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Bjorn Engquist
Laurence Halpern

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

FAR FIELD BOUNDARY CONDITIONS FOR
COMPUTATION OVER LONG TIME

Björn Engquist [1)]

Department of Mathematics
University of California
Los Angeles, Ca 90024

and

Laurence Halpern [2)]

Centre de Mathématiques Appliquées
Ecole Polytechnique
91128 Palaiseau Cedex, France

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Abstract

A new class of computational far field boundary conditions for hyperbolic partial differential equations is developed. These boundary conditions combine properties of absorbing boundary conditions for transient solutions and properties of far field boundary conditions for steady state problems. The conditions can be used to limit the computational domain when both travelling waves and evanescent waves are present. Boundary conditions for scalar wave equations are derived and analyzed. Extensions to systems of equations are discussed and results from numerical experiments are presented.

1. Introduction. Computational far field boundary conditions are used in order to restrict the spatial domain in the numerical approximation of partial differential equations. This is often necessary if the original domain is infinite or too large for practical calculations. The research in the area of computational far field boundary conditions has been very active during the last few years. These types of boundary conditions are in the literature also called absorbing, artificial, radiation and transparent, see eg. [1], [2], [9 - 13], [15], [22], [23]. For applications in seismology see eg. [5], [6].

The computational far field boundary conditions should have three important properties:

1) It should, together with the partial differential equation and possibly also together with other boundary conditions form a well-posed problem.

2) The far field boundary condition should also be well satisfied by a relevant class of solutions to the original problem.
3) Finally, it should be possible to implement efficient computational algorithms based on the boundary conditions.

The first two properties guarantee that the error between the original solution and the solution over the restricted domain with the new boundary conditions is small, [10], [12].

In [9], [10] a class of computational far field boundary conditions were developed for the numerical simulation of waves. These boundary conditions were designed such that artificial reflections in the computational boundaries should be avoided. The resulting initial-boundary value problem was shown to be well-posed and the boundary conditions were well satisfied for travelling waves impinging on the boundary, see also [22]. For constant coefficient hyperbolic problems the original solution can be exactly represented on the bounded computational domain by using nonlocal boundary conditions based on Fourier transforms. The third desired property, discussed above, was achieved in [9], [10] by local approximations of the nonlocal boundary conditions. Pseudo-differential operators were replaced by differential operators. The approximations given in [9] and [10] are better suited for a transient solution than for a steady or almost steady solution.

The steady state problem is considered in [15], [23]. The computational far field boundary conditions for elliptic problems in these papers are not local. The interior numerical method is implicit so it is practically possible to use boundary conditions that couple all computational boundary values. There are also local boundary conditions for elliptic problems based on asymptotic expansions at infinity,[1]. The computational domain must then be taken somewhat larger. Non local boundary conditions for steady state solutions of the Euler equations are given in [11].

In this paper we shall integrate the time-dependent and the steady techniques. We shall construct a class of boundary conditions which can be used in both the transient regime and when the solution approaches steady state. This means that these boundary conditions can be applied when a time-dependent formulation is used for computations to steady state or when evanescent as well as travelling waves are present in the time-dependent calculation. The class of solutions discussed above under property 2) is thus wider than for the standard techniques aimed at travelling waves or steady state separately. In order to be computationally practical the conditions are local in time and local or non-local in space.

We illustrate our results by the following set of boundary conditions at $x = 0$ for the simple dispersive wave equation

$$u_{tt} - u_{xx} + u = f(x), \qquad t,x \geq 0. \tag{1.1}$$

We assume that u initially, and f have compact support in $(0,\infty)$. We are seeking conditions at $x = 0$ such that the solution on the semi-bounded domain is close to the solution on the unbounded domain. The first and second order absorbing boundary conditions for transient solutions are, [9], (1.2) $u_x - u_t = 0, \qquad x = 0,$

(1.3) $u_{xt} - u_{tt} - \frac{1}{2} u = 0, \qquad x = 0.$

The transparent boundary condition which is correct for steady solutions is

(1.4) $u_x - u = 0, \qquad x = 0.$

The first and second order new boundary conditions have the form

(1.5) $u_x - u_t - u = 0, \qquad x = 0,$

(1.6) $u_{xt} - u_{tt} + u_x - u_t - u = 0, \quad x = 0.$

The steady solution does not satisfy the equations (1.2) and (1.3) but it satisfies (1.4), (1.5) and (1.6). The solution to the wave equation (1.1) with (1.5) or (1.6) converges as $t \to \infty$ to the steady solution on the unbounded domain. The condition (1.4) gives strong artificial reflections.
In the following section the motivation and the principles for the new boundary conditions are outlined. The derivation is carried out for the dispersive wave equation. This equation is simple enough for a theoretical investigation and complex enough to generate interesting results.

Section 3 contains the special case of one space dimension. Explicit examples of the general principles are given.

The new type of boundary conditions for the dispersive wave equation is analyzed in section 4. Results of well-posedness and convergence to steady state as $t \to \infty$ are established. The representation of the non-local boundary operator is discussed. The results are given in forms of theorems. The proofs, when technical, are only indicated. For the details, see [8].

Extensions to general hyperbolic systems are briefly discussed in section 5, and in the last section results from numerical experiments are presented. Solutions of the one dimensional problem (1.1) with different types of far field boundary conditions are compared.

2. <u>Boundary conditions for the dispersive wave equation</u>. Let us consider the scalar dispersive wave equation in all of R^n,

(2.1a) $u^*_{tt} - \Delta u^* + \alpha^2 u^* = f(x)$ in $R^n x(0,T)$,

(2.1b) $u^*(x,0) = u^0(x)$, $u^*_t(x,0) = u^1(x)$ in R^n.

The functions f, u^0 and u^1 are assumed to have compact support and α is positive.

For the results in section 4 we will need the following regularity assumptions

(2.2) $f \in L^2(R^n)$, $u^0 \in H^1(R^n)$, $u^1 \in L^2(R^n)$,

where $H^1(R^n)$ is the standard Sobolev space.
The problem (2.1) has a unique solution on any interval $(0,T)$ with u^* in $H^1(R^n)$ and u^*_t in $L^2(R^n)$.

For a proof of this classical result see eg. [18].

It is well known that u^* converges locally as $t \to \infty$ to a steady state v^*, $(|u^* - v^*| = O(t^{-n/2}))$. The solution v^* is defined by

(2.3a) $-\Delta v^* + \alpha v^* = f(x)$ in R^n,

(2.3b) $v^* \in H^1(R^n)$.

The result can be found in [19] and is based on Fourier transform and stationary phase arguments.

Let Ω be a bounded convex domain in R^n with a boundary Γ which is at least C^1 and let n denote the exterior normal to Γ. We assume that Ω contains the support of the data. The domain Ω will be our computational domain and we are interested in boundary conditions on Γ such that solutions to the dispersive wave equation on the bounded domain are close to u^* for $x \in \Omega$.

A first order absorbing boundary condition for the wave equation (2.1) can be written as

$$\text{(2.4)} \qquad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} + a(x)u = 0, \qquad x \in \Gamma .$$

For a straight boundary the traditional condition is obtained when a is zero, [9]. For $\Omega = \{x, |x| < R\}$, $a = R^{-1}$ in three dimensions and $a = (2R)^{-1}$ in two. This follows from both [2] and [9] even if the arguments are different in the two papers. If $a(x)$ is any positive reasonably smooth function defined on the boundary Γ , (2.4) yields an absorbing boundary condition, [12]. Furthermore it is well-posed on any finite time interval $(0,T)$ as elementary energy estimates can demonstrate. Nevertheless the solution of the wave equation with a forcing term (2.1a) defined in a bounded domain with the boundary condition (2.4) will not converge to the steady state given by (2.3) as $t \to \infty$. This is obviously due to the fact that the steady state does not in general satisfy the boundary condition

$$\text{(2.5)} \qquad \frac{\partial v}{\partial n} + a(x)v = 0, \qquad x \in \Gamma .$$

The solution of the wave equation (2.1a) coupled with any boundary condition of the form

$$\text{(2.6)} \qquad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} + Bu = 0, \qquad x \in \Gamma$$

for a wide class of operators B converges as $t \to \infty$ to the solution of (2.3a) in Ω , coupled with the boundary condition

$$\text{(2.7)} \qquad \frac{\partial v}{\partial n} + Bv = 0, \qquad x \in \Gamma.$$

Therefore u can converge to v^* only if v^* satisfies (2.7). This proves that the only useful boundary condition on the form (2.6) is

$$\text{(2.8)} \qquad \frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} + Ku = 0, \qquad x \in \Gamma,$$

where

$$\text{(2.9)} \qquad \frac{\partial v}{\partial n} + Kv = 0, \qquad x \in \Gamma,$$

is the transparent boundary condition for the steady state v^*.

