UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

Fokker-Planck Equation for Interacting Waves. Dispersion Law Renormalization and Wave Turbulence.

S. J. Putterman
P. H. Roberts

March 1988

CAM Report 88-07

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

Contents

1. Introduction

This work is directed towards the study of systems whose hamiltonians are nearly integrable. More precisely, the hamiltonian will depend on a parameter ϵ in such a way that the system is non-integrable for $\epsilon \neq 0$ but integrable for $\epsilon = 0$. We shall study the case of small ϵ. The motion of a system that has an integrable hamiltonian is characterized by degrees of freedom, here labelled by an integer i, to which correspond frequencies ω_i. The presence of a coupling between modes, which we shall assume is non-integrable, leads to the appearance of harmonics, $\sum_{m_i} m_i \omega_i$ (the m_i are integers), and to the exchange of energy between the modes. The goal of our work is to formulate the Fokker-Planck equation which describes these non-integrable processes, and to elucidate the relationship between the irreversibility intrinsic to that equation and the reversibility of the Liouville equation from which it is developed.

This paper will focus on a bath of interacting oscillators. Our reason for selecting this system is that its behavior parallels that of a sea of interacting waves. Such a sea is encountered in a number of physical situations, such as sound waves and surface gravity waves of finite amplitude.

Generally speaking, progress in the quantum theory of the spectral evolution of a sea of interacting waves [Landau and Rumer (1937), Landau and Khalatnikov (1949), Peierls (1929, 1955)] has preceded corresponding advances in the classical theory [Litvak (1960), Hasselmann (1960, 1962), Benney and Saffman (1966), Newell and Aucoin (1971)].

One of the earliest predictions (Landau and Rumer, 1937) was the decay rate, B, at which the number of phonons, $n(\omega)$, at frequency ω in a sound wave relaxes back to the thermal population due to the (non-integrable) interactions of the wave with the modes of higher frequency. In second order quantum perturbation theory, applied to an isotropic dielectric material, their approach yields

$$n(\omega, t) = n(\omega, 0)\, e^{-B\,t}, \tag{1.1}$$

where

$$B = \pi\,|\omega|\, G^2\, U \,/\, \rho v^2. \tag{1.2}$$

Here U is the integrated energy density of the modes of frequency greater than ω, v is the sound speed, ρ is density, and

$$G = 1 + \frac{\rho}{v}\frac{dv}{d\rho}. \tag{1.3}$$

is the Grüneisen constant, a nonlinear coefficient that measures the coupling between different modes of the system. Result (1.2) applies to a medium for which $d^2\omega/dk^2 > 0$, where $\omega(k)$ is the dispersion law for the frequency as a function of the wavenumber, k, for small amplitude oscillations. For fluid dielectrics at low temperatures, Landau and Rumer expressed their result (1.2) in terms of the thermal equilibrium energy density:

$$U_E = \frac{\pi^2}{30}\, k_B T \left(\frac{k_B T}{\hbar v}\right)^3. \tag{1.4}$$

Here T is the temperature, k_B is Boltzmann's constant, and $\hbar$ is Planck's constant divided by 2π.

According to the tenets of the old quantum theory, changes in a quantum number such as $n(\omega)$ are, to within the scale factor $\hbar$, equal to changes in the correponding adiabatic invariant $I(\omega)$ of the classical system having the same hamiltonian. Thus, the existence of a quantum kinetic equation for $n(\omega)$ implies the existence of a corresponding kinetic equation for the action $I(\omega)$ of the classical sound field. This formal connection was exploited by Hasselmann (1966), and by Westervelt (1976) who particularly emphasized that, because the decay rate B can be expressed in the form (1.2), which is independent of Planck's constant, it must also follow from purely classical reasoning. [Although the form (1.4) of U_E cannot be deduced from classical theory, U_E can be determined by classical thermodynamic measurements.] Thus, the experimental verification of (1.2) through measurements of the attenuation of propagating sound waves in He^4 [Roach *et al* (1972), Abraham *et al* (1969)] provides strong evidence for the physical relevance of kinetic equations for interacting classical waves.

In addition to Landau-Rumer processes, data has been gathered and analysed from other situations [Phillips (1977), Forristall (1981), Russell (1972)], to which the classical theory of interacting waves may be applied. These include the statistically steady, off-equilibrium power spectrum of surface gravity waves [Hasselmann (1960, 1962, 1966, 1967) Larraza (1987), Larraza and Putterman (1987)] and the Alfvén waves present in the solar wind [Larraza (1987), Zakharov (1984)]. We shall refer to such off-equilibrium steady states as "wave turbulence". Classical wave kinetic theory has also provided a classical interpretation of the two-fluid theory of superfluid flow (Putterman and Roberts, 1983a, b), as well as a basis for second sound phenomena in classical wave turbulence (Larraza and Putterman (1986); $1/f$ noise has also been interpreted in terms of the highly anharmonic limit of wave turbulence (Larraza *et al*, 1986).

A common limitation of all experiments and kinetic theories discussed so far is that they describe measurements and calculations of the expected power spectrum of the interacting modes; the quantum theories describe $\langle n(\omega)\rangle$ and the classical theories describe $\langle I(\omega)\rangle$. Off-equilibrium information about higher moments, such as $\langle I(\omega)\, I(\omega')\rangle$, is not devoid of interest but it is completely lacking. Such information could, however, be extracted from the distribution function, $P(\mathbf{I};\boldsymbol{\theta};t)$, of the system. [Here $\mathbf{I}$ and $\boldsymbol{\theta}$ are abbreviations for the set of all N action and angle coordinates,I_i and θ_i.]

Using the method of multiple timescales, we shall derive the Fokker-Planck equation governing the time evolution of $P(\mathbf{I};\boldsymbol{\theta};t)$. There are some very significant differences between our approach to the Fokker-Planck equation and that of previous authors, such as Prigogine and Henin (1960) and Prigogine (1962). First, the method of multiple timescales avoids the necessity of summing an infinite sequence of diagrams. Second, it readily allows a more general Fokker-Planck equation to be derived, one in which the dependence of the distribution function $P(\mathbf{I};\boldsymbol{\theta};t)$ on $\boldsymbol{\theta}$ need not be discarded, as happens in the random wave approximation. This angle dependence is needed for calculating a number of averages of physical interest, such as the renormalization of mode frequencies (or of the speed of sound) created in one mode by the presence of excitations in the other modes of the system. It is also needed if equations of motion are required for the evolution of the average of the conjugate coordinates q_i and p_i, related to I_i and θ_i by canonical transformation.

In §2 we introduce a model hamiltonian for the system of interacting oscillators in the form

$$H = H_0 + \epsilon H_1, \tag{1.5}$$

where

$$H_0 = \sum_{i=1}^{N} \omega_i I_i, \tag{1.6}$$

$$H_1 = 8 \sum_{i,j,k=1}^{N} c_{ijk} (I_i I_j I_k)^{\frac{1}{2}} \cos\theta_i \cos\theta_j \cos\theta_k, \tag{1.7}$$

ϵ is a small (positive) parameter used to scale the size of the perturbation, and the c_{ijk} are coupling coefficients. From the kinetic equations

$$\frac{dI_i}{dt} = -\frac{\partial H}{\partial \theta_i}, \qquad \frac{d\theta_i}{dt} = \frac{\partial H}{\partial I_i}, \tag{1.8}$$

we see that I_i is a "slow" variable, i.e. $\dot{I}_i$ is of order $\epsilon \ll 1$. If, instead of θ_i, we employ

$$\phi_i = \theta_i - \omega_i t, \tag{1.9}$$

we see from (1.8) that this variable is also slow. We therefore frequently find it convenient in what follows to work with $P(\mathbf{I};\, \boldsymbol{\phi};\, t)$ rather than $P(\mathbf{I};\, \boldsymbol{\theta};\, t)$. In §2 we shall derive a theory governing $P(\mathbf{I};\, \boldsymbol{\phi};\, t)$ valid to order ϵ^2.

In §3 we shall discuss the transition from this theory, which is valid for a finite number, N, of degrees of freedom and which is therefore reversible in time, to a continuous ($N \to \infty$) system described by the irreversible Fokker-Planck equation. The irreversibility arises formally because $\partial P \,/\, \partial t$ is given in terms of an integral over ω that involves $(e^{i\omega t} - 1)\,/\, i\omega$ and, by the method of stationary phase, the effect of this term in the integrand is, in the limit $t \to \infty$, identical to that of the transformation

$$\frac{e^{i\omega t} - 1}{i\omega} \longrightarrow \pi\,\delta(\omega) + i\,\mathcal{P}\left(\frac{1}{\omega}\right), \tag{1.10}$$

where $\delta(\omega)$ is the Dirac delta function, and $\mathcal{P}(1/\omega)$ signifies that, after the division by ω, the principal part of the resulting integral is taken. The restrictions that P must obey before the replacement (1.10) is legitimate are discussed in §3. The key requirement is that there should be many modes within a bandwidth, $B(\omega)$, of modes that determines the rate of change of P, or

$$\frac{1}{\sigma(\omega)} \ll B(\omega) \ll \omega, \tag{1.11}$$

where $\sigma(\omega)$ is the density of states. The requirement $B(\omega) \ll \omega$ has been added to (1.11) since this restriction is required in order that ϵ be small enough for the perturbation expansion to be valid; the requirement $1/\sigma(\omega) \ll B(\omega)$ actually places a lower bound on ϵ, though not a very demanding one in most situations. Provided (1.11) is obeyed, the Fokker-Planck equation is valid over timescales for which $Bt = O(1)$, provided that P is also smooth over a frequency range of width at least $O(1/t)$ about the ω of interest.

Random waves in a uniform continuous medium can be treated as readily as oscillators. It is necessary only to insist that the underlying translational invariance of such a material is reflected in the coupling constants c_{ijk}. Letting i, j, k now label the wavenumber vectors $\mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_k$ (or $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ for short) of propagating modes, we find that translational invariance requires

$$c_{ijk} \propto \delta_{\mathbf{i}+\mathbf{j}+\mathbf{k}}, \tag{1.12}$$

where the Kronecker-delta, $\delta_{\mathbf{m}}$, is here defined to be zero unless $\mathbf{m} = 0$, when it is zero. Using the expression for c_{ijk} appropriate for interacting sound waves, we derive in §8 the Fokker-Planck equation determining $\partial P / \partial t$ for this system. On taking the first moment of that equation with respect to one particular action I_p, that is forming

$$\frac{d}{dt}\langle I_p\rangle = \iint I_p \frac{\partial P}{\partial t}\, \mathrm{d}\mathbf{I}\, \mathrm{d}\boldsymbol{\theta}, \tag{1.13}$$

we obtain the basic kinetic equation for interacting waves:

$$\frac{d}{dt}\langle I_p\rangle = \frac{G^2}{16\pi^2\rho c^2} \sum_{s_p s_j s_k} \iint_{-\infty}^{\infty} \delta(s_p\omega_p + s_j\omega_j + s_k\omega_k)\, \delta(s_p\mathbf{p} + s_j\mathbf{j} + s_k\mathbf{k}) \times$$
$$s_p\left(s_p\langle I_j\rangle\langle I_k\rangle + s_j\langle I_k\rangle\langle I_p\rangle + s_k\langle I_j\rangle\langle I_p\rangle\right) \omega_p\, \omega_j\, \omega_k\, \mathrm{d}^3 j\, \mathrm{d}^3 k, \tag{1.14}$$

where the constants s_p, s_j and s_k take the values ± 1 in the summations.

In order to realize the irreversibility implied by (1.10) and to arrive at (1.14), we have had to make use of the closure relation

$$\langle I_j\, I_k\rangle = \langle I_j\rangle\, \langle I_k\rangle.$$

In §5 we show that the Fokker-Planck equation, though not requiring such a factorization, will maintain it over times long compared with the time required for the system to attain equilibrium (assuming, of course, that the factorization is true at the initial instant). In fact, deviations from such a factorization, as measured by the appropriate cumulant, build up from zero at a rate which is smaller, by a factor of N^{-1}, than that over which equilibrium is attained.

The connection between (1.14) and (1.2) is established by setting

$$\langle I(\omega)\rangle = A(t)\, \delta(\omega - \omega_0) + \langle \bar{I}(\omega)\rangle, \tag{1.15}$$

and solving for $A(t)$ for a given fixed noise distribution $\bar{I}(\omega)$, which defines the energy density

$$U = \frac{1}{V}\int \omega\, \langle \bar{I}(\omega)\rangle\, \sigma(\omega)\, \mathrm{d}\omega,$$

where V is the volume of the system.

The kinetic equation (1.14) includes not only direct collisions, which remove energy from channel ω and are responsible for result (1.2), but also restituting collisions that restore energy to channel ω and allow the possibility of a steady state energy distribution. The isotropic wave turbulent steady state can be obtained by rewriting (1.14) in the form of a local cascade of energy (see Appendix A)

$$\frac{\partial \tilde{E}(\omega)}{\partial t} + \frac{\partial \tilde{Q}(\omega)}{\partial \omega} = 0, \tag{1.16}$$

where

$$\tilde{E}(\omega) = \frac{1}{V}\omega\,\langle I(\omega)\rangle\,\sigma(\omega)$$

is the energy per unit volume per unit frequency interval. The solution of (1.16) for which the energy flux $\tilde{Q}(\omega)$ is a constant ($\tilde{Q}_0$, say), this being the rate per unit volume at which external sources deliver energy to the long wavelength components of the wave spectrum, is [Zakharov (1965, 1984), Zakharov and Sagdeev (1970), Kraichnan (1968), Sagdeev (1979)]

$$\langle I(\omega)\rangle = K\rho\,c^5\left(\frac{\tilde{Q}_0}{\rho\,c^2\,G^2\epsilon^2}\right)^{\frac{1}{2}}\omega^{-\frac{9}{2}}, \tag{1.17}$$

where K is a numerical factor of order unity and where, for the case of non-dispersive sound waves, we have taken

$$\sigma(\omega) = \frac{V\,\omega^2}{2\pi^2c^3}.$$

The global equilibrium solution ($\tilde{Q}_0 = 0$) of (1.14) is

$$\langle I(\omega)\rangle_E = \frac{k_BT}{\omega}. \tag{1.18}$$

The correlations for all moments of a particular action I_p, for either the wave-turbulent or equilibrium states, can be obtained to order ϵ^2 from the reduced one-mode distribution function,

$$P^1(I_p;\,t) = \iint P(\mathbf{I};\,\boldsymbol{\theta};\,t)\,\mathrm{d}_p\mathbf{I}\,\mathrm{d}\boldsymbol{\theta}, \tag{1.19}$$

which is studied in §5. The notation $\mathrm{d}_p\mathbf{I}$ means the product of $\mathrm{d}I_i$ over all modes i with the exception of I_p. We show that, in steady state conditions, the $\mathrm{O}(\epsilon^0)$ contribution, $\bar{P}^1(I_p)$, to $P^1(I_p)$ is

$$\bar{P}^1(I_p) = \frac{1}{\langle I_p\rangle}\,\mathrm{e}^{-I_p/\langle I_p\rangle}. \tag{1.20}$$

Comparison of (1.17) and (1.18) with the distribution (1.20) suggests that the wave turbulent state may be characterized by a frequency-dependent effective temperature, $T^\dagger$, such that

$$k_BT^\dagger = K\,\rho\,c^5\left(\frac{\tilde{Q}_0}{\rho\,c^2\,G^2\,\epsilon^2}\right)^{\frac{1}{2}}\omega^{-\frac{7}{2}}. \tag{1.21}$$

Depending on the precise form of c_{ijk}, other stationary distributions are possible [see §5 and Appendix A]. Here we have quoted the physically relevant case of turbulence of non-dispersive acoustic waves.

In §6, the reduced Fokker-Planck equation governing the generalization,

$$P^1(I_p;\,\theta_p;\,t) \;=\; \iint P(\mathbf{I};\,\boldsymbol{\theta};\,t)\,\mathrm{d}_p\mathbf{I}\,\mathrm{d}_p\boldsymbol{\theta},$$

of the reduced one-mode distribution (1.19) that includes not only the action of mode p but also its angle. This allows us to study, at the $\mathrm{O}(\epsilon^2)$ level, the renormalization of oscillator frequencies. This corresponds, in the case of waves studied in §8, to a renormalization in the speed of sound, which comes about because any particular sound wave must propagate across a medium whose compressibility is changed by the presence of a random sea of other acoustic waves.

In §4, general properties of the Fokker-Planck equation are studied, an "H-theorem" is proved, and the approach of a closed system to the microcanonical ensemble is exhibited.

Although the time development of all correlations is implicit in the Fokker-Planck equation, an alternative view of the response of a particular oscillator can be obtained from its Langevin equation. In §7, we discuss the criteria under which the equation of motion for the particular p-oscillator takes the form

$$\ddot{q}_p \;+\; B_p\dot{q}_p \;+\; \Omega_p^2 q_p \;=\; F'_p, \tag{1.22}$$

where B_p is a friction coefficient and Ω_p is the renormalized frequency. Both B_p and $\Omega_p - \omega_p$ are due to the interaction of the particular oscillator with the remaining oscillators of the "bath", and are $\mathrm{O}(\epsilon^2)$; $F'_p(t)$ is the random force to which the particular oscillator is subjected through the other oscillators of the bath;

$$q_p \;=\; \left(\frac{2I_p}{\omega_P}\right)^{\frac{1}{2}} \cos\theta_p,$$

is the coordinate of oscillator p. In §7, we derive the spectral distribution of the random force, i.e. $\langle F'_p(\omega)\,F'_p(\omega')\rangle^B$, where $F'_p(\omega)$ is the Fourier transform of $F'_p(t)$ and the superscript ^{B} indicates that the average should be carried out over the probability distribution for the oscillators of the bath alone. We show that, when the bath probability distribution is independent of the angles, $\langle F'_p(t_1)\,F'_p(t_2)\rangle^B$ is even in $t_1 - t_2$. When the distribution of actions is an equilibrium (E) thermal distribution, we deduce the fluctuation dissipation relation

$$\langle F'_p(\omega)\,F'_p(\omega')\rangle^B_E \;=\; 2k_BT\,B_{p,E}\,\delta(\omega\,+\,\omega'). \tag{1.23}$$

A similar relation with T replaced by $T^\dagger$ applies to the wave-turbulent steady state.

On multiplying (1.22) by $\dot{q}_p$, we obtain a Langevin equation for the action,

$$\dot{I}_p \;+\; B_p\,I_p \;-\; A_p \;=\; G'_p, \tag{1.24}$$

where the restituting collisions are contained in

$$A_p = \langle \dot{q}_p F'_p / \omega_p \rangle^B. \tag{1.25}$$

Properties of G'_p are derived in §7.

As indicated by the Langevin equation (1.22), the linear frequency, ω_p, of an oscillator acquires a renormalization, $\Omega_p - \omega_p$, through the nonlinear couplings. For a sound wave, the ω-renormalization implies a renormalization in the speed of sound, v. This effect is calculated in §8, for the case of interacting sound waves governed by the dispersion law

$$\omega^2 = v^2 k^2 + \gamma k^4, \tag{1.26}$$

where γ is the dispersion constant. The renormalized sound velocity, v_p is found to be

$$v_p = v + \frac{2\,G^2 U}{\rho\, v} \log\left(\frac{k_\gamma}{3.88\ldots k_T}\right), \tag{1.27}$$

where $k_\gamma \equiv v/|\gamma|^{1/2}$ and k_T is an average of the spectral distribution of sound wave energy. The renormalization is independent of the sign of γ, but the attenuation coefficient (1.2) applies only when $\gamma > 0$.

2. Multiple Timescale Expansion of the Liouville Equation for a Bath of Discrete Oscillators

We consider a set of N oscillators, the hamiltonian of the i^{th} oscillator being $\frac{1}{2}(p_i^2+\omega_i^2 q_i^2)$, where (q_i, p_i) are its generalized coordinate and momentum, and ω_i is its frequency. In the absence of interaction between the oscillators, the hamiltonian of the system would be

$$H_0 = \frac{1}{2}\sum_{i=1}^{N}(p_i^2 + \omega_i^2 q_i^2). \tag{2.1}$$

Such a system would not thermalize: each oscillator would retain the energy it initially possessed. We therefore add a weak interaction between oscillators, and consider the hamiltonian

$$H = H_0 + \epsilon H_1, \tag{2.2}$$

where

$$H_1 = \sum_{i,j,k} d_{ijk} q_i q_j q_k. \tag{2.3}$$

The sum is over all i, j, k and the constants d_{ijk} are real and symmetric in their indices:

$$d_{ijk} = d_{jik} = d_{ikj}. \tag{2.4}$$

For simplicity, we exclude terms invoving repeated suffices, i.e. we assume that $d_{iik} = 0$ (not summed). The cubic interaction is perhaps the simplest that allows genuine thermalization. Had we assumed instead a quadratic interaction with only two q's in the sum (2.3), we could have rediagonalized H into a form like (2.1) with slightly modified q_i . Such a system would be as incapable of thermalization as (2.1). We shall not consider here the effect of canonical transformation, which allows the ϵH_1 to be removed from (2.2) at the expense of introducing an $\epsilon^2 H_2$ term, a process of postponing the non-integrability that can be repeated indefinitely. Suffice it to say here that canonical transformation does not affect the central issue: by the addition of ϵH_1, with H_1 given by (2.3), we have transformed the integrable hamiltonian (2.1) into the non-integrable hamiltonian (2.2).

As motivated in §1, we introduce action-angle coordinates, by setting

$$q_i = \left(\frac{2I_i}{\omega_i}\right)^{\frac{1}{2}} \cos\,\theta_i, \qquad p_i = -(2\omega_i I_i)^{\frac{1}{2}} \sin\,\theta_i, \tag{2.5}$$

or more succinctly as

$$q_i = \sum_{s_i} q_I, \qquad p_i = \sum_{s_i} i s_i \omega_i q_I, \tag{2.6a}$$

where

$$q_I = \left(\frac{I_i}{2\omega_i}\right)^{\frac{1}{2}} e^{i s_i \theta_i}, \tag{2.6b}$$

and the sum over s_i involves only $s_i = \pm 1$. We now have

$$H_0 = \sum_{i=1}^{N} \omega_i I_i, \tag{2.7}$$

$$H_1 = \sum_{IJK} c_{ijk}\, I_{ijk}^{1/2}\, e^{i\theta_{IJK}}. \tag{2.8}$$

In (2.6) and (2.8) we have introduced some abbreviations. Capital letters such as K denote composite labels (k, s_k), and $-K$ will mean $(k, -s_k)$. A sum over K, as in (2.8), is in reality a double sum over k and both values of s_k. Also appearing in (2.8) are

$$c_{ijk} = d_{ijk}(8\omega_i\omega_j\omega_k)^{-\frac{1}{2}}, \quad I_{ijk} = I_i I_j I_k,$$

$$\theta_{IJK} = s_i\theta_i + s_j\theta_j + s_k\theta_k = -\theta_{-I,-J,-K}.$$

Here c_{ijk}, like d_{ijk}, obeys (2.4) and vanishes if $i = j$, etc.

A single system governed by (2.2) follows a trajectory in phase space, i.e. the $2N$-dimensional $\mathbf{I}\boldsymbol{\theta}$-space of actions and angles, in obedience to Hamilton's equations:

$$\frac{dI_i}{dt} = -\frac{\partial H}{\partial \theta_i}, \qquad \frac{d\theta_i}{dt} = \frac{\partial H}{\partial I_i}. \tag{2.9, 2.10}$$

For given initial actions and angles, say

$$I_i(0) = I_{0i}, \qquad \theta_i(0) = \theta_{0i}, \tag{2.11}$$

(2.9) and (2.10) determine the $\mathbf{I}\boldsymbol{\theta}$-trajectory for all t.

The partial differential equation associated with (2.9) and (2.10) is Liouville's equation,

$$i\frac{\partial P}{\partial t} = LP, \tag{2.12}$$

where L is the Liouville operator:

$$L = -i\sum_i \left(\frac{\partial H}{\partial I_i}\frac{\partial}{\partial \theta_i} - \frac{\partial H}{\partial \theta_i}\frac{\partial}{\partial I_i} \right). \tag{2.13}$$

Clearly iLP is the Poisson bracket of H and P.

