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1 Introduction

The random vortex method is a numerical scheme which was introduced by Chorin [8] for the calculation of slightly viscous, incompressible fluid flows past solid boundaries; several modifications of the scheme have been proposed (see, especially, [9]). The basic idea of the method in its various forms is that vortex elements are generated at the boundary and allowed to diffuse randomly into the fluid; once in the fluid, these vortices are convected by the velocity fields which they induce and undergo random walks (to approximate diffusion). It has been found that the random vortex method is a very attractive numerical scheme for the calculation of flows in parameter regions of particular physical and numerical interest, namely moderate and high Reynolds number flows. An important characteristic common to these flow regimes is that large coherent vortex structures are shed and travel downstream, as found, for example, in flow past immersed bodies (such as cylinders and airfoils) and in flow over backwards facing steps in channels. Vortex shedding which looks similar

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to that seen experimentally has been observed in a number of interesting calculations by the random vortex method [7,12,23,25,27]. The question that interests us in this paper is the determination of the sense in which these computed solutions may represent accurate approximations to exact solutions of the Navier-Stokes equations.

The convergence of the vortex method has been studied, on the one hand, by numerical experimenters doing parameter studies, and, at the other extreme, by analysts who have made some progress in providing a convergence theory. However, we believe that there are important questions concerning the nature of the convergence which have not been addressed, and which are not faced in usual convergence analysis. These problems arise because of instabilities of solutions of the Navier-Stokes equations. The issues we discuss are well-known to, and partly understood by, dynamical systems theorists in other contexts. These issues concern the phase portrait of Navier-Stokes flow and perturbations of this dynamical system. The perturbed dynamical systems of interest are the dynamical systems corresponding to discretizations of the Navier-Stokes equations at various levels of refinement.

One point we wish to make is that convergence results of the classical variety must fail to provide a satisfactory theoretical foundation for numerical calculations, as they are usually carried out, of many important flows. For, as we shall see, it is likely (when the Reynolds number is sufficiently large) that the random vortex method, when applied to many problems of interest and given initial and boundary conditions natural to these problems, converges to solutions which are not seen in the laboratory. Thus, we are led to the unpleasant conclusion that sufficiently fine parameter refinements may yield unphysical solutions. This problem is related to the choice of initial and boundary conditions. A choice of conditions which seems less natural leads to a more reasonable problem from the numerical analytic point of view—one for which the notions of mathematical convergence and of correct physical simulation coincide. We should stress at this point that the issues under discussion here are not peculiar to the vortex method—they are features of any convergent numerical scheme applied to these fluid flow problems and to many other physical phenomena modeled by differential equations—but we focus our discussion on the random vortex method in this article.

Although the issues we examine apply to a number of different flow geometries, we choose to focus our discussion on the flow around impulsively started, two-dimensional, circular cylinders. Due to its fundamental interest and simple geometry, such flow has been the object of many numerical calculations and physical experiments.

The governing equations are the incompressible Navier-Stokes equations, which, in terms of the velocity $\mathbf{u}$ and pressure p variables have the form

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \frac{1}{R}\nabla^2 \mathbf{u} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{0} \quad \text{for } \mathbf{x} \text{ on solid boundaries}$$

$$\mathbf{u}(\mathbf{x}, t) = (U, 0) \quad \text{for } \mathbf{x} \text{ at infinity and } t > 0$$

The velocity field at time $t = 0$ is taken to be potential flow (with velocity $(U, 0)$ at infinity). R is the Reynolds number, which is defined from the cylinder radius a, the velocity of the cylinder U, and the fluid's kinematic viscosity ν by the relation $R = 2aU/\nu$.

2 A Conjectured Phase Portrait for Flow Past a Cylinder

In order to understand the behavior of numerical schemes used to calculate flow past an impulsively started cylinder, it is of course helpful to understand the major qualitative features of solutions of the Navier-Stokes equations for this flow. Unfortunately, this problem is very hard for analysts, even as a linear stability problem, and very little is known about it. However, on the basis of experimental results, and by comparison with other flows simpler to analyze, a possible phase portrait of the dynamics has been conjectured. These dynamics involve a Hopf bifurcation, and at much higher Reynolds numbers, a succession of bifurcations leading to turbulence [6]. In this section, we review some of the experimental results which have been obtained about flow past a cylinder, and then discuss the implications of these results for the dynamics in phase space. The book [10], among others, can be consulted for more about stability and bifurcation for fluid flows. A discussion of linear stability analysis for flow in channels and pipes can be found in [24].

