,k}\Big)-f(u_{j-1/2,k})\right]$$

$$= \text{a.e. }\lim_{\Delta x\to 0}\left[h\Big(u_{j-1/2,k}+\frac{\Delta x}{2}\sigma_{j-1/2,k},u_{j+1/2,k}-\frac{\Delta x}{2}\sigma_{j+1/2,k}\Big)-f\Big(u_{j-1/2,k}-\frac{\Delta x}{2}\sigma_{j-1/2,k}\Big)\right]$$

$$+ \text{a.e. }\lim_{\Delta x\to 0}\left[f\Big(u_{j-1/2,k}-\frac{\Delta x}{2}\sigma_{j-1/2,k}\Big)-f(u_{j-1/2,k})\right] = 0$$

Theorem 4.4: Suppose that Φ_{jk}, $L^x_\pm\Phi_{jk}$ and $D^y_\pm\Phi_{jk}$ converge boundedly a.e. as $\Delta x,\Delta y\to 0^+$ to Φ, $\partial_x\Phi$ and $\partial_y\Phi$ respectively.

Further assume that f is convex and that h is a Lipschitz continuous E-flux consistent with f .

Then Φ is a weak solution of (TSD-limited-scheme) which satisfies the entropy inequality (4.2).

Proof: Using Theorem 1.3, we only have to prove that Φ satisfies the distribution inequality (4.2). By Lemma 4.1, we have:

$$\partial_t V_{jk}+D^x_+F^{(1)}_{j-1/2,k}+D^y_+F^{(2)}_{j,k-1/2} = \boxed{\text{II}} + \boxed{\text{III}} \le \boxed{\text{III}}$$

Let Ψ be a nonnegative test function with compact support $\subset\Omega\times[0,T]$ and set $\Psi_{jk}(t)=T^\Delta\Psi(t)$.

Multiplying (4.3) by $u_{j-1/2,k}\Delta x\,\Delta y = D^x_-\Phi_{jk}\Delta x\,\Delta y$, summing over j,k and using Lemma 4.2 yield:

$$\sum_{j,k}\Delta x\,\Delta y\,\Psi_{jk}\left[\partial_t V_{jk}+D^x_+F^{(1)}_{j-1/2,k}+D^y_+F^{(2)}_{j,k-1/2}\right] \le 0$$

Using the summation by parts formulas of Lemma 2.2, the fact that Ψ has compact support and integrating w.r. to t over $[0,\infty)$ yields:

$$-\int\limits_{0}^{\infty} dt \sum_{j,k} \Delta x \Delta y \left[[\partial_t \Psi_{jk}] V_{jk} + [D_+^x \Psi_{jk}] F_{j-1/2,k}^{(1)} + [D_+^y \Psi_{jk}] F_{j,k-1/2}^{(2)} \right] \leq 0$$

Taking the limit as Δx, $\Delta y \rightarrow 0^+$ and using Lemma 4.3 and Lebesgue's Dominated Convergence Theorem yield:

$$-\iiint\limits_{0 \ \Omega}^{\infty} dt \, dx \, dy \left[[\partial_t \Psi] V(\vec{\nabla}\Phi) + [\partial_x \Psi] F^{(1)}(\vec{\nabla}\Phi) + [\partial_y \Psi] F^{(2)}(\vec{\nabla}\Phi) \right] \leq 0$$

i.e. the difference approximations Φ_{jk} satisfy the entropy condition (4.2) in the limit.

This theorem ensures that, if the difference approximations (TSD-limited-scheme) converge, then the limit solution is the unique (physical) solution of the problem.

5. TIME DISCRETIZATION OF THE TSD SCHEME

In this section, we present an implicit, forward Euler-type time discretization of the method (TSD-scheme) introduced in Section 1 and of its TVD extension (TSD-limited-scheme) introduced in Section 4.

This time discretization uses time-splitting in a fashion similar to that of the Alternate Direction Implicit (ADI) method.