An interesting advantage of the action-angle formulation over the qp-alternative may be explained by a simple analogy: random walk in two dimensions. If the probability density, $P_n(x, y)$, that the moving point lies at (x, y) after n steps is required, there is no need to distinguish between the polar angles $\theta + 2m\pi$ (where m is an integer, positive, negative or zero) corresponding to (x, y), and θ may be restricted to (say) the interval $0 \le \theta < 2\pi$. If however the probable number of times the wandering point has encircled its starting point (the origin) is sought, it is necessary to distinguish between the points $(r, \theta + 2m\pi)$, and to work with a probability density $p_n(r, \theta)$ defined on $-\infty < \theta < \infty$, which generally takes different values at $\theta + 2m\pi$. The determination of p_n is known as "the winding number problem"; see for instance Lévy (1940) or Spitzer (1958). It is simple to obtain P_n from p_n but not the reverse:

$$P_n(r,\theta) = \sum_{m=-\infty}^{\infty} p_n(r, \theta + 2m\pi).$$

When determining moments of single-valued functions of θ, it is unnecessary to employ p_n but, if for example we need a moment of a non-single-valued function of θ (such as θ^2), p_n must be used.

In a similar way, we may for most purposes regard P in (2.12) as a function of $(\mathbf{q}, \mathbf{p})$, and ignore differences between P at $\theta_i + 2m_i\pi$. When we want moments of functions that differ at $\theta_i + 2m_i\pi$, we must regard θ_i as defined over $-\infty < \theta_i < \infty$, and introduce p to distinguish between probability densities for the different integers m_i.

Corresponding to the initial state (2.11) is

$$P(\mathbf{I};\ \boldsymbol{\theta};\ 0) = \prod_i \left[\delta(I_i - I_{0i})\ \delta(\theta_i - \theta_{0i})\right]. \tag{2.14}$$

The trajectory $[\mathbf{I}(t),\ \boldsymbol{\theta}(t)]$ obtained by solving (2.9) and (2.10) subject to (2.11) then gives the solution of (2.12) satisfying (2.14):

$$P(\mathbf{I};\ \boldsymbol{\theta};\ t) = \prod_i \left[\delta(I_i - I_i(t))\ \delta(\theta_i - \theta_i(t))\right]. \tag{2.15}$$

In what follows, we shall be more interested in considering an ensemble of trajectories, with P measuring the probability of the occurrence of each. Time reversibility is reflected by the statement that, if $P(\mathbf{I};\ \boldsymbol{\theta};\ t)$ is a solution of (2.12), so is $P(\mathbf{I}; -\boldsymbol{\theta}; -t)$.

Corresponding to (2.2), we divide L into two operators,

$$L = L_0 + \epsilon L_1, \tag{2.16}$$

where

$$L_0 = -i \sum_i \omega_i \frac{\partial}{\partial \theta_i}, \tag{2.17}$$

$$L_1 = -3 \sum_{IJK} c_{ijk}\, I_{ijk}^{1/2}\, e^{i\theta_{IJK}} \left(s_i \frac{\partial}{\partial I_i} + \frac{i}{2I_i} \frac{\partial}{\partial \theta_i} \right), \tag{2.18}$$

and (2.12) becomes

$$i \left[\frac{\partial P}{\partial t} + \sum_i \omega_i \frac{\partial P}{\partial \theta_i} \right] = \epsilon L_1 P. \tag{2.19}$$

The form of (2.19) suggests that we might benefit from a change of variables. Let

$$\phi_i = \theta_i - \omega_i t, \tag{2.20}$$

and replace $P(\mathbf{I};\ \boldsymbol{\theta};\ t)$ by $P(\mathbf{I};\ \boldsymbol{\phi};\ t)$. Note that $\boldsymbol{\phi}$ coincides with $\boldsymbol{\theta}$ at $t = 0$. We may now write (2.19) as

$$i\frac{\partial P}{\partial t} = \epsilon L_1 P, \tag{2.21}$$

where by (2.18)

$$L_1 = -\sum_{IJK} c_{ijk} I_{ijk}^{1/2} \mathrm{e}^{i(\phi_{IJK}+\omega_{IJK}t)} \nabla_{IJK}. \tag{2.22}$$

The partial derivative in (2.21) is here and below taken at constant ϕ_i, rather than at constant θ_i. In (2.22) we have introduced further abbreviations:

$$\begin{aligned}
\phi_{IJK} &= s_i\phi_i + s_j\phi_j + s_k\phi_k = -\phi_{-I,-J,-K}, \\
\omega_{IJK} &= s_i\omega_i + s_j\omega_j + s_k\omega_k = -\omega_{-I,-J,-K}, \\
\nabla_{IJK} &= s_i\nabla_I + s_j\nabla_J + s_k\nabla_K = -\nabla^*_{-I,-J,-K}, \\
\nabla_I &= \frac{\partial}{\partial I_i} + \frac{is_i}{2I_i}\frac{\partial}{\partial \phi_i} = \nabla^*_{-I}.
\end{aligned}$$

[Because of (2.4) we could, as in (2.18), write (2.22) as a triple sum involving $3s_i\nabla_I$ in place of ∇_{IJK}. The greater symmetry of (2.22) has, however, technical advantages.] Note that ∇_I commutes with $I_i^{1/2}\,\mathrm{e}^{is_i\theta_i}$ and that $\nabla_I\left(I_i^{1/2}\,\mathrm{e}^{-is_i\theta_i}\right) = I_i^{-1/2}\,\mathrm{e}^{-is_i\theta_i}$.

If $\epsilon = 0$, $\mathbf{I}$ and $\boldsymbol{\phi}$ are constants of the motion. We may expect that, for finite but small ϵ, these variables will change their values secularly, on a timescale of order ϵ^{-2}. Thus we are motivated to seek a solution to (2.21) in the limit $\epsilon \to 0$ by the method of multiple timescales. We introduce

$$\tau_0 = t, \quad \tau_1 = \epsilon t, \quad ,\tau_2 = \epsilon^2 t, \tag{2.23}$$

and expand P as a power series in ϵ:

$$P = P_0(\mathbf{I};\ \boldsymbol{\phi};\ \boldsymbol{\tau}) + \epsilon\ P_1(\mathbf{I};\ \boldsymbol{\phi};\ \boldsymbol{\tau}) + \epsilon^2\ P_2(\mathbf{I};\ \boldsymbol{\phi};\ \boldsymbol{\tau}), \tag{2.24}$$

with $\boldsymbol{\tau} = (\tau_0,\ \tau_1,\ \tau_2)$. We substitute (2.24) into (2.21) and equate like powers of ϵ. An infinite sequence of equations results, the first three of which are

$$i\frac{\partial P_0}{\partial \tau_0} = 0, \tag{2.25}$$

$$i\frac{\partial P_1}{\partial \tau_0} = -i\frac{\partial P_0}{\partial \tau_1} + L_1\ P_0, \tag{2.26}$$

$$i\frac{\partial P_2}{\partial \tau_0} = -i\frac{\partial P_0}{\partial \tau_2} - i\frac{\partial P_1}{\partial \tau_1} + L_1\ P_1. \tag{2.27}$$

The initial state $P(\mathbf{I};\ \boldsymbol{\phi};\ 0)$ divides similarly:

$$P_0(\mathbf{I};\ \boldsymbol{\phi};\ 0) = P(\mathbf{I};\ \boldsymbol{\phi};\ 0), \tag{2.28}$$

$$P_1(\mathbf{I};\ \boldsymbol{\phi};\ 0) = 0, \tag{2.29}$$

$$P_2(\mathbf{I};\ \boldsymbol{\phi};\ 0) = 0. \tag{2.30}$$

By (2.25) and (2.28), we have

$$P_0 \;=\; P_0(\mathbf{I};\ \boldsymbol{\phi};\ \tau_1,\ \tau_2) \;=\; P(\mathbf{I};\ \boldsymbol{\phi};\ 0). \tag{2.31}$$

The solution of (2.26) and (2.27) subject to (2.29) and (2.30) is effected by Laplace transformation. Let us write

$$\tilde{P}_i(\mathbf{I};\ \boldsymbol{\phi};\ s,\ \tau_1,\ \tau_2) = \int_0^\infty \mathrm{e}^{-s\tau_0} P_i(\mathbf{I};\ \boldsymbol{\phi};\ \tau_0,\ \tau_1,\ \tau_2)\ d\tau_0,$$

and note that, if $\mathcal{L}$ denotes the Laplace transform operator, then

$$\mathcal{L}\left(\mathrm{e}^{i\omega_{IJK}t} P_i(\tau_0)\right) = \tilde{P}_i(s\ -\ i\omega_{IJK}). \tag{2.32}$$

It follows at once from (2.26), (2.29) and (2.31) that

$$\tilde{P}_1 \;=\; -\frac{1}{s^2}\frac{\partial P_0}{\partial \tau_1} \;+\; i\sum_{IJK} \frac{c_{ijk}\ I_{ijk}^{1/2}}{s(s\ -\ i\omega_{IJK})}\ \mathrm{e}^{i\phi_{IJK}} \nabla_{IJK}\ P_0, \tag{2.33}$$

which on inversion gives

$$P_1 \;=\; -\tau_0 \frac{\partial P_0}{\partial \tau_1} \;+\; i\sum_{IJK} \Delta^+(\omega_{IJK})\ c_{ijk}\ I_{ijk}^{1/2}\ \mathrm{e}^{i\phi_{IJK}}\ \nabla_{IJK}\ P_0. \tag{2.34}$$

We have here introduced one of the three *dephasing functions* defined, for real h, by

$$\Delta^+(h) \;=\; \frac{(\mathrm{e}^{ih\tau_0}\ -\ 1)}{ih}, \qquad \Delta^-(h) \;=\; \frac{(\mathrm{e}^{-ih\tau_0}\ -\ 1)}{-ih}, \tag{2.35a}$$

$$\Delta(h) \;=\; \frac{1}{2}\left(\Delta^+(h)\ +\ \Delta^-(h)\right) \;=\; \frac{\sin h\tau_0}{h}. \tag{2.35b}$$

We note that

$$\Delta^-(h) \;=\; \Delta^+(-h) \;=\; \Delta^+(h)^* \;=\; \mathrm{e}^{-ih\tau_0}\Delta^+(h) \tag{2.35c.}$$

The multiple timescale method is a convenient means of removing secularities, i.e. contributions to P that increase, rather than oscillate, with time. The first term on the right-hand side of (2.34) is secular in τ_0. Other secularities arise from "accidental resonances", to be denoted by 'AR'. These originate from modes for which $\omega_{IJK} \;=\; 0$ in the sum appearing in (2.34). Such accidental resonances are easily imagined. If, for instance, $\omega_k \;=\; k\omega_0$, where k is a positive integer and ω_0 is some constant frequency, then ω_{j+k} and $\omega_{|j-k|}$ resonate with ω_j and ω_k, and there will be approximately $\frac{1}{4}N^2$ such resonances. This situation is physically realistic, and would arise for sound traversing a one-dimensional non-dispersive medium.

Let $\sum^{AR}$ denote the sum over the AR-terms of the (2.34) sum, and let $\sum^{NR}$ be the sum over the remaining non-resonant terms. Then secularities are removed from (2.34) by taking

$$\frac{\partial P_0}{\partial \tau_1} = i \sum_{IJK}^{AR} c_{ijk}\, I_{ijk}^{1/2}\, e^{i\phi_{IJK}}\, \nabla_{IJK}\, P_0, \tag{2.36}$$

and substituting that back into (2.34) we find

$$P_1 = i \sum_{IJK}^{NR} \Delta^+(\omega_{IJK})\, c_{ijk}\, I_{ijk}^{1/2}\, e^{i\phi_{IJK}}\, \nabla_{IJK}\, P_0. \tag{2.37}$$

In a similar fashion (2.27), (2.30) and (2.37) yield

$$\tilde{P}_2 = -\frac{1}{s^2}\frac{\partial P_0}{\partial \tau_2} - \sum_{IJK}^{NR}\sum_{LMN}^{NR} \frac{1}{s(s - i\omega_{LMN})(s - i\omega_{IJKLMN})} \times$$
$$c_{lmn} I_{lmn}^{1/2} e^{i\phi_{LMN}} \nabla_{LMN}\big[c_{ijk} I_{ijk}^{1/2} e^{i\phi_{IJK}} \nabla_{IJK} P_0\big]. \tag{2.38}$$

Even though the individual sums in (2.38) are non-resonant, secularities can arise in two principal ways:

$$(i,\ j,\ k) = \text{permutation of } (l,\ m,\ n); \tag{2.39}$$

$$\omega_{IJKLMN} = 0, \quad \text{but} \quad (i,\ j,\ k) \neq \text{permutation of } (l,\ m,\ n). \tag{2.40}$$

The former are essential resonances ('ER') that necessarily arise regardless of the dependence of ω on i. The latter are again of accidental type. From (2.38), we now obtain

$$\frac{\partial P_2}{\partial \tau_0} + \frac{\partial P_0}{\partial \tau_2} = 6 \sum_{IJK}^{NR} \Delta^-(\omega_{IJK})\, c_{ijk}^2\, \nabla_{IJK}^*\big[I_{ijk}\, \nabla_{IJK}\, P_0\big]$$
$$- \left\{\sum_{IJK}^{NR}\sum_{LMN}^{NR}\right\}^{AR} \Delta^+(\omega_{LMN})\, c_{lmn}\, I_{lmn}^{1/2}\, e^{i\phi_{LMN}} \nabla_{LMN}\big[c_{ijk}\, I_{ijk}^{1/2}\, e^{i\phi_{IJK}} \nabla_{IJK} P_0\big]$$
$$- \left\{\sum_{IJK}^{NR}\sum_{LMN}^{NR}\right\}^{NR} e^{i\omega_{LMN}\tau_0} \Delta^+(\omega_{IJK})\, c_{lmn}\, I_{lmn}^{1/2}\, e^{i\phi_{LMN}} \nabla_{LMN}\big[c_{ijk}\, I_{ijk}^{1/2}\, e^{i\phi_{IJK}} \nabla_{IJK} P_0\big]. \tag{2.41}$$

The first sum on the right-hand side of (2.41) arises from essential resonances; use has been made of the identity

$$\nabla_I\big(I_i^{1/2} e^{i\phi_I}\big) = 0. \tag{2.42}$$

The secularity is removed from (2.41) by requiring that

$$\frac{\partial P_0}{\partial \tau_2} = 6 \sum_{IJK}^{NR} \frac{1}{i\omega_{IJK}}\, c_{ijk}^2\, \nabla_{IJK}^*\big[I_{ijk}\, \nabla_{IJK}\, P_0\big]$$
$$+ \left\{\sum_{IJK}^{NR}\sum_{LMN}^{NR}\right\}^{AR} \frac{1}{i\omega_{LMN}} c_{lmn}\, I_{lmn}^{1/2}\, e^{i\phi_{LMN}} \nabla_{LMN}\big[c_{ijk}\, I_{ijk}^{1/2}\, e^{i\phi_{IJK}} \nabla_{IJK} P_0\big]. \tag{2.43}$$

Equations (2.31), (2.37) and (2.43) may now be summarized by the following equations, valid up to order ϵ^2:

$$P = \bar{P} + \hat{P}, \qquad \hat{P} = \hat{P}_1 + \hat{P}_2 + ..., \tag{2.44}$$

where

$$\frac{\partial \bar{P}}{\partial t} = 6\epsilon^2 \sum_{IJK} \Delta^-(\omega_{IJK})\; c_{ijk}^2 \nabla_{IJK}^* \big[I_{ijk}\; \nabla_{IJK}\; \bar{P}\big], \tag{2.45}$$

$$\hat{P}_1 = i\epsilon \sum_{IJK} \Delta^+(\omega_{IJK})\; c_{ijk}\; I_{ijk}^{1/2}\; \mathrm{e}^{i\phi_{IJK}} \nabla_{IJK} \bar{P}, \tag{2.46}$$

$$\hat{P}_2 = -\,\epsilon^2 \sum_{\substack{IJKLMN \\ IJK \neq -(LMN)}} \left\{ \frac{\Delta^+(\omega_{IJKLMN}) \;-\; \Delta^+(\omega_{LMN})}{i\omega_{IJK}} \right\} \times \; c_{ijk} c_{lmn} I_{ijklmn}^{1/2}\; \mathrm{e}^{i\phi_{IJKLMN}}\; \nabla_{IJK} \nabla_{LMN} \bar{P}. \tag{2.47}$$

The AR terms have been excluded from (2.45)–(2.47) since we shall consider only those systems for which accidental resonances are negligibly few; the superfix NR, being then superfluous, has been omitted. Because of the $\mathrm{e}^{-i\omega_{IJK}t}$ contained in $\Delta^-(\omega_{IJK})$, the $\bar{P}$ appearing in (2.45) includes not only P_0, but also part of the rapidly oscillating P_2; see (2.41). The remainder of P_2 is the $\hat{P}_2$ of (2.47). It should not be forgotten in interpreting the left-hand side of (2.45) that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau_0} + \epsilon \frac{\partial}{\partial \tau_1} + \epsilon^2 \frac{\partial}{\partial \tau_2}. \tag{2.48}$$

The τ_0 of the definitions (2.35) of $\Delta^\pm$ is replaced in (2.45)–(2.47) by t.

Equations (2.45)-(2.47) form a closed system that determines the motion of the "ensemble fluid" to order ϵ^2. To that order they are reversible, i.e. if we use them to determine $P\ (=\ \bar{P} + \hat{P}_1 + \hat{P}_2)$ at time t from an arbitrary initial state, and we then reverse $\boldsymbol{\theta}$ and integrate for a further time t, we recover to order ϵ^2 that initial state (with reversed $\boldsymbol{\theta}$).

A main use of the probability distribution is that of calculating moments of physical quantities that are functions of $\mathbf{I}$ and $\boldsymbol{\theta}$, rather than $\mathbf{I}$ and $\boldsymbol{\phi}$:

$$\langle f(\mathbf{I};\ \boldsymbol{\theta})\rangle(t) = \iint f(\mathbf{I};\ \boldsymbol{\theta})\ P(\mathbf{I};\ \boldsymbol{\theta};\ t)\ d\mathbf{I}\ d\boldsymbol{\theta}. \tag{2.49}$$

It is then found (see §3) to be useful to restate (2.45) – (2.47) in terms of $\boldsymbol{\theta}$ rather than $\boldsymbol{\phi}$:

$$\frac{\partial \bar{P}}{\partial t} + \sum_i \omega_i \frac{\partial \bar{P}}{\partial \theta_i} = 6\epsilon^2 \sum_{IJK} \Delta^-(\omega_{IJK}) c_{ijk}^2 \nabla_{IJK}^* \{ I_{ijk} \nabla_{IJK} \bar{P} \}, \tag{2.50}$$

$$\hat{P}_1 = i\epsilon \sum_{IJK} \Delta^-(\omega_{IJK}) c_{ijk} I_{ijk}^{1/2} \mathrm{e}^{i\theta_{IJK}} \nabla_{IJK} \bar{P}, \tag{2.51}$$

$$\hat{P}_2 = -\epsilon^2 \sum_{\substack{IJKLMN \\ IJK \neq -(LMN)}} \left\{ \frac{\Delta^-(\omega_{IJKLMN}) - \Delta^-(\omega_{LMN})}{-i\omega_{IJK}} \right\} c_{ijk} c_{lmn} I_{ijklmn}^{1/2} \mathrm{e}^{i\theta_{IJKLMN}} \nabla_{IJK} \nabla_{LMN} \bar{P}. \tag{2.52}$$

Here ∇_I and ∇_{IJK} are defined as below (2.22), with $\partial/\partial\phi_i$ replaced by $\partial/\partial\theta_i$. In future transformations between $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ we shall not need to distinguish between these two forms of derivative.

Corresponding to the expansion (2.44) for P, the average of some function $f(\mathbf{I}, \boldsymbol{\theta})$ is, as in (2.49), given by

$$\langle f \rangle = \langle f \rangle_0 + \langle f \rangle_1 + \langle f \rangle_2, \tag{2.53}$$

where

$$\langle f \rangle_0 = \iint f(\mathbf{I}, \boldsymbol{\theta}) \, \bar{P}(\mathbf{I}; \boldsymbol{\theta}; t) \, \mathrm{d}\mathbf{I} \, \mathrm{d}\boldsymbol{\theta}, \tag{2.54}$$

$$\langle f \rangle_1 = \iint f(\mathbf{I}, \boldsymbol{\theta}) \, \hat{P}_1(\mathbf{I}; \boldsymbol{\theta}; t) \, \mathrm{d}\mathbf{I} \, \mathrm{d}\boldsymbol{\theta}, \tag{2.55}$$

$$\langle f \rangle_2 = \iint f(\mathbf{I}, \boldsymbol{\theta}) \, \hat{P}_2(\mathbf{I}; \boldsymbol{\theta}; t) \, \mathrm{d}\mathbf{I} \, \mathrm{d}\boldsymbol{\theta}. \tag{2.56}$$

3. Continuous Systems: Irreversibility and the Fokker-Planck Equation

The theory developed in §2, as exemplified by (2.50)–(2.52) for instance, is reversible in time, but it predicts that the the structure of P will become increasingly irregular with time. For example, according to (2.46), even if the initial $P(= P_0)$ is independent of $\boldsymbol{\theta}$, P will, through the dependence of $\hat{P}_1$ on $\boldsymbol{\phi}$ $(= \boldsymbol{\theta} - \boldsymbol{\omega}t)$, develop a rapid variation in t. This rapid variation is an essential ingredient of the phenomenon of time reversibility, and any smoothing operation introduced for $t > 0$, such as the one we use below to convert the truncated Liouville system into a Fokker-Planck equation, will introduce time irreversibility.

We shall argue below that, when the spacing between modes is sufficiently small, the behavior of a discrete system mimics that of the corresponding continuous system, in which a smoothing operation has taken place, and which therefore evolves irreversibly.

The continuous system is governed by equations analagous to (2.50)–(2.52), but with the *dephasing factors*, $\Delta^{\pm}(\omega)$ replaced by

$$\Delta^{+}(\omega) \;\rightarrow\; \pi\delta^{+}(\omega) \;\equiv \pi\delta(\omega) + i\mathcal{P}(\omega^{-1}), \tag{3.1a}$$

$$\Delta^{-}(\omega) \;\rightarrow\; \pi\delta^{-}(\omega) \;\equiv \pi\delta(\omega) - i\mathcal{P}(\omega^{-1}), \tag{3.1b}$$

so that

$$\Delta(\omega) \;\rightarrow\; \pi\delta(\omega). \tag{3.1c}$$

Here '$\mathcal{P}$ 'signifies that the principal part should be taken in the integrals that arise below. The equations analogous to (2.50)–(2.52) are

$$\frac{\partial\bar{P}}{\partial t} + \sum_i \omega_i \frac{\partial\bar{P}}{\partial\theta_i} = 6\pi\epsilon^2 \sum_{IJK} \delta^{-}(\omega_{IJK}) c_{ijk}^2 \nabla^{*}_{IJK}\{I_{ijk}\nabla_{IJK}\bar{P}\}, \tag{3.2}$$

$$\hat{P}_1 = i\pi\epsilon \sum_{IJK} \delta^{-}(\omega_{IJK}) c_{ijk} I_{ijk}^{1/2} e^{i\theta_{IJK}} \nabla_{IJK}\bar{P}, \tag{3.3}$$

$$\hat{P}_2 = -\pi\epsilon^2 \sum_{IJKLMN}{}' \left\{ \frac{\delta^{-}(\omega_{IJKLMN}) - \delta^{-}(\omega_{LMN})}{-i\omega_{IJK}} \right\} c_{ijk} c_{lmn} I_{ijklmn}^{1/2} e^{i\theta_{IJKLMN}} \nabla_{IJK}\nabla_{LMN}\bar{P}, \tag{3.4}$$

where to order ϵ^2

$$P = \bar{P} + \hat{P}_1 + \hat{P}_2. \tag{3.5}$$

Equations (3.1) – (3.5) have been written in a form that brings out the similarity between the continuous and discrete cases, but it is to be understood first that the $\partial/\partial I_i$ parts of ∇_I and ∇_{IJK} have become functional derivatives. Second, the sums are now to be interpreted as integrals over the continuous labelling variable i $(0 \le i < \infty)$ according to the prescription

$$\sum_{i=1}^{N} Q_i \longleftrightarrow \int_0^{\infty} di\; \sigma(i) Q(i), \tag{3.6}$$

where $\sigma(i)$ is the density of states. In this limit, the measure of the surface on which $\omega_{IJK} = 0$ is vanishingly small. We assume that in the continuous limit, the contributions from accidental.resonances have vanishing weight, but note however that there exist very special systems of lower dimensionality for which this assumption would not be justified.