For the explicit form of the new boundary condition in the one dimensional case, see the following section and (1.5). The multi-dimensional case is discussed together with the analysis in section 4.

The boundary condition (2.8) is local in time and for dimensions $n > 1$ it is global in space. The nonlocal operator K can be approximated by local operators based on far field asymptotic expansions [1],[2]. As an example with $n=2$ and $\Omega = \{x, |x|<R\}$ we have the approximation

$$(2.10) \qquad K \approx \alpha + \frac{1}{2R} ,$$

from the first term in a far field expansion of the K_0 Bessel function. For $n = 3$ we have $K \approx \alpha + \frac{1}{R}$.

It is possible to improve the accuracy of (2.8) by a higher order approximation in time. This can be done by composing a higher order absorbing boundary condition for transient solutions with the transparent boundary condition (2.9). See theorem 5.5 in section 4 and for the one dimensional case the derivation of (3.9).

3. <u>The dispersive wave equation in one space dimension</u>. We shall consider the wave equation

(3.1a) $$u^*_{tt} - u^*_{xx} + \alpha^2 u^* = f(x), \qquad \text{in } R\times(0,T),$$

(3.1b) $$u^*(x,0) = u^0(x), \quad u^*_t(x,0) = u^1(x), \quad \text{in } R.$$

All the data u^0, u^1 and f are assumed to be compactly supported in an interval (a,b). The constant α is positive.

The equation (3.1) has a unique solution if u^0 is in H^1 and if u^1, f are in L^2. We shall assume enough smoothness on the data for the higher order boundary conditions to be well defined.

<u>Remark 3.1</u>. Equation (3.1) can be regarded as the Fourier transform of the multidimensional wave equation. The constant α^2 does then contain the dual variables to the coordinate directions tangent to the computational boundary.

Let us introduce the steady problem

(3.2a) $$-v^*_{xx} + \alpha^2 v^* = f(x), \qquad \text{in } R$$

(3.2b) $$v^* \to 0, \quad x \to \pm\infty \quad .$$

It is well known that u^* converges to v^* on any bounded interval as $t \to \infty$, [19]. Since f has support in (a,b) it is easy to see that on (a,b) v^* is the solution of

(3.3a) $$-v_{xx} + \alpha^2 v = f(x), \qquad 0 \le x \le b,$$

(3.3b) $$S_a v = -v_x + \alpha v = 0, \qquad x = a$$

(3.3c) $$S_b v = v_x + \alpha v = 0, \qquad x = b.$$

The boundary conditions (3.3bc) correspond to the general transparent condition (2.9) in section 2. The operator K is here αI.

We are interested in solving the original equation (3.1) on the bounded interval (a,b) with boundary conditions such that the new solution is a good approximation of the solution u^*. We also want the new solution to converge to v as $t \to \infty$. Let us write the new problem as follows,

(3.4a) $$u_{tt} - u_{xx} + \alpha^2 u = f(x), \qquad a \le x \le b, \quad t \ge 0$$

(3.4b) $$u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), \qquad 0 \le x \le b$$

(3.4c) $$B_a u = 0, \qquad x = a$$

(3.4d) $$B_b u = 0, \qquad x = b.$$

Following (2.8) the new first order far field boundary conditions are given by the following definitions of the operators B_a and B_b,

(3.5a) $$B_a^1 u = u_x - u_t - \alpha u = 0, \qquad x = a$$

(3.5b) $$B_b^1 u = u_x + u_t + \alpha u = 0, \qquad x = b.$$

We can compare with the standard first order absorbing boundary conditions [9]

(3.6a) $$A_a^1 u = u_x - u_t = 0, \qquad x = a,$$

(3.6b) $$A_b^1 u = u_x + u_t = 0, \qquad x = b,$$

and the second order absorbing boundary condition

(3.7a) $$A_a^2 u = u_{xt} - u_{tt} - \frac{\alpha^2}{2} u = 0, \qquad x = a,$$

(3.7b) $$A_b^2 u = u_{xt} + u_{tt} + \frac{\alpha^2}{2} u = 0, \qquad x = b.$$

These boundary conditions can be derived as approximations of the nonlocal transparent boundary conditions which are based on a Fourier transform in time [9]. From (3.1) we have

$$\hat{u}_{xx} - (\alpha^2-\omega^2)\hat{u} = 0,$$

with the relevant boundary conditions

(3.8a) $\hat{u}_x - (\alpha^2-\omega^2)^{1/2}\hat{u} = 0, \qquad x = a,$

(3.8b) $\hat{u}_x + (\alpha^2-\omega^2)^{1/2}\hat{u} = 0, \qquad x = b,$

$$(\alpha^2-\omega^2)^{1/2} = \begin{cases} \sqrt{\alpha^2-\omega^2}, & \alpha^2 \geq \omega^2, \\ i\omega\sqrt{1-(\alpha/\omega)^2}, & \alpha^2 < \omega^2. \end{cases}$$

The conditions (3.6) and (3.7) are first and second order approximations respectively of (3.8) in the limit of $\omega^2 \to \infty$.

Note that the steady boundary conditions in (3.3) can be seen as an approximation of (3.8) for ω^2 small.

Higher order far field conditions (3.4c,d) can be regarded as approximation of (3.8) for the full range of ω. They should also be exact for $\omega = 0$. One way to achieve this is to derive the higher order boundary operators as compositions of the lower order ones, [13].

The operator A^2 can be derived from $A^1 \cdot A^1$ if the differential equation is used to define the quantity

$$\frac{\partial}{\partial x} u_x = u_{tt} - \alpha^2 u, \qquad x = a,b.$$

For $x = a$ we have, see [13],

$$(\frac{\partial}{\partial x} - \frac{\partial}{\partial t})^2 u = \frac{\partial}{\partial x} u_x - 2u_{xt} - u_{tt} =$$

$$= -\frac{1}{2}(u_{xt} - u_{tt} - \frac{\alpha^2}{2} u) = -\frac{1}{2} A^2 u.$$

Analogously we can get higher order conditions of the new type by compositions on the form $(A^1)^m S$. For $m=1$ we have the second order far field boundary operator B^2. For $x=a$ we get

$$(\frac{\partial}{\partial x} - \frac{\partial}{\partial t})(-\frac{\partial}{\partial x} + \alpha)u = -\frac{\partial}{\partial x}u_x + u_{xt} + \alpha u_x - \alpha u_t$$

$$= u_{xt} - u_{tt} + \alpha u_x - \alpha u_t - \alpha^2 u.$$

The second order new boundary conditions are thus

$$(3.9a) \qquad u_{xt} - u_{tt} + \alpha u_x - \alpha u_t - \alpha^2 u = 0, \qquad x = a,$$

$$(3.9b) \qquad u_{xt} + u_{tt} + \alpha u_x + \alpha u_t + \alpha^2 u = 0, \qquad x = b.$$

The behaviour of the different computational boundary conditions can be studied in numerical experiments, see section 6. Another useful quantitative study is to compare the dispersion relations of the computational boundary conditions to the exact relations (3.8). This is done in figures 1 and 2. The dual with respect to x is denoted by k. It is clear from the figures that the new boundary conditions based on B^1 and B^2 give good approximations to the exact conditions for a wide range of ω.

The behaviour of the boundary conditions can also be studied qualitatively in terms of well-posedness and convergence to the correct steady state as $t \to \infty$. Let us refer to section 4 and to [8] for this type of analysis of the first order conditions in the general multidimensional case.

We shall finish this section with an energy estimate for the second order boundary conditions. Let us first write problem (3.4) with the boundary operators B_a^2, B_b^2 on variational form.

