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A PROOF OF THE NONEMPTINESS OF THE CORE OF TWO SIDED MATCHING MARKETS

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Abstract

In this paper we prove the nonemptiness of the core in two-sided matching markets, or, "central assignment games" in the terminology of Kaneko (1982). The proof is of interest because it is different than that presented by Kaneko.

1. Introduction

A two sided matching market (TSMM) is a game in which there are two types of agents, and the essential coalitions are singletons and doubletons containing one agent of each type. Over the years, they have become an important part of economic theory. One reason for this is that they have been deemed worthy models of economic markets with indivisible goods. As for back as 1962, Gale and Shapley modeled college admissions as a matching market. A decade later, Shapley and Shubik (1972) adopted a similar model for their housing market. And still later, Crawford and Knoer (1981) defined labor markets in these terms.

In all these instances, the relevant solution concept is that of the core. Simply put, the core is the set of economic allocations where no coalition of agents can improve their lot on their own. Herein lies another reason for the study of these games; the fact that their cores have many "nice" properties. For instance, their cores are always nonempty (Kaneko 1982, Quinzii 1984). Indeed, much of the literature (Gale & Shapley, Shapley & Shubik, Crawford & Knoer, Kelso & Crawford 1982) relates relatively simple algorithms which calculate core points.

This paper too considers the issue of proving existence of core points. The model is the "central assignment game" of Kaneko. However, we feel our proof of "balancedness" [which implies core nonemptiness] is simpler than his, or than Quinzii's generalization of his result. In fact, our proof is merely a simple mathematical induction argument on the size of the set of balancing coalitions, and uses only one outside theorem from the theory of linear algebra.

The paper is organized as follows. In Section 2, we go over the class of two-sided matching markets, or "central assignment games". In Section 3, we present our proof.

2. The Model[1]

Let us review in detail the definition of a two-sided matching market, or, equivalently, of a "central assignment game". First, there are two types of players in the game, indexed by $i \epsilon I$ and $j \epsilon J$. Call these types employers and employees respectively. Let $|I| = m$ and $|J| = n$, and let $N = I \cup J$ be the set of all $m + n$ players in the game. Next, let V be a characteristic function whose domain $\mathcal{S}$ is the set of all possible coalitions, i.e., subsets of N. For each $S \epsilon \mathcal{S}$, $V(S)$ represents the set of possible utility vectors that the members of S can attain for themselves without outside intervention. Let $\Re^S$ denote the $|S|$-dimensional Euclidean space whose coordinates are indexed by S's agents.

Assume for all nonempty $S \epsilon \mathcal{S}$ that[2]

1) $V(S)$ is a closed set in $\Re^S$.

2) If $\underline{w} \epsilon \Re^S$ and $\underline{y} \epsilon V(S)$ with $y_i \geq w_i\ \forall i \epsilon S$, then $\underline{w} \epsilon V(S)$.

3) The set of vectors in $V(S)$ where each player $i \epsilon S$ receives at least $x_i = \max_{v \epsilon V(\{i\})} v$ is nonempty and bounded.

We say that a nonempty coalition S can block a vector $x \epsilon V(N)$ if there is a

[1] This section is taken almost directly from Scarf (1967), Section 1, and Kaneko (1982), Section 2.

[2] These conditions arise because Scarf (1967) needs them in his proof that a balanced game has a nonempty core.

vector $y \epsilon V(S)$ with $y_i > x_i \; \forall i \epsilon S$. The core is the set of vectors in $V(N)$ which cannot be blocked by any coalition.

Up to now, we have yet to describe the special structure of central assignment games, other than to state that there are two types of agents. To remedy this, set

$$\pi = \{S \epsilon \mathcal{S} : |S| = 1 \text{ or } [|S| = 2, |S \cap I| = 1, |S \cap J| = 1]\}$$

In other words, π consists of basic coalitions, i.e., those which are singletons or doubletons containing one agent of each type. As we shall see, basic coalitions are the "building blocks" of two-sided matching markets.

Next, define a π-partition $p_s = \{S_1, ..., S_k\}$ of S as any partition of S satisfying $S_t \epsilon \pi$ for all $t \epsilon 1, ..., k$. Let $P(S)$ be the set of all π-partitions of S.

At this point, we are ready to define a central assignment game, or two-sided matching market. Specifically, these are games which satisfy

$$V(S) = \bigcup_{p_s \epsilon P(S)} \bigotimes_{S_t \epsilon p_s} V(S_t) \; \forall S \epsilon \mathcal{S} \tag{2.1}$$

Thus, the value of a coalition is determined solely by the value of basic coalitions in π-partitions of it. *All essential coalitions are basic*[3]. Thus, the interpretation is that utility is *not transferable*, except possibly between members of a formed basic coalition.