The similarities between the continuous and discrete cases is, in one respect, both deceptive and ambiguous. For finite N, it is incorrect to include the "diagonal" terms, $IJK = -(LMN)$ in (2.52), but in the limit $N \to \infty$ the "off-diagonal" terms, $IJK \neq -(LMN)$, are infinitely more numerous, and it might appear that negligible errors would result from including the $\omega_{IJKLMN} = 0$ surface in (3.4). This however is not the case. The matter is discussed in Appendix B, where a prescription for overcoming this difficulty is developed. Meanwhile we add a prime above the $IJKLMN$-summations as a warning.

It may be wondered why the equations that arise in the continuous case resemble (2.50)–(2.52) rather than (2.45)–(2.47). The reason has been touched on at the end of §2. Of greatest interest is the asymptotic form of moments like (2.49) in the limit $t \to \infty$, with $\epsilon^2 t$ ($= \tau_2$) *and* $\boldsymbol{\theta}$, *not* $\boldsymbol{\phi}$, held fixed. The situation may be clarified by an example. Let $\bar{P}$ depend on $\mathbf{I}$, and of course on τ_2, but not on $\boldsymbol{\theta}$ (or equivalently $\boldsymbol{\phi}$), and suppose that we wish to find the average of a $\boldsymbol{\theta}$-dependent function such as

$$f(\mathbf{I}\,;\,\boldsymbol{\theta}) = g(\mathbf{I})\;\mathrm{e}^{-i\theta_{PQR}}.$$

Equation (2.54) shows that $\bar{P}$ does not contribute to the average, i.e. $\langle f \rangle$ is $\mathrm{O}(\epsilon)$ and is given to that order by (2.51) and (2.55). The only the terms that remain after $\boldsymbol{\theta}$ integration are those for which IJK is a permutation of PQR, so that

$$\langle f(\mathbf{I};\,\boldsymbol{\theta})\rangle \;=\; 6i\epsilon \int g(\mathbf{I})\;\Delta^-(-\omega_{PQR})\;c_{pqr}\;I_{pqr}^{1/2}\;\nabla_{pqr}\;\bar{P}\;d\mathbf{I},$$

where the lower case letters pqr on ∇_{pqr} imply that, since $\bar{P}$ is independent of $\boldsymbol{\theta}$, only the $\partial/\partial I_p$ part of ∇_P is effective in ∇_{PQR}, i.e.

$$\nabla_{pqr} \;=\; s_p\nabla_p \;+\; s_q\nabla_q \;+\; s_r\nabla_r, \quad \text{where} \quad \nabla_p \;=\; \frac{\partial}{\partial I_p}.$$

The limiting form of $\langle f(\mathbf{I};\,\boldsymbol{\theta})\rangle$ as $t \to \infty$ is

$$\langle f(\mathbf{I};\,\boldsymbol{\theta})\rangle \;=\; 6i\pi\epsilon \int g(\mathbf{I})\;\delta^-(\omega_{PQR})\;c_{pqr}\;I_{pqr}^{1/2}\;\nabla_{pqr}\;\bar{P}\;d\mathbf{I}. \tag{3.7}$$

This is the expression that would be obtained by formally taking the limit $t \to \infty$ in (2.51) — that is with $\boldsymbol{\theta}$ fixed — using (3.1b), and then computing the moment $\langle f \rangle$ subsequently. If however we took the corresponding limit ($t \to \infty$ with $\boldsymbol{\phi}$ fixed) in (2.46), using (3.1a) before computing the moment, we would obtain

$$\langle f(\mathbf{I};\,\boldsymbol{\theta})\rangle \;=\; 6i\pi\epsilon \int g(\mathbf{I})\;\delta^+(\omega_{PQR})\;\mathrm{e}^{-i\omega_{PQR}t}\;c_{pqr}\;I_{pqr}^{1/2}\;\nabla_{pqr}\;\bar{P}\;d\mathbf{I}. \tag{3.8}$$

Incorrect results such as (3.8) arise whenever the limit $\Delta^+ \to \pi\delta^+$ is taken, disregarding the fact that Δ^+ is multiplied by a rapidly oscillating term, in this case $e^{-i\omega_{PQR}t}$. The guiding principle is that the $t \to \infty$ limit of the Liouville theory of §2 should be such that it gives the correct moments in the same limit.

We shall call (3.2) the *Fokker-Planck equation* for our model hamiltonian (2.2), and on occasion we shall prefer to use the equivalent ϕ-forms of (3.2)–(3.4), which are

$$\frac{\partial \bar{P}}{\partial t} = 6\pi\epsilon^2 \sum_{IJK} \delta^-(\omega_{IJK}) c_{ijk}^2 \nabla_{IJK}^* \{I_{ijk} \nabla_{IJK} \bar{P}\}, \tag{3.9}$$

$$\hat{P}_1 = i\pi\epsilon \sum_{IJK} \delta^-(\omega_{IJK}) c_{ijk} I_{ijk}^{1/2} e^{i(\phi_{IJK}+\omega_{IJK}t)} \nabla_{IJK} \bar{P}, \tag{3.10}$$

$$\hat{P}_2 = -\pi\epsilon^2 \sum_{IJKLMN}{}' \left\{ \frac{\delta^-(\omega_{IJKLMN}) - \delta^-(\omega_{LMN})}{-i\omega_{IJK}} \right\} c_{ijk} c_{lmn} I_{ijklmn}^{1/2} e^{i(\phi_{IJKLMN}+\omega_{IJKLMN}t)} \nabla_{IJK} \nabla_{LMN} \bar{P}. \tag{3.11}$$

Subsequent sections will describe properties of (3.2)–(3.5). The remainder of this section will be devoted to identifying circumstances in which the limiting $t \to \infty$ forms (3.2)–(3.5) describe the behavior of a discrete system for which (2.50)–(2.52) hold. This question is not examined in most analyses of interacting waves where it is assumed from the outset that the spectrum of normal modes is continuous, and in which therefore some subtelties (such as the one concerning $\langle f \rangle$ just considered) need not be faced. The question deserves further attention however. Continuous spectra pertain only to unbounded systems, and no physical system is in reality infinite, though if large its spectrum of normal modes will be tightly packed. It is then natural to suppose that, in some range of parameters, the discrete system will behave not very differently from the corresponding continuous system, i.e. the system obtained by allowing the volume V it occupies to increase indefinitely, with a corresponding decrease in mode spacing to zero. An examination of this prejudice inevitably poses mathematical questions concerning the interchange of two limits: $t \to \infty$ (with $\tau_2 = \epsilon^2 t$ held fixed), and $V \to \infty$. In the usual treatments V is allowed to tend to ∞ first, so that the spectrum is continuous; the waves then interact irreversibly on the slow (τ_2) timescale. It is apparent from §2 however that, while V is finite, P remains reversible no matter how large t becomes. This non-uniformity in the mathematics gives rise to subtelties in the physics, such as the paradoxes of Loschmidt and Zermelo (e.g. see Chandrasekhar, 1943).

The issue may be clarified by means of an example. For simplicity, suppose (2.45) is replaced by

$$\frac{\partial \bar{P}}{\partial t} = \epsilon^2 \sum_{j,k}^{N} \Delta^+(\omega_j - \omega_k) f_{jk} \bar{P}. \tag{3.12}$$

We derive criteria under which this becomes

$$\frac{\partial \bar{P}}{\partial t} = \pi\epsilon^2 \sum_{j,k} \delta^+(\omega_j - \omega_k) f_{jk} \bar{P} \tag{3.13}$$

in the continuous limit. As usual, the summation in (3.13) is to be understood as an integral over the labels j and k.

First suppose that $\bar{P}$ is smooth, or is represented by an average ($\mathcal{F}$, say) that is smooth, and does not change appreciably over the frequency scale Ω. On the one hand, the limit (3.13) of (3.12) cannot be justified unless

$$\Omega t \gg 1. \tag{3.14}$$

This lower bound on t is determined by the *graininess* of $\mathcal{F}$. On the other hand, when t becomes large compared with the reciprocal spacing between modes, the peak of $\Delta^+(\omega_j - \omega_k)$ will in general lie "between" the discrete points of j or k summation, and the replacement of $\Delta^+(\omega_j - \omega_k)$ by $\delta^+(\omega_j - \omega_k)$ will be unjustified. It is essential that many of the discrete modes lie within a "bandwidth"of ω_j or ω_k. This bandwidth, $B(\omega)$, is proportional to the rate at which $\bar{P}$ is removed from the mode of frequency ω by the non-integrable interactions. In the case of model (3.12),

$$B(\omega) = \mathrm{O}\left(\epsilon^2 f(\omega, \omega)\sigma(\omega)\right), \tag{3.15}$$

where f sets the scale of f_{jk} and $\sigma(\omega)$ is the density of states, i.e. $\sigma(\omega)d\omega$ is the number of modes with frequencies lying between ω and $\omega + d\omega$. The bandwidth condition requires that

$$B(\omega) \gg 1/\sigma(\omega). \tag{3.16}$$

By (3.15), this places a lower bound on ϵ; unless $\partial\mathcal{F}/\partial t$ is sufficiently large, $\mathcal{F}$ will not have changed substantially before t^{-1} becomes small compared with the mode spacing. The lower bound on ϵ placed by (3.16) does not violate the upper bound necessary for the perturbation expansion to be valid:

$$B(\omega) \ll \omega. \tag{3.17}$$

Returning to the Fokker-Planck equation (3.9), we may regard B as the rate at which $\bar{P}$ varies with time, and Ω as a measure of the smoothness of $\bar{P}$ in I_i as i varies. In general, physical results are obtained as moments, by multiplying P by some function of $\mathbf{I}$ and $\boldsymbol{\phi}$, and integrating over all $\mathbf{I}$ and $\boldsymbol{\phi}$. Contributions which do not exactly "match off "in $\boldsymbol{\phi}$ integrate to zero, as in the example considered below (3.6). Thus, when P is expanded as

$$P(\mathbf{I};\ \boldsymbol{\phi};\ t) = \sum_{\mathbf{n}} P_{\mathbf{n}}(\mathbf{I};\ t)\ \mathrm{e}^{i\mathbf{n}\cdot\boldsymbol{\phi}}, \tag{3.18}$$

it is only the dependence of $P_{\mathbf{n}}$ on $\mathbf{I}$ that is relevant in the inequalities (3.14), (3.16) and (3.17).

While we believe that the inequalities (3.14), (3.16) and (3.17) are necessary to justify the transition (3.1) from the reversible Liouville system (2.45) – (2.47) to the irreversible Fokker-Planck system (3.9) – (3.11), they are not necessarily sufficient. One key question remains concerning the assumed smoothness of P: if at some initial instant, P is sufficiently smooth, so that (3.14) holds true and, if H_1 is "sufficiently non-integrable" so that (3.16) is met, will the Liouville system lead to a smooth P at later times? This question is evaded when, as is customary, the $\sigma \to \infty$ limit precedes the $t \to \infty$ limit. A complete answer to the question would be tantamount to proving the second law of thermodynamics. Nothing so ambitious is attempted here, but some speculations may be in order.

Whenever a measurement is made of the state of a system, an interaction is introduced with another "measuring system", that of the "observer", which is characterized by its own P. If the P of an extended observer is smooth, it is reasonable to suppose that during its interaction it will impose its own smoothness on the P of the observed system. This assumption does not remove the difficulty; it transfers it to a more remote level, that of explaining why the observer's P should be smooth. This scenario, and its shortcoming, runs parallel to the familiar difficulty of measurement in quantum theory. In this respect one is tempted to regard P as a physical variable that is the counterpart of the wavefunction in quantum mechanics, with the smoothing of P through measurement paralleled by the collapse of the wavefunction. (Note also the linearity of Liouville's equation and that of Schrödinger's equation.) There is of course one important difference: the minimum scale for irreversibility in quantum mechanics is set by $\hbar$ and is finite, whereas in classical mechanics it is infinitesimal. (See chapter 1 of Landau and Lifshitz, 1969.)

From now onwards we shall set these questions on one side, assume that $\bar{P}$ is smooth and that (3.16) and (3.17) hold so that, over sufficiently long times as dictated by (3.14), P is governed by the Fokker-Planck system (3.2) – (3.5).

4. General Consequences of the Fokker-Planck Equation

In this section, general consequences of the Fokker-Planck ("FP") equation (3.9) governing $\bar{P}$ are presented.

The full Liouville equation (2.12) admits any differentiable function ($g(H)$, say) of the Hamiltonian (2.2) as a steady state solution. It is not difficult to show that the truncated Liouville system (2.45) – (2.47) shares the same property to $O(\epsilon^2)$. More precisely, recalling that $\boldsymbol{\phi}$ and $\boldsymbol{\theta}$ coincide at $t = 0$, and solving (2.45) subject to the $\boldsymbol{\theta}$-dependent initial condition

$$P_0 = g(H_0) + \epsilon g'(H_0)H_1(\mathbf{I};\boldsymbol{\phi}) + \frac{1}{2}\epsilon^2 g''(H_0)H_1(\mathbf{I};\boldsymbol{\phi})^2 \approx g(H), \tag{4.1a}$$

substituting the resulting $\bar{P}$ into (2.46) and (2.47) to obtain $\hat{P}_1$ and $\hat{P}_2$, and simplifying with the help of (2.42), we find that to $O(\epsilon^2)$ for all t,

$$P = g(H_0) + \epsilon g'(H_0)H_1(\mathbf{I};\boldsymbol{\theta}) + \frac{1}{2}\epsilon^2 g''(H_0)H_1(\mathbf{I};\boldsymbol{\theta})^2 \approx g(H). \tag{4.1b}$$

If, in contrast to (4.1a), we start at $t = 0$ with a $\boldsymbol{\theta}$-independent probability such as

$$P_0 = g(H_0),$$

which differs from (4.1a) by $O(\epsilon)$, we find that P varies rapidly at all subsequent times: to order ϵ^2,

$$P = g(H - \epsilon \sum_{IJK} c_{ijk}\, I_{ijk}^{1/2}\, e^{i(\theta_{IJK} - \omega_{IJK} t)}).$$

We shall shortly discuss an analogous but irreversible property of the FP-equation: starting from an arbitrary P_0,

$$\bar{P} \to g(H_0), \qquad P \to g(H), \qquad t \to \infty, \tag{4.2}$$

for some function g; see (4.14) below.

The FP-equation maintains the positivity of $\bar{P}$. To see this, assume that $\bar{P} \geq 0$ for all $\mathbf{I}$ and $\boldsymbol{\phi}$ at $t = 0$. Then (3.9) shows that the same is true for all t. For suppose otherwise. Then as t increases, a time will be reached at which $\bar{P} = 0$ at one $(\mathbf{I}, \boldsymbol{\phi})$, say at $(\mathbf{I}_0, \boldsymbol{\phi}_0)$. Being a minimum of $\bar{P}$, all first derivatives of $\bar{P}$ vanish at $(\mathbf{I}_0, \boldsymbol{\phi}_0)$, and second derivatives will be non-negative in all "directions" from that point. Thus at $(\mathbf{I}_0, \boldsymbol{\phi}_0)$ we have

$$\nabla^*_{IJK}[I_{ijk}\, \nabla_{IJK}\, \bar{P}] = I_{ijk}\Big\{\big(s_i\frac{\partial}{\partial I_i} + s_j\frac{\partial}{\partial I_j} + s_k\frac{\partial}{\partial I_k}\big)^2\bar{P} + \frac{1}{4}\big(\frac{1}{I_i}\frac{\partial}{\partial \phi_i} + \frac{1}{I_j}\frac{\partial}{\partial \phi_j} + \frac{1}{I_k}\frac{\partial}{\partial \phi_k}\big)^2\bar{P}\Big\}, \tag{4.3}$$

which is positive. When we substitute (4.3) into (3.9), we find that, because the rest of the integrand is even in $\mathbf{s}$, the principal part of $\delta^-(\omega_{IJK})$ makes no contribution and may be omitted. Using (4.3), we see that $\partial\bar{P}\ /\partial t\ >\ 0$, so that the zero of $\bar{P}$ disappears, leaving $\bar{P}$ again positive for all $(\mathbf{I},\ \boldsymbol{\phi})$.

On multiplying (3.9) by $g'(\bar{P})$ and integrating, we find that

$$\frac{\partial}{\partial t}\iint g(\bar{P})\ d\mathbf{I}\ d\boldsymbol{\phi}\ =\ -6\pi\epsilon^2\iint g''(\bar{P})\sum_{IJK}\ \delta(\omega_{IJK})\ c_{ijk}^2\ I_{ijk}\ \left|\ \nabla_{IJK}\ \bar{P}\ \right|^2\ \mathrm{d}\mathbf{I}\ \mathrm{d}\boldsymbol{\phi}. \quad (4.4)$$

Again, the principal part makes no contribution, this time because $|\nabla_{IJK}\ \bar{P}|^2$ is even in $\mathbf{s}$. In writing (4.4) it has been suppose that $g(\bar{P})$ is such that the integrated part vanishes at $I_i\ =\ 0$ and ∞. Particular cases of (4.4) are

$$\frac{\partial}{\partial t}\iint \bar{P}\ \mathrm{d}\mathbf{I}\ \mathrm{d}\boldsymbol{\phi}\ =\ 0, \quad (4.5)$$

$$\frac{\partial}{\partial t}\iint \bar{P}^2\mathrm{d}\mathbf{I}\ \mathrm{d}\boldsymbol{\phi}\ =\ -12\pi\epsilon^2\iint\sum_{IJK}\ \delta(\omega_{IJK})\ c_{ijk}^2\ I_{ijk}\ \left|\ \nabla_{IJK}\ \bar{P}\ \right|^2\ \mathrm{d}\mathbf{I}\ \mathrm{d}\boldsymbol{\phi}. \quad (4.6)$$

Equation (4.5) ensures that the normalization of P is maintained: if at $t\ =\ 0$

$$\iint P\ \mathrm{d}\mathbf{I}\ \mathrm{d}\boldsymbol{\phi}\ =\ \iint \bar{P}\ \mathrm{d}\mathbf{I}\ \mathrm{d}\boldsymbol{\phi}\ =\ 1, \quad (4.7)$$

then the same is true for all t. [Note that the integrals of $\hat{P}_1$ and $\hat{P}_2$ over $\mathbf{I}$ and $\boldsymbol{\phi}$ are zero by (3.10) and (3.11).] From (4.7) and the fact that, by (4.6),

$$\frac{\partial}{\partial t}\iint \bar{P}^2\ \mathrm{d}\mathbf{I}\ \mathrm{d}\boldsymbol{\phi}\ \leq\ 0, \quad (4.8)$$

$\bar{P}$ must ultimately equilibrate to a distribution, $\bar{P}_E$ say, that makes the right-hand side of (4.6) vanish.

To elucidate the nature of $\bar{P}_E$, it is convenient to note first that, because of its single-valuedness, $\bar{P}$ may be Fourier expanded:

$$\bar{P}\ =\ \sum_{\mathbf{n}}\bar{P}_{\mathbf{n}}(\mathbf{I},\ t)\ \mathrm{e}^{i\mathbf{n}\cdot\boldsymbol{\phi}}, \quad (4.9)$$

where, because of the reality of $\bar{P}$,

$$\bar{P}_{-\mathbf{n}}(\mathbf{I},\ t)\ =\ \bar{P}_{\mathbf{n}}^*(\mathbf{I},\ t). \quad (4.10)$$

Substituting (4.9) into (4.6) we obtain the "H-theorem":

$$\frac{\partial}{\partial t}\iint \bar{P}^2\mathrm{d}\mathbf{I}\ \mathrm{d}\boldsymbol{\phi}\ =\ -12\pi\epsilon^2\iint\sum_{IJK}\sum_{\mathbf{n}}\ \delta(\omega_{IJK})\ c_{ijk}^2\ I_{ijk}\ \times$$
$$\left\{\left|\ [s_i\frac{\partial}{\partial I_i}+s_j\frac{\partial}{\partial I_j}+s_k\frac{\partial}{\partial I_k}]\ \bar{P}_{\mathbf{n}}\ \right|^2+\frac{1}{4}\left|\ [\frac{n_i}{I_i}+\frac{n_j}{I_j}+\frac{n_k}{I_k}]\ \bar{P}_{\mathbf{n}}\ \right|^2\right\}\mathrm{d}\mathbf{I}\ \mathrm{d}\boldsymbol{\phi}. \quad (4.11)$$

In equilibrium the two negative definite contributions to the right-hand side of (4.11) must vanish separately. Thus only the $\mathbf{n} = 0$ term in (4.9) survives. In fact the development of a $\boldsymbol{\phi}$-independent $\bar{P}$ must take place rather rapidly, since these terms in (4.11) are associated with decay rates proportional to n^2. One may conclude that after a short time, any surviving $\mathbf{n}$-dependence in $\bar{P}$ must be confined to relatively small n.

According to (4.11), in equilibrium $\bar{P}_E = \bar{P}_E(\mathbf{I})$ and

$$\left| \left[s_i \frac{\partial}{\partial I_i} + s_j \frac{\partial}{\partial I_j} + s_k \frac{\partial}{\partial I_k}\right] \bar{P}_E \right|^2$$

must vanish whenever $\omega_{IJK} = 0$. It seems plausible that therefore

$$\left[s_i \frac{\partial}{\partial I_i} + s_j \frac{\partial}{\partial I_j} + s_k \frac{\partial}{\partial I_k}\right] \bar{P}_E \propto \omega_{IJK}, \tag{4.12}$$

which is satisfied if

$$\bar{P}_E = \bar{P}_E(H_0). \tag{4.13}$$

Conversely, it is clear that (4.13) satisfies (3.9). Corresponding to (4.13), we have, by (3.10) and (3.11),

$$P_E = \bar{P}_E(H_0) + \epsilon \bar{P}_E'(H_0) H_1(\mathbf{I};\boldsymbol{\theta}) + \frac{1}{2}\epsilon^2 \bar{P}_E''(H_0) H_1(\mathbf{I};\boldsymbol{\theta})^2, \tag{4.14}$$

which is the expansion to $O(\epsilon^2)$ of

$$P_E = P_E(H). \tag{4.15}$$

Compare (4.13) and (4.15) with (4.2).

A subtle point arises here, which concerns the diagonal terms defined below (3.6) and discussed in Appendix B. There is no justification, at the $O(\epsilon^2)$ accuracy of the present theory, for including the diagonal part of $H_1(\mathbf{I};\boldsymbol{\theta})^2$ in (4.14). More precisely, one should replace (4.14) by

$$P_E = \bar{P}_E(H_0) + \epsilon \bar{P}_E'(H_0) H_1(\mathbf{I};\boldsymbol{\theta}) + \frac{1}{2}\epsilon^2 \bar{P}_E''(H_0) \left[H_1(\mathbf{I};\boldsymbol{\theta})^2 - 48 \sum_{ijk} c_{ijk}^2 I_{ijk} \right]. \tag{4.14$'$}$$

At first sight, the difference between (4.14) and (4.14′) appears to be inconsequential; $H_1(\mathbf{I};\boldsymbol{\theta})^2$ involves a six-fold integration whereas the summation of $c_{ijk}^2 I_{ijk}$ involves only three integrations, and seems comparatively negligible. If however we use (4.14) to obtain the moment of a function of $\mathbf{I}$ alone, we obtain an $O(\epsilon^2)$ contribution that (4.14′) would not produce.