<u>Lemma 3.1</u>. If w is a solution of (3.4ab) with $f \equiv 0$ and with the boundary condition (3.9) then for any ϕ belonging to $H^1(a,b)$ we have,

$$(3.10) \quad (w_{tt},\phi)+\alpha(w_{tt},\phi)+[w_{tt},\phi]_a^b+(w_{xt},\phi_x)+\alpha(w_x,\phi_x)+\alpha^2(w_t,\phi)$$

$$+ \alpha[w_t\,\phi]_a^b+\alpha^3(w,\phi)+\alpha^2[w\phi]_a^b = 0.$$

Proof: We first integrate equation (3.4a) against a test function ϕ and integrate by parts,

(3.11) $$(w_{tt},\phi) + (w_x,\phi_x) + \alpha^2(w,\phi) - [w_x\phi]_a^b = 0.$$

We then differentiate (3.4a) in time and integrate it against ϕ :

(3.12) $$(w_{ttt},\phi) + (w_{xt},\phi_x) + \alpha^2(w_t,\phi) - [w_{xt}\phi]_a^b = 0.$$

We multiply (3.11) by α and add the resulting equality to (3.12),

$$(w_{ttt},\phi)+\alpha(w_{tt},\phi)+(w_{xt},\phi_x)+\alpha(w_x,\phi_x)+\alpha^2(w_t,\phi)+\alpha^3(w,\phi) = [\alpha w_x+w_{xt},\phi]_a^b .$$

On the boundary, $\alpha w_x + w_{xt}$ is equal to $\pm\,(w_{tt} + \alpha^2 w + \alpha w_t)$ and we get (3.10).

We are now able to prove an a priori estimate. The boundary $x=a,b$ is here denoted by Γ, $|u|_\Gamma^2 = [u^2]_b^a$.

Theorem 3.1: If u is a solution of (3.4) with the second order boundary condition (3.9), we have

(3.13) $$E(u,t) + \alpha|u|_\Gamma^2 \le \frac{C(1+\alpha^2)}{\alpha^2}\,(\|u^0\|_{H^2} + \|u^1\|_{H^1} + \|f\|_{L^2})$$
$$E(u,t) = \|u_t\|^2 + \|u_x\|^2 + \alpha^2\|u\|^2.$$

Proof : Let v be the steady solution defined by (3.3) and let w be the difference, $w = u - v$. We first choose $\phi = w_{tt}$ in (3.10) and using the identity

$$g g_{tt} = \frac{1}{2}\frac{d^2}{dt^2} g^2 - g_t^2$$

for any function $g(t)$, we get:

(3.14) $$\frac{1}{2}\frac{d^2}{dt^2}(\alpha\|w_x\|^2+\alpha^3\|w\|^2+\alpha^2|w|_\Gamma^2)+\frac{1}{2}\frac{d}{dt}(\|w_{tt}\|^2+\|w_{xt}\|^2+\alpha^2\|w_t\|^2+\alpha|w_t|_\Gamma^2)$$
$$+ \|w_{tt}\|^2+|w_{tt}|_\Gamma^2-\alpha\|w_{xt}\|^2-\alpha^3\|w_t\|^2-\alpha^2|w_t|_\Gamma^2 = 0.$$

We now take $\phi = w_t$, so that

$$(3.15)\quad \frac{1}{2}\frac{d^2}{dt^2}\|w_t\|^2+\frac{1}{2}\frac{d}{dt}[\alpha\|w_t\|^2+|w_t|_\Gamma^2+\alpha\|w_x\|^2+\alpha^3\|w\|^2+\alpha^2|w|_\Gamma^2]$$

$$-\|w_{tt}\|^2+\|w_{xt}\|^2+\alpha^2\|w_t\|^2+\alpha|w_t|_\Gamma^2 = 0.$$

We multiply (3.15) by α and add to (3.14):

$$\frac{\alpha}{2}\frac{d^2}{dt^2}[\|w_x\|^2+\|w_t\|^2+\alpha^2\|w\|^2+\alpha|w|_\Gamma^2]$$

$$+\frac{1}{2}\frac{d}{dt}[\|w_{tt}\|^2+\|w_{xt}\|^2+2\alpha^2\|w_t\|^2+2\alpha|w_t|_\Gamma^2+\alpha^2\|w_x\|^2+\alpha^4\|w\|^2+\alpha^3|w|_\Gamma^2]$$

$$+|w_{tt}|_\Gamma^2 = 0$$

This can be rewritten by defining an energy which takes the boundary into account.

$$(3.16)\quad E_\Gamma(w,t) = E(w,t) + \alpha|w|_\Gamma^2,$$

$$(3.17)\quad \frac{\alpha}{2}\frac{d^2E_\Gamma}{dt^2}(w,t) + \frac{1}{2}\frac{d}{dt}[E_\Gamma(w_t,t)+\alpha^2E_\Gamma(w,t)] + |w_{tt}|_\Gamma^2 = 0.$$

We integrate this expression in time and get,

$$\frac{\alpha}{2}\frac{d}{dt}E_\Gamma(w,t) + \frac{1}{2}E_\Gamma(w_t,t) + \frac{\alpha^2}{2}E_\Gamma(w,t) + \int_0^t |w_{tt}|_\Gamma^2 =$$

$$= \frac{\alpha}{2}\frac{d}{dt}E_\Gamma(w,0) + \frac{1}{2}E_\Gamma(w_t,0) + \frac{\alpha^2}{2}E_\Gamma(w,0).$$

We can then bound $\frac{d}{dt}E_\Gamma(w,t) + \alpha E_\Gamma(w,t)$ by an expression E_w, negative or positive, depending only on w at time $t = 0$,

$$(3.18)\quad \frac{d}{dt}E_\Gamma(w,t) + \alpha E_\Gamma(w,t) \le E_w .$$

$$E_w = \frac{d}{dt}E_\Gamma(w,0) + \frac{1}{\alpha}E_\Gamma(w_t,0) + \alpha E_\Gamma(w,0).$$

If the expression E_w is negative, the energy is decreasing in time. If the expression is positive, (3.18) can be rewritten as

$$\frac{d}{dt}(E_\Gamma(w,t)e^{\alpha t}) \le E_w e^{\alpha t}$$

and integrated in time, so that

$$E_\Gamma(w,t) \le \frac{E_w}{\alpha}(1 - e^{-\alpha t}) + E_\Gamma(w,0).$$

The estimate (3.13) can now be derived from $u = w+v$, the inequalities for w above and the following estimate for v,

$$\|v_x\|^2 + \frac{\alpha^2}{2}\|v\|^2 + |v|_\Gamma^2 \le \frac{1}{2\alpha^2}\|f\|^2 .$$

This inequality is immediate from multiplying (3.3a) by v and integrating by parts.

4. Analysis of the new boundary conditions. Let us consider the dispersive wave equation (2.1) with a general boundary condition of the form (2.6)

(4.1a) $$u_{tt} - \Delta u + \alpha^2 u = f(x), \qquad \text{in } \Omega \times R_+,$$

(4.1b) $$u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), \quad \text{in } \Omega,$$

(4.1c) $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} + Bu = 0, \qquad \text{on } \Gamma \times R_+,$$

where B is a boundary operator such that

(4.2a) $$B \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma)),$$

(4.2b) $$\forall \phi \in H^{1/2}(\Gamma),\ < B\ \phi\ ,\ \phi >_\Gamma\ \geq 0.$$

See [18] for the notation. Let v be the solution of the associated steady problem

(4.3a) $$-\Delta v + \alpha^2 v = f(x) \quad \text{in } \Omega,$$

(4.3b) $$\frac{\partial v}{\partial n} + Bv = 0 \qquad \text{on } \Gamma.$$

The results established here are slight modifications of those proved in [4]. Let us first state the well-posedness of the steady problem.

Theorem 4.1: For any f belonging to $L^2(\Omega)$, problem (4.3) has a unique solution v in $H^1(\Omega)$, and the following estimate holds

(4.4) $$||\nabla v||^2 + \frac{\alpha^2}{2}\ ||v||^2 + \quad <Bv,v>_\Gamma \quad \leq \frac{1}{2\alpha^2}\ ||f||^2.$$

This result is an easy consequence of the Lax-Milgram theorem. —

Let w be equal to $u - v$. The difference w is a solution to the homogeneous wave equation:

$$w_{tt} - \Delta w + \alpha^2 w = 0 \qquad \text{in} \quad \Omega \times R_+$$

$$w(x,0) = u^0 - v, \quad w_t(x,0) = u^1 \qquad \text{in } \Omega$$

$$w_t + \frac{\partial w}{\partial n} + Bw = 0.$$

An easy consequence of semi-group theory is the well-posedness of the problem above, and thus of (4.1).