The ADI method has been used extensively for implementing first order accurate finite difference schemes for the TSD equation (see e.g. [20], [15] and [6]).

Others have used an Approximate Factorization (AF) method instead of the ADI method (see e.g. [1]).

The implementation of the time-splitting algorithms presented here was done by the author.

Numerical results are the object of Section 6.

5.1. Time-Splitting Algorithm

The *unlimited* version of our time-splitting method can be written as:

x–sweep: $\quad 2k \, L_-^x \dfrac{\Phi_{jk}^* - \Phi_{jk}^n}{\Delta t} + L_-^x \dfrac{h(L_-^x \Phi_{jk}^n, L_+^x \Phi_{jk}^n) + h(L_-^x \Phi_{jk}^*, L_+^x \Phi_{jk}^*)}{2} - D_-^y D_+^y \Phi_{jk}^n = 0$ $\qquad$ (5.1a)

y–sweep: $\quad 2k \, L_-^x \dfrac{\Phi_{jk}^{n+1} - \Phi_{jk}^*}{\Delta t} - \dfrac{1}{2} D_-^y D_+^y (\Phi_{jk}^{n+1} - \Phi_{jk}^n) = 0$ $\qquad$ (5.1b)

This method is linearly stable as proved in Theorem 5.1 below.

5.2. Linear Stability Analysis

Using the same notations as in Section 2, the amplification coefficient of the time-splitting algorithm (5.1) is given by:

$$G = \frac{1-z}{1+z} \frac{1+z'}{1-z'} \quad \text{where:} \quad \begin{cases} z = \dfrac{\Delta t}{4k} \left[\dfrac{\partial h}{\partial u_-}(u,u)\hat{L}_-^x + \dfrac{\partial h}{\partial u_+}(u,u)\hat{L}_+^x \right] \\[2mm] z' = \dfrac{\Delta t}{4k} (\hat{L}_-^x)^{-1} \hat{D}_+^y \hat{D}_-^y \end{cases}$$

and the expressions for $\hat{L}_\pm^x$ and $\hat{D}_\pm^y$ are given in Lemma 2.3.

Thus $\quad |G|^2 = \dfrac{1 - 2\text{Re}z + |z|^2}{1 + 2\text{Re}z + |z|^2} \cdot \dfrac{1 + 2\text{Re}z' + |z'|^2}{1 - 2\text{Re}z' + |z'|^2}$

We have $\text{Re}z' \leq 0$. Moreover, if $\dfrac{\partial h}{\partial u_-} - \dfrac{\partial h}{\partial u_+} \geq 0$, then $\text{Re}z \geq 0$ thus $|G| \leq 1$.

This yields the following theorem:

Theorem 5.1: The method (5.1) is linearly stable unconditionally for any differentiable consistent numerical flux h which satisfies the condition:

$$\frac{\partial h}{\partial u_-} - \frac{\partial h}{\partial u_+} \geq 0$$

5.3. Time-Splitting Algorithm with TVD Flux Limiters

The *limited* version of our time-splitting method can be written as:

x–sweep:
$$2k\, L_-^x\, \frac{\Phi_{jk}^* - \Phi_{jk}^n}{\Delta t} + L_-^x\, \frac{h(\underline{L}_-^x\Phi_{jk}^n\,,\underline{L}_+^x\Phi_{jk}^n) + h(\underline{L}_-^x\Phi_{jk}^*\,,\underline{L}_+^x\Phi_{jk}^*)}{2} - D_-^y D_+^y \Phi_{jk}^n = 0 \qquad (5.2a)$$

y–sweep:
$$2k\, L_-^x\, \frac{\Phi_{jk}^{n+1} - \Phi_{jk}^*}{\Delta t} - \frac{1}{2}\, D_-^y D_+^y (\Phi_{jk}^{n+1} - \Phi_{jk}^n) = 0 \qquad (5.2b)$$

6. RESULTS OF NUMERICAL EXPERIMENTS

In this section, we present the results of some of the numerical experiments we carried to compute steady state approximate solutions to the TSD equation.