Ironically, it is (4.14) that gives the physically correct moment at $O(\epsilon^2)$. The Liouville and Fokker-Planck equations, truncated at the $O(\epsilon^2)$ of the present theory, cannot reproduce this term. We presume that, if we generalized our expansion to the $O(\epsilon^4)$ level, we would obtain a truncated Liouville equation for $\bar{P}$ of the form

$$\frac{\partial \bar{P}}{\partial t} = \epsilon^2 \mathcal{L}_2 \bar{P} + \epsilon^4 \mathcal{L}_4 \bar{P},$$

in place of (2.45), and that the resulting post-Fokker-Planck equation, replacing (3.9) in the continuous limit, would, when solved for the equilibrium distribution, supply the missing diagonal terms in (4.14′). This is, however, a matter of conjecture: the fact is that, when we take moments of $\boldsymbol{\theta}$-independent functions, we obtain answers from (4.14′) that are physically valid only to order ϵ.

If the initial energy, E, of the system is known with certainty, then for some F (that for finite N need depend on only $2N - 1$ actions and angles),

$$P(\mathbf{I};\ \boldsymbol{\theta};\ 0) \ = \ F(\mathbf{I};\ \boldsymbol{\theta})\,\delta(H_0 \ - \ E). \tag{4.16}$$

This suggests that we should seek solutions of (3.2) of the form

$$\bar{P}(\mathbf{I};\ \boldsymbol{\theta};\ t) \ = \ F(\mathbf{I};\ \boldsymbol{\theta};\ t)\,\delta(H_0 \ - \ E). \tag{4.17}$$

On substituting this into (3.2), we discover that the factor $\delta(H_0 \ - \ E)$ factors out, leaving

$$\frac{\partial F}{\partial t} + \sum_i \omega_i \frac{\partial F}{\partial \theta_i} = 6\pi\epsilon^2 \sum_{IJK} \delta^-(\omega_{IJK}) c_{ijk}^2 \nabla_{IJK}^* \{ I_{ijk} \nabla_{IJK} F \}, \tag{4.18}$$

The arguments that led to (4.13) may now be repeated to show that

$$F \ \rightarrow \ F_E \ = F(H_0), \qquad \text{as} \qquad t \ \rightarrow \ \infty, \tag{4.19}$$

which, in view of (4.17), implies that

$$\bar{P} \ \rightarrow \ \bar{P}_E \ = \ C\,\delta(H_0 \ - \ E), \qquad \text{as} \qquad t \ \rightarrow \ \infty, \tag{4.20}$$

so that

$$P \ \rightarrow \ P_E \ = \ C\,\delta(H \ - \ E), \qquad \text{as} \qquad t \ \rightarrow \ \infty, \tag{4.21}$$

where C is a constant. This is the microcanonical distribution: amongst the states of the given energy, all $\mathbf{I}$ and $\boldsymbol{\theta}$ become equally probable .

It is perhaps worth mentioning that only minor modifications are required to generalize the arguments of this section to the multivalued probability $p(\mathbf{I};\ \boldsymbol{\phi};\ t)$ described below (2.13). The demand that p vanishes for $|\theta_i| \ \rightarrow \infty$ stands in place of the assumed periodicity of P in θ_i. The Fourier sum (4.9) is replaced by a Fourier integral over a continuous variable $\mathbf{n}$, and similarly an integral over $\mathbf{n}$ replaces the sum over $\mathbf{n}$ in (4.11).

5. Reduced Distributions, Kinetic Equations and the Stosszahl Ansatz

In this section we shall use the Fokker-Planck equation to follow the evolution of mean values, $\langle f(\mathbf{I};\, \boldsymbol{\theta})\rangle(t)$, of bounded functions $f(\mathbf{I};\, \boldsymbol{\theta})$; see (2.49).

Consider first the simplest case, $f = f(\mathbf{I})$, in which f is independent of the angles $\boldsymbol{\theta}$, so that only the $\mathbf{n} = \mathbf{0}$ terms contribute to $\langle f\rangle$. Then, as regards the secular, $O(\epsilon^0)$, contribution to the average of $f(\mathbf{I})$ as given by (2.54), we may disregard the dependence of $\bar{P}$ on $\boldsymbol{\theta}$. If, in fact, $\bar{P}$ is independent of $\boldsymbol{\theta}$ then $\langle f\rangle_1 = \langle f\rangle_2 = 0$, because the $\boldsymbol{\theta}$-dependent terms in $\hat{P}_1$ and $\hat{P}_2$ do not affect $\langle f\rangle$ in this case. In this section, we shall consider only $\langle f(\mathbf{I})\rangle_0$, and can therefore ignore both the subscript $_0$ and the $\boldsymbol{\theta}$-dependence of $\bar{P}$:

$$\langle f(\mathbf{I})\rangle = \iint f(\mathbf{I})\; \bar{P}(\mathbf{I})\; \mathrm{d}\mathbf{I}\; \mathrm{d}\boldsymbol{\theta}. \tag{5.1}$$

[The integration over $\boldsymbol{\theta}$ is cosmetic, introducing the factors of 2π necessary to maintain the normalization $\langle 1\rangle = 1$; see (4.7).] Since the $\boldsymbol{\theta}$-derivatives in ∇_I and ∇_{IJK} act on $\boldsymbol{\theta}$-independent functions we may, in the notation introduced above (3.7), replace ∇_{IJK} in (3.9) by ∇_{ijk} $(= \nabla^*_{ijk})$. The principal part term in (3.9) is now odd in $\mathbf{s}$, so that δ may stand in place of δ^-. On differentiating (5.1) by t, applying (3.9) and integrating twice by parts, we obtain

$$\frac{d}{dt}\langle f(\mathbf{I})\rangle = 6\pi\epsilon^2 \sum_{IJK} \delta(\omega_{IJK})\; c_{ijk}^2\; \langle \nabla_{ijk}(I_{ijk}\nabla_{ijk} f)\rangle. \tag{5.2}$$

[To make the integrated parts zero, f must be bounded at $\mathbf{I} = 0$, and $\bar{P}$ must vanish sufficiently rapidly for $|\mathbf{I}| \to \infty$.]

When $f = f(I_p)$ is a function of one particular action I_p alone, (5.2) simplifies further:

$$\frac{d}{dt}\langle f(I_p)\rangle = 18\pi\epsilon^2 \sum_{s_p JK} \delta(\omega_{PJK})\; c_{pjk}^2 s_p \big[s_p\langle I_j I_k (I_p f')'\rangle + s_j\langle I_k I_p f'\rangle + s_k\langle I_j I_p f'\rangle\big]. \tag{5.3}$$

In particular

$$\frac{d}{dt}\langle I_p\rangle = 18\pi\epsilon^2 \sum_{s_p JK} \delta(\omega_{PJK})\; c_{pjk}^2 s_p \big[s_p\langle I_j I_k\rangle + s_j\langle I_k I_p\rangle + s_k\langle I_j I_p\rangle\big], \tag{5.4}$$

$$\frac{d}{dt}\langle I_p^2\rangle = 36\pi\epsilon^2 \sum_{s_p JK} \delta(\omega_{PJK})\; c_{pjk}^2 s_p \big[2s_p\langle I_j I_k I_p\rangle + s_j\langle I_k I_p^2\rangle + s_k\langle I_j I_p^2\rangle\big]. \tag{5.5}$$

Equations (5.4) and (5.5) are not closed kinetic equations, since it is in general impossible to express the mean of products, such as $\langle I_j I_k\rangle$, as products of means, such as $\langle I_j\rangle\langle I_k\rangle$. What can be demonstrated however is that, if I_j and I_k are initially uncorrelated, they will develop correlations at a rate that is "thermodynamically small", i.e. at a rate that vanishes in the $V \to \infty$ limit of a continuous system.

In order to examine this question, we must return to the discrete system of §3, and again examine the transition to the continuous case. Suppose that at $t = 0$, some finite number n ($<< N$) of actions are uncorrelated, so that

$$\langle g_1(I_1)g_2(I_2)....g_n(I_n)\rangle = \langle g_1(I_1)\rangle\langle g_2(I_2)\rangle....\langle g_n(I_n)\rangle. \tag{5.6}$$

We aim to show that, although (5.6) is, according to (3.9), untrue for any $t > 0$, deviations from (5.6) build up at a rate proportional to N^{-1}, or more precisely $\sigma(\omega)^{-1}$. By (3.16), this is small compared with $[\gamma(\omega)]^{-1}$, the timescale of thermal equilibration in which we are primarily interested.

We illustrate the argument through the special case of (5.6) in which all g_i are unity, except $g_p(I_p) = I_p$ and $g_q(I_q) = I_q$ where $p \neq q$. By (5.2) we have

$$\frac{d}{dt}\langle I_pI_q\rangle = 18\pi\epsilon^2 \sum_{s_pJK} \delta(\omega_{PJK})\, c^2_{pjk}\, s_p\, [s_p\langle I_jI_kI_q\rangle + s_j(1+\delta_{jq})\langle I_kI_pI_q\rangle + s_k(1+\delta_{kq})\langle I_jI_pI_q\rangle] + (p \leftrightarrow q), \tag{5.7}$$

where by '$(p \leftrightarrow q)$' we mean that the previous expression on the right-hand side of (5.7) is repeated, with p and q interchanged. When we combine (5.7) with (5.4), the cumulant

$$\mathcal{C}_{jk,q} = \langle I_jI_kI_q\rangle - \langle I_jI_k\rangle\langle I_q\rangle, \tag{5.8}$$

arises. Clearly $\mathcal{C}_{jk,q} = \mathcal{C}_{kj,q}$, and for convenience we shall take $\mathcal{C}_{qk,q} = \mathcal{C}_{jq,q} = 0$ (not summed). [Strictly Δ should replace $\pi\delta$ in (5.7) and (5.9) below, but its effect in the later limit $N \to \infty$ is that of the $\pi\delta$ shown.] By (5.2) and (5.7)

$$\begin{aligned}\frac{d}{dt}[\langle I_pI_q\rangle - \langle I_p\rangle\langle I_q\rangle] =\\ 18\pi\epsilon^2\Big\{ \sum_{\substack{s_pJK\\ j,k\neq q}} \delta(\omega_{PJK})\, c^2_{pjk}\, s_p\, [s_p\mathcal{C}_{jk,q} + s_j\mathcal{C}_{kp,q} + s_k\mathcal{C}_{jp,q}]\\ + 2\sum_{s_ps_qJ} \delta(\omega_{PQJ})\, c^2_{pqj}\, s_p\, [s_p\{\langle I_jI_p^2\rangle - \langle I_jI_p\rangle\langle I_p\rangle\} + s_q\{2\langle I_jI_pI_q\rangle - \langle I_jI_p\rangle\langle I_q\rangle\}\\ + s_j\{\langle I_pI_q^2\rangle - \langle I_pI_q\rangle\langle I_q\rangle\}]\Big\} + (p \leftrightarrow q).\end{aligned} \tag{5.9}$$

The first summation in braces is a double sum over J and K, or (in view of the δ-function) a single sum over J or K. The second summation is a single sum over J, or (in view of the δ-function) a sum essentially only over two nonzero terms at the sum and difference frequencies $\omega_j = \omega_p + \omega_q$ and $\omega_j = |\omega_p - \omega_q|$. The first summation in braces is therefore formally larger than the second by a factor of order N. Suppose now that (5.6) holds initially, so that $\mathcal{C}_{jk,q} = 0$ at $t = 0$. Then $\langle I_pI_q\rangle - \langle I_p\rangle\langle I_q\rangle$ evolves at a rate that vanishes in the limit $N \to \infty$. But $\langle I_pI_q\rangle$ and $\langle I_p\rangle\langle I_q\rangle$ evolve at rates determined by the nonlinearity, i.e. at rates proportional to ϵ^2. Thus, if at $t = 0$,

$$\langle I_pI_q\rangle = \langle I_p\rangle\langle I_q\rangle, \tag{5.10}$$

then this continues to hold with negligible error throughout the subsequent thermalization.

This argument may be generalized to a cumulant of order n ($<< N$). The principal terms that lead to the development of cross-correlations vanish when the cumulants are zero, but they are formally N/n times more numerous than the secondary terms that are present because of the finiteness of the system. The latter can be regarded as "thermodynamically insignificant" in the limit $V \to \infty$. Henceforward we consider only continuous system, and omit the thermodynamically small terms without comment.

The lack of correlation epitomized by the factorization (5.10), or more generally (5.6), was named "the Stosszahl Ansatz" by the Ehrenfests (1912). It is a common tool in theories of wave interaction in continuous systems, e.g. Hasselmann (1960), Litvak (1960), Newell and Aucoin (1971), Putterman and Roberts (1983a,b). It has been the object of special study by Benney and Saffman (1966) and by Newell (1968); see also Hasselmann (1967) and Saffman (1967).

With the help of factorizations such as (5.6) and (5.10), we may now close kinetic equations such as (5.3) – (5.5). For example, we may write (5.3) as

$$\frac{d}{dt}\langle f(I_p)\rangle \;=\; A_p\langle \frac{\partial}{\partial I_p}(I_p\frac{\partial f(I_p)}{\partial I_p})\rangle \;-\; B_p\langle I_p\frac{\partial f(I_p)}{\partial I_p}\rangle, \tag{5.11}$$

where

$$A_p \;=\; 18\pi\epsilon^2 \sum_{s_p JK} \delta(\omega_{PJK})\; c^2_{pjk}\langle I_j\rangle\langle I_k\rangle \;>\; 0, \tag{5.12}$$

$$B_p \;=-18\pi\epsilon^2 \sum_{s_p JK} \delta(\omega_{PJK})\; c^2_{pjk}s_p[s_j\langle I_k\rangle \;+s_k\langle I_j\rangle]. \tag{5.13}$$

In particular, taking $f(I_p) = I_p$ in (5.11), we obtain in place of (5.4),

$$\frac{d}{dt}\langle I_p\rangle \;=\; 18\pi\epsilon^2 \sum_{s_p JK} \delta(\omega_{PJK})\; c^2_{pjk}s_p\big[s_p\langle I_j\rangle\langle I_k\rangle \;+\; s_j\langle I_k\rangle\langle I_p\rangle \;+\; s_k\langle I_j\rangle\langle I_p\rangle\big], \tag{5.14}$$

as indeed would be obtained by direct factorization of the products of moments in (5.4).

The factorization (5.6), which has led to simple kinetic equations like (5.11), indicate that P itself can be factored. To study this question, we introduce reduced probability densities, such as

$$\bar{P}^1(I_p;\; t) \;=\iint \bar{P}\;\mathrm{d}_p\mathbf{I}\;\mathrm{d}\boldsymbol{\theta}, \tag{5.15a}$$

$$\bar{P}^2(I_p,\; I_q;\; t) \;=\iint \bar{P}\;\mathrm{d}_{pq}\mathbf{I}\;\mathrm{d}\boldsymbol{\theta}, \tag{5.15b}$$

which only involve a small number, n ($<< N$), of modes. Here

$$d_p\mathbf{I} = \prod_{i\neq p}\mathrm{d}I_i = \mathrm{d}\mathbf{I}/\mathrm{d}I_p, \qquad \mathrm{d}_{pq}\mathbf{I} = \prod_{i\neq p,q}\mathrm{d}I_i = \mathrm{d}\mathbf{I}/\mathrm{d}I_p\mathrm{d}I_q. \tag{5.16}$$

It is clear that

$$\langle f(I_p)\rangle \;=\; \int f(I_p)\,\bar{P}^1(I_p)\,\mathrm{d}I_p, \tag{5.17}$$

$$\bar{P}^1(I_p) \;=\; \int \bar{P}^2(I_p,\,I_q)\,\mathrm{d}I_q. \tag{5.18}$$

It is also convenient to introduce averages taken over all modes *except* p or *except* p and q:

$$\langle f(\mathbf{I})\rangle_p \;=\; \iint f(\mathbf{I})\,\bar{P}(\mathbf{I})\,\mathrm{d}_p\mathbf{I}\,\mathrm{d}\boldsymbol{\theta}, \tag{5.19}$$

$$\langle f(\mathbf{I})\rangle_{pq} \;=\; \iint f(\mathbf{I})\,\bar{P}(\mathbf{I})\,\mathrm{d}_{pq}\mathbf{I}\,\mathrm{d}\boldsymbol{\theta}. \tag{5.20}$$

In this sense

$$\bar{P}^1(I_p) \;=\; \langle 1\rangle_p, \qquad \bar{P}^2(I_p,\,I_q) \;=\; \langle 1\rangle_{pq}, \tag{5.21}$$

$$\langle I_p\frac{\partial f(I_p)}{\partial I_p}\rangle \;=\; -\int f(I_p)\,I_p\frac{\partial \bar{P}^1(I_p)}{\partial I_p}\,\mathrm{d}I_p, \tag{5.22}$$

$$\langle\frac{\partial}{\partial I_p}(I_p\frac{\partial f(I_p)}{\partial I_p})\rangle \;=\; \int f(I_p)\frac{\partial}{\partial I_p}(I_p\frac{\partial \bar{P}^1(I_p)}{\partial I_p})\,\mathrm{d}I_p, \tag{5.23}$$

the integrated part vainshing for the usual reasons. On substituting into (5.11), we obtain

$$\int f(I_p)\Big\{\frac{\partial \bar{P}^1}{\partial t} \;-\; \frac{\partial}{\partial I_p}\big[I_p(A_p\frac{\partial \bar{P}^1}{\partial I_p} \;+\; B_p\bar{P}^1)\big]\Big\}\,\mathrm{d}I_p \;=\; 0,$$

and, since this holds for arbitrary $f(I_p)$,

$$\frac{\partial \bar{P}^1}{\partial t} \;=\; \frac{\partial}{\partial I_p}\big[I_p(A_p\frac{\partial \bar{P}^1}{\partial I_p} \;+\; B_p\bar{P}^1)\big]. \tag{5.24}$$

Alternatively, we may argue as follows. Integrate (3.9) over all $\boldsymbol{\theta}$, and over all $\mathbf{I}$ except I_p. This gives

$$\frac{\partial \bar{P}^1(I_p)}{\partial t} \;=\; 18\pi\epsilon^2\sum_{s_p JK}\delta(\omega_{PJK})\;c_{pjk}^2 s_p\frac{\partial}{\partial I_p}\;I_p\Big\{s_p\frac{\partial}{\partial I_p}\langle I_jI_k\rangle_p \;-\; \big[s_j\langle I_k\rangle_p \;+\; s_k\langle I_j\rangle_p\big]\Big\}, \tag{5.25}$$

and, apart from thermodynamically small terms, we find similarly that

$$\frac{\partial \bar{P}^2(I_p,\,I_q)}{\partial t} \;=\; 18\pi\epsilon^2\sum_{s_p JK}\delta(\omega_{PJK})\;c_{pjk}^2 s_p\frac{\partial}{\partial I_p}\;I_p$$
$$\Big\{s_p\frac{\partial}{\partial I_p}\langle I_jI_k\rangle_{pq} \;-\; \big[s_j\langle I_k\rangle_{pq} \;+\; s_k\langle I_j\rangle_{pq}\big]\Big\} + \;(p\leftrightarrow q). \tag{5.26}$$

If at $t = 0$

$$\bar{P}^2(I_p,\ I_q) = \bar{P}^1(I_p)\,\bar{P}^1(I_q), \tag{5.27}$$

then, recalling that when f is independent of I_p and I_q,

$$\langle f\rangle_{pq} = \langle f\rangle_p\,\bar{P}^1(I_q) = \langle f\rangle_q\,\bar{P}^1(I_p), \tag{5.28}$$

we have (taking successively $f = I_j$ and $f = I_j I_k$) by (5.25) and (5.26)

$$\frac{\partial}{\partial t}\left[\bar{P}^2(I_p,\ I_q) - \bar{P}^1(I_p)\bar{P}^1(I_q)\right] = 0, \tag{5.29}$$

i.e. the factorization (5.27) is maintained. To the extent that the effect of a single mode p to an average $\langle f\rangle$ over all $N(>> 1)$ modes is negligibly small,

$$\langle f\rangle_p = \langle f\rangle\,\bar{P}^1(I_p). \tag{5.30}$$

On using this result in (5.25), we again obtain (5.24).

We now turn our attention to steady states, and equilibrium states. According to (5.11), the mode p cannot be steady unless

$$B_p > 0. \tag{5.31}$$

For, setting $f(I_p) = I_p$ in (5.11), we see that in steady conditions

$$\langle I_p\rangle = A_p/B_p, \tag{5.32}$$

the left-hand side of which is necessarily positive, as is A_p. We shall find that (5.31) is obeyed in thermal equilibrium. By setting $f(I_p) = I_p^n$ in (5.11), we obtain

$$\frac{d}{dt}\langle I_p^n\rangle = n^2 A_p\langle I_p^{n-1}\rangle - nB_p\langle I_p^n\rangle, \tag{5.33}$$

so that

$$\frac{d}{dt}\left[\langle I_p^2\rangle - 2\langle I_p\rangle^2\right] = -2B_p\left[\langle I_p^2\rangle - 2\langle I_p\rangle^2\right]. \tag{5.34}$$

Even if only the first moment has its equilibrium value, the second moment will evolve and, according to (5.34), will attain its equilibrium in a time of order B_p^{-1}. More generally it follows from (5.33) that, when all moments have become time independent,

$$\langle I_p^n\rangle = n!\,\langle I_p\rangle^n, \tag{5.35}$$

so that (assuming the series involved are convergent)

$$\langle f(I_p)\rangle = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\langle I_p^n\rangle = \sum_{n=0}^{\infty} f^{(n)}(0)\langle I_p\rangle^n. \tag{5.36}$$

The result (5.35) reflects the fact that, when mode-p is statistically steady,

$$\bar{P}^1(I_p) \;=\; \frac{1}{\langle I_p \rangle}\, e^{-I_p/\langle I_p \rangle}, \tag{5.37}$$

which also satisfies the steady-state form of (5.24).

Suppose now that *all* modes are statistically steady. According to (5.14),

$$0 \;=\; 18\pi\epsilon^2 \sum_{s_p JK} \delta(\omega_{PJK})\; c_{pjk}^2 s_p \big[s_p \langle I_j \rangle \langle I_k \rangle \;+\; s_j \langle I_k \rangle \langle I_p \rangle \;+\; s_k \langle I_j \rangle \langle I_p \rangle\big]. \tag{5.38}$$

There appear to be just two ways in which the right-hand side of (5.38) can be made zero. First, one can arrange that, for all i, j, k,

$$s_i \langle I_j \rangle \langle I_k \rangle \;+\; s_j \langle I_k \rangle \langle I_i \rangle \;+\; s_k \langle I_i \rangle \langle I_j \rangle \;\propto\; \omega_{IJK},$$

by taking $\langle I_i \rangle \;\propto\; \omega_i^{-1}$. In this case of thermal equilibrium, the constant of proportionality is $k_B T$, where T is temperature and k_B is Boltzmann's constant:

$$\langle I_i \rangle_E \;=\; k_B T / \omega_i. \tag{5.39}$$

By (5.32), $B_p \;>\; 0$. When (5.39) holds, both A_p and B_p become infinite in the limit $N \;\rightarrow\; \infty$. Nevertheless, (5.32) is still justified, since it applies term by term to every summand in expressions (5.12) and (5.13) for A_p and B_p. That is, when (5.39) holds,

$$\delta(\omega_{PJK}) \Big\{ \omega_p \langle I_j \rangle \langle I_k \rangle \;+\; k_B T\; s_p \big[s_j \langle I_k \rangle \;+\; s_k \langle I_j \rangle\big] \Big\} \;=\; 0. \tag{5.40}$$

Infinite sums can also be evaded by the time-honored device of cutting-off high frequencies.