Theorem 4.2: If u^0 belongs to $H^1(\Omega)$ and u^1 belongs to $L^2(\Omega)$, with a support strictly included in Ω, problem (4.1) has a unique solution such that

$$u(.,t) \in H^1(\Omega), \; u_t(.,t) \in L^2(\Omega).$$

We define the energy in a classical way

(4.5) $$E(w)(t) = ||\nabla w||^2 + \alpha^2||w||^2 + < Bw, w >_\Gamma + ||w_t||^2.$$

Theorem 4.3: With the assumptions of theorem 4.2, the solution of problem (4.1) converges exponentially as $t \to \infty$ to the steady state defined in (4.3): There exist two constants M and μ such that

(4.6) $$E(u-v)(t) \leq M\, e^{-\mu t} E(u^0-v,u^1) \qquad \forall t \geq 0$$

$$(E(u^0-v,u^1) = ||\nabla(u^0-v)||^2 + \alpha^2||u^0-v||^2 + <B(u^0-v),u^0-v>_\Gamma + ||u^1||^2).$$

The proof of this theorem is a simple extension of one due to Chen when $\alpha = 0$ and B is a real positive number, see[4] . This proof is rather technical. It is based on the use of multipliers introduced by K. Morawetz, see [19] and on the equivalence for $B = 0$ between exponential decay and exact controllability for the wave equation under Neumann action.

It is now clear that the only boundary condition (4.1c) which can force the solution u of (4.1) to converge to the steady-state v^* is the one with B such that v^* satisfies (4.3b). It will be our new boundary condition.

Let us now consider the transparent boundary conditions for v^*. Our purpose is to derive and study two equivalent formulations of the new boundary conditions. We shall work in the space H which is the Sobolev space $H^1(\Omega)$ equipped with the following scalar product which is equivalent to the usual one

(4.7) $$(u_1,u_2)_H = \alpha^2(u_1,u_2) + (\nabla u_1,\nabla u_2).$$

The usual scalar product in $L^2(\Omega)$ is denoted by $(\cdot,\cdot)$. We shall write the relation satisfied by v^* on the boundary explicitly. For that purpose we introduce the boundary operator K,

(4.8) $$K : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$
$$K\phi = -\frac{\partial w}{\partial n},$$

where w is the solution of the exterior steady problem in Ω^C

(4.9a) $$-\Delta w + \alpha^2 w = 0 \quad \text{in} \quad \Omega^C$$

(4.9b) $$w = \phi \;; \quad x \in \Gamma .$$

The problem above is obviously well posed (using the Lax Milgram theorem). The operator K is then defined as

$$\forall \psi \in H^{1/2}(\Gamma) \; < K\phi,\psi >_\Gamma = (w,\tilde{w})_H$$

where $\tilde{w}$ is any extension of ψ in $H^1(\Omega^C)$. In theorem 4.4 we shall see that K is the transparent boundary operator. We shall first need the following result.

Lemma 4.1: K is a continuous coercive operator on $H^{\frac{1}{2}}(\Gamma)$

(4.10) $$< K\phi, \phi >_\Gamma \geq C||\phi||^2_{H^{1/2}(\Gamma)}$$

Proof:

$$||K\phi||_{H^{-1/2}(\Gamma)} = \sup_{\psi\in H^{1/2}(\Gamma)} \frac{<K\phi,\ \psi>_\Gamma}{||\psi||_{H^{1/2}(\Gamma)}} = \sup_{\psi\in H^{1/2}(\Gamma)} \frac{(w,\tilde{w})_H}{||\psi||_{H^{1/2}(\Gamma)}} .$$

Here $\tilde{w}$ is any extension of ψ to $H^1(\Omega^c)$ such that

$$||\tilde{w}||_H \leq C||\psi||_{H^{1/2}(\Gamma)}$$

We thus have:

$$||K\phi||^2_{H^{-1/2}(\Gamma)} \leq C||w||^2_H \equiv C < K\phi,\ \phi >_\Gamma$$

$$||K\phi||^2_{H^{-1/2}(\Gamma)} \leq C||K\phi||_{H^{-1/2}(\Gamma)}||\phi||_{H^{1/2}(\Gamma)}.$$

Moreover,

$$< K\phi,\phi >_\Gamma = ||w||^2_H \geq C||\phi||^2_{1/2,\Gamma}$$

the constant $C = \min(1,\alpha^2)$.
The operator K defines the transparent boundary condition for the steady state.

Theorem 4.4: The problem

(4.11a) $\quad -\Delta v + \alpha^2 v = f(x) \quad$ in Ω,

(4.11b) $\quad \frac{\partial v}{\partial n} + Kv = 0 \quad$ on Γ,

has a unique solution v in $H^1(\Omega)$. It is the restriction of v^* to Ω. Moreover the following estimate holds, $(C = \min(1,\alpha^2))$,

(4.12) $\quad ||v||^2_H + C|v|^2_\Gamma \leq ||v||^2_H + < Kv,v>_\Gamma \leq \frac{1}{\alpha^2}\ ||f||^2$

The boundary condition (4.1c) with $B = K$ is then our new far field boundary condition. This boundary condition can be formulated as a composition of an absorbing boundary condition of type (2.4) and the transparent boundary condition for the steady state (4.11b).

Theorem 4.5: If u satisfies the wave equation (4.1a), we have on the boundary Γ :

$$(4.13) \qquad \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial n} + \frac{\partial u}{\partial t} + Ku\right) = \left(\frac{\partial}{\partial n} + \frac{\partial}{\partial t} + (n-1)S\right)\left(\frac{\partial u}{\partial n} + Ku\right)$$

where S is the mean curvature on Γ .

The proof will not be given here and can be found in [8]. It is technical and relies on some results in differential geometry. the main difficulty is to define $\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n} + Ku\right)$, since $\frac{\partial u}{\partial n} + Ku$ is defined only on the boundary. Let us recall that

$$\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n} + Ku\right) = \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n} - \frac{\partial w}{\partial n}\right), \quad \text{where } w \text{ is}$$

the solution of the exterior problem

$$(4.14a) \qquad -\Delta w + \alpha^2 w = 0 \quad \text{in} \quad \Omega^C,$$

$$(4.14b) \qquad w = u, \quad x \in \Gamma .$$

The expression $\frac{\partial}{\partial n}\left(\frac{\partial u}{\partial n} - \frac{\partial w}{\partial n}\right)$ can be defined as $\frac{\partial^2 u}{\partial n^2} - \frac{\partial^2 w}{\partial n^2}$ and formulated in terms of the tangential and time derivatives using the differential equations (4.1a),(4.14a). A Green formula on the boundary then leads to (4.13).

Due to the compact support of the initial values the two sides of (4.13) are equivalent to

$$(4.15) \qquad \frac{\partial u}{\partial n} + \frac{\partial u}{\partial t} + Ku = 0.$$

Remark 4.1: The absorbing boundary condition in [9] and [2] differs from the one used to build (4.13), (4.15) by the lower order term. It is the same for flat boundaries.

We thus have derived an initial boundary value problem

(4.16a) $$\frac{\partial^2 u}{\partial t^2} - \Delta u + \alpha^2 u = f(x) \quad \text{in} \quad \Omega \times [0,T],$$

(4.16b) $$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) \quad \text{in} \quad \Omega$$

(4.16c) $$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial n} + Ku = 0 \quad \text{on} \quad \Gamma \quad ,$$

the solution of which converges exponentially to the steady state. Unfortunately the boundary condition is based on K, an integral operator along the boundary. Except in some very special cases the operator K is not known explicitely. If Γ is a hyperplane in R^n or if Γ is a sphere, K can be decomposed by Fourier integrals or Fourier series along Γ. For example, in the two dimensional case of a circle the dependence in r can be expressed in Bessel functions. These techniques have been used for stationary problems in connection to both finite element, [16], and finite difference methods, [11].

The new boundary condition can also be expressed in a variational formulation. This formulation can then be the basis for a numerical finite element and boundary integral method. We shall follow the technique by Johnson and Nedelec [15] for stationary problems. They formulate a problem whose solution is u and its normal derivative on the boundary.