Finally, one should note two other ramifications of (2.1) which we shall need in the next section. First is the fact that any subgame of a two-sided matching market is itself a two-sided matching market. Also, the following superadditivity property

[3] An essential coalition S is one which exhibits *strict* superadditivity for every split up into smaller nonempty coalitions,i.e.,

$$\bigcup_{(S_1,S_2) \epsilon 2(S)} [V(S_1) \times V(S_2)] \subset V(S)$$

$[2(S) = \{(S_1, S_2) : S_1, S_2 \epsilon \mathcal{S}, \; S_1, S_2 \neq \emptyset, \; S_1 \cup S_2 = S, \text{ and } S_1 \cap S_2 = \emptyset\}.]$

should be apparent:

$$\underline{w}^{S_1} \epsilon V(S_1),\ \underline{w}^{S_2} \epsilon V(S_2), \Longrightarrow \underline{w}^{S_1 \cup S_2} \epsilon V(S_1 \cup S_2) \tag{2.2}$$

for all $\underline{w} \epsilon \Re^{I \cup J}$, $S_1, S_2 \epsilon S$ with $S_1 \cap S_2 = 0$.

[$\underline{w}^S$ is the projection of $\underline{w}$ onto $\Re^S$.]

3. The Proof

Theorem: *The core of a two-sided matching market is nonempty.*

Proof: Once again, we show that the game is balanced. Let T be any balanced set of coalitions, i.e., $\exists$ scalars $\delta_S \geq 0$ for $S \epsilon T$ with $\sum_{S \epsilon T} \delta_S \underline{a}^S = \underline{1}$. [$\underline{a}^S$ is a $m+n$-vector with $a_k^S = 1$ if agent $k \epsilon S$, $a_k^S = 0$ otherwise. $\underline{1}$ is a vector of 1's.] Also, suppose $\underline{w}$ is a utility vector with $\underline{w}^S \epsilon V(S)$ for every $S \epsilon T$. [$\underline{w}^S$ is the projection of $\underline{w}$ onto S.] If we can show that $\underline{w} \epsilon V(N)$, then Scarf's theorem (1967) will imply the nonemptiness of the core.

Claim: *Without loss of generality, we can assume T contains only basic coalitions. That is, for every balanced set T and for all $\underline{w} \epsilon \Re^N$ satisfying $\underline{w}^S \epsilon V(S)$ $\forall S \epsilon T$, there is a "basic" balanced set $\tilde{T}$ with $\underline{w}^S \epsilon V(S)$ $\forall S \epsilon \tilde{T}$.*

Proof: Consider any $S \epsilon T$. Since $\underline{w}^S \epsilon V(S)$, there exists a π-partition $\pi_{\underline{w}^S} = \{S_1, ..., S_k\}$ of S under which $\underline{w}^S$ is attainable for the members of S.

Replace S in T by $\{S_1, ..., S_k\}$, and let $\delta_{S_t} = \delta_S$ for all $t \epsilon 1, ..., k$.

Repeat this process for every $S \epsilon T$, obtaining $\tilde{T}$. Then,

1) $\tilde{T}$ is balanced.

2) $\tilde{T}$ contains only 1-agent and 1-employer-1-employee coalitions.

3) $\underline{w}^S \epsilon V(S)$ for every $S \epsilon \tilde{T}$.

These three facts imply the Claim.

Next, we prove $\underline{w} \epsilon V(N)$ by induction on the number of elements in T. It is obvious when $|T| = 1$. So assume $|T| = z$. Let A be the $(m+n) \times z$ "0-1" matrix formed by the $\underline{a}_S$'s, i.e., $A_{kt} = 1$ iff the kth agent is in the tth coalition. Thus, T balanced means that $\Delta = \{\underline{\delta} : A\delta = \underline{1}, \underline{\delta} \geq 0\}$ is nonempty.

Case 1: $\delta_{\hat{S}} = 0$ for some $\hat{S} \epsilon T$ and some $\underline{\delta} \epsilon \Delta$.