The second way that all modes can be maintained steadily is by "pumping" energy into the oscillators in a band surrounding some low frequency at a constant rate, $\tilde{Q}_0$, and removing it at the same rate at high frequencies, e.g. by invoking dissipative processes. The lack of dissipation between the injection and ejection frequencies in this "wave turbulence" is analogous to the lossless "cascade down the spectrum" occurring in the inertial range of vortex turbulence. And, in the same way, energy in the inertial range of wave turbulence follows a power law:

$$\langle I_i \rangle^\dagger \;=\; \frac{K}{\sigma_0} \left(\frac{\tilde{Q}_0}{\sigma_0 c_0^2} \right)^{\frac{1}{2}} \omega_i^{-\frac{3}{2}(2\beta + d)}, \tag{5.41}$$

where † is used to distinguish this steady cascade solution; d (> 1) is the dimensionality of the system, which enters because the density of states is of the form

$$\sigma \;=\; \sigma_0\, \omega^{d\,-\,1}, \tag{5.42}$$

where σ_0 is constant; the constants β and c_0 appear in c_{ijk}, for which it is assumed that

$$\epsilon c_{ijk} \;=\; c_0 (\omega_i \omega_j \omega_k)^\beta; \tag{5.43}$$

K is a dimensionless constant of order unity. That (5.41) obeys (5.38) was first established by Zakharov (1965) for the case $d = 3$, $\beta = 0$, for which $\langle I_i \rangle^\dagger \propto \omega_i^{-\frac{9}{2}}$. The more general result (5.41) was obtained by Larraza *et al* (1985); see also Appendix A below. By analogy with (5.39), we may use (5.41) to define an effective temperature, $T_i^\dagger$, associated with mechanical noise at frequency ω_i:

$$T_i^\dagger = \frac{K}{k_B \sigma_0} [\frac{\tilde{Q}_0}{\sigma_0 c_0^2}]^{\frac{1}{2}} \omega_i^{-\frac{1}{2}(3d + 6\beta - 2)}. \tag{5.44}$$

This increases as ω_i decreases and, at sufficiently low frequencies, the fluctuations due to mechanical noise greatly exceeds thermal noise.

A word of caution may be in order. The simple exponential form of (5.37) for a single degree of freedom implies by (5.6) that the reduced distribution function for n ($<< N$) modes is also exponential. This in no way implies that the global distribution function $\bar{P}$ for the full N degrees of freedom is also exponential. For example, it was proved at the end of §4 that the microcanonical distribution, in which

$$\bar{P}_E \propto \delta(H_0 - E), \tag{5.45}$$

where E is the total energy of the system, is a solution of the FP-equation (3.9), but it is not of exponential form. Nevertheless (5.6), (5.27) and (5.37) are true. There may be many other global distributions $\bar{P}$ that are not exponential but which imply those equations. A noteworthy difference between the stationary state (5.41) and the equilibrium state (5.39) is that the canonical distribution,

$$\bar{P}(\mathbf{I}) = \prod_{i=1}^{N} \bar{P}^1(I_i), \tag{5.46}$$

is a steady state solution of (3.9) when $\langle I_i \rangle$ takes the form (5.39) appropriate to thermal equilibrium; nevertheless, (5.46) is untrue for off-equilibrium stationary states. The requirement $n << N$ attached to (5.6) is especially significant for states out of equilibrium.

6. Angle Dependence and Frequency Renormalization

Because of nonlinearities, the frequency of an oscillator is a function of its amplitude. In our model, direct self-interactions are ignored ($c_{iik} = 0$, see §2), and the frequency shift is created by indirect interactions with the other oscillators of the bath in which oscillator p influences oscillators j and k at order ϵ, and those oscillators return that influence to oscillator p at order ϵ^2. We shall exhibit this this effect in two ways.

First, according to (2.2) and (2.7) - (2.10), we have

$$\dot{\phi}_p = \frac{3}{2}\epsilon \sum_{s_p JK} c_{pjk}\Big(\frac{I_j I_k}{I_p}\Big)^{\frac{1}{2}} e^{-i\theta_{PJK}}, \tag{6.1}$$

and

$$\langle\dot{\phi}_p\rangle = \iint \dot{\phi}_p[\bar{P} + \hat{P}_1]\, \mathrm{d}\mathbf{I}\, \mathrm{d}\boldsymbol{\theta} \tag{6.2}$$

gives $\langle\dot{\phi}_p\rangle$ correctly to order ϵ^2. Even when $\bar{P}$ is independent of $\boldsymbol{\theta}$, P possesses (through $\hat{P}_1$) $\boldsymbol{\theta}$-dependent parts, so that the average (6.2) of $\dot{\phi}_p$ is nonzero. This systematic contribution to $\langle\dot{\phi}_p\rangle$, which remains when $\bar{P}$ is independent of $\boldsymbol{\theta}$, is by (3.3)

$$\langle\dot{\phi}_p\rangle = 9\pi i\epsilon^2 \sum_{s_p JK} \iint \delta^-(\omega_{PJK})\, c_{pjk}^2 I_j I_k \nabla_{pjk}\bar{P}\, \mathrm{d}\mathbf{I}\,\mathrm{d}\boldsymbol{\theta}. \tag{6.3}$$

On integrating by parts and assuming that the factorization (5.10) is valid, one obtains

$$\langle\dot{\phi}_p\rangle = -\frac{1}{2}B_p', \tag{6.4}$$

where

$$B_p' = 18\epsilon^2 \sum_{s_p JK} \mathcal{P}\Big(\frac{1}{\omega_{PJK}}\Big)\, c_{pjk}^2\big[s_j\langle I_k\rangle + s_k\langle I_j\rangle\big]. \tag{6.5}$$

By (2.20), we may also write (6.4) as

$$\langle\dot{\theta}_p\rangle = \Omega_p, \tag{6.6}$$

where

$$\Omega_p = \omega_p - \frac{1}{2}B_p', \tag{6.7}$$

i.e. the bath has renormalized the frequency of oscillator p to be Ω_p.

The second method of deriving these results is to generalize the reduced distributions, $\bar{P}^1$, $\bar{P}^2, \ldots$, defined in §5 to include angles. We replace (5.15) by

$$\bar{P}^1(I_p;\ \theta_p;\ t) = \iint \bar{P}\, \mathrm{d}_p\mathbf{I}\, \mathrm{d}_p\boldsymbol{\theta}, \tag{6.8}$$

$$\bar{P}^2(I_p, I_q;\ \theta_p, \theta_q;\ t) = \iint \bar{P}\, \mathrm{d}_{pq}\mathbf{I}\, \mathrm{d}_{pq}\boldsymbol{\theta}, \tag{6.9}$$

where in analogy with (5.16)

$$\mathrm{d}_p\boldsymbol{\theta} \;=\; \prod_{i\,\neq\,p} d\theta_p \;=\; \mathrm{d}\boldsymbol{\theta}/\mathrm{d}\theta_p, \qquad \mathrm{d}_{pq}\boldsymbol{\theta} \;=\; \prod_{i\,\neq\,p,q} d\theta_i \;=\; \mathrm{d}\boldsymbol{\theta}/\mathrm{d}\theta_p d\theta_q. \tag{6.10}$$

When $\bar{P}$ is $\boldsymbol{\theta}$-independent, (6.8) and (6.9) reduce to (5.15), apart from factors of 2π.
When repeated here, the arguments of §5 about factorization of $\bar{P}$ lead to

$$\frac{\partial \bar{P}^1}{\partial t} \;+\; \omega_p \frac{\partial \bar{P}^1}{\partial \theta_p} = \;\sum_{s_p} \nabla_P^*\big[I_p\big(A_P \nabla_P \bar{P}^1 \;+\; B_P \bar{P}^1\big)\big], \tag{6.11}$$

where

$$A_P \;=\; 18\pi\epsilon^2 \sum_{JK} \delta^-(\omega_{PJK})\; c_{pjk}^2 \langle I_j\rangle\langle I_k\rangle \;=\; A_{-P}^*, \tag{6.12}$$

$$B_P \;=-18\pi\epsilon^2 \sum_{JK} \delta^-(\omega_{PJK})\; c_{pjk}^2 s_p\big[s_j\langle I_k\rangle \;+\; s_k\langle I_j\rangle\big] \;=\; B_{-P}^*. \tag{6.13}$$

It may be noted that, according to (5.12), (5.13) and (6.5),

$$A_p \;=\; \sum_{s_p} A_P, \qquad B_p \;=\; \sum_{s_p} B_P, \qquad B_p' \;=\; \frac{1}{i}\sum_{s_p} s_p B_P. \tag{6.14}$$

The operations on the right-hand side of (6.11) do not destroy the reality of $\bar{P}^1$, and this is also evident when that equation is rewritten in the form

$$\frac{\partial \bar{P}^1}{\partial t} \;+\; \Omega_p \frac{\partial \bar{P}^1}{\partial \theta_p} = \; \frac{\partial}{\partial I_p}\big[I_p\big(A_p \frac{\partial \bar{P}^1}{\partial I_p} \;+\; B_p \bar{P}^1\big)\big] \;+\; \frac{A_p}{4I_p}\frac{\partial^2 \bar{P}^1}{\partial \theta_p^2}. \tag{6.15}$$

Before we apply (6.15) to the computation of $\langle\dot{\theta}_p\rangle$, we should note that we shall have to average non-single valued functions of $\boldsymbol{\theta}$, and that, when we multiply such functions into (6.15) and integrate over $\boldsymbol{\theta}$, the integrated part will not vanish. The first step in the argument is therefore to replace $\bar{P}$ by the $\bar{p}$ introduced below (2.13), and to perform the $\boldsymbol{\theta}$ integrations from $-\infty$ to ∞. The integrated parts then vanish, because $\bar{p} \to 0$, as $\boldsymbol{\theta} \to \pm\infty$; under those conditions, equation (6.15) is as valid for $\bar{p}^1$ as it is for $\bar{P}^1$. Thus, defining $\langle\theta_p\rangle$ by

$$\langle\theta_p\rangle \;=\; \iint \theta_p\; \bar{p}^1\; \mathrm{d}I_p\; \mathrm{d}\theta_p, \tag{6.16}$$

multiplying (6.15) (with $\bar{p}^1$ in place of $\bar{P}^1$) by θ_p and integrating, we recover (6.6). [Since the right-hand side of (6.1) is single-valued in $\boldsymbol{\theta}$, the first method of computing $\langle\dot{\theta}_p\rangle$ did not require us to introduce $\bar{p}^1$.]

If one regards the oscillator hamiltonian (2.7) as a discretization of the sound field in a continuous medium, then p labels a particular normal mode, and the renormalized frequency (6.7) corresponds to a renormalized speed of sound. In §8, this fact is used to compute the temperature dependence of the speed of sound in a dielectric at low temperatures.

In conclusion, we observe that, in analogy with the expansion (3.5) of the full probability distribution P, the reduced probability distribution $\bar{P}^1$ is the first term in a similar expansion,

$$P^1(I_p;\ \theta_p;\ t) = \bar{P}^1(I_p;\ \theta_p;\ t) + \hat{P}^1_1(I_p;\ \theta_p;\ t) + \hat{P}^1_2(I_p;\ \theta_p;\ t), \tag{6.17}$$

of the full reduced probability density:

$$P^1(I_p;\ \theta_p;\ t) = \iint P(\mathbf{I};\ \boldsymbol{\theta};\ t)\ \mathrm{d}_p\mathbf{I}\ \mathrm{d}_p\boldsymbol{\theta}, \tag{6.18}$$

[cf. (6.8)]. It will be recalled that, after $\bar{P}$ has been obtained by solving the full FP-equation (3.2), $\hat{P}_1$ and $\hat{P}_2$ can be derived by integrations (3.3) and (3.4). Analogously, once $\bar{P}^1$ has been derived by solving (6.15), $\hat{P}^1_1$ and $\hat{P}^1_2$ can be obtained by integration:

$$\hat{P}^1_1(I_p;\ \theta_p;\ t) = 3i\pi\epsilon \sum_{s_p JK} \delta^-(\omega_{PJK})\ d_{pjk}\ \langle q_J q_K\rangle_0\ s_p q_P\ \nabla_P \bar{P}^1, \tag{6.19}$$

$$\hat{P}^1_2(I_p;\ \theta_p;\ t) = -9\pi\epsilon^2 \sum_{s_p s'_p JKMN}{}' \left\{ \frac{\delta^-(\omega_{PJKP'MN}) - \delta^-(\omega_{P'MN})}{-i\omega_{PJK}} \right\} d_{pjk}\ d_{p'mn}\ \langle q_J q_K q_M q_N\rangle_0\ s_p q_P\ s_{p'} q_{P'}\ \nabla_{PP'} \bar{P}^1. \tag{6.20}$$

We have here used representation (2.6) and definition (2.54). Results (6.19) and (6.20) will be useful in §7.

7. Langevin Equations and Fluctuation-Dissipation Relations

As well as leading to kinetic equations, such as (5.11) and (5.14), the FP-equation can be used to trace the time development of the primitive variables **q** and **p**. This is possible because our Fokker-Planck equation governs a probability density that depends not only on the actions but also on the angles. Of particular interest is the Langevin equation for one particular oscillator. It is supposed that the coordinates (q_p, p_p) of that oscillator are accurately known at $t = 0$, in the sense that

$$\bar{P}^1(I_p;\ \theta_p;\ 0) \ = \ \delta(I_p - I_{0p})\ \delta(\theta_p - \theta_{0p}). \tag{7.1}$$

In contrast to (7.1), the probability distribution for all other oscillators will be assumed to be **smooth**, i.e. $\bar{P}$ is smooth in ${}_p\mathbf{I}$ and ${}_p\boldsymbol{\theta}$, a notation we use to mean all I_i and θ_i except I_p and θ_p. This smoothness characterizes the state of the bath, and because of it $P^1(I_p;\ \theta_p;\ t)$ will not maintain its initial sharp distribution. The interaction of the p-oscillator with the bath destroys the sharp initial condition and leads to the diffusion of P^1 in (I_p, θ_p) space. This diffusion, and therefore all moments of q_p and p_p, can then be calculated from (6.17) – (6.20) or from (3.2) – (3.5). An equivalent way of viewing this diffusion is through the Langevin equation, which describes the fluctuations about the average time development that would result from the initial condition (7.1) were there no fluctuations. We develop this picture below, but emphasize at the outset that it contains no new information beyond what is already contained in (6.17) – (6.20).

To obtain the Langevin equation for oscillator p, we recall that, by (2.1) – (2.3) and the canonical equations governing q_i and p_i,

$$\frac{d^2}{dt^2} q_p \ + \ \omega_p^2 q_p \ = \ F_p, \tag{7.2}$$

where

$$F_p \ = \ - \ 3\epsilon \sum_{jk} d_{pjk} q_j q_k. \tag{7.3}$$

In the framework of the Langevin picture, F_p is to be regarded as the "external" force that the bath provides through the current configuration of its oscillators. This force will have a systematic part, proportional to q_p, as well as a fluctuating part which is independent of the instantaneous value of q_p, and thus depends only on the bath coordinates.

To clarify the averaging procedure appropriate for the Langevin approach, we appeal to §5 and factorize $\bar{P}$:

$$\bar{P} \ = \ \bar{P}^1(I_p;\ \theta_p;\ t)\ \bar{P}^B({}_p\mathbf{I};\ {}_p\boldsymbol{\theta};\ t). \tag{7.4}$$

Here $\bar{P}^B$, which is $O(\epsilon^0)$, is the distribution function for the oscillators of the bath. To order ϵ^1,

$$P \ = \ [\bar{P}^1(I_p;\ \theta_p;\ t) \ + \ \hat{P}_1^1(\mathbf{I};\ \boldsymbol{\theta};\ t)]\ [\bar{P}^B({}_p\mathbf{I};\ {}_p\boldsymbol{\theta};\ t) \ + \ \hat{P}_1^B(\mathbf{I};\ \boldsymbol{\theta};\ t)], \tag{7.5}$$

where by (3.3)

$$\hat{P}_1^1(\mathbf{I};\ \boldsymbol{\theta};\ t) \ = \ 3i\pi\epsilon \sum_{s_p JK} \delta^-(\omega_{PJK})\ c_{pjk}\ I_{pjk}^{1/2}\ e^{i\theta_{PJK}}\ s_p\ \nabla_P \bar{P}^1, \tag{7.6}$$

and

$$\hat{P}_1^B = \hat{P}_{1,B}^B + \hat{P}_{1,p}^B, \tag{7.7}$$

where

$$\hat{P}_{1,B}^B = i\pi\epsilon \sum_{IJK \neq P} \delta^-(\omega_{IJK})\, c_{ijk}\, I_{ijk}^{1/2}\, \mathrm{e}^{i\theta_{IJK}}\, \nabla_{IJK} \bar{P}^B, \tag{7.8}$$

$$\hat{P}_{1,p}^B = 3i\pi\epsilon \sum_{s_p JK} \delta^-(\omega_{PJK})\, c_{pjk}\, I_{pjk}^{1/2}\, \mathrm{e}^{i\theta_{PJK}}\, \left(s_j \nabla_J + s_k \nabla_K\right) \bar{P}^B. \tag{7.9}$$

Here $\hat{P}_1^B$ is the O(ϵ) part of the probability distribution of the bath alone. It consists of a part $\hat{P}_{1,p}^B$ that is proportional q_p – see (7.9) – and a part $\hat{P}_{1,B}^B$ that is independent of the state of the p oscillator and concerns only processes in which the bath interacts with itself – see (7.8).

Consider an instant in time when q_p and p_p are measured with sufficient accuracy so that (7.1) is a reasonably accurate description of $\bar{P}^1$. On averaging the total probability distribution P over a small domain $(\Delta I_p, \Delta\theta_p)$ surrounding (I_{0p}, θ_{0p}), where $\Delta I_p \ll I_{0p}$ and $\Delta\theta_p \ll 2\pi$, we obtain

$$\frac{1}{\Delta I_p\, \Delta\theta_p} \iint_{\Delta I_p\, \Delta\theta_p} P\, \mathrm{d}I_p \mathrm{d}\theta_p = \bar{P}^B + \hat{P}_1^B.$$

This follows because the O(ϵ) term, which according to expression (7.6) for $\hat{P}_1^1$ is proportional to the derivative of a δ-function from the initial state (7.1), integrates to zero. Thus, $\bar{P}^B + \hat{P}_1^B$ is the distribution function for the reservoir that provides both the systematic (long timescale) and fluctuating (fast timescale) contributions to the motion of the p-oscillator. For instance, the average force on the p oscillator due to the bath is

$$\langle F_p \rangle^B = \iint F_p \left[\bar{P}^B({}_p\mathbf{I};\ {}_p\boldsymbol{\theta};\ t) + \hat{P}_1^B(\mathbf{I};\ \boldsymbol{\theta};\ t)\right] \mathrm{d}_p\mathbf{I}\, \mathrm{d}_p\boldsymbol{\theta}, \tag{7.10}$$

an expression that is valid to O(ϵ^2), since F_p is already O(ϵ). It is convenient to divide $\langle F_p \rangle^B$ into two parts. One is

$$F_p^\dagger = \iint F_p \left[\bar{P}^B + \hat{P}_{1,B}^B\right] \mathrm{d}_p\mathbf{I}\, \mathrm{d}_p\boldsymbol{\theta}, \tag{7.11}$$

which is that part of the average force which the bath exerts on the p-oscillator that is independent of the state of that oscillator. The other,

$$F_p^\ddagger = \iint F_p \hat{P}_{1,p}^B\, \mathrm{d}_p\mathbf{I}\, \mathrm{d}_p\boldsymbol{\theta}, \tag{7.12}$$

is the average force that the bath exerts on the p-oscillator through the influence of that oscillator on the bath, i.e. it depends on the coordinate of the p^{th} oscillator. It follows from (7.3) and (7.9) that

$$F_p^\ddagger = -\tilde{B}_p \dot{q}_p - 2\omega_p \left(\tilde{\Omega}_p - \omega_p\right) q_p, \tag{7.13}$$

where

$$\tilde{B}_p = -18\pi\epsilon^2 \sum_{s_p JKM} \delta(\omega_{PJK})\, c_{pjk}\, s_p$$
$$\left[s_j\, c_{pjm}\, \langle I_{km}^{1/2}\, e^{i\theta_{KM}}\rangle + s_k\, c_{pkm}\, \langle I_{jm}^{1/2}\, e^{i\theta_{JM}}\rangle\right], \tag{7.14}$$
$$\tilde{\Omega}_p = \omega_p - 9\epsilon^2 \sum_{s_p JKM} \mathcal{P}\left(\frac{1}{\omega_{PJK}}\right) c_{pjk}$$
$$\left[s_j\, c_{pjm}\, \langle I_{km}^{1/2}\, e^{i\theta_{KM}}\rangle + s_k\, c_{pkm}\, \langle I_{jm}^{1/2}\, e^{i\theta_{JM}}\rangle\right]. \tag{7.15}$$

(Here and henceforward the superscipt ^{B} is omitted from bath averages $\langle \ldots \rangle^B$ whenever such terms are already $O(\epsilon^2)$.) We may now write

$$F_p = \langle F_p \rangle^B + F_p', \tag{7.16a}$$

or as

$$F_p = F_p^\dagger + F_p^\ddagger + F_p', \tag{7.16b}$$

F_p' being the fluctuating force on the p-oscillator. On substituting (7.16) into (7.2), we obtain

$$\ddot{q}_p + \tilde{B}_p \dot{q}_p + \tilde{\Omega}_p^2 q_p = F_p^\dagger + F_p'. \tag{7.17}$$

In the Langevin approach, the randomness in q_p is attributed to the fluctuations of the bath, i.e. to F_p', so that the autocorrelation of this force is of central interest. From (7.3), (7.10) and (7.11), one finds that

$$\langle F_p'(t_1) F_p'(t_2)\rangle^B = 9\epsilon^2 \sum_{jkmn} d_{pjk} d_{pmn} \left[\langle q_j(t_1) q_k(t_1) q_m(t_2) q_n(t_2)\rangle\right.$$
$$\left. - \langle q_j(t_1) q_k(t_1)\rangle \langle q_m(t_2) q_n(t_2)\rangle\right]. \tag{7.18}$$

Equation (7.17), supplemented by (7.11), (7.14), (7.15) and (7.18), determines the general Langevin-type motion of the p-oscillator. Further simplification follows from taking $\bar{P}^B$ independent of angle; then $F_p^\dagger = 0$, and only terms for which $M = -J$ or $-K$ make a nonzero contribution to the sum (7.18), so that (7.14) and (7.15) lead to (5.13) and (6.7). All the $\tilde{}$ may then be removed from (7.17). It follows that

$$\ddot{q}_p + B_p \dot{q}_p + \Omega_p^2 q_p = F_p', \tag{7.19}$$

and (7.18) reduces to

$$\langle F_p'(t_1) F_p'(t_2)\rangle^B = 36\epsilon^2 \omega_p \sum_{JK} c_{pjk}^2 \langle I_j I_k\rangle\, e^{i\omega_{JK}(t_1 - t_2)}, \tag{7.20}$$

where we have used the $O(\epsilon)$ part of (2.6b) to relate averages at t_1 to averages at t_2. This is equivalent to taking

$$q_I(t_2) = q_I(t_1) e^{-i s_i \omega_i (t_1 - t_2)}, \tag{7.21}$$

i.e. to ignoring the slow time-dependence in q_I, which does not affect the validity of (7.20) at $O(\epsilon^2)$. The response described by (7.20), or for that matter (7.18), is stationary, i.e., on the fast timescale, $\langle F_p'(t_1)F_p'(t_2)\rangle^B$ depends only on $t_1 - t_2$. Also, the autocorrelation (7.20) is even in $t_1 - t_2$. Had $\bar{P}^B$ depended on angles, the response would no longer have been even in $t_1 - t_2$; reversibility on the fast timescale would have been lost. In terms of the Fourier transform,

$$F_p'(\omega) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} F_p'(t) e^{i\omega t} dt, \tag{7.22}$$

of $F_p'(t)$, (7.20) is

$$\langle F_p'(\omega)F_p'(\omega')\rangle^B = F_p'^2(\omega)\delta(\omega' + \omega), \tag{7.23}$$

where

$$F_p'^2(\omega) = 72\pi\epsilon^2\omega_p \sum_{JK} c_{pjk}^2 \langle I_j I_k\rangle \delta(\omega + s_j\omega_j + s_k\omega_k). \tag{7.24}$$

In thermal equilibrium $\langle I_i\rangle = \langle I_i\rangle_E = k_B T/\omega_i$ and, according to (5.40),

$$\langle I_j I_k\rangle_E = k_B T \frac{[s_j\langle I_k\rangle_E + s_k\langle I_j\rangle_E]}{(s_j\omega_j + s_k\omega_k)}. \tag{7.25}$$

In this case, therefore, $F_p'^2 = {F'_{p,E}}^2$ where

$${F'_{p,E}}^2 = -36\pi\epsilon^2 k_B T \frac{\omega_p}{\omega} \sum_{s_p JK} \delta(s_p\omega + s_j\omega_j + s_k\omega_k)\, c_{pjk}^2\, s_p\, [s_j\langle I_k\rangle_E + s_k\langle I_j\rangle_E]. \tag{7.26}$$

The response of the p oscillator to the fluctuating force, $F_p'(t)$, may, in the case in which $\bar{P}^B$ is independent of angles, be found from (7.19):

$$q_p'(\omega) = -\frac{F_p'(\omega)}{\omega^2 - \Omega_p^2 + i\omega B_{p,E}}. \tag{7.27}$$

Since the response of the oscillator to the fluctuating force is concentrated within a bandwidth $B_{p,E}$ about Ω_p, it is, to the ϵ^2 accuracy considered here, only the value of ${F'_{p,E}}^2(\omega)$ at $\omega = \omega_p$ that has any real significance, and by (7.26) this is

$${F'_{p,E}}^2(\omega_p) = 2k_B T B_{p,E}, \tag{7.28}$$

which is the fluctuation-dissipation relation for thermal equilibrium.