If u is the solution of (4.16) we set

$$\lambda = \frac{\partial u}{\partial n} .$$

Multiplying (4.16a) by a test function w in $H^1(\Omega)$, integrating by parts and using the boundary condition (4.16c) yields

(4.17) $$(u_{tt},w)+\langle u_t,w\rangle_\Gamma+(u,w)_H+\langle\lambda,w\rangle_\Gamma=(f,w).$$

We now use the exterior problem and λ is then defined as $-\frac{\partial\tilde{u}}{\partial n}$, where $\tilde{u}$ is the solution to (4.9) with $\tilde{u} = u$ on Γ. Thus $\tilde{u}$ can be expressed on the boundary by, see [7],

(4.18) $$\frac{1}{2}\, u(x) = \int_\Gamma u(y) G_n(x,y)\, d\gamma_y - \int_\Gamma \lambda(y) G(x,y)\, d\gamma_y,$$

where G is the Green function for $-\Delta + \alpha^2$, and G_n is

(4.19) $$G_n(x,y) = \frac{\partial}{\partial n_y} G(x,y).$$

Let us recall that G is given in two dimensions by

(4.20) $$G(x,y) = \frac{1}{2\pi} K_0(\alpha|x-y|),$$

where K_0 is a modified Bessel function, and in three dimensions by,

(4.21) $$G(x,y) = \frac{1}{2\pi} \frac{e^{-\alpha|x-y|}}{|x-y|}.$$

We formally apply $\mu \in H^{-1/2}(\Gamma)$ to (4.17) and since $\tilde{u} = u$ on Γ we get

(4.22) $$b(\lambda,\mu) + \langle G_n u, \mu\rangle_\Gamma - \frac{1}{2}\langle u,\mu\rangle_\Gamma = 0,$$

where

(4.23) $$b(\lambda,\mu) = -\int_\Gamma \int_\Gamma G(x,y)\lambda(y)\mu(x)\, d\gamma_x d\gamma_y,$$

(4.24) $$G_n u(x) = \int_\Gamma u(x) G_n(x,y)\, d\gamma_y.$$

We thus couple (4.17) and (4.22) and define a variational formulation as:

Find (u,λ) such that

$$u \in L^\infty(0,T,H^1(\Omega)),\ u_t \in L^\infty(0,T,L^2(\Omega)),\ \lambda \in L^\infty(0,T,H^{-1/2}(\Gamma))$$

(4.25a) $(u_{tt},w) + \langle u_t,w\rangle_\Gamma + (u,w)_H + \langle\lambda,w\rangle_\Gamma = (f,w), \quad \forall w \in H^1(\Omega),$

(4.25b) $b(\lambda,\mu) + \langle G_n u,\mu\rangle_\Gamma - \frac{1}{2}\langle\mu,w\rangle_\Gamma = 0 \qquad \forall\mu \in H^{-1/2}(\Gamma)$

with the initial values

(4.25c) $\quad u(0) = u^0, \qquad u_t(0) = u^1.$

We recall that b is bilinear, continuous and coercive on $H^{-1/2}(\Gamma)$ [15]. We have the following well-posedness result.

Theorem 4.6: If u^0, u^1, f have the regularity prescribed in (2.2) the variational problem (4.25) has a unique solution.

Proof: The existence follows from theorem 4.2. For the uniqueness, let us assume that $u^0 = u^1 = f = 0$. From (4.25a) we deduce that

$$u_{tt} - \Delta u + \alpha^2 u = 0 \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial n} + \frac{\partial u}{\partial t} + \lambda = 0 \quad \text{on } \Gamma.$$

Let $\tilde{u}$ be the extension of u in Ω^C such that

$$-\Delta\tilde{u} + \alpha^2\tilde{u} = 0 \quad \text{in } \Omega^C,$$

$$\tilde{u} = u, \quad x \in \Gamma,$$

and let $\tilde{\lambda}$ be defined by $\tilde{\lambda} = -\frac{\partial\tilde{u}}{\partial n}$.

We can then write

$$b(\tilde{\lambda},\mu) + \langle G_n u,\mu\rangle_\Gamma - \frac{1}{2}\langle\mu,u\rangle_\Gamma = 0, \qquad \forall\mu \in H^{-1/2}(\Gamma)$$

which implies that $b(\lambda-\tilde{\lambda},\mu) = 0$ for each μ in $H^{-1/2}(\Gamma)$, and thus

$\lambda = \tilde{\lambda}$, due to the coercivenes of b. We now have

(4.26) $$u_{tt} - \Delta u + \alpha^2 u = 0 \qquad \text{in } \Omega ,$$

(4.27) $$u_t + \frac{\partial u}{\partial n} - \frac{\partial \tilde{u}}{\partial n} = 0 \qquad \text{on } \Gamma ,$$

$$u = \tilde{u} \qquad \text{on } \Gamma ,$$

(4.28) $$-\Delta \tilde{u} + \alpha^2 \tilde{u} = 0 \qquad \text{in } \Omega^C.$$

We multiply (4.26) by u_t and integrate on Ω. We get, due to the boundary condition (4.27)

(4.29) $$\frac{1}{2}\frac{d}{dt}(\|u_t\|^2 + \|u\|_H^2) + \int_\Gamma u_t^2 - \int_\Gamma \frac{\partial \tilde{u}}{\partial n} u_t = 0.$$

We now multiply (4.28) by $\tilde{u}_t$ and integrate on Ω^C

(4.30) $$\frac{1}{2}\frac{d}{dt}(\|\tilde{u}\|^2_{H^1(\Omega^C)}) + \int_\Gamma \frac{\partial \tilde{u}}{\partial n} \tilde{u}_t = 0.$$

On the boundary $u_t = \tilde{u}_t$. We can thus add (4.29) and (4.30) and conclude that u vanishes in Ω and u in Ω^C.

Theorem 4.6 shows that the variational solution is the one from theorem 4.2.

5. Hyperbolic systems. Let us first consider the absorbing boundary conditions for a hyperbolic system on the form

(5.1a) $$u_t + Au_x + Bu = f(x) \qquad \text{in } R\times R_+,$$

(5.1b) $$u(x,0) = u^0(x) \qquad \text{in } R,$$

where the matrix A has distinct eigenvalues $\neq 0$. Assume the supports of u^0 and f are in (a,b) and let T_0 diagonalize A,

(5.2) $$T_0AT_0^{-1} = D, \qquad D = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix}$$

$$D_{11} = \text{diag}(d_j), \quad d_j > 0,$$

$$D_{22} = \text{diag}(d_j), \quad d_j < 0.$$

In the characteristic variables $v = T_0u$ the system can be written

(5.3) $$v_x + D^{-1}v_t + D^{-1}\tilde{B}v = D^{-1}\,T_0f(x),\ \tilde{B} = T_0\,BT_0^{-1}$$

We restrict the domain of x in (5.1) to (a,b). The first order absorbing boundary conditions are then [9],

(5.4a) $$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T_0u = P_a\,T_0\,u = 0,\ x = a,$$

(5.4b) $$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} T_0u = P_b\,T_0\,u = 0,\ x = b.$$

The projection matrices are here partitioned as the matrix D in (5.2).

The higher order absorbing boundary conditions are derived from the Fourier transform of (5.3),

(5.5) $$\hat{v}_x + i\omega(D^{-1} + \frac{1}{i\omega}D^{-1}\tilde{B})\hat{v} = D^{-1}\,T_0\hat{f},$$

using a diagonalizing transformation T of the coefficient matrix in (5.5),

$$T(\omega)^{-1}(D^{-1} + \frac{1}{i\omega} D^{-1}B)T(\omega) = \begin{pmatrix} D_{11}(\omega) & 0 \\ 0 & D_{22}(\omega) \end{pmatrix},$$

and a high frequency expansion

(5.6) $$T(\omega) = T_0 + T_1 \frac{1}{i\omega} + 0\left(\frac{1}{|\omega|^2}\right).$$

The second order absorbing boundary conditions are

(5.7a) $$P_a\left(T_0 \frac{\partial}{\partial t} + T_1\right)u = 0, \qquad x = a,$$

(5.7b) $$P_b\left(T_0 \frac{\partial}{\partial t} + T_1\right)u = 0, \qquad x = b.$$

Remark 5.1: The boundary conditions can also be applied in the integral form

$$(\text{Projection}) \cdot \left(T_0 + \int_0^t T_1\right)u = 0.$$

Let us now turn to the steady problem

(5.8a) $$Au_x + Bu = f(x), \qquad \text{in } R,$$

(5.8b) $$u \to 0, \quad x \to \pm\infty .$$

Again we diagonalize the system

(5.9a) $$u_x + A^{-1}Bu = A^{-1}f,$$

(5.9b) $$w_x + Rw = \tilde{f}.$$

The function w is given by $w = Su$,

$$S A^{-1}B S^{-1} = R = \begin{pmatrix} R_{11} & 0 \\ 0 & R_{22} \end{pmatrix},$$

$$R_{11} = \text{diag}(r_j), \qquad r_j > 0,$$

$$R_{22} = \text{diag}(r_j), \qquad r_j < 0.$$

The matrix $A^{-1}B$ is assumed to have distinct eigenvalues $\neq 0$.