For Reynolds numbers less than the so-called critical Reynolds number $R_c \approx 50$, steady solutions are observed experimentally. The wakes in these flows take the form of two oppositely signed concentrations of vorticity, symmetric about the horizontal axis of the cylinder. For larger Reynolds numbers, vortex shedding occurs; that is, large vortical structures are shed, alternately, from the top and bottom of the cylinder and travel downstream. If the cylinder is impulsively started, the shedding of the vortices at these higher Reynolds numbers does not occur instantly. For short times the solution is symmetric, with two vortices found to the rear of the cylinder and growing in length. After some amount of time has passed, the flow becomes asymmetric and vortices begin to be shed. The time of onset of asymmetry in the laboratory depends on the experiment—the more uniform the incoming flow is, the later is the onset of shedding. At higher Reynolds numbers, further dramatic transitions occur; the wake becomes turbulent, the separation point shifts far back on the cylinder, and asymmetric vortex shedding disappears.

Let us now make guesses about the structure of the flow in state space. For all Reynolds numbers, there is a steady, symmetric solution which is called the *basic flow*. For Reynolds numbers greater than some very small value (which may be zero [20]), the basic flows take the form of symmetric, recirculating wakes which grow as the Reynolds number increases (Fornberg [11] has found that up to $R = 600$, the length grows roughly proportionally to R).

At the critical Reynolds number R_c, the basic flow becomes unstable by a Hopf bifurcation. (A recent numerical study by Jackson [18] finds evidence for this assertion.) Below the critical Reynolds number, the basic flow is stable. For Reynolds numbers above R_c, there is a time-periodic solution of the Navier-Stokes equations which is stable and attracting (flow conditions which are sufficiently close to the periodic orbit approach this orbit as time increases). The periodic flow is the von Kármán vortex street [22].

Consider the infinite dimensional space $S \times \Re^+$, where S is the state space (consisting of incompressible velocity fields which satisfy the boundary conditions) and $\Re^+ = [0, \infty)$ is the one-dimensional parameter space of Reynolds numbers. Solutions of the Navier-Stokes equations, over the time interval $[t_0, t_1]$, at Reynolds number R, are represented as trajectories $u(t)$, $t_0 \leq t \leq t_1$, which lie in the slice $(S, \{R\})$ of $S \times \Re^+$. Steady flows correspond to fixed points $u(t) \equiv u(t_0)$. The manifold of basic flows is a one-dimensional submanifold in $S \times \Re^+$ which consists of the points

$$(u_0(R), R),$$

where $u_0(R)$ is the (steady) basic flow at Reynolds number R.

If the Navier-Stokes dynamics is correctly described by a Hopf bifurcation, then the periodic orbits (vortex streets) form a two-dimensional submanifold of $S \times \Re^+$ which intersects the space of basic flows at R_c. For fixed Reynolds numbers below critical, the basic flow is attracting, so that nearby velocity fields evolve under the Navier-Stokes equations toward the basic flow. Above critical, the basic flow is unstable and there is a manifold of codimension two in S, called the *stable manifold*, of states which evolve toward the basic flow. On the complement of this set, states