The physical realization of the fluctuation-dissipation relation for our system may be seen in the following way. First, let the distinguished oscillator be displaced far from equilibrium, and be held in some state (I_{0p}, θ_{0p}) while the bath comes into equilibrium. (By "far from equilibrium", we mean only that q_{0p} is large compared with a typical fluctuation in q_p due to the bath.) Let the oscillator now be "released". Due to "collisions" of the p oscillator with the bath, q_p will (averaged over many repetitions of the same experiment) decay from its initial value, q_{0p}, to zero at the rate B_p, which according to (7.26) then determines the spectral intensity of the fluctuating force that describes deviations in q_p from its averaged evolution, $\langle q_p\rangle^B$.

The decay of $\langle q_p\rangle^B$ from some initial value, q_{0p}, to zero is the result of direct collisions between oscillator p and the bath, as represented by B_p. Also present are restituting collisions in which two oscillators of the bath interact to restore energy to oscillator p. This effect is apparent in the kinetic equation (5.14) through the term A_p. The Langevin equation that demonstrates this effect can be obtained by multiplying (7.19) by $\dot{q}_p$ to obtain

$$\frac{dI_p}{dt} + B_p I_p = G_p, \tag{7.29}$$

where

$$G_p = \frac{\dot{q}_p F_p'}{\omega_p} + \frac{1}{2\omega_p}\left\{B_p\left(\omega_p^2 q_p^2 - \dot{q}_p^2\right) - \left(\Omega_p^2 - \omega_p^2\right)\frac{d}{dt}q_p^2\right\}. \tag{7.30}$$

Again, we may write

$$G_p = \langle G_p\rangle^B + G_p'. \tag{7.31}$$

The fact that G_p depends explicitly on q_p means that some care must be exercised in evaluating the bath-average of (7.29). In the framework of the Langevin equation, one must include the dependence of q_p on the initial conditions as well as on the bath coordinates, which determine F_p'. And so one writes the solution of (7.19) in the form

$$q_p = q_p^H + q_p', \tag{7.32}$$

where

$$q_p' = q_p'(F_p'), \tag{7.33}$$

is obtained by inverting (7.27), and q_p^H solves the homogeneous equation associated with (7.19), i.e. the equation obtained by setting the right-hand side of (7.19) to zero; q_p^H is required so that q_p given by (7.32) satisfies the initial conditions. Since (see above) the initial displacement q_{0p} is large compared with the fluctuations, q_p^H is large compared with $q_p(F_p')$, and so must in the first approximation coincide with q_{0p} at $t = 0$:

$$q_p^H(t) = q_{0p}\, e^{-\frac{1}{2}B_p t}\cos\Omega_p t. \tag{7.34}$$

From (7.30) we now have

$$\langle G_p\rangle^B = \frac{\langle \dot{q}_p' F_p'\rangle^B}{\omega_p} + \frac{1}{2}q_{0p}^2\, e^{-B_p t}\left[B_p\omega_p\cos 2\Omega_p t - \left(\Omega_p^2 - \omega_p^2\right)\sin 2\Omega_p t\right]. \tag{7.35}$$

In order to obtain (7.35), we have used

$$\langle q_p^H q_p'\rangle^B = \langle \omega_p^2 q_p'^2 - \dot{q}_p'^2\rangle^B = \frac{d}{dt}\langle q_p'^2\rangle^B = 0.$$

To reduce (7.35) further, we take from (7.27)

$$\dot{q}'_p(\omega) = \frac{i\omega\, F'_p(\omega)}{\omega^2 - \Omega_p^2 + i\omega B_p}, \tag{7.36}$$

and hence obtain

$$\begin{aligned}\frac{\langle \dot{q}'_p\, F'_p\rangle^B}{\omega_p} &= \frac{1}{2\,\pi\,\omega_p}\iint_{-\infty}^{\infty} \langle \dot{q}'_p(\omega)\, F'_p(\omega')\rangle^B \mathrm{e}^{-i(\omega+\omega')t}\, \mathrm{d}\omega\, \mathrm{d}\omega' \\ &= \frac{1}{2\,\pi\,\omega_p}\int_{-\infty}^{\infty} \frac{i\omega F_p'^2(\omega)}{\omega^2 - \Omega_p^2 + i\omega B_p}\, \mathrm{d}\omega. \end{aligned} \tag{7.37}$$

By using (7.24) and contour integration, we may reduce (5.12) simply to

$$\frac{\langle \dot{q}'_p\, F'_p\rangle^B}{\omega_p} = A_p. \tag{7.38}$$

Omitting terms in (7.34) having a zero time average taken over a period of the oscillator, we now obtain from (7.29)

$$\frac{d}{dt}\langle I_p\rangle^B + B_p\,\langle I_p\rangle^B = A_p. \tag{7.39}$$

While $\langle I_p\rangle^B$ is governed by the kinetic equation (5.14), the fluctuations about this evolution are determined by G'_p, the fluctuating force central to the Langevin equation governing I_p. Unlike the familiar forms of Langevin equations (such as that governing q_p), the Langevin equation governing I_p is not homogeneous in I_p. It was shown in §5 that, the correlations in the actions of initially uncorrlated oscillators grow at a thermodynamically negligible rate. In the context of the Langevin equation, there correspond properties of the fluctuating force G'_p. If one forms the equal time correlations of G'_p with the actions I_p and I_i $(i \neq p)$, one finds

$$\langle I_p\, G'_p\rangle^B = A_p\,\langle I_p\rangle^B, \tag{7.40}$$

$$\langle I_i\, G'_p\rangle^B = 18\,\pi\,\epsilon^2 \sum_{s_p\,JK} \delta(\omega_{PJK})\, c_{pjk}^2\, \mathcal{C}_{jk,i}, \tag{7.41}$$

where the cumulant $\mathcal{C}_{jk,i}$ is defined in (5.8). At $t = 0$, when the probability density factorizes, $\mathcal{C}_{jk,i} = 0$. (It may be recalled that $\mathcal{C}_{qk,q} = \mathcal{C}_{jq,q} = 0$ by definition.) Thus, the correlation (7.41) is thermodynamically small compared with correlation (7.40). The correlation $\langle I_p\, G'_p\rangle$ is, however, very significant. It accounts for the fact that $\langle I_p^2\rangle \neq \langle I_p\rangle^2$; see (5.35).

In order to derive (7.40), it is necessary to divide the action I_p into fluctuating, homogeneous and "mixed" contributions:

$$\begin{aligned} I_p &\equiv \frac{1}{2\,\omega_p}\left(\dot{q}_p^2 + \omega_p^2\, q_p^2\right) \\ &= I_p^H + I'_p + \frac{1}{\omega_p}\left(\dot{q}'_p\,\dot{q}_p^H + \omega_p^2\, q'_p\, q_p^H\right), \end{aligned} \tag{7.42}$$

where

$$I'_p \equiv I_p(q'_p), \qquad I^H_p \equiv I_p(q^H_p), \tag{7.43}$$

so that

$$\langle I_p \rangle^B = I^H_p + \langle I'_p \rangle^B, \tag{7.44}$$

where we have recalled that, in the present situation, $\bar{P}^B$ is independent of angle. Continuing to neglect terms that are $O(\epsilon^2)$ or that are purely oscillatory at frequency $2\Omega_p$, we now have

$$G'_p = \frac{\dot{q}^H_p F'_p}{\omega_p} + \frac{\dot{q}'_p F'_p}{\omega_p} - A_p. \tag{7.45}$$

From the relations

$$\langle I'_p \frac{\dot{q}^H_p F'_p}{\omega_p} \rangle^B = 2\, A_p \, \langle I_p \rangle^B, \tag{7.46}$$

$$\langle I^H_p \, G'_p \rangle^B = 0, \tag{7.47}$$

$$\langle \frac{1}{\omega_p} \left(\dot{q}'_p \, \dot{q}^H_p + \omega^2_p \, q'_p \, q^H_p \right) \frac{\dot{q}^H_p F'_p}{\omega_p} \rangle^B = A_p \, I^H_p, \tag{7.48}$$

we now obtain (7.40).

The equation of motion (7.19) can be related to the reduced Fokker-Plank equation derivied in §6. By multiplying (6.15) by q_P, and integrating over all (I_p, θ_p), one obtains

$$\frac{d}{dt}\langle q_P \rangle_0 - i s_p \Omega_p \langle q_P \rangle_0 + \frac{1}{2} B_p \langle q_P \rangle_0 = 0, \tag{7.49}$$

where the suffix 0 is added because $\langle q_P \rangle_0$ is only the first term in the complete average. More generally, to $O(\epsilon^2)$, the average of a function $f(I_p,\ \theta_p)$ is, as in (6.17),

$$\langle f \rangle = \langle f \rangle_0 + \langle f \rangle_1 + \langle f \rangle_2, \tag{7.50}$$

where

$$\langle f \rangle_0 = \iint f(I_p,\ \theta_p)\ \bar{P}^1(I_p;\ \theta_p;\ t)\ \mathrm{d}I_p\ \mathrm{d}\theta_p, \tag{7.51}$$

$$\langle f \rangle_1 = \iint f(I_p,\ \theta_p)\ \hat{P}^1_1(I_p;\ \theta_p;\ t)\ \mathrm{d}I_p\ \mathrm{d}\theta_p, \tag{7.52}$$

$$\langle f \rangle_2 = \iint f(I_p,\ \theta_p)\ \hat{P}^1_2(I_p;\ \theta_p;\ t)\ \mathrm{d}I_p\ \mathrm{d}\theta_p. \tag{7.53}$$

On substituting (7.50) into (7.49), we obtain, to $O(\epsilon^2)$,

$$\frac{d}{dt}\langle q_P \rangle - i s_p \Omega_p \langle q_P \rangle + \frac{1}{2} B_p \langle q_P \rangle = \left[\frac{d}{dt} - i s_p \omega_p \right] \{ \langle q_P \rangle_1 + \langle q_P \rangle_2 \}. \tag{7.54}$$

On taking the time derivative of (7.54), and summing the result over s_p, we recover (7.19) but without the random force F'_p, because the average is taken over the probability density $\hat{P}_1^1$.

Finally we consider the nature of the fluctuation spectrum when the bath is in the turbulent stationary state characterized by (5.41) and (5.44). The Langevin equation (7.19) still applies, with the spectral intensity of the fluctuating force given by (7.22). And (7.28) still holds with the coefficients A_p and B_p taking the values deduced in Appendix A; see (A18) and (A19):

$$F_p'^2(\omega_p)^\dagger \;=\; 2\, k_B\, T^\dagger\, B_p^\dagger. \tag{7.55}$$

8. Fokker-Planck Equation for Interacting Waves. Renormalized Sound Speed

As stated in §1, the present paper is strongly motivated by the study of interacting waves. Yet in order to simplify the analysis, we have so far considered only interacting oscillators. Never far underlying our discussion of oscillators has, however, been the study of waves. We have, for instance, had in mind density of states (5.42), which is the natural choice for an isotropic distribution of waves, for which there are of order $\omega^{d-1}\mathrm{d}\omega$ modes with frequencies lying between ω and $\omega + \mathrm{d}\omega$ for a bounded container in d-dimensions. In this section, we shall explicitly study the case of interacting waves in three dimensions.

Some small but significant technical points should be resolved at the outset. First, the labelling must be refined; each of the modes mentioned above must carry an index that indicates both its frequency and direction of propagation. This is most conveniently done by using the wavenumber vector, $\mathbf{k}$, of the wave as its label, and by taking its frequency ω_k (> 0) to be a known function of $\mathbf{k}$: in isotropic materials at rest, ω_k depends only on the magnitude k ($= |\mathbf{k}|$) of $\mathbf{k}$ and not on its direction $\hat{\mathbf{k}}$.

The nature of wave interactions depends on whether the waves themselves disperse normally, anomalously or semi-dispersively. In normal dispersion, the phase speed, v_k ($= \omega_k/k$), decreases as k increases; in anomalous dispersion, it increases with k; for semi-dispersion, it is constant. As before, we introduce $s_k = \pm 1$ to distinguish between the $\mathrm{e}^{i\omega_k t}$ and $\mathrm{e}^{-i\omega_k t}$ time dependencies, but now the composite label will be $K = (\mathbf{k}, s_k)$ with $-K = (-\mathbf{k}, -s_k)$. Three waves $I = (\mathbf{i}, s_i)$, $J = (\mathbf{j}, s_j)$ and $K = (\mathbf{k}, s_k)$ interact strongly (resonate) if

$$\mathbf{i} + \mathbf{j} + \mathbf{k} = 0, \tag{8.1}$$

$$\omega_{IJK} \equiv s_i\omega_i + s_j\omega_j + s_k\omega_k = 0. \tag{8.2}$$

In the case of anomalous dispersion, there are, for each I, an infinity of resonating J and K waves, i.e. these equations can easily be satisfied simultaneously. At low wave amplitudes, such "3-wave interactions" dominate the processes of energy exchange between waves. In the case of normal dispersion, (8.1) and (8.2) cannot be simulaneously satisfied, and 4-wave interactions dominate the exchange processes. The semi-dispersive case, in which 3-wave interactions are possible, but only when the wave vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel or antiparallel to each other, will be used to motivate (8.18) below.

Our objectives will be to derive the Fokker-Planck equation for acoustic waves in a uniform barotropic fluid with small anomalous dispersion, and to determine the renormalized speed of sound, i.e. the change in the velocity with which a particular sound wave travels, due to the finite amplitude of the other sound waves traversing the material. The renormalized sound speed will in fact apply to both the cases of anomalous and normal dispersion. We shall first, as in §2, derive the FP-equation in the discrete case where the container has finite volume V. We shall then, as in §3, make the transition to the continuous limit, $V \to \infty$.

In a semi-dispersive barotropic material, the internal energy density per unit mass, e, is a function of the density, ρ and the hamiltonian is

$$H = \int \rho\Big[\frac{1}{2}(\boldsymbol{\nabla}\Phi)^2 + e(\rho)\Big]\, \mathrm{d}^3x, \tag{8.3}$$

where $\mathbf{\nabla}\Phi$ is the fluid velocity; ρ and Φ are conjugate variables. For small departures, $\rho' = \rho - \rho_0$, from the equilibrium state $\rho = \rho_0$, we may expand $H' = H - H_0$ as

$$H' = \int [\frac{1}{2}\rho_0(\mathbf{\nabla}\Phi)^2 + \frac{v_0^2}{2\rho_0}\rho'^2] \mathrm{d}^3x + \int [\frac{1}{2}\rho'(\mathbf{\nabla}\Phi)^2 + \frac{v_0^2}{3\rho_0^2}(\frac{\rho}{v}\frac{dv}{d\rho} - \frac{1}{2})_0\, \rho'^3] \mathrm{d}^3x, \tag{8.4}$$

where $v^2 = dp/d\rho$, $p = \rho^2 de/d\rho$; v is the speed of sound and $v_0 = v(\rho_0)$. The suffix $_0$ is henceforward omitted from v_0 and ρ_0. In terms of the Fourier components

$$\rho'(\mathbf{x}) = \sum_{\mathbf{k}} \rho_{\mathbf{k}} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}}, \qquad \Phi(\mathbf{x}) = \sum_{\mathbf{k}} \Phi_{\mathbf{k}} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}},$$
$$\rho_{\mathbf{k}} = \frac{1}{V}\int_V \rho'(\mathbf{x})\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\, \mathrm{d}^3x, \qquad \Phi_{\mathbf{k}} = \frac{1}{V}\int_V \Phi(\mathbf{x})\, \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\, \mathrm{d}^3x, \tag{8.5}$$

where

$$\rho_{-\mathbf{k}} = \rho_{\mathbf{k}}^*, \qquad \Phi_{-\mathbf{k}} = \Phi_{\mathbf{k}}^*, \tag{8.6}$$

and (8.4) is

$$H = H_0 + \epsilon H_1, \tag{8.7}$$

where

$$H_0 = \frac{V}{2}\sum_{\mathbf{k}}\Big\{\frac{v^2}{\rho}|\rho_{\mathbf{k}}|^2 + \rho|\mathbf{k}\Phi_{\mathbf{k}}|^2\Big\}, \tag{8.8}$$

$$\epsilon H_1 = \frac{V}{6}\sum_{\mathbf{ijk}}\Big\{\frac{v^2}{\rho^2}(\frac{2\rho}{v}\frac{dv}{d\rho} - 1)\ \rho_{\mathbf{i}}\rho_{\mathbf{j}}\rho_{\mathbf{k}} - (\mathbf{j}\cdot\mathbf{k})\rho_{\mathbf{i}}\Phi_{\mathbf{j}}\Phi_{\mathbf{k}} - (\mathbf{k}\cdot\mathbf{i})\rho_{\mathbf{j}}\Phi_{\mathbf{k}}\Phi_{\mathbf{i}} - (\mathbf{i}\cdot\mathbf{j})\rho_{\mathbf{k}}\Phi_{\mathbf{i}}\Phi_{\mathbf{j}}\Big\}\delta_{\mathbf{i}+\mathbf{j}+\mathbf{k}}. \tag{8.9}$$

The Kronecker-δ is here defined by

$$\delta_{\mathbf{m}} = \begin{cases} 1, & \text{if } \mathbf{m} = 0 \\ 0, & \text{if } \mathbf{m} \neq 0. \end{cases}$$

In terms of the hamiltonian density, $\mathcal{H} \equiv H\,/\,V$, the canonical equations are

$$\dot{\rho}_{\mathbf{k}} = \frac{\partial\mathcal{H}}{\partial\Phi_{-\mathbf{k}}}, \qquad \dot{\Phi}_{\mathbf{k}} = -\frac{\partial\mathcal{H}}{\partial\rho_{-\mathbf{k}}}. \tag{8.10}$$

We transform to action-angle coordinates by writing

$$\rho_{\mathbf{k}} = (\frac{\rho\omega_k}{Vv^2})^{\frac{1}{2}}\sum_{s_k} I_K^{1/2}\mathrm{e}^{is_k\theta_K}, \tag{8.11}$$

$$\Phi_{\mathbf{k}} = (\frac{\omega_k}{V\rho k^2})^{\frac{1}{2}}\sum_{s_k} is_k I_K^{1/2}\mathrm{e}^{is_k\theta_K}. \tag{8.12}$$

A significant difference between these representations and (2.5) should be noted. Since $\rho_{\mathbf{k}}$ and $\Phi_{\mathbf{k}}$ are complex and obey (8.6),

$$I_{-K} = I_K, \qquad \theta_{-K} = \theta_K, \tag{8.13}$$

but the action-angle pairs $(I_{(\mathbf{k},1)}, \theta_{(\mathbf{k},1)})$ and $(I_{(\mathbf{k},-1)}, \theta_{(\mathbf{k},-1)})$ correspond to waves travelling in antiparallel directions, and are therefore independent of each other. Substituting (8.11) and (8.12) into (8.8) and (8.9), we find

$$H_0 = \sum_K \omega_k I_K, \tag{8.14}$$

$$H_1 = \sum_{IJK} c_{IJK}\, I_{IJK}^{1/2}\, e^{i\theta_{IJK}}, \tag{8.15}$$

where

$$c_{IJK} = \left(\frac{8\pi^3}{V}\right)^{\frac{1}{2}} C_{IJK}\, \delta_{\mathbf{i}+\mathbf{j}+\mathbf{k}}, \tag{8.16}$$

$$\epsilon C_{IJK} = \frac{1}{6}\left\{\frac{\omega_i\omega_j\omega_k}{8\pi^3\rho v^2}\right\}^{\frac{1}{2}}\left\{\frac{2\rho}{v}\frac{dv}{d\rho} - 1 + s_j s_k \hat{\mathbf{j}}\cdot\hat{\mathbf{k}} + s_k s_i \hat{\mathbf{k}}\cdot\hat{\mathbf{i}} + s_i s_j \hat{\mathbf{i}}\cdot\hat{\mathbf{j}}\right\}. \tag{8.17}$$

This notation is chosen to parallel that of §2; the principal difference is that I_{IJK}, θ_{IJK} and c_{IJK} now depend on $s_i s_j s_k$ and therefore carry capital letters as suffices; these are used to redefine θ_{IJK} and later ∇_{IJK}:

$$\theta_{IJK} = s_i\theta_I + s_j\theta_J + s_k\theta_K = -\theta_{-I,-J,-K},$$
$$\nabla_{IJK} = s_i\nabla_I + s_j\nabla_J + s_k\nabla_K = -\nabla^*_{-I,-J,-K},$$
$$\nabla_I = \frac{\partial}{\partial I_I} + \frac{is_i}{2I_I}\frac{\partial}{\partial\theta_I} = \nabla^*_{-I}.$$

(Discussion of the small technical difficulties that arise because every term appears twice in (8.14), first as I_K and then as I_{-K}, is postponed to the end of this section.)

The present case, in which e is a function of ρ (rather than a functional of ρ) is semi-dispersive and, if $|\mathbf{i}|$ (say) exceeds $|\mathbf{j}|$ and $|\mathbf{k}|$, then by (8.1) $\mathbf{i} = \mathbf{j} + \mathbf{k}$ so that $\hat{\mathbf{j}}\cdot\hat{\mathbf{k}} = -\hat{\mathbf{k}}\cdot\hat{\mathbf{i}} = -\hat{\mathbf{i}}\cdot\hat{\mathbf{j}} = 1$, and (8.2) requires $s_i = -s_j = -s_k$. Then (8.17) simplifies to

$$\epsilon C_{IJK} = \frac{G}{3}\left(\frac{\omega_i\omega_j\omega_k}{8\pi^3\rho v^2}\right)^{\frac{1}{2}}, \tag{8.18}$$

where G is the Grüneisen coefficient:

$$G = 1 + \frac{\rho}{v}\frac{dv}{d\rho}. \tag{8.19}$$

In the cases of anomalous and normal dispersion, (8.17) does not simplify to (8.18); see, for example, Putterman and Roberts (1983b). We shall, however, continue to use (8.18) in all cases, i.e. we shall neglect nonlinear terms arising from amplitude dependence of dispersion.