The following boundary conditions for the steady problem on the bounded domain (a,b) are satisfied by the steady solution on the unbounded domain

(5.10a) $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} S\, u = S_a\, u = 0, \qquad x = a,$

(5.10b) $\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} S\, u = S_b\, u = 0, \qquad x = b.$

for the decay condition (5.8b) to be valid. This is clear since the general solution of (5.9) outside the support of f is

$$w(x) = \begin{pmatrix} e^{-R_{11}(x-a)} w_1(a) \\ e^{-R_{22}(x-a)} w_2(a) \end{pmatrix}, \qquad x \leq a,$$

$$w(x) = \begin{pmatrix} e^{-R_{11}(x-b)} w_1(b) \\ e^{-R_{22}(x-b)} w_2(b) \end{pmatrix}, \qquad x \geq b,$$

where $w = (w_1,w_2)^T$ is partitioned in the same way as R.

Following the derivation of the new boundary conditions for the scalar wave equation we get the first order far field boundary conditions from a combination of the absorbing and steady boundary conditions

(5.11a) $P_a\,(T_0 \frac{\partial}{\partial t} + S_a)u = 0, \qquad x = a$

(5.11b) $P_b\,(T_0 \frac{\partial}{\partial t} + S_b)u = 0, \qquad x = b.$

Note that (5.11) is only defined up to a factor in front of S_a and S_b. The boundary conditions for the Euler equations in [21] are of a similar type.

Higher order boundary conditions can formally be derived analogously

(5.12a) $P_a\,(T_0 \frac{\partial^2}{\partial t^2} + T_1 \frac{\partial}{\partial t} + S_a)u = 0, \quad x = a,$

(5.12b) $$P_b \left(T_0 \frac{\partial^2}{\partial t^2} + T_1 \frac{\partial}{\partial t} + S_b\right)u = 0, \qquad x = b.$$

The well-posedness of the new initial-boundary value problems follows from the form of the leading term in the boundary conditions. The ingoing characteristic quantities are set to zero. Steady solutions to (5.1) with the boundary conditions (5.11) or (5.12) clearly satisfy the steady equation (5.8) for $0 \leq x \leq b$. General criteria for convergence to steady state will not be studied here. Instead we shall consider two explicit examples.

Example 5.1: Consider the simple 2×2-system

(5.13a) $$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix},$$

(5.13b) $$u(x,0) = u^0(x), \quad v(x,0) = v^0(x),$$

where u^0, v^0, f and g have compact support in $(0,1)$. The exact solution is given by

(5.14a) $$v(x,t) = e^{-t}\left(v^0(x+t) + \int_0^t e^{\tau} g(x+t-\tau)\,d\tau\right)$$

(5.14b) $$u(x,t) = e^{-t}\left(u^0(x-t) + \int_0^t e^{\tau}\left(f(x-t+\tau) - v(x-t+\tau,\tau)\right) d\tau\right).$$

The first order absorbing boundary conditions for the restriction to the interval $0 \leq x \leq 1$ are

(5.15a) $$u = 0, \qquad x = 0$$

(5.15b) $$v = 0, \qquad x = 1.$$

The steady or transparent boundary conditions are given by the reduced equation

$$\frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix},$$

and the matrix S transforming the system to diagonal form

$$S = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix},$$

which implies,

(5.16a) $u + 1/2\, v = 0, \qquad x = 0,$

(5.16b) $v = 0, \qquad x = 1.$

The new boundary condition at $x = 0$ combines equations (5.15a) and (5.16a) in analogy to (5.11a)

(5.17a) $u_t + u + 1/2\, v = 0, \qquad x = 0.$

Since (5.15b) is the same as (5.16b) we can use it as the right boundary condition

(5.17b) $v = 0, \qquad x = 1.$

In this case, conditions on the form (5.7b): $v_t + \alpha v = 0$ are equivalent to (5.17b) for any α.

It is easy to see that the initial-boundary value problem (5.13) with the new boundary conditions (5.17) has a solution which converges as $t \to \infty$, in $0 \le x \le 1$, to the solution of the steady unbounded problem. The second component v is independent of u so (5.17b) and thus v are exactly given by (5.14a) for all $t \geq 0$. The solution of (5.17a) will then converge to the correct boundary value for u as $t \to \infty$ and hence u will also converge to the steady solution of the original problem.

For this problem it is possible to show the limit as $t \to \infty$ in (5.14),

(5.18a) $$v(x) = \int_x^{\infty} e^{x-\xi} g(\xi)\, d\xi,$$

(5.18b) $$u(x) = \int_{-\infty}^{x} e^{\xi-x} (f(\xi) - v(\xi))\, d\xi.$$

Remark 5.2. For the example above it is possible to show by explicit calculation that the boundary condition (5.17a) is satisfied by the solution (5.14).

Example 5.2. Let us consider the dispersive wave equation and study the connection between the derivation in this section and the earlier results. The wave equation

$$u_{tt} - u_{xx} + u = f(x) \tag{5.19}$$

can be written as a system

$$u_t + v_x + w = 0 \tag{5.20a}$$

$$v_t + u_x = 0 \tag{5.20b}$$

$$w_t - u = -f. \tag{5.20c}$$

With $U = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 \\ 0 \\ -f \end{pmatrix}$,

we have

$$U_t + AU_x + BU = F. \tag{5.21}$$

Note that this system is not on the standard form discussed above. The matrix A has a zero eigenvalue. Therefore, we need to modify our technique slightly. First we transform the system to diagonal form

$$V = TU, \quad T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad TAT^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V_t + TAT^{-1}V_x + TBT^{-1}V = TF. \tag{5.22}$$

If we consider the domain $0 \le x < \infty$, $t > 0$ the first order absorbing boundary condition at $x = 0$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} TU = PTU \tag{5.23}$$

Since (5.19) is derived from (5.20) after differentiation, for comparison we write (5.23) as

$$(5.24) \qquad PTU_t = 0, \qquad x = 0.$$

Using the original variables we get $u_t + v = 0$ and from (5.20b) we have

$$(5.25) \qquad u_t - u_x = 0, \quad x = 0,$$

which is the same absorbing boundary condition as before, (1.2).

The new boundary condition should have the form

$$(5.26) \qquad PTU_t + SU = 0, \qquad x = 0,$$

where S is derived from the steady state. Again, since (5.19) can be defined from (5.20) after differentiation there is in general no steady solution to (5.20) so we define S such that (5.26) is satisfied for a steady solution to (5.19). Such a solution in terms of u, v and w has the form

$$u = a(x)$$
$$v = b(x) - ta_x(x)$$
$$w = -b_x(x) + t(a(x)-f(x))$$

for some functions a and b. The first equation in (5.26) implies

$$-a_x + s_{11}a + s_{12}(a+tb) + s_{13}(-a_x+tb_x) = 0,$$

which is valid at $x = 0$, in general, only if $s_{12} = s_{13} = 0$ and since $a(0) = a_x(0)$, $(u(0) = u_x(0))$ if $s_{11} = 1$.

The new boundary condition (5.26) is then

$$(5.27) \qquad u_t + v_t + u = 0, \qquad x = 0,$$

and from (5.20b) we get

$$(5.28) \qquad u_t - u_x + u = 0,$$

which is equal to (1.5).

Let us finally note that the boundary condition (5.11) can be formally extended to simple multidimensional cases

$$u_t + Au_x + Bu_y + Cu = f(x,y),$$

$$x \geq 0, \; -\infty < y < \infty, \quad t \geq 0.$$

After a Fourier transform in y we get

$$\hat{u}_t + A\hat{u}_x + (ik_y B+C)\hat{u} = \hat{f},$$

which can be treated in the same way as (5.1).

6. A numerical example. We shall solve the following wave equation numerically

(6.1a) $$u_{tt} - u_{xx} + u = f(x), \qquad 0 \le x \le 1, \quad t \ge 0,$$

(6.1b) $$u(x,0) = u^0(x), \quad u_t(x,0) = 0, \qquad 0 \le x \le 1.$$

The forcing function is defined by

$$f(x) = \frac{1}{2}(1+\cos(\pi(x - \frac{a+b}{2})/(\frac{b-a}{2} - \varepsilon)) \quad \text{on} \quad (a+\varepsilon, b-\varepsilon),$$

$$f(x) = 0 \quad \text{elsewhere}.$$

The steady state v is given by

(6.2a) $$-v_{xx} + v = f(x), \qquad -\infty < x < \infty,$$

(6.2b) $$v \to 0, \quad x \to \pm\infty .$$

The steady state which is known is closed form is computed using the trapezoidal rule approximation of

$$v = \frac{1}{2}(\int_x^1 f(\xi)e^{x-\xi}d\xi + \int_0^x f(\xi)e^{-x+\xi}d\xi), \quad 0 \le x \le 1.$$

The following boundary conditions for (6.4) from section 3 have been tested

$$u_x = c_1u_t + c_2u, \qquad x = 0,$$

$$u_x = -c_1u_t - c_2u, \qquad x = 1,$$

(6.3a) $c_1 = 0, \quad c_2 = 1,$ (Steady),

(6.3b) $c_1 = 1, \quad c_2 = 0,$ (Absorbing, 1st order),

(6.3c) $c_1 = 1, \quad c_2 = 1,$ (New, 1st order).