The numerical experiments presented here were carried on a Sun 3/50 workstation with a 68020 cpu board using a Motorola 68881 floating point processor to compile the Fortran 77 codes.

The semi-discrete, continuous in time approximations used are described in Section 1 for the *unlimited* version and in Section 4 for the *limited* version.

The time-discretized algorithm for both versions is described in Section 5.

The flow over a circular-arc airfoil of thickness ratio 6% were simulated by our time splitting algorithm.

Iterations were carried in double precision until the maxnorm of two successive time steps fell below a chosen tolerance factor (typically 10^{-5} or 10^{-6}) or until it was reduced by two orders of magnitude or so, thus approximating steady state. The time step was adjusted at the end of each iteration to satisfy a preset CFL condition. Each iteration of the unlimited algorithm incurs an average cost of 20 seconds of cpu time (including overhead of all types). This figure becomes 25 seconds when the limiters are inserted. This should be compared to an average cost of 15 seconds per iteration for the first order algorithm.

Various meshes were tested. The results presented here used the default mesh in NASA's XTRAN2L code (plotted in Graph 0) which is smooth enough to prevent grid-induced oscillations. This computational grid extends 20 chord lengths upstream and downstream of the airfoil and 25 chord lengths in the direction transversal to the airfoil and has 80 points in the streamwise direction, 60 grid points in the normal direction and 50 grid points spread along the airfoil chord.

Graph 0: XTRAN2L default grid
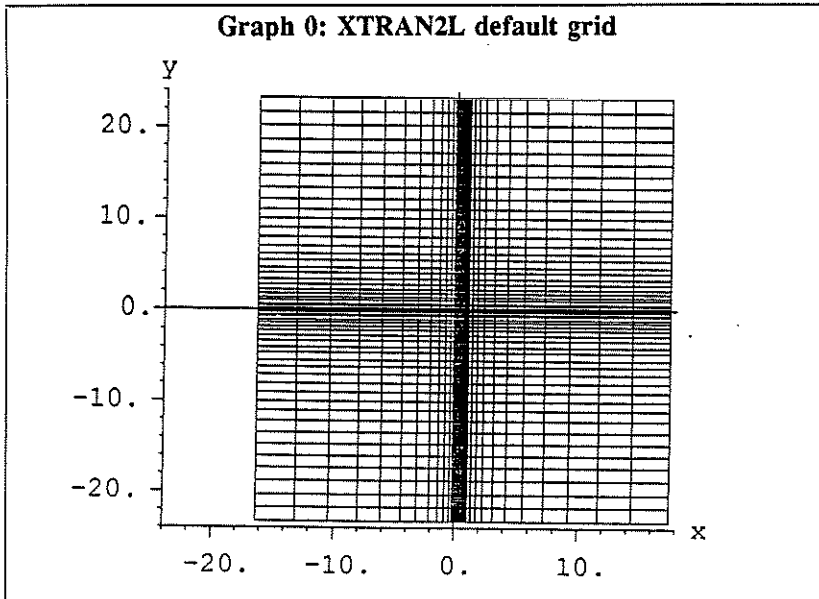
The graphs of the following pages are for a Mach number in $\{0.707, 0.908, 0.961\}$ (Graphs 1, 2, 3 respectively):