To leading order in the limit $k \to 0$, dispersion arises through the coefficient γ in the relationship,

$$\omega^2 = v^2 k^2 + \gamma k^4, \tag{8.20}$$

between frequency and wavenumber; $\gamma > 0$ gives anomalous dispersion and $\gamma < 0$ gives normal dispersion. The cases of small dispersion we have in mind are those for which

$$\frac{|\gamma|\, k^2}{v^2} \ll 1, \tag{8.21}$$

for the wavenumbers k of interest. Although a term $\gamma\,[\nabla^2\rho']^2$ must then be added into the hamiltonian (8.4), H_0 may still be represented in the action-angle form (8.14). The leading order (3-phonon) nonlinearity (8.15) is, however, no longer given by (8.17). In particular, terms that depend on $d\gamma\,/\,d\rho$ might appear. We will regard these terms as sufficiently small to neglect, which seems from (8.21) to be a reasonable assumption. For small dispersion, the dominant contributions to $\delta^-(\omega_{IJK})$ in what follows arise from nearly collinear interactions, so that the approximation (8.18) is also a physically sensible simplification.

The steps leading to the truncated Liouville equation, governing now $P(I_K;\ \theta_K;\ t)$, follow as in §2. We have

$$P = \bar{P} + \hat{P}_1 + \hat{P}_2, \tag{8.22}$$

where

$$\frac{\partial \bar{P}}{\partial t} + \sum_I \omega_i \frac{\partial \bar{P}}{\partial \theta_I} = 6\epsilon^2 \left(\frac{8\pi^3}{V}\right) \sum_{IJK} \Delta^-(\omega_{IJK})\, \delta_{\mathbf{i+j+k}}\, C^2_{IJK}\ \nabla^*_{IJK}\{I_{IJK}\nabla_{IJK}\bar{P}\}, \tag{8.23}$$

$$\hat{P}_1 = i\epsilon \left(\frac{8\pi^3}{V}\right)^{\frac{1}{2}} \sum_{IJK} \Delta^-(\omega_{IJK})\delta_{\mathbf{i+j+k}}\ C_{IJK}\ I^{1/2}_{IJK} e^{i\theta_{IJK}} \nabla_{IJK}\bar{P}, \tag{8.24}$$

$$\hat{P}_2 = -\epsilon^2 \left(\frac{8\pi^3}{V}\right) \sum_{\substack{IJKLMN \\ IJK \neq -(LMN)}} \left\{\frac{\Delta^-(\omega_{IJKLMN}) - \Delta^-(\omega_{LMN})}{-i\omega_{IJK}}\right\} \delta_{\mathbf{i+j+k}}\delta_{\mathbf{l+m+n}} \times$$
$$C_{IJK}C_{LMN}\, I^{1/2}_{IJKLMN} e^{i\theta_{IJKLMN}} \nabla_{IJK}\nabla_{LMN}\bar{P}. \tag{8.25}$$

In deriving (8.18), we have used the fact that $\delta_{\mathbf{m}}\delta_{\mathbf{m}} = \delta_{\mathbf{m}}$. [The small technical difficulty referred to below (8.17) has repercussions for the interpretation of P; these are described also at the end of this section.]

The transition to the $V \to \infty$ limit follows the pattern of §3. The $\mathbf{k}$ part of the discrete label K is replaced by a continuous label $\mathbf{k}$. Instead of derivatives with respect to I_K and θ_K, functional derivatives for the same variables now appear. As in §3, we continue to use summation signs for integrals over continuous labels, following the prescriptions:

$$\sum_{K} \cdots \to \frac{V}{8\pi^3} \sum_{s_k} \iiint_{-\infty}^{\infty} \mathrm{d}^3k \; \ldots, \tag{8.26}$$

$$\delta_{\mathbf{i}+\mathbf{j}+\mathbf{k}} \;\to\; \frac{8\pi^3}{V}\, \delta(\mathbf{i}+\mathbf{j}+\mathbf{k}). \tag{8.27}$$

Thus, for example, though (8.5) hold, they are, when written out in full,

$$\begin{aligned} \rho'(\mathbf{x}) &= \frac{V}{8\pi^3} \iiint_{-\infty}^{\infty} \rho_{\mathbf{k}} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3k, & \Phi(\mathbf{x}) &= \frac{V}{8\pi^3} \iiint_{-\infty}^{\infty} \Phi_{\mathbf{k}} \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3k, \\ \rho_{\mathbf{k}} &= \frac{1}{V} \iiint_{-\infty}^{\infty} \rho'(\mathbf{x}) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3x, & \Phi_{\mathbf{k}} &= \frac{1}{V} \iiint_{-\infty}^{\infty} \Phi(\mathbf{x}) \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \mathrm{d}^3x, \end{aligned} \tag{8.28}$$

where (8.6) holds. Equations (8.11) and (8.12) are unaltered and, with (8.26) used to replace $\mathbf{k}$-sums by $\mathbf{k}$-integrals, (8.14) and (8.15) are also unchanged. While (8.17) and (8.18) are unaltered, (8.16) must be interpreted using (8.27):

$$c_{IJK} = \left(\frac{8\pi^3}{V}\right)^{\frac{3}{2}} C_{IJK}\, \delta(\mathbf{i}+\mathbf{j}+\mathbf{k}). \tag{8.29}$$

The continuous forms of (8.23)-(8.25) follow in a similar way:

$$\frac{\partial \bar{P}}{\partial t} + \sum_{I} \omega_i \frac{\partial \bar{P}}{\partial \theta_I} = 6\pi\epsilon^2 \left(\frac{8\pi^3}{V}\right)^2 \sum_{IJK} \delta^-(\omega_{IJK}) \delta(\mathbf{i}+\mathbf{j}+\mathbf{k}) C_{IJK}^2 \nabla_{IJK}^* \{I_{IJK} \nabla_{IJK} \bar{P}\}, \tag{8.30}$$

$$\hat{P}_1 = i\pi\epsilon \left(\frac{8\pi^3}{V}\right)^{\frac{3}{2}} \sum_{IJK} \delta^-(\omega_{IJK}) \delta(\mathbf{i}+\mathbf{j}+\mathbf{k}) C_{IJK} I_{IJK}^{1/2} \mathrm{e}^{i\theta_{IJK}} \nabla_{IJK} \bar{P}, \tag{8.31}$$

$$\begin{aligned} \hat{P}_2 = -\pi\epsilon^2 \left(\frac{8\pi^3}{V}\right)^3 \sum_{IJKLMN}{}' & \left\{ \frac{\delta^-(\omega_{IJKLMN}) - \delta^-(\omega_{LMN})}{-i\omega_{IJK}} \right\} \delta(\mathbf{i}+\mathbf{j}+\mathbf{k}) \delta(\mathbf{l}+\mathbf{m}+\mathbf{n}) \\ & \times\; C_{IJK} C_{LMN} I_{IJKLMN}^{1/2} \mathrm{e}^{i\theta_{IJKLMN}} \nabla_{IJK} \nabla_{LMN} \bar{P}. \end{aligned} \tag{8.32}$$

As in §3, the prime over the summation sign in (8.32) refers to the absence of non-diagonal ($IJK \neq -(LMN)$) terms; see Appendix B.

We comment on a few of the many parallels that exist between the consequences of (8.23)–(8.25) or (8.30)–(8.32) and the corresponding results deduced from (2.50)–(2.52) or (3.2)–(3.4). Factorization of $\bar{P}$ is possible under the same circumstances as described in §5, and after multiplying (8.23) by I_P and, integrating over all $\mathbf{I}$ and $\boldsymbol{\theta}$, we obtain the analogue of (5.14), namely

$$\frac{d}{dt}\langle I_P\rangle = 18\pi\epsilon^2\frac{8\pi^3}{V}\sum_{JK}\delta(\omega_{PJK})\,\delta_{\mathbf{p}+\mathbf{j}+\mathbf{k}}C^2_{PJK}s_p\big[s_p\langle I_J\rangle\langle I_K\rangle + s_j\langle I_K\rangle\langle I_P\rangle + s_k\langle I_J\rangle\langle I_P\rangle\big], \tag{8.33}$$

which [using now integral signs instead of summation signs – see(8.26)] is

$$\frac{d}{dt}\langle I_P\rangle = \frac{G^2}{4\pi^2\rho v^2}\sum_{s_j\,s_k}\iint_{-\infty}^{\infty}\delta(\omega_{PJK})\,\delta(\mathbf{p}+\mathbf{j}+\mathbf{k})\times$$
$$s_p\big[s_p\langle I_J\rangle\langle I_K\rangle + s_j\langle I_K\rangle\langle I_P\rangle + s_k\langle I_J\rangle\langle I_P\rangle\big]\omega_p\omega_j\omega_k \mathrm{d}^3j\,\mathrm{d}^3k. \tag{8.34}$$

We may write (8.34) in a more accessible form by introducing

$$I_{\mathbf{k}} = I_K + I_{-K} = 2\,I_{(\mathbf{k},1)} \tag{8.35}$$

as the total action in mode $\mathbf{k}$, and by noting that

$$I_{(\mathbf{k},-1)} = I_{(-\mathbf{k},1)} = \frac{1}{2}\,I_{-\mathbf{k}},$$

so that generally

$$I_K = \frac{1}{2}\,I_{s_k\mathbf{k}}. \tag{8.36}$$

Setting $s_p = 1$ in (8.34) and making the changes of variables $s_j\mathbf{j}\to\mathbf{j}$ and $s_k\mathbf{k}\to\mathbf{k}$, we obtain

$$\frac{d}{dt}\langle I_{\mathbf{p}}\rangle = \frac{G^2}{4\pi^2\rho v^2}\sum_{s_j\,s_k}\iint_{-\infty}^{\infty}\delta(\omega_p+s_j\omega_j+s_k\omega_k)\,\delta(\mathbf{p}+s_j\mathbf{j}+s_k\mathbf{k})\times$$
$$\big[\langle I_{\mathbf{j}}\rangle\langle I_{\mathbf{k}}\rangle + s_j\langle I_{\mathbf{k}}\rangle\langle I_{\mathbf{p}}\rangle + s_k\langle I_{\mathbf{j}}\rangle\langle I_{\mathbf{p}}\rangle\big]\,\omega_p\omega_j\omega_k\,\mathrm{d}^3j\,\mathrm{d}^3k, \tag{8.37}$$

from which we obtained the result (1.14) quoted in the introduction. Equation (8.37) should also be compared with the corresponding result obtained by directly averaging the kinetic equations. [For example, see Putterman and Roberts (1983b), eq. (4.16). Note that their $n^{s_k}(\mathbf{k})$ corresponds to $I_K/8\pi^3$ used here; see (8.51) below. We wish to take this opportunity of correcting an inconsequential error in their paper. Their equation (4.12) should read

$$\langle a^s(\mathbf{k},t)\,a^{s'}(\mathbf{k}',t)\rangle = \frac{\rho}{c^2}\,E^s(\mathbf{k},t)\,\delta^{ss'}\,\delta(\mathbf{k}+\mathbf{k}').\,]$$

We may also compute the renormalized dispersion law, using the same methods as in §6. We find

$$\langle\dot{\theta}_P\rangle = \Omega_P = \omega_p - \frac{1}{2}B'_P, \tag{8.38}$$

where

$$B'_P \;=\; 18\epsilon^2 \frac{8\pi^3}{V} \sum_{JK} \mathcal{P}\big(\frac{1}{\omega_{PJK}}\big)\, \delta_{\mathbf{p}+\mathbf{j}+\mathbf{k}}\, C^2_{PJK}\big[s_j\langle I_K\rangle \;+\; s_k\langle I_J\rangle\big]. \tag{8.39}$$

In the continuous limit, the renormalized speed of long wavelength sound of mode P is therefore (written again with integral signs)

$$v_P \;=\; v\Big\{1 \;-\; \frac{G^2}{8\pi^3\rho v^2} \sum_{s_j\, s_k} \iint_{-\infty}^{\infty} \mathcal{P}\big(\frac{1}{\omega_{PJK}}\big)\, \delta(\mathbf{p}+\mathbf{j}+\mathbf{k})\, \big[s_j\langle I_K\rangle \;+\; s_k\langle I_J\rangle\big]\; \omega_j\omega_k\; \mathrm{d}^3j\mathrm{d}^3k\Big\}. \tag{8.40}$$

Using (8.18), we write (8.39) as

$$B'_P \;=\; \frac{G^2\omega_p}{2\,\pi^3\,\rho\, v^2}\,\frac{8\pi^3}{V} \sum_{s_j\,,\,K} \mathcal{P}\left(\frac{s_j\,\omega_{|\mathbf{p}+\mathbf{k}|}\,\omega_k}{s_p\omega_p \;+\; s_j\,\omega_{|\mathbf{p}+\mathbf{k}|} \;+\; s_k\omega_k}\right)\langle I_K\rangle. \tag{8.41}$$

We simplify (8.41) for the case of greatest physical interest, in which the bulk of the wave energy is concentrated at wavenumbers much greater than that of the impressed wave, p. Then, to first order in p,

$$\omega_{|\mathbf{p}+\mathbf{k}|} \;-\; \omega_k \;=\; \mathbf{p}\cdot\mathbf{v}_{Gk} \;=\; p\,v_{Gk}\,\mu, \tag{8.42}$$

where μ is the cosine of the angle between $\mathbf{p}$ and the group velocity of mode $\mathbf{k}$, namely

$$\mathbf{v}_{Gk} \;=\; \frac{d\omega_k}{d\mathbf{k}}. \tag{8.43}$$

The four choices of sign in (8.41) are

$$\begin{aligned}
(a)\quad & s_j \;=\; -s_k \;=\; -s_p,\\
(b)\quad & s_j \;=\; -s_k \;=\; s_p,\\
(c)\quad & s_j \;=\; s_k \;=\; -s_p,\\
(d)\quad & s_j \;=\; s_k \;=\; s_p.
\end{aligned}$$

No small or zero denominators arise for choice (d) and, since $\omega_{|\mathbf{p}+\mathbf{k}|} \approx \omega_k \gg \omega_p$, the same is true of choice (c). For both of these we may set $\omega_{|\mathbf{p}+\mathbf{k}|} \;=\; \omega_k$ in the $\mathcal{P}$-part of (8.41). For choices (a) and (b), $s_j \;=\; -s_k$, and the difference between $\omega_{|\mathbf{p}+\mathbf{k}|}$ and ω_k is influential. In this way we obtain

$$\begin{aligned}
B'_P \;=\; \frac{G^2\omega_p}{2\,\pi^3\rho\, v^2}\;\frac{8\,\pi^3}{V}\sum_{\mathbf{k}}\omega_k\,\Big[\,&\tfrac{1}{2}\langle I_{(\mathbf{k},s_p)} \;+\; I_{(\mathbf{k},-s_p)}\rangle\\
&-\;\mathcal{P}\Big\{\frac{\omega_k \;+\; p\,v_{Gk}\mu}{\omega_p \;-\; p\,v_{Gk}\mu}\Big\}\,\langle I_{(\mathbf{k},s_p)}\rangle \;+\; \mathcal{P}\Big\{\frac{\omega_k \;+\; p\,v_{Gk}\mu}{\omega_p \;+\; p\,v_{Gk}\mu}\Big\}\,\langle I_{(\mathbf{k},-s_p)}\rangle\Big].
\end{aligned} \tag{8.44}$$

We now introduce $I_{\mathbf{k}}$ defined by (8.35) and further reduce (8.44) by assuming that the wave field is isotropic, so that

$$I_{-\mathbf{k}} = I_{\mathbf{k}} = I_k \text{ say.} \tag{8.45}$$

Then (8.44) shows that

$$B'_{(\mathbf{p},s_p)} = B'_{(\mathbf{p},-s_p)} = B'_p, \tag{8.46}$$

where, in integral form,

$$\begin{aligned} B'_p &= \frac{G^2\omega_p}{4\pi^3\rho v^2} \iiint \left(1 - \mathcal{P}\left\{\frac{\omega_k + p\,v_{Gk}\mu}{\omega_p - p\,v_{Gk}\mu}\right\} + \mathcal{P}\left\{\frac{\omega_k + p\,v_{Gk}\mu}{\omega_p + p\,v_{Gk}\mu}\right\}\right) \omega_k \langle I_k\rangle\, k^2\, \mathrm{d}k\, \mathrm{d}\mu\, \mathrm{d}\phi \\ &= \frac{G^2\omega_p}{4\pi^3\rho v^2} \int \left(3 + \frac{\omega_p}{p\,v_{Gk}} \ln\left|\frac{\omega_p - p\,v_{Gk}}{\omega_p + p\,v_{Gk}}\right|\right) \omega_k \langle I_k\rangle\, \mathrm{d}^3 k. \end{aligned} \tag{8.47}$$

In the case of normal dispersion ($\gamma < 0$), there is no value of μ for which the denominators in (8.44) vanish and the $\mathcal{P}$ symbols are superfluous. For anomalous dispersion ($\gamma > 0$), the denominators vanish for $\mu = s_p\,\omega_p / p\,v_{Gk}$; the principal parts must then be taken, but the final result is the same.

If we assume small dispersion, as in (8.20), we find that, to leading order in γ, (8.47) is

$$B'_p = -\frac{G^2\omega_p}{4\pi^3\rho v^2} \int \left[\ln\left(\frac{4v^2}{3k^2|\gamma|}\right) - 3\right] \omega_k \langle I_k\rangle\, \mathrm{d}^3 k. \tag{8.48}$$

The reduced form of (8.40), for long wavelength sound, is

$$v_p = v\left\{1 + \frac{G^2}{8\pi^3\rho v^2} \int \left[\ln\left(\frac{4v^2}{3k^2|\gamma|}\right) - 3\right] \omega_k \langle I_k\rangle\, \mathrm{d}^3 k\right\}. \tag{8.49}$$

In the semidispersive limit, $\gamma \to 0$, the expression (8.49) for v_p diverges. Such a divergence does not arise in a theory, such as that of Newell and Aucoin (1971), that confines itself to kinetic equations. Presumably, the divergence of (8.49) would not arise in a self-consistent treatment which recognized the dependence of wavespeed on amplitude by replacing ω_{PJK} by Ω_{PJK}. Such a procedure could be justified only through a Fokker-Planck equation that included terms up to $O(\epsilon^4)$ and which is outside the scope of this paper.

The temperature dependence of the dispersion of sound in liquid He^4 at low temperatures has been calculated, and interpreted in terms of expressions like (8.49); see Khalatnikov (1963, Ch.22) and Maris (1973). Further reduction of (8.49) requires that the distribution $\langle I_k\rangle$ be provided. If we replace k in the logarithm of (8.49) by k_T, an average over a thermal distribution of waves, we obtain

$$v_p = v + \frac{2G^2U}{\rho v} \ln\left(\frac{k_\gamma}{3.88\ldots k_T}\right), \tag{8.50}$$

where $k_\gamma \equiv v / |\gamma|^{1/2}$ is a characteristic wavenumber for separating regimes of weak and strong dispersion. [According to (8.21), law (8.20) is inappropriate if $k > k_\gamma$, and (8.50) will be tenable only if $k_T < k_\gamma$.] In (8.50), U is the wave energy per unit volume which, by (8.14) and (8.26) is

$$U = \frac{1}{8\pi^3} \int \omega_k(I_k)\, \mathrm{d}^3 k. \tag{8.51}$$

Finally, as promised earlier, we make a few remarks about the dependence of P on both K and $-K$. The use of K instead of ($\mathbf{k}$ and s_k is technically a convenient abbreviation, but it exacts the penalty that, by (8.13), the summation (8.14) is over pairs of equal terms; the triple sum (8.15) similarly consists of pairs of equal combinations involving $(I,\, J\, K)$ and $(-I,\, -J\, -K)$. The canonical equations preserve (8.13) and, when we adopt the equivalent Liouville formalism, we must insist that P is function of $(I_K,\, \theta_K)$ and $(I_{-K},\, \theta_{-K})$ that is unaltered when, for any K, $(I_K,\, \theta_K)$ and $(I_{-K},\, \theta_{-K})$ are exchanged; the canonical equations ensure that, if this is true initially, it is true for all t. The correlation between $(I_K,\, \theta_K)$ and $(I_{-K},\, \theta_{-K})$ is complete at all times, so that

$$P(\ldots, I_K, \ldots, I_{-K}, \ldots; \ldots, \theta_K, \ldots, \theta_{-K}, \ldots; t) = \breve{P}(\ldots, I_K, \ldots; \ldots, \theta_K, \ldots; t) \prod_{\mathbf{k}} \left[\delta(I_K - I_{-K})\, \delta(\theta_K - \theta_{-K})\right], \tag{8.52}$$

where $\breve{P}$ involves only those K for which $s_k = 1$. To obtain a mean value, for example that of $I_{\mathbf{p}}$ defined in (8.35), one must multiply (8.52) by $\frac{1}{2} I_{(\mathbf{p},1)}$ and integrate over all $(I_K,\, \theta_K)$, even including $(I_{-P},\, \theta_{-P})$; equivalently, this is the integral of $I_{\mathbf{p}}\, \breve{P}$ over all I_K and θ_K for which $s_k = 1$. Similarly, $\breve{P}$ is easily obtained from P by integrating (8.52) over all $(I_K,\, \theta_K)$ for which $s_k = -1$.

Acknowledgements

This work was commenced in the summer of 1985 when one of the authors (SJP) visited the other at his then home University (Newcastle upon Tyne, England). The authors are grateful to the Science Research Council of Great Britain for grant GR/C/74904 which made that collaboration possible. Later support was provided to SJP by the Grant DE-F603-87ER13686 awarded by the Department of Energy, Office of Basic Energy Sciences, Division of Engineering and Geosciences, and to PHR by the Office of Naval Research, under contract N00014-86-K-0641.

References

Abraham, B.M., Eckstein, Y., Ketterson, J.B., Kuchnir, M. and Vignos, J., 1969, *Sound Propagation in Liquid ^4He*, Phys. Rev. **181**, 347-373.

Benney, D. J. and Saffman, P. G. 1966, *Nonlinear Interactions of random Waves in a dispersive Medium*, Proc. R. Soc. Lond., A**289**, 301-320.

Chandrasekhar, S. 1943, *Stochastic Processes in Physics and Astronomy*, Rev. Mod. Phys. **15**, 1-88. Reprinted in Wax, N. 1954, (Ed.) *Selected Papers on Noise and stochastic Processes*, Dover Publications, New York.

Ehrenfest, P. and Ehrenfest, T. 1912, *The conceptual Foundations of the statistical Approach in Mechanics*, No. 6 in "Encyclopädie der mathematischen Wissenschaften, **IV,2,II**", B. G. Teubner, Leipzig. English translation, Cornell University Press, Ithica, 1959.

Forristall, G. Z. 1981, *Measurements of a saturated Range on Ocean Wave Spectra*, J. Geophys. Res., **86**, 8075-8084.

Hasselmann, K. 1960, *Grundgleichungen der Seegangsvoraussage*, Shiffstech., **7**, 191-195.

Hasselmann, K. 1962, *On the non-linear Energy Transfer in a Gravity-Wave Spectrum. Part 1. General Theory.* J. Fluid Mech., **12**, 481-500.

Hasselmann, K. 1966, *Feynman Diagrams and interaction Rules of Wave-Wave scattering Processes.* Rev. Geophys., **4**, 1-32.

Hasselmann, K. 1967, *Nonlinear Interactions treated by the Methods of Theoretical Physics (with Application to the Generation of Waves by the Wind)*, Proc. R. Soc. Lond., A**299**, 77-100.

Khalatnikov, I.M. 1965, *Introduction to the Theory of Superfluidity*, Benjamin, New York.

Kraichnan, R. 1968, *Lagrangian-History Statistical Theory for Burger's Equation*, Phys. Fluids, **11**, 265-277.

Landau, L.D. and Khalatnikov I. M., 1949, *Theory of Viscosity of Helium II. I. Collision of elementary Excitations in Helium II*, Zh. Eksper. Teor. Fiz., **19**, 637-650.

Landau, L.D. and Khalatnikov I. M., 1949, *Theory of Viscosity of Helium II. II. Calculation of the Viscosity Coefficient*, Zh. Eksper. Teor. Fiz., **19**, 709-726.

Landau, L.D. and Lifshitz, E.M. 1969, *Statistical Physics*, 2nd edition. Pergamon, Oxford.