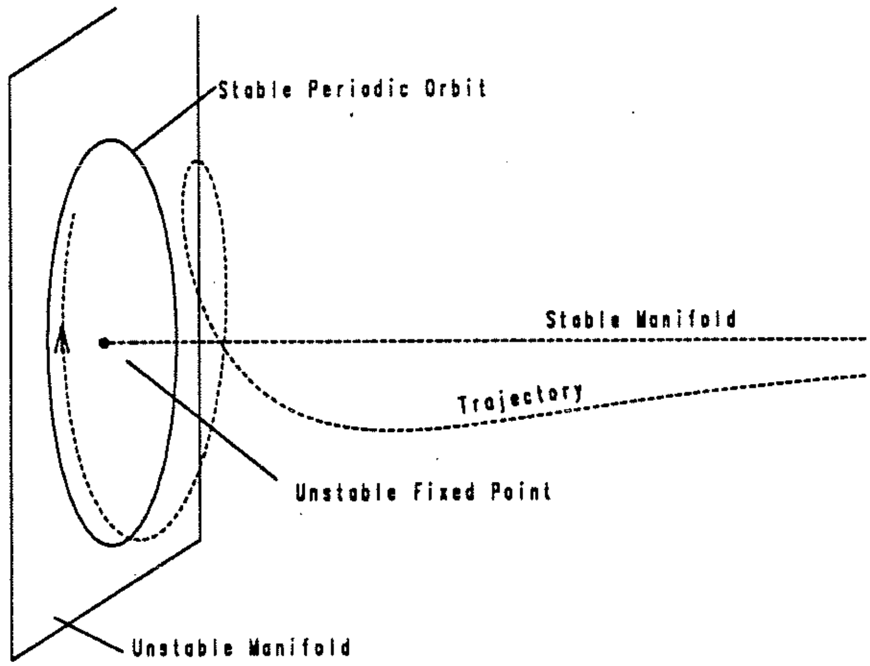

Figure 1: Depicted here are the stable and unstable manifolds of an unstable fixed point, and a trajectory which begins near the stable manifold of this point and approaches a stable periodic solution. The dynamical system is a system of three ordinary differential equations.

evolve away from the basic flow (eventually). In Figure 1, we depict these dynamics schematically by plotting the actual dynamics of a three-dimensional system of ordinary differential equations.

Hopf bifurcations are divided into two classes, subcritical and supercritical. In a supercritical Hopf bifurcation, the periodic solutions exist only for Reynolds numbers above critical and are stable. In a subcritical bifurcation, the periodic solutions exist for Reynolds numbers above and below critical. Figures 2 and 3 can be consulted for elaboration, again schematic (but exact for the dynamical systems—three-dimensional ordinary differential equations—pictured there). The book [?] contains a more detailed discussion of the Hopf bifurcation.

Further bifurcations take place at higher Reynolds numbers. These are poorly understood; we shall not attempt to describe them.

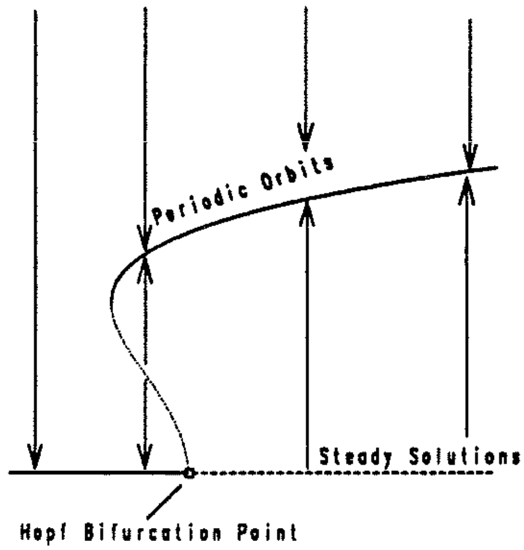

Figure 2: This plot is a bifurcation diagram of a subcritical Hopf bifurcation. The horizontal coordinates are parameter values and the vertical axis is a one-dimensional projection of state space. Solid lines indicate stable solutions; broken curves indicate unstable ones. The vertical arrows indicate trajectories.

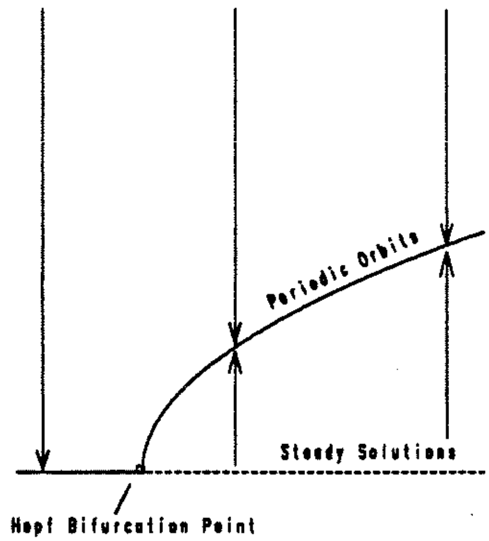

Figure 3: This plot is a bifurcation diagram of a supercritical Hopf bifurcation. The markings have the same meaning as in Figure 2.