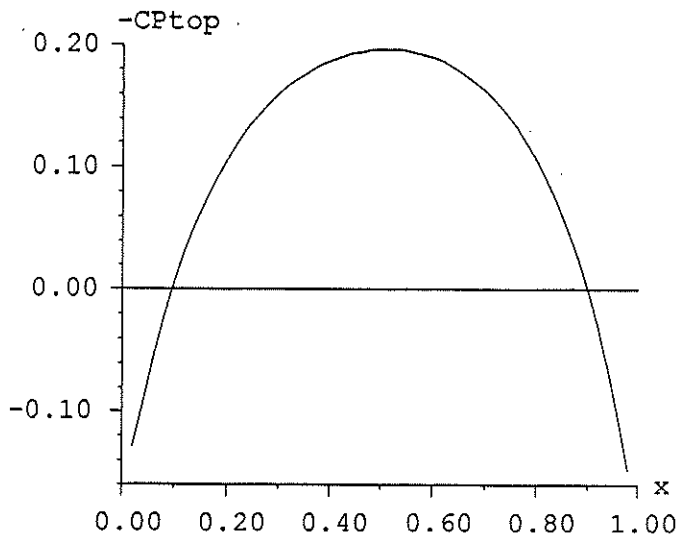Graphs 1a & 2a are for the first order Engquist-Osher (E.-O.) scheme.

Graphs 1b, 2b & 3b are for the unlimited method (5.1) with the E.-O. flux used as building block.

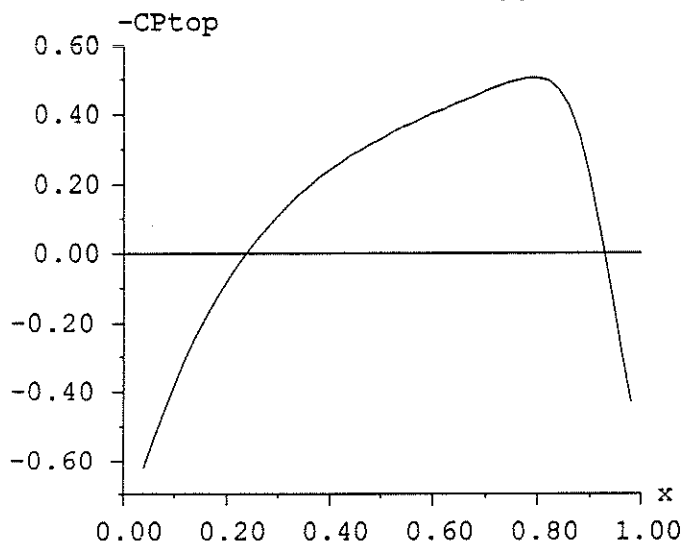Graph 3c is for the limited method (5.2) with the E.-O. flux used as building block.

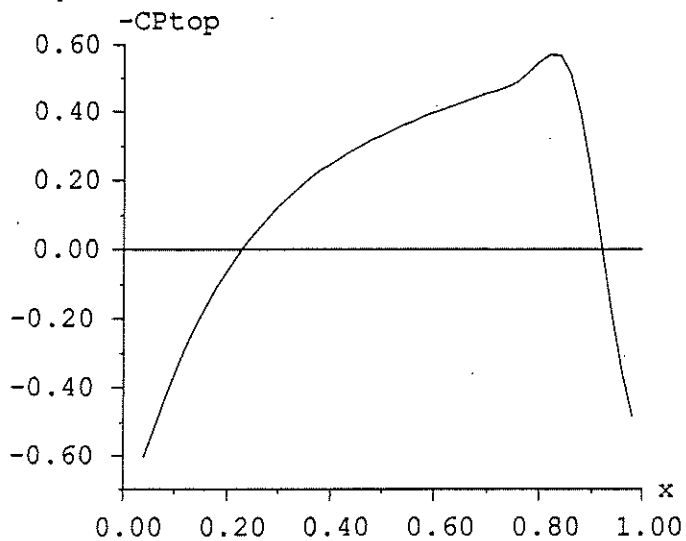Graph 1a: $M_\infty = 0.707$; First order Engquist-Osher flux
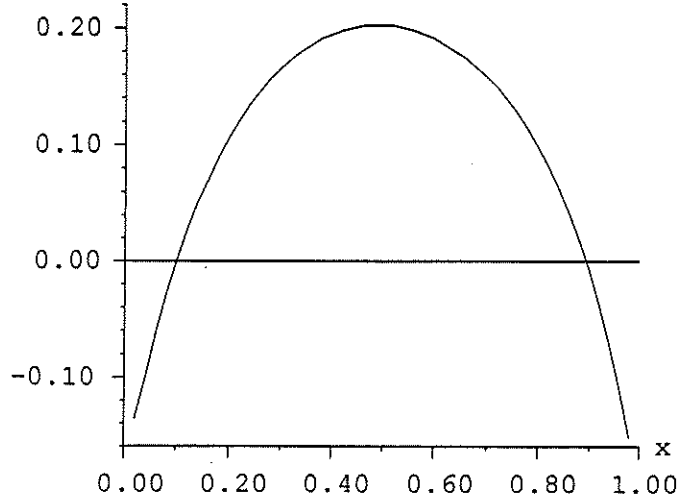
Graph 1b: $M_\infty = 0.707$; Second order flux without limiters