Landau, L.D. and Rumer, G. 1937, *Über Schallabsorption in Festen Körpern*, Phys. Zeits. Sov., **11**, 18-25. Engl. Trans: *On the Absorption of Sound in Solids*, in ter Haar (1965), pp.187-192.

Larraza, A. 1987, *Universal Power Spectra for Wave Turbulence: Applications to Wind Waves, Flicker Noise, solar Wind Spectum, and classical second Sound*, Ph. D. Thesis, Dept. Phys. UCLA.

Larraza, A. and Putterman, S. 1986, *Second Sound in Wave Turbulence: A Clue to the Cause of anomalous Plasma Diffusivity*, Phys. Rev. Lett. **57**, 2810-2813.

Larraza, A. and Putterman, S. 1987, *Universal Power Spectra for Wave Turbulence: Applications to Wind Waves and $1/f$ Noise*, in *Irreversible Phenomena and Dynamical Systems Analysis in Geosciences*, Eds. C. Nicolis and G. Nicolis. NATO ASI Series C, **192**, pp. 139-144. Dordrecht: Elsevier (1987)

Larraza, A., Putterman, S. and Roberts, P.H. 1985, *A universal $1/f$ Power Spectrum as the Accumulation Point of Wave Turbulence*, Phys. Rev. Lett., **55**, 897-900.

Lévy, P. 1940, *Le Mouvement Brownién Plan*, Am. J. Math., **62**, 487-550.

Litvak, M. M. 1960, *A Transport Equation for Magnetohydrodynamic Waves*, AVCO Research Report **92**.

Maris, H. J. 1973, *Hydrodynamics of superfluid Helium below 0.6°K. II. Velocity and Attenuation of ultrasonic Waves*, Phys. Rev. A. **8**, 2629-2639.

Newell, A.C. 1968, *The Closure Problem in a System of random Gravity Waves*, Rev. Geophys. **6**, 1-13.

Newell, A.C. and Aucoin, P.J. 1971, *Semidispersive Wave Systems*, J. Fluid Mech., **49**, 593-609.

Peierls, R. 1929, *Zur kinetischen Theorie der Wärmeleitung in Kristallen*, Ann. Phys. **3** (Ser. 5), 1055-1101.

Peierls, R. 1955, *Quantum Theory of Solids*, Oxford: Clarendon Press.

Phillips, O.M. 1977, *Dynamics of the Upper Ocean*, Cambridge: University Press.

Prigogine, I., 1962, *Non-equilibrium Statistical Mechanics*, New York: Wiley Interscience.

Prigogine, I. and Henin, F. 1960, *On the general Theory of the Approach to Equilibrium. 1. Interacting normal Modes*, J. Math. Phys., **1**, 349-371.

Putterman, S.J. and Roberts, P.H. 1983a, *Nonlinear Hydrodynamics and a one Fluid Theory of Superfluid He^4*, Phys. Lett., **89**A, 444-447.

Putterman, S.J. and Roberts, P.H. 1983b, *Classical non-linear Waves in dispersive nonlocal Media, and the Theory of Superfluidity*, Physica, **117A**, 369-397.

Roach, P.R., Ketterson, J.B. and Kuchnir, M. 1972, *Ultrasonic Attenuation in Liquid ^4He under Pressure*, Phys. Rev. A**5**, 2205-2214.

Russell, C. T. 1972, *Comments on the Measurement of Power Spectra of the interplanetary magnetic Field*, in *Solar Wind*, NASA Report SP-308. (Eds. C.P.Sonett, P.J. Coleman, Jr., and J.M.Wilcox), pp. 365-374.

Saffman, P. G. 1967, *Discussion on "Nonlinear Interactions treated by the Methods of Theoretical Physics (with Application to the Generation of Waves by the Wind)" by K. Hasselmann*, Proc. R. Soc. Lond., A**299**, 101-103.

Sagdeev, R. Z. 1979, *The 1976 Oppenheimer Lectures: Critical Problems in Plasma Astrophysics. I. Turbulence and nonlinear Waves*, Rev. Mod. Phys., **51**, 1 -9.

Spitzer, F. 1958, *Some Theorems concerning 2-dimensional Brownian motion*, Am. Math. Soc. Trans., **87**, 187-197.

ter Haar, D. (Editor) 1965, *Collected Papers of L. D. Landau*, Oxford: Pergamon Press.

Westervelt, P.J. 1976, *Absorption of Sound by Sound*, J. Acoust. Soc. Am., **59**, 760-764.

Zakharov, V. E. 1965, *Weak Turbulence in Media with a Decay Spectrum*, J. Appl. Mech. & Tech. Phys. **6**(4), 22-24. (Russian Original: Zh. Prik. Mek. Tek. Fiz. (4), 35-39.)

Zakharov, V.E. 1984, *Kolmogorov Spectra in weak Turbulence Problems*, Ch. 5.1 in Galeev, A.A. and Sudan, R. N.(Eds.), *Basic Plasma Physics*, **2**, pp 3-36. New York: Elsevier.

Zakharov, V.E. and Sagdeev, R.Z. 1970, *Spectrum of acoustic Turbulence*, Sov. Phys. Dokl., **15** 439-441.

Appendix A: The Wave Turbulent Steady State

When energy is supplied at low frequencies and withdrawn at high frequencies, the kinetic equation (5.14) admits stationary, off-equilibrium solutions of the form (5.41). We aim in this Appendix to demonstrate that fact.

Written out in full in the continuous case, (5.14) is

$$\begin{aligned}\frac{dI_i}{dt} &= 36\pi\epsilon^2 \int_0^\infty\int_0^\infty c_{ijk}^2\,\delta(\omega_i-\omega_j-\omega_k)\,[I_jI_k - I_kI_i - I_iI_j]\,\sigma_j\,\sigma_k\,\mathrm{d}\omega_j\,\mathrm{d}\omega_k \\ &+ 36\pi\epsilon^2 \int_0^\infty\int_0^\infty c_{imn}^2\,\delta(\omega_i-\omega_m+\omega_n)\,[I_mI_n - I_nI_i + I_iI_m]\,\sigma_m\,\sigma_n\,\mathrm{d}\omega_m\,\mathrm{d}\omega_n \\ &+ 36\pi\epsilon^2 \int_0^\infty\int_0^\infty c_{ipq}^2\,\delta(\omega_i+\omega_p-\omega_q)\,[I_pI_q + I_qI_i - I_iI_p]\,\sigma_p\,\sigma_q\,\mathrm{d}\omega_p\,\mathrm{d}\omega_q, \qquad \text{(A1)}\end{aligned}$$

where dummy suffices have been relabelled, the average signs, $\langle\ \ \rangle$, have been removed from the mean actions $\langle I_s\rangle$, and the density of states $\sigma(\omega_s)$ has been abbreviated by σ_s. The three contributions to dI_i/dt in (A1) represent the sum total of all processes that take energy out of, and bring energy into, mode i.

Now perform the following transformations on (A1):

$$\begin{aligned}\omega_m &= \frac{\omega_i^2}{\omega_j}, & \omega_n &= \frac{\omega_i}{\omega_j}\omega_k, \\ \omega_p &= \frac{\omega_i}{\omega_k}\omega_j, & \omega_q &= \frac{\omega_i^2}{\omega_k},\end{aligned} \qquad \text{(A2)}$$

and use (5.42) and (5.43), which are

$$\sigma = \sigma_0\,\omega^{d-1}, \qquad \epsilon c_{ijk} = c_0(\omega_i\omega_j\omega_k)^\beta. \qquad \text{(A3)}$$

We find that

$$\begin{aligned}\frac{dI_i}{dt} &= 36\pi\epsilon^2 \int_0^\infty\int_0^\infty c_{ijk}^2\delta(\omega_i-\omega_j-\omega_k)\Big\{[I_jI_k - I_kI_i - I_iI_j] + \\ &\Big(\frac{\omega_i}{\omega_j}\Big)^{6\beta+3d-1}[I_mI_n - I_nI_i + I_iI_m] + \Big(\frac{\omega_i}{\omega_k}\Big)^{6\beta+3d-1}[I_pI_q + I_qI_i - I_iI_p]\Big\}\,\sigma_j\sigma_k\mathrm{d}\omega_j\mathrm{d}\omega_k.\end{aligned} \qquad \text{(A4)}$$

The form of (A4) suggests that the stationary off-equilibrium distribution is of similarity form:

$$I_i = a_0\,\omega_i^\gamma, \qquad \text{(A5)}$$

where a_0 is a constant. Then (A4) may be written as

$$\begin{aligned}\frac{dI_i}{dt} &= 36\pi\epsilon^2\,a_0^2 \int_0^\infty\int_0^\infty c_{ijk}^2\ \delta(\omega_i-\omega_j-\omega_k)\ [I_jI_k - I_kI_i - I_iI_j]\ \times \\ &\Big\{1 - \Big(\frac{\omega_i}{\omega_j}\Big)^{6\beta+3d+2\gamma-1} - \Big(\frac{\omega_i}{\omega_k}\Big)^{6\beta+3d+2\gamma-1}\Big\}\ \sigma_j\sigma_k\mathrm{d}\omega_j\mathrm{d}\omega_k.\end{aligned} \qquad \text{(A6)}$$

We see that a stationary solution, in which the inflow and outflow of energy into and out of every mode i are balanced, is achieved when $6\beta + 3d + 2\gamma - 1 = -1$, i.e. when, as in (5.41),

$$\gamma = -\frac{3}{2}(2\beta + d). \tag{A7}$$

We may rewrite (5.14) as an equation of energy conservation in frequency space

$$\frac{\partial \tilde{E}}{\partial t} + \frac{\partial \tilde{Q}}{\partial \omega} = 0, \tag{A8}$$

where $\tilde{E}$ is energy per unit frequency interval and $\tilde{Q}$ is the corresponding energy flux:

$$\tilde{E} = \omega \sigma I, \tag{A9}$$

$$\tilde{Q} = -\int_0^{\omega} \omega_i \left[\int_0^{\omega_i} U_1 \,\mathrm{d}\omega_j - \int_{\omega_i}^{\infty} U_2 \,\mathrm{d}\omega_j\right] \mathrm{d}\omega_i, \tag{A10}$$

where

$$U_1 = 36\,\pi\,\epsilon^2\, c_{i,j,i-j}\sigma_i\sigma_j\sigma_{i-j}\left[I_j I_{i-j} - I_i I_j - I_i I_{i-j}\right], \tag{A11}$$

$$U_2 = 72\,\pi\,\epsilon^2\, c_{i,j,j-i}\sigma_i\sigma_j\sigma_{j-i}\left[I_i I_{j-i} - I_i I_j - I_j I_{j-i}\right], \tag{A12}$$

and the suffix $_{i-j}$ denotes values corresponding to $\omega_i - \omega_j$.

Since , as we know, the steady kinetic equation is satisfied when (A7) holds, the two integrals in square brackets in (A10) cancel in this case, and it might be thought that, by the choice of lower limit of ω_i integration in (A10), $\tilde{Q}$ is zero for all ω. This is in fact not the case. When $\tilde{Q}$ is defined by (A10), it tends to a finite limit as $\omega \to 0$. Indeed, it is clear from (A8) that, when $\tilde{E}$ is time independent, the energy flux, $\tilde{Q}$, takes the same value ($\tilde{Q}_0$, say) for all ω. To determine $\tilde{Q}_0$, we first substitute (A3) and (A5) into (A11) and (A12) to obtain

$$\int_0^{\omega_i} U_1 \,\mathrm{d}\omega_j = 36\,\pi\,\epsilon^2 a_0^2\, c_0^2\, \sigma_0^3\, \omega_i^{2\gamma+6\beta+3d-2} \int_0^1 x^{2\beta+d-1}(1 - x)^{2\beta+d-1}\left[x^{\gamma}(1 - x)^{\gamma} - x^{\gamma} - (1 - x)^{\gamma}\right] \mathrm{d}x, \tag{A13}$$

$$\int_{\omega_i}^{\infty} U_2 \,\mathrm{d}\omega_j = 36\,\pi\,\epsilon^2 a_0^2\, c_0^2\, \sigma_0^3\, \omega_i^{2\gamma+6\beta+3d-2} \int_0^1 \left[x^{-2\gamma-6\beta-3d+1} + (1 - x)^{-2\gamma-6\beta-3d+1}\right] x^{2\beta+d-1}(1 - x)^{2\beta+d-1}\left[x^{\gamma}(1 - x)^{\gamma} - x^{\gamma} - (1 - x)^{\gamma}\right] \mathrm{d}x. \tag{A14}$$

[In (A13), $x = \omega_j/\omega_i$; in (A14), $x = \omega_i/\omega_j$.]

Now let

$$\gamma = -\frac{3}{2}(2\beta + d) + \delta. \tag{A15}$$

Then, in the limit $\delta \to +0$,

$$\int_0^{\omega_i} U_1 \, d\omega_j \; - \; \int_{\omega_i}^{\infty} U_2 \, d\omega_j \; = \; -72\,\pi\,\epsilon^2 a_0^2\, c_0^2\, \sigma_0^3\, \omega_i^{2\delta-2} \; \times$$

$$2\delta \int_0^1 \frac{\left[1 \; - \; x^{\frac{3}{2}(2\beta+d)} \; - \; (1-x)^{\frac{3}{2}(2\beta+d)}\right]}{x^{\frac{1}{2}(2\beta+d)}\;(1-x)^{\frac{1}{2}(2\beta+d)+1}} \ln x \; dx. \tag{A16}$$

Substituting into (A10), we find

$$\tilde{Q}_0 \; = \; 72\,\pi\,\epsilon^2 a_0^2\, c_0^2\, \sigma_0^3 \int_0^1 \frac{\left[1 \; - \; x^{\frac{3}{2}(2\beta+d)} \; - \; (1-x)^{\frac{3}{2}(2\beta+d)}\right]}{x^{\frac{1}{2}(2\beta+d)}\;(1-x)^{\frac{1}{2}(2\beta+d)+1}} \ln x \; dx. \tag{A17}$$

The integral converges for $\beta \; < \; \frac{1}{2}(4-d)$. Equation (A17) affords the means of relating the hitherto unknown amplitude a_0 in (A5) to the energy flux, $\tilde{Q}_0$. The most significant fact is, of course, that $a_0 \; \propto \; \tilde{Q}_0^{\frac{1}{2}}$.

Equation (A1) tells us that, when (A5) and (A7) hold, the constants A_p and B_p which were introduced in (5.12) and (5.13), and which measure the rates at which the direct and restituting collisions transfer energy out of, and into, mode i, are related as in (5.32) by

$$I_i^\dagger \; = \; A_i^\dagger / B_i^\dagger, \tag{A18}$$

and also that

$$A_i^\dagger \; = \; 36\,\pi\,\epsilon^2 \int_0^\infty \int_0^\infty c_{ijk}^2\, \delta(\omega_i - \omega_j - \omega_k)\, I_j\, I_k \; \times$$

$$\left[1 \; + \; \left(\frac{\omega_i}{\omega_j}\right)^{\gamma-1} \; + \; \left(\frac{\omega_i}{\omega_k}\right)^{\gamma-1}\right] \sigma_j\, \sigma_k \, d\omega_j \, d\omega_k. \tag{A19}$$

We may deduce the value of $B_i^\dagger$ from (5.41), by using (A18) and (A19).

Use of (5.41) in (A19) yields an expression that diverges at the low frequency limit. Relation (A18) is still meaningful however, in the same sense as the corresponding expression for equipartition was meaningful [see discussion above (5.40)]. Throughout the frequency range, the integrand for $A_i^\dagger$ is, term by term, $I_i^\dagger$ times the corresponding term in the integrand for $B_i^\dagger$. It should also be noted that the spectrum (5.41) cannot be extended down to zero frequency; there is, in practice, some lowest injection frequency determined by the nature of the external forces that excite the oscillations. Below this frequency $I_i^\dagger$ decreases rapidly, so that the physical $A_i^\dagger$ converges.

For acoustic wave turbulence, such as discussed in §8, a maintained injection of energy density leads to the power law (1.17), as can also be established by the methods of this Appendix.

Appendix B: Transition from the Discrete to the Continuous

The principal use of the Fokker-Planck equation is the calculation of moments such as (2.54) – (2.56). In the discrete case, $\langle f \rangle_2$ is evaluated from the expression (2.52) for $\hat{P}_2$, in which the $IJK = -(LMN)$ terms are excluded from the summation. At first sight it might appear that this restriction is unnecessary in the continuous limit, for it involves a subset of zero measure in the six-fold integration over $ijklmn$. And yet, by ignoring the restriction $IJK \neq -(LMN)$, one would be guilty of double-counting, for the same terms have already played an essential rôle in (3.2). Moreover, one would obtain unjustified values for the moments of $\boldsymbol{\theta}$-independent quantities.

An unambiguous procedure, that correctly performs the transition from (2.50)–(2.52) to (3.2)–(3.4), is through Fourier transformation:

$$\bar{P} = \sum_{\mathbf{n}} \bar{P}_{\mathbf{n}}\, e^{i\mathbf{n}.\boldsymbol{\theta}}, \tag{B1}$$

where $\mathbf{n}$ stands for a set of N positive or negative integers n_i. (The probability density, $\bar{p}$, that distinguishes between values of θ_i differing by multiples of 2π, would analogously be expressed as a Fourier integral over a set $\mathbf{n}$ of continuous n_i.) Differentiation with respect to θ_i in ∇_I becomes now an algebraic multiplication, so that the operators appearing on the right-hand side of (2.50) involve $\partial/\partial I_i$ but are $\boldsymbol{\theta}$-independent. For example, acting on an n_i harmonic of mode i, ∇_I is effectively

$$\nabla_{In_i} = \frac{\partial}{\partial I_i} + \frac{i\, s_i\, n_i}{2\, I_i}. \tag{B2}$$

Thus, according to (2.50), each Fourier component $\bar{P}_{\mathbf{n}}$ of $\bar{P}$ evolves from its inital state without being influenced by the other Fourier components; explicitly we have

$$\frac{\partial \bar{P}_{\mathbf{n}}}{\partial t} + i(\mathbf{n}.\boldsymbol{\omega})\, P_{\mathbf{n}} = 6\,\epsilon^2 \sum_{IJK} \Delta^-(\omega_{IJK})\, c_{ijk}^2\, \nabla^*_{IJKn_in_jn_k} \left[I_{ijk}\, \nabla_{IJKn_in_jn_k}\, \bar{P}_{\mathbf{n}} \right]. \tag{B3}$$

In taking the continuous limit, one replaces Δ^- by $\pi\delta^-$, the label i and the summation being interpreted as in §3. From (B3) one obtains in this limit

$$\frac{\partial \bar{P}_{\mathbf{n}}}{\partial t} + i(\mathbf{n}.\boldsymbol{\omega})\, P_{\mathbf{n}} = 6\,\pi\,\epsilon^2 \sum_{IJK} \delta^-(\omega_{IJK})\, c_{ijk}^2\, \nabla^*_{IJKn_in_jn_k} \left[I_{ijk}\, \nabla_{IJKn_in_jn_k}\, \bar{P}_{\mathbf{n}} \right]. \tag{B4}$$

Equation (3.2) may then be regarded as the resynthesis of $\bar{P}$ from its Fourier components after the transition from (B3) to (B4) has been made.

The continuous, irreversible limit can evidently be taken without difficulty for each of the Fourier components of $\bar{P}$ but, because these components do not mix, the restriction $IJK \neq -(LMN)$ is as effective after the limit has been taken as it was before the limit was taken. In the Fourier analysed form of (3.2) – (3.4), the meaning of the condition $IJK \neq -(LMN)$ is unequivocal: the Fourier expansion of $\bar{P}$ generates a like expansion in $\hat{P}_2$, but no term generated from $IJK = -(LMN)$ may appear in the $\hat{P}_2$ expansion. Operationally, this prescription is summarized by the following statement:

$$\langle f \rangle_{CL} = \int \left\{ \lim_{\Delta \to \delta} \left[\int f(\mathbf{I}; \boldsymbol{\theta}) \, P(\mathbf{I}; \boldsymbol{\theta}; t) \, \mathrm{d}\boldsymbol{\theta} \right] \right\} \mathrm{d}\mathbf{I}, \tag{B5}$$

in which $_{CL}$ stands for 'continuous limit'. We note particularly that the $\boldsymbol{\theta}$ integrations are performed prior to the passage to the continuous limit, $N \to \infty$ (which is indicated in (B5) by $\Delta \to \delta$).

As an example, consider the explicit Fourier representation of $\bar{P}$ and $\hat{P}_2$ in terms of their dependence on zero, one, two ... nonzero integers n_i:

$$\bar{P}(\mathbf{I}; \boldsymbol{\theta}; t) = \bar{P}_0(\mathbf{I}; t) + \sum_{i\, n_i} \bar{P}_{i\, n_i}(\mathbf{I}; t)\, \mathrm{e}^{i\, n_i \theta_i} + \sum_{ij\, n_i n_j} \bar{P}_{ij\, n_i n_j}(\mathbf{I}; t)\, \mathrm{e}^{i(n_i\theta_i + n_j\theta_j)} + \ldots, \tag{B6}$$

$$\hat{P}_2(\mathbf{I}; \boldsymbol{\theta}; t) = \hat{P}_{2,0}(\mathbf{I}; t) + \sum_{i\, n_i} \hat{P}_{2,i\, n_i}(\mathbf{I};\, t)\, \mathrm{e}^{i\, n_i \theta_i} + \sum_{ij\, n_i n_j} \hat{P}_{2,ij\, n_i n_j}(\mathbf{I}; t)\, \mathrm{e}^{i(n_i\theta_i + n_j\theta_j)} + \ldots, \tag{B7}$$

where we may assume that, whenever two indices are equal, the corresponding coefficient, $\bar{P}_{ij\ldots n_i n_j \ldots}$, vanishes; for instance

$$\bar{P}_{ii\, n_i n_i} = 0, \qquad \hat{P}_{2,ii\, n_i n_i} = 0. \tag{B8}$$

The continuous limit of (2.52) can now be found using prescription (B5). For example, the angle-independent part of $\hat{P}_2$ is

$$\begin{aligned} \hat{P}_{2,0}(\mathbf{I}, t) &= -36\pi\epsilon^2 \sum_{IJKN} F_{IJ-K-I-JN} \nabla_{IJ-K00s_k} \nabla_{-(IJN00s_n)} \bar{P}_{kns_k - s_n} \\ &- 216\pi\epsilon^2 \sum_{IJKMN} F_{I-J-K-IMN} \nabla_{I-J-K0s_j s_k} \nabla_{-(IMNK0s_m s_n)} \bar{P}_{jkmns_j s_k - s_m - s_n} \\ &- 720\pi\epsilon^2 \sum_{IJKLMN} F_{-I-J-KLMN)} \nabla_{-I-J-Ks_i s_j s_k} \nabla_{-(LMNs_l s_m s_n)} \bar{P}_{ijklmns_i s_j s_k - s_l - s_m - s_n}, \end{aligned} \tag{B9}$$

where

$$F_{IJKLMN} = \left\{ \frac{\delta^-(\omega_{IJKLMN}) - \delta^-(\omega_{LMN})}{-i\omega_{IJK}} \right\} c_{ijk} c_{lmn} I^{1/2}_{ijklmn}. \tag{B10}$$

We see from (B9) that only terms in the $\bar{P}$ expansion (B6) that involve two, four, and six angles contribute to the angle independent part of $\hat{P}_2$. The angle independent part, $\bar{P}_0$, of $\bar{P}$ makes no contribution to $\hat{P}_{2,0}$. This is a consequence of the requirement $IJK \neq -(LMN)$. The exclusion of the "diagonal" terms, $IJK = -(LMN)$, apparently makes a negligible difference to the sums over $\mathbf{n}$, that involve the far more numerous terms for which $IJK \neq -(LMN)$. However, those "off-diagonal" terms make zero contributions to several moments of physical significance (for example, moments of $\boldsymbol{\theta}$-independent quantities), and the present analysis indicates that to include them in the $O(\epsilon^2)$ framework of the present Fokker-Planck theory would be unjustified.