3 The Random Vortex Method

As we have stated, the issues on which we focus in this article arise for all numerical schemes; we discuss the vortex method as a particular example. (The vortex method is of special interest, however, because of its stochastic nature.) In this section, we review the random vortex method very briefly, and mention some of the theoretical and computational studies which have been reported concerning the accuracy of the scheme. The reader is referred to [?] for a more complete introduction to the method.

The curl of (1), because of (2), is given by

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega = \frac{1}{R}\nabla^2 \omega, \tag{3}$$

where $\omega = \nabla \times \mathbf{u}$ is the vorticity. The curl operator can be inverted in the following sense. Let ω be a distribution of vorticity. Then a velocity field $\mathbf{u}$ satisfying $\nabla \times \mathbf{u} = \omega$ and which is tangent to the boundary of the solid body in the fluid is given by

$$\mathbf{u} = \mathbf{K} * \omega + \nabla\phi. \tag{4}$$

The kernel $\mathbf{K}$ is called the Biot-Savart kernel; it has the form

$$\mathbf{K}(\mathbf{x}) = \frac{(-x_2, x_1)}{2\pi \mid \mathbf{x} \mid^2}.$$

The potential term $\nabla\phi$, whose vorticity is zero, is added to satisfy the boundary condition.

In the vortex method, pieces of vorticity are followed in time, and their velocities are calculated by a discretization of the Biot-Savart law (4). Given the vortex strengths Γ_i, and the vortex positions $\mathbf{x}_i^n$ at the nth time step, these vortices are convected by the velocities they induce on each other and undergo random jumps for the simulation of diffusion:

$$\mathbf{x}_i^{n+1} = \mathbf{x}_i^n + \Delta t \sum_j \mathbf{K}_\epsilon(\mathbf{x}_i^n - \mathbf{x}_j^n)\Gamma_j + \nabla\phi^n(\mathbf{x}_i^n) + \eta_i, \tag{5}$$

where the η_i are pairs of independent Gaussian random variables of mean 0 and variance $2\Delta t/R$, $\mathbf{K}_\epsilon$ is a truncated version (the singularity at the origin is removed) of the Biot-Savart kernel, and $\nabla\phi^n$ is the potential correction at the nth time step. After the convection-diffusion step (5), the tangential component of the velocity field on the boundary no longer vanishes, and so new pieces of vorticity are introduced at the boundary in order to represent this slip flow. These vortex elements are then diffused into the fluid, and the steps are repeated.

No fully discrete convergence theorem for the random vortex method for flows past solid boundaries has been proved, but a number of partial convergence results have been obtained. Most of these results apply to approximations of solutions to the inviscid Euler equations ($R = \infty$) in free space. In this context, vortex methods have the form of (5), but without the random walk or potential terms (the latter, presumably, cause no great theoretical difficulty). The strongest results in the two-dimensional, inviscid case seem to be those of Hald [16], who proved convergence assuming only Hölder continuous initial vorticity data. For the Navier-Stokes equations (also in the case without boundaries), Goodman [13] has proved that the random vortex method gives convergent approximations of the velocity field (with probability arbitrarily close to one). Boundaries have been taken into account, but spatial discretization ignored, in the work of Benfatto and Pulvirenti [2], who have shown that the continuous version of the random vortex method converges for flow over a flat plate. That is, they proved that if one, at each time step, generates the right amount of vorticity at the boundary and then solves over one time step the heat equation and then the Euler equations, one obtains the solution of the Navier-Stokes equations in the limit of zero time step sizes.

Numerical studies have also been carried out as assessments of the accuracy of random vortex method simulations. These studies usually entail the evaluation of the behavior of certain functionals and qualitative features of the computed flows under parameter refinement. The desired goal, of course, is the demonstration of the repeatability of these qualities and quantities, and the establishment that the calculated results agree with experimental results when the latter are available. Sethian and Ghoniem [23] have measured the reattachment point for flow in a channel over a backwards-facing step at low Reynolds numbers and have achieved results consistent with experiment. Cheer [7] has successfully calculated lift and drag coefficients for a circular cylinder in uniform flow. Puckett has studied the behavior of the vortex sheet method [9] in the boundary layer above a flat plate and has found numerical convergence to Blasius flow [21].