Graph 2a: $M_\infty = 0.908$; **First order Engquist-Osher flux**

Graph 2b: $M_\infty = 0.908$; **Second order flux without limiters**

Graph 3b: $M_\infty = 0.961$; Second order flux without limiters
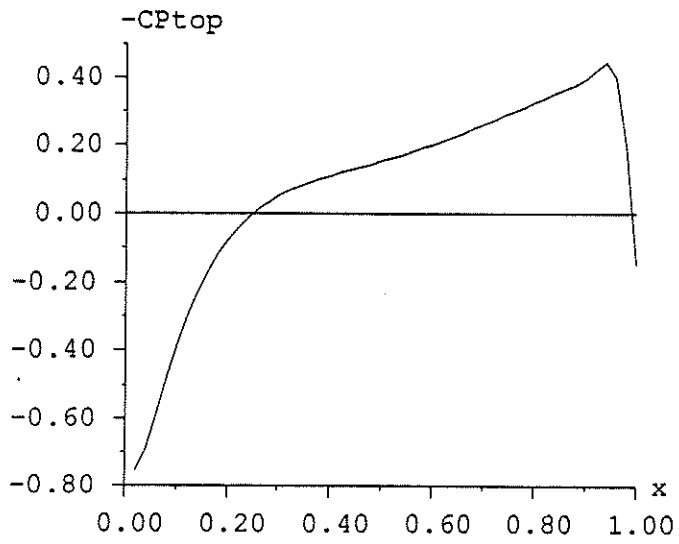
Graph 3c: $M_\infty = 0.961$; Second order flux with limiters
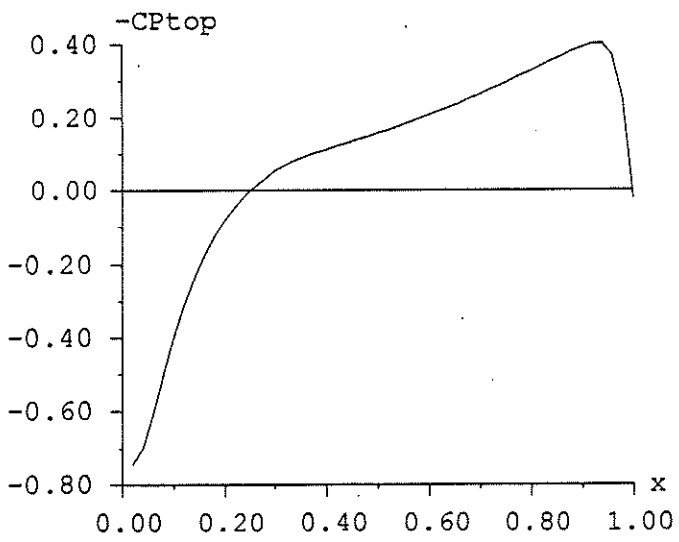
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