The second order absorbing and new far field boundary conditions are respectively

$$(6.3d)\quad \begin{aligned} &u_{tx} - u_{tt} - \tfrac{1}{2}u = 0, && x = 0, \\ &u_{tx} + u_{tt} - \tfrac{1}{2}u = 0, && x = 1, \end{aligned}$$

$$(6.3e)\quad \begin{aligned} &u_{xt} - u_{tt} + u_x - u_t - u = 0, && x = 0, \\ &u_{xt} + u_{tt} + u_x + u_t + u = 0, && x = 1. \end{aligned}$$

The standard second order Galerkin method with piecewise linear elements are used for (6.1a). The boundary conditions (6.3a), (6.3b) and (6.3c) are approximated by the box-scheme. The second order boundary conditions (6.3d)-and (6.3e) are also approximated by the box scheme applied to their factored forms (3.17) and (3.19). The stepsizes are chosen small in order to see the errors due to different boundary conditions and not the discretization errors ($\Delta x=0.01$, $\Delta t=0.009$).

In figure 3 the solutions with different boundary conditions are compared at $x = 0$ to the exact solution of (6.1a) over $(-\infty,\infty)$. The plots mainly display the behaviour in the transient phase. The initial value u^0 is identically zero.

The steady state solution (6.2) is given in figure 4 and the convergence to this solution as $t \to \infty$ is described in figures 5a - 5d. The initial value u^0 is defined as $v+0.05$.
The numerical results give quantitative evidence that the new type of far field boundary conditions are useful both for short and long times. In figure 3 we see that the absorbing and the new boundary conditions give solutions in the transient phase which are close to the exact solution in the unbounded domain. The steady condition produces a larger error. It is also clear that the second order conditions are better than the first order ones.
The difference between the different boundary conditions is even more striking for computations over long time. The steady condition does not result in a converging solution as $t \to \infty$, figure 5a. The second order absorbing condition converges to the wrong steady state, figure 5b.

The first order absorbing condition gives a similar picture which is not displayed. The new boundary conditions produce solutions converging to the steady state for the unbounded domain as $t \to \infty$, figures 5c, 5d.

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Figure Captions

<u>Figure 1</u>: Dispersion relations for different boundary conditions, real k. Square (3.3), triangle (3.5), circle (3.9), filled square (3.8).

<u>Figure 2</u>: Dispersion relations for different boundary conditions, imag k. Square (3.3), triangle (3.5), circle (3.9), filled square (3.8).

<u>Figure 3</u>: Comparaison at $x = 0$ between the exact solution and the solutions with different boundary conditions for 500 time steps. Dashed line : exact solution, A1 (6.3b), A2 (6.3d), B1 (6.3c), B2 (6.3e), ST (6.3a).

<u>Figure 4</u>: Steady state solution of (6.2).

<u>Figure 5a</u>: L^2-error between the solution with boundary condition (6.3d) and the steady state solution as a function of time.

<u>Figure 5 b</u>: L^2-error for the solution with boundary condition (6.3d).

<u>Figure 5c</u>: L^2-error for the solution with boundary condition (6.3c).

<u>Figure 5d</u>: L^2-error for the solution with boundary condition (6.3e).