4 Problems with Convergence when Natural Initial Conditions are Applied

It is likely that a full convergence theorem will be proved in the future. Such a theorem might state that, for a fixed time interval, as the resolution on the boundary increases appropriately, and for boundaries which are sufficiently smooth (perhaps), the velocity fields calculated in some sufficiently accurate version of the vortex method approach the solutions of the Navier-Stokes equations to arbitrary accuracy (in some probabilistic sense, only, because of the randomness in the algorithm). This might seem to provide the necessary theoretical justification for the scheme, when physically natural boundary conditions are imposed, but that is not the case. In the discussion which follows, we assume features of the phase portrait for flow past a cylinder described in Section 2; we would like to emphasize, however, that our arguments are not sensitive to the particular details of this phase portrait.

A numerical method used to compute the time evolution of flow past an impulsively started cylinder and intended to provide physically meaningful results should reproduce the experimentally observed features (at least for flows in which three-dimensional effects are not important). For Reynolds numbers below critical, computations should yield symmetric wakes, while for higher Reynolds numbers, periodic, asymmetric vortex shedding should be calculated.

One obvious difficulty with the satisfaction of this requirement is that the phase portrait associated with the dynamics of the numerical scheme at a given level of refinement is unlikely to be so similar to that of the underlying exact equations that the critical Reynolds number is the same for both systems. Thus, it is likely that for some range of Reynolds numbers, the basic flow is stable in one system and unstable in the other. An example of this kind of phenomenon is discussed in [4], in the context of discretizations of ordinary differential equations. These authors find dynamics of the discrete systems which are very different from those of the continuous system, and which persist for arbitrarily small mesh sizes. In our problem one can expect difficulties in an interval about the critical Reynolds number, although the length of this interval should go to zero as the discretization becomes increasingly refined.

Let us assume that a Reynolds number away from critical is fixed, and that the dynamics in state space of a discrete scheme "converge" to those of the Navier-Stokes equations as the numerical scheme becomes infinitely refined. For an impulsively started cylinder, the initial condition is symmetric and (in all likelihood) lies on the stable manifold of the basic flow. If $R < R_c$, one can expect the computed solution to evolve toward the basic flow, since the basic flow is attracting. For $R > R_c$, but not too large, each sufficiently refined discrete system has a stable periodic orbit (by hypothesis), and trajectories on the stable manifold of the basic flow are unstable to generic perturbations. At a fixed level of refinement, typical perturbations due to numerical errors lead to trajectories which leave the stable manifold, as long as the computational scheme does not force solutions back to the stable manifold (for example, by enforcing symmetry). As time increases, these trajectories should approach the stable, periodic, discrete orbit. (An interesting theorem has been proved which states that for suitably chosen finite element schemes, if an initial condition which is close enough to an attracting orbit is chosen, and if the Navier-Stokes equations satisfy certain reasonable properties, then convergence to this orbit is attained as time goes to infinity [17].)

Unfortunately, the time which is required for these perturbed trajectories to depart significantly from the stable manifold and approach the periodic solution depends on the level of discretization. Over a fixed time interval, the trajectory starting at a point on the stable manifold will converge to a trajectory on the stable manifold as the numerical method is refined. Thus, given any fixed time interval, we can insure that by choosing a sufficiently fine discretization, the solution will be within some small distance from the stable manifold (and correspondingly large distance from the stable, periodic flow with vortex shedding) over this time interval. If one follows the procedure of validating the numerical method by fixing time and refining the numerical parameters, one will obtain convergence to an unphysical solution.

Thus, convergence to an exact solution of the Navier-Stokes equations which exhibits shedding is not obtained, except in the following unsatisfactory sense. Suppose one wants to obtain a computed periodic solution with an error bounded by some positive value δ. For a sufficiently fine scheme, the difference between the continuous and the discrete periodic orbits can be bounded by $\delta/2$. If one computes for a sufficiently long time so that the computed solution is within $\delta/2$ of the discrete, periodic solution, then one obtains a computed periodic flow within the desired tolerance of an exact, periodic flow. Of course, the length of time required for this approach to the periodic flow approaches infinity as δ approaches zero.