Then $T - \hat{S}$ is balanced, and

$$\underline{w}^S \epsilon V(S) \text{ for every } S \epsilon T \Longrightarrow \underline{w}^S \epsilon V(S) \text{ for every } S \epsilon T - \hat{S}$$

$$\Longrightarrow \underline{w} \epsilon V(N) \text{ by inductive hypothesis.}$$

Case 2: $\delta_{\hat{S}} = 1$ for some $\hat{S} \epsilon T$ and some $\underline{\delta} \epsilon \Delta$.

Let $\hat{S}^c$ be the complement of $\hat{S}$, i.e., the set of agents not contained in $\hat{S}$. Then $T - \hat{S}$ is a balanced set over $\hat{S}^c$. Also, the subgame of V defined over the set of agents in $\hat{S}^c$ is itself a TSMM. Thus,

$$\underline{w}^S \epsilon V(S) \text{ for every } S \epsilon T \Longrightarrow \underline{w}^S \epsilon V(S) \text{ for every } S \epsilon T - \hat{S}$$

$$\Longrightarrow \underline{w}^{\hat{S}^c} \epsilon V(\hat{S}^c) \text{ by inductive hypothesis.}$$

But this, combined with $\underline{w}^{\hat{S}} \epsilon V(\hat{S})$ and relation (2.2), implies $\underline{w} \epsilon V(N)$ as desired.

Lemma: It is always true that Case 1 or Case 2 holds.

Note that the proof of the Lemma suffices to prove the Theorem.

Proof: Assume we are not in Case 1 or Case 2. Then, it must be true that $0 < \delta_S < 1$ for every $S \epsilon T$. This implies every player in the game is a member of at least two coalitions in T.

Possibility 1: There is at least one coalition in T with only one agent.

In this instance, set

$$C = \text{number of columns in } A$$

$$R = \text{number of rows in } A$$

$$\overline{O} = \text{number of ones in } A$$

By the Claim and the assumption of Possibility 1, $\overline{O} < 2C$. Also, since every agent is in at least two coalitons, $\overline{O} \geq 2R$. Hence, $R < C$. But this in conjunction with Δ's nonemptiness implies

$$A\underline{\delta} = \underline{1}, \underline{\delta} \geq \underline{0}, \delta_S = 0 \text{ for some } S$$

has a solution.[4] So in reality, we were in Case 1.

Possibility 2: T consists solely of 1-employer-1-employee coalitions.

Now $\overline{O} = 2C$ and $\overline{O} \geq 2R$, but if $\overline{O} > 2R$, then $R < C$ and the argument from Possibility 1 shows we were actually in Case 1. So assume $\overline{O} = 2R = 2C$. Then, in T,

1) Every employer is in coalitions with exactly 2 employees.

2) Every employee is in coalitions with exactly 2 employers.

3) There are exactly z employers and z employees in coalitions of T.

So, without loss of generality, assume $\{(1\ 1), (1\ 2), (2\ 1)\} \subset T$. Then $\delta_{(1\ 2)} = \delta_{(2\ 1)} = 1 - \delta_{(1\ 1)}$.

I) Suppose $(2\ 2) \epsilon T$.

In this case, $\delta_{2\ 2} = \delta_{1\ 1}$, and we can replace $\{\delta_{(1\ 1)}, \delta_{(1\ 2)}, \delta_{(2\ 1)}, \delta_{(2\ 2)}\}$ by $\{1, 0, 0, 1\}$ and $\underline{\delta}$ still is a vector of balancing weights. But then we would be in Case 1 (or Case 2).

II) So assume $(2\ 2) \notin T$.

Then, without loss of generality, suppose $\{(2\ 3), (3\ 2)\} \subset T$.

A) Suppose $(3\ 3) \epsilon T$.

Again, we can replace $\{\delta_{(1\ 1)}, \delta_{(1\ 2)}, \delta_{(2\ 1)}, \delta_{(2\ 3)}, \delta_{(3\ 2)}, \delta_{(3\ 3)}\}$ by $\{1, 0, 0, 1, 1, 0\}$ and still have balancing weights. So we are back in Case 1.

[4] see Gale, p. 50.

B) So assume $(3\ 3) \notin T$.

Then assume $\{(3\ 4), (4\ 3)\} \subset T$...

Continuing in this fashion, we must eventually reach a $\hat{z} \leq z$ for which $(\hat{z}\ \hat{z}) \epsilon T$. Again, we can replace $\{\delta_{(1\ 1)}, \delta_{(1\ 2)}, ..., \delta_{(\hat{z}\ \hat{z}-1)}, \delta_{(\hat{z}\ \hat{z})}\}$ by $\{1, 0, 0, 1, 1, ...\}$ and these are balancing weights. So again Case 1 applies.

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