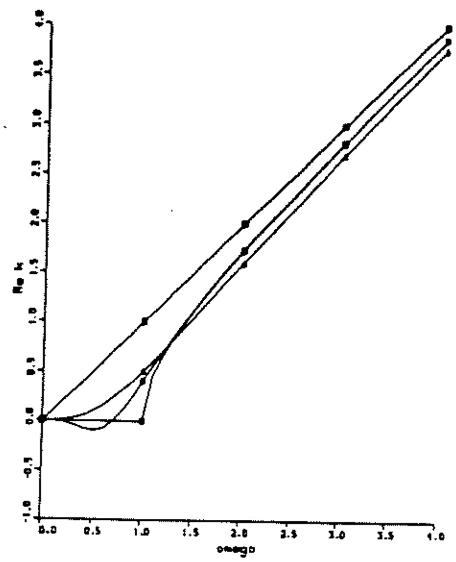

figure 1

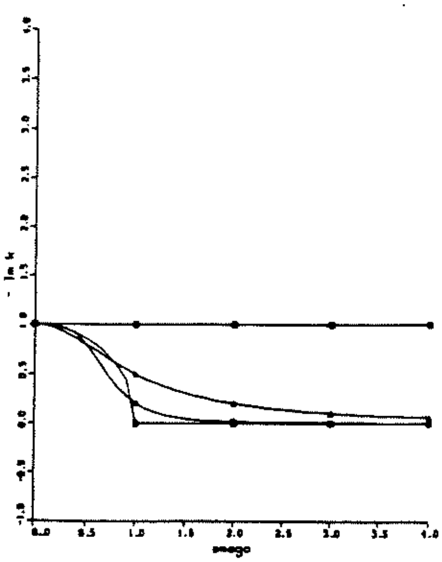

figure 2

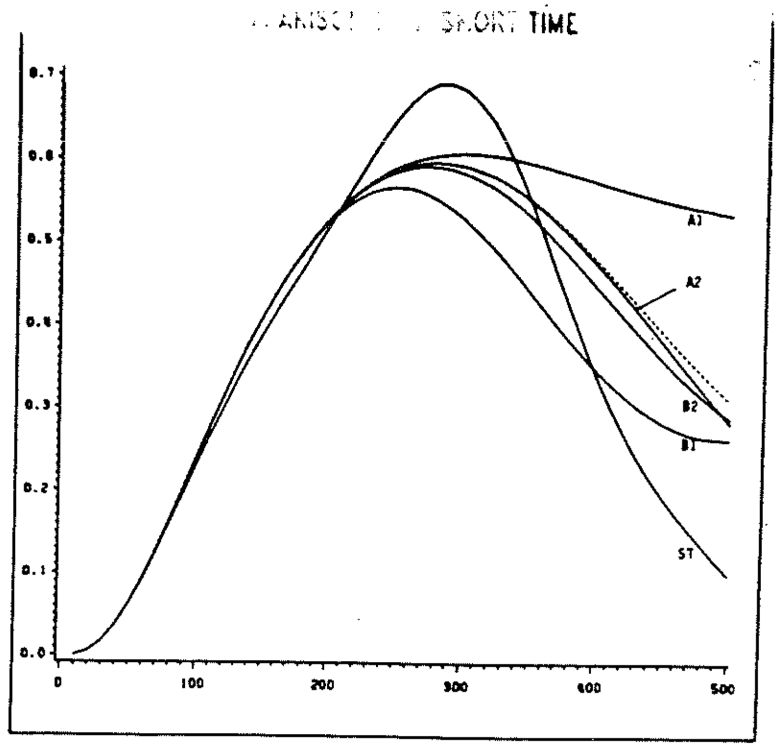

figure 3

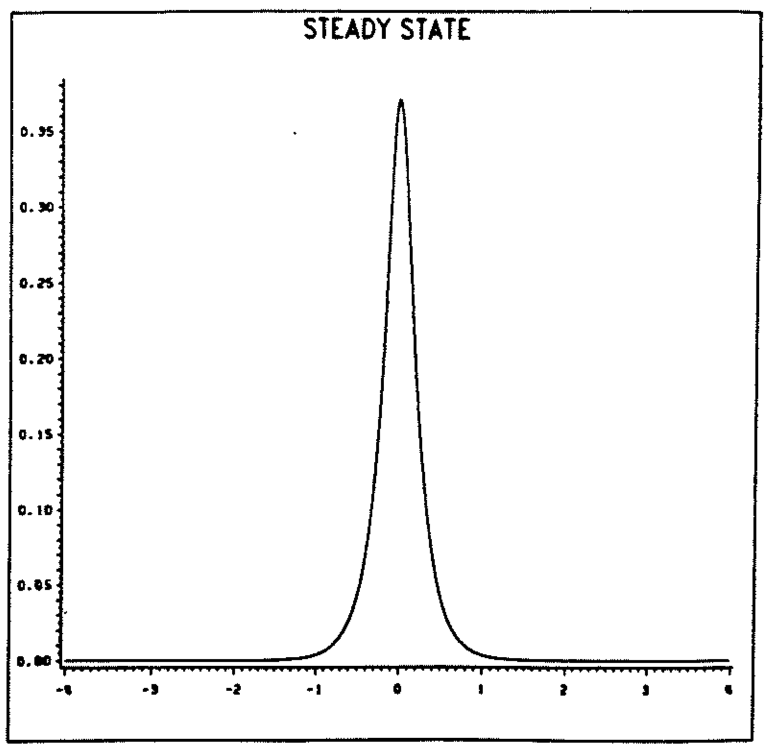

figure 4

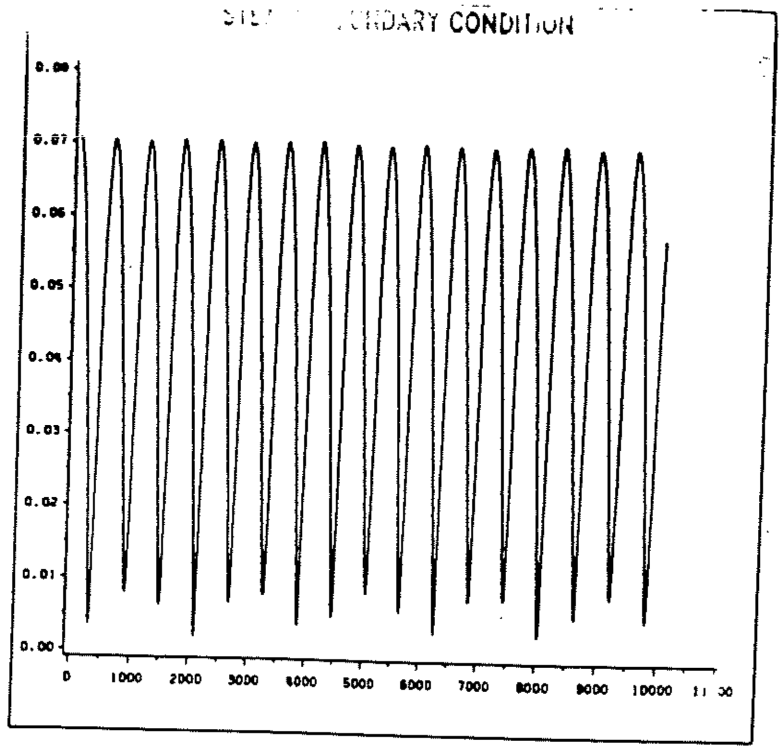

figure 5a

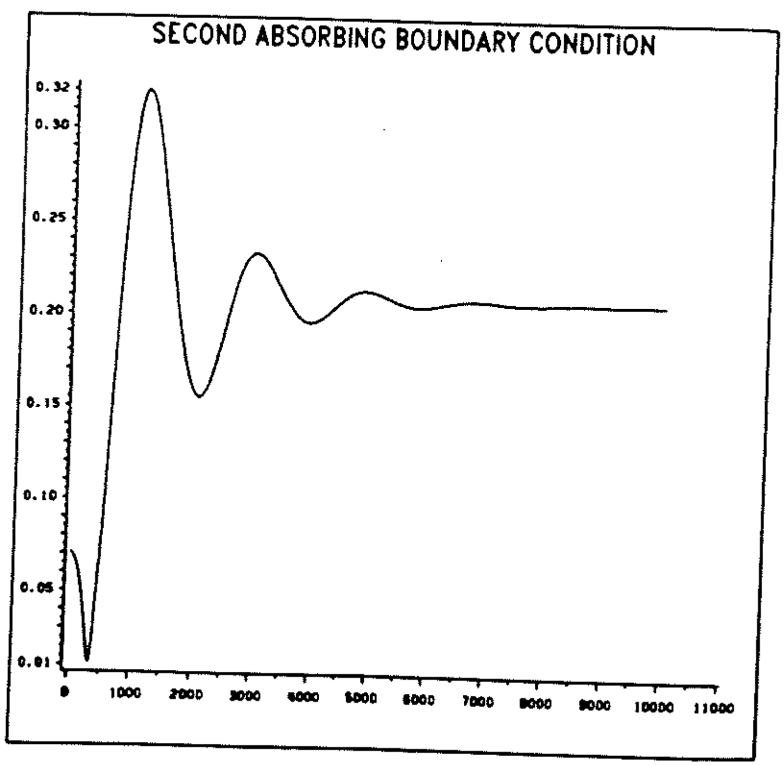

figure 5b

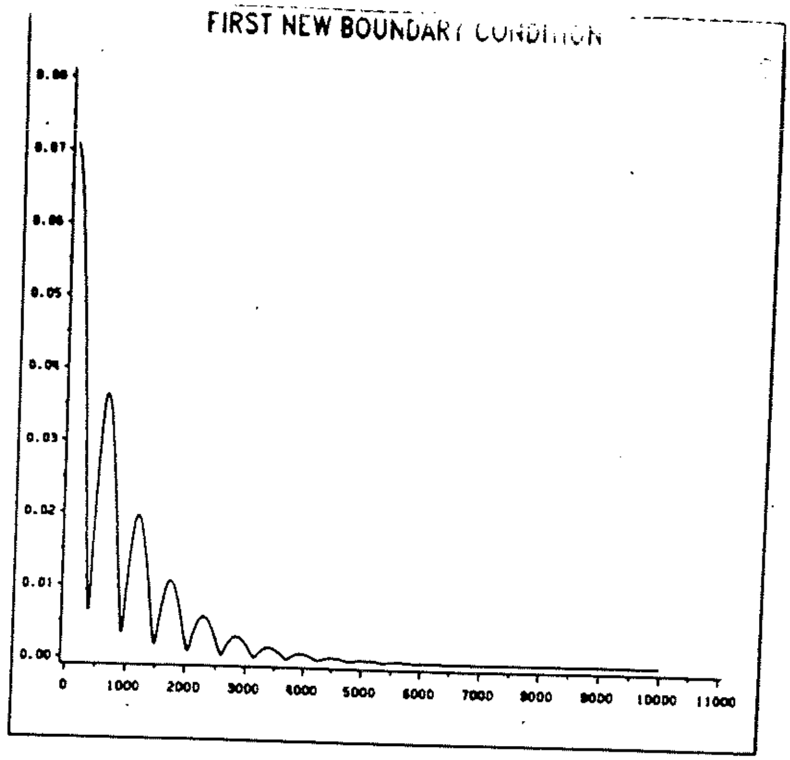

figure 5c

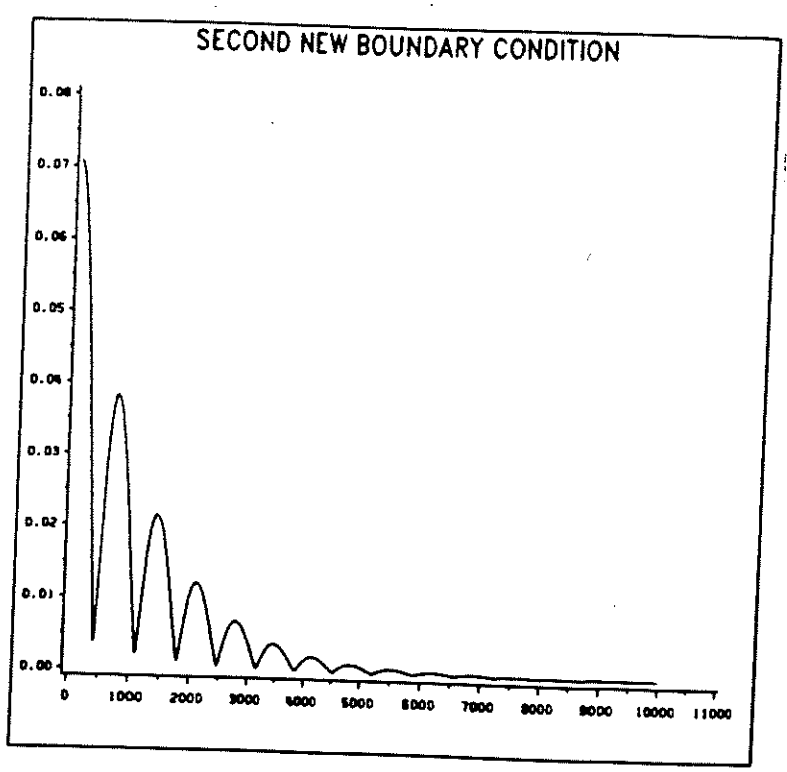

figure 5d