The previous discussion ignores the randomness of the vortex method, and actually applies more directly to non-stochastic schemes. In the random vortex method, because the scheme is not deterministic, there is no discrete periodic orbit associated with the algorithm. However, one might conjecture that there is a temporally periodic distribution of vorticity which, with high probability, computed solutions at a (sufficiently resolved) given level of refinement approach and to which they remain near. Again it is plausible that this periodic vorticity distribution converges to the stable, periodic solution of the Navier-Stokes equations.

The flow geometry we have been discussing is very special because of its symmetry and the consequent symmetry of the basic flow. Nevertheless, as we mentioned in the introduction, this discussion also applies to many other flows. In the channel flow over a step investigated in [23], for example, the basic flow seems to consist of a recirculating eddy. When the Reynolds number is sufficiently large, it is observed that eddies break off and travel downstream, and new eddies are repeatedly generated which themselves break off and travel downstream. This flow behavior is probably due to a bifurcation, where for Reynolds numbers above a critical value, the basic flow is unstable and a stable flow with vortex shedding exists. If uniform initial and inflow conditions are applied, we expect that the same problems with convergence will arise—if one fixes a time interval and refines the numerical parameters, one will obtain convergence to the basic flow (no shedding of eddies) even above the critical Reynolds number.

In summary, it may be possible, when $R > R_c$, to compute stable, periodic flows with arbitrary accuracy, even if one does not forcibly perturb the computation, as long as one uses a numerical scheme which behaves in a reasonable way (symmetry should not be enforced in the computation, for example) and one is willing to compute for arbitrarily long times. However, one cannot compute the transition to these flows in a convergent way if one uses initial conditions which lie on the stable manifold of the basic flow and if one assumes uniform incoming flow.

A remedy to this situation is to impose initial conditions which do not lie on the stable manifold. If the initial condition is in the domain of attraction of the periodic orbit, then the solution should tend to a flow which involves vortex shedding, and convergence study by parameter refinement could be carried out. (The introduction of perturbations with the aim of inducing vortex shedding has been used in numerical computations [19].) However, there are difficulties in the forcible perturbation approach, for the details of the transition to asymmetric periodic flows may depend on the type of perturbation. We have assumed a simplified model in which there is a single attracting, periodic solution. There could certainly be more attractors present, and, if this is the case, the ones to which solutions tend depend on the nature of the perturbation. For flow past a cylinder, there is experimental evidence of a multiplicity of periodic solutions. Tritton [26] has observed two very different modes of shedding, which he calls *low mode* and *high mode*. Work by later researchers suggests that the selection of the different modes depends on the level of turbulence in the oncoming flow [3].

Numerically, one can impose different kinds of perturbations and study the evolution of the induced flows. Each of these problems (except for non-generically perturbed ones) admits convergent numerical approximations which lead to experimentally realizable flows. Thus, by choosing different initial conditions, or by continually perturbing, one can study accurately the transition to periodicity. Unfortunately, the magnitude of the perturbations required to trigger asymmetry in physical experiments seems to be extremely small. Thus, it may be very expensive to resolve a realistic perturbation and its development computationally, and so the problem of using physically realistic noise in order to calculate real flows properly is a difficult one.

Calculations which address questions of this sort are impossibly time consuming with the vortex method when the number of computations per time step is proportional to the square of the number of vortices. Recently, L. Greengard and Rokhlin [14] (see also the improvement [5]) have introduced an algorithm which computes the velocity field in time which is approximately linear in the number of vortices (for a different fast vortex method, see [1] or the article by Baden in this collection), which makes long time computations with the vortex method possible. The first two authors of this article, in collaboration with L. Greengard and Rokhlin, are currently applying the method in [14] to implement a fast random vortex method for calculating flow past an impulsively started cylinder and investigating some of the questions addressed above.

5 Conclusion

For flow past a cylinder, under the assumption that a numerical method has a phase portrait which is similar to that of the actual flow, we have concluded that a convergence study of this flow with the natural initial condition of an impulsive start is very likely to lead to an unsatisfactory result—convergence to an non-physical solution—for Reynolds numbers above critical. At a fixed level of resolution, on the other hand, calculations which run for long enough time could approach physically meaningful solutions, because of numerical noise. If one intentionally induces fixed *a priori* perturbations, however, one should be able to obtain solutions, to any desired degree of precision, which have physical counterparts. Of course, different flow evolutions will result from different perturbations, and not all artificial perturbations correspond to realistic physical noise.

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