# UCLA
## COMPUTATIONAL AND APPLIED MATHEMATICS

# Domain Decomposition Preconditioners for General Second Order Elliptic Problems

Tony F. Chan
Thomas Y. Hou

June 1988

CAM Report 88-16

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

SIAM J. SCI. STAT. COMPUT.
Vol. 12, No. 6, pp. 1471-1479, November 1991


014

# TIMELY COMMUNICATIONS

*Under the "timely communications" policy for the SIAM Journal on Scientific and Statistical Computing, papers that have significant timely content and do not exceed five pages automatically will be considered for a separate section of the journal with an accelerated reviewing process. It will be possible for the note to appear approximately six months after the date of acceptance.*

# EIGENDECOMPOSITION OF DOMAIN DECOMPOSITION INTERFACE OPERATORS FOR CONSTANT COEFFICIENT ELLIPTIC PROBLEMS*

TONY F. CHAN† AND THOMAS Y. HOU‡

**Abstract.** In this paper the authors derive the exact eigendecomposition of the interface operators arising in domain decomposition methods for general five-point discretizations to constant coefficient elliptic equations on rectangular domains. The special case of convection-diffusion problems is studied in some detail, including both central and upwind differencing for the convection term and flows normal and tangential to the interface. It is shown that preconditioners based on the diffusion operator alone may give very slow convergence when the cell Reynolds number is of order $O(1)$.

**Key words.** domain decomposition, interface operator, eigendecomposition, convection-diffusion

**AMS(MOS) subject classifications.** 65N20, 65F10

**1. Introduction.** In nonoverlapping domain decomposition methods, the key idea is to reduce the differential operator on the whole domain to an operator governing the solution on a reduced set of variables on the interfaces between the subdomains and perhaps a set of cross-points on which a coarse grid discretization is available [2], [13]. The understanding of the properties of the reduced operator is fundamental to domain decomposition methods.

In this paper, we follow the approach in [4], [5] and derive the exact eigendecomposition of the interface operators arising in domain decomposition methods for general five-point discretizations to constant coefficient elliptic equations on rectangular domains. We study in some detail the special case of convection-diffusion problems, including both central and upwind differencing for the convection term and flows normal and tangential to the interface.

One of our motivations is to study the effect of the size of the convection term on the performance of preconditioners based on the diffusion operator only [1], [2], [4], [5], [9], [11]. In particular, we show that convergence can be slow when the cell Reynolds number is of order $O(1)$.

Another possible use of the eigendecompositions is as direct preconditioners for variable coefficient problems (e.g., by averaging coefficients) on more general domains (e.g., by approximating irregular domains by domains with regular geometries sharing the same interface [6]). Although we do not have theoretical justifications for such

* Received by the editors June 24, 1988; accepted for publication (in revised form) May 15, 1991.

† Department of Mathematics, University of California, Los Angeles, Califoria 90024. The work of this author was supported in part by the Department of Energy under contract DE-FG03-87ER25037, by the Army Research Office under contract DAAL03-88-K-0085, and by the National Science Foundation under contracts NSF-DMS87-14612 and ASC 9003002.

‡ Courant Institute, 251 Mercer Street, New York, New York 10012. The work of this author was supported in part by the Air Force Office of Scientific Research under University Research Initiative grant AFOSR 86-0352, the National Science Foundation under contract DMS-9003202, and a Sloan fellowship.

a procedure, the motivation is that by taking into account the convection term in the preconditioners, the rate of convergence will be less sensitive to the cell Reynolds number. For some numerical experiments along this line, see [8]. We note that some of the known preconditioners in the literature, in particular the Neumann–Dirichlet preconditioner [1] and Schwarz-type methods [3], do implicitly account for the first-order terms.

**2. Formulation.** Since we are primarily interested in deriving interface preconditioners, we need only consider the simple case of a domain split into two subdomains with one interface. For example consider the following problem: $Lu = f$ on $\Omega$ with boundary conditions $u = u_b$ on $\partial\Omega$, where $L$ is a linear elliptic operator and the domain $\Omega$ is decomposed into two subdomains $\Omega_1$ and $\Omega_2$ by an interface $\Gamma$. If we order the unknowns for the internal points of the subdomains first and those in the interface $\Gamma$ last, then the discrete solution vector $u = (u_1, u_2, u_3)$ satisfies the linear system $Au = b$, which can be expressed in block form as

$$\begin{pmatrix} A_{11} & & A_{13} \\ & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

By block-Gaussian elimination, the reduced operator on $\Gamma$ (i.e., the Schur complement of $A_{33}$ in $A$) is given by

$$C = A_{33} - A_{31}A_{11}^{-1}A_{13} - A_{32}A_{22}^{-1}A_{23}. \tag{2.1}$$

**3. General five-point stencil.** We consider the general second-order elliptic equations with constant coefficients

$$Lu = \alpha\frac{\partial^2 u}{\partial x^2} + 2\beta\frac{\partial^2 u}{\partial x \partial y} + \gamma\frac{\partial^2 u}{\partial y^2} + \delta\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \rho u = g \quad \text{on} \quad \Omega \tag{3.1}$$

with boundary condition $u = 0$ on $\partial\Omega$, where $\Omega$ is a two-dimensional rectangle.

Without loss of generality, we may assume that $\beta = 0$, $\alpha > 0$, and $\gamma > 0$.

We use a uniform mesh with grid size $h$ on $\Omega$ and with $n$ internal grid points in the $x$-direction, i.e., $h = 1/(n+1)$. We assume that the interface $\Gamma$ is parallel to the $x$-axis and that $\Omega_1$ has $m_1$ internal grid points and $\Omega_2$ has $m_2$ internal grid points in the $y$-direction, i.e., the heights of $\Omega_1$ and $\Omega_2$ are given by $l_1 = (m_1+1)h$ and $l_2 = (m_2+1)h$. Let $x_i = ih$, $y_k = kh$, and denote the approximation to $u(x_i, y_k)$ by $u_{i,k}$. Suppose that equation (3.1) is approximated by a discrete finite-difference scheme of the following form:

$$au_{i-1,k} + bu_{i,k} + cu_{i+1,k} + du_{i,k+1} + eu_{i,k-1} = g_{i,k} \tag{3.2}$$

with boundary conditions given by $u_{0,k} = u_{i,0} = u_{n+1,k} = u_{i,n+1} = 0$, where we assume that the coefficients $a$, $c$, $d$, and $e$ are nonzero. The idea now is to generalize the result in [3], [5], [7] to the more general five-point stencil formula (3.2), by deriving an exact eigendecomposition of the corresponding capacitance matrix $C$. This result is summarized in the following theorem.

THEOREM 3.1. *Define $\tilde{W} = DW$, where $D$ is defined by*

$$D = diag\left[1, \sqrt{\frac{a}{c}}, \cdots, \sqrt{\frac{a}{c}}^{\,n-1}\right]$$

*and $W$ is an orthogonal matrix whose columns are given by*

$$w_j = \sqrt{2h}(\sin j\pi h, \sin 2j\pi h, \cdots, \sin nj\pi h)^T.$$

*Then we have $\tilde{W}^{-1}C\tilde{W} = diag(\lambda_1, \cdots, \lambda_n)$, where the eigenvalues $\lambda_j$ are given by*

$$\lambda_j = -\left(\frac{1+\gamma_j^{m_1+1}}{1-\gamma_j^{m_1+1}} + \frac{1+\gamma_j^{m_2+1}}{1-\gamma_j^{m_2+1}}\right)\sqrt{\frac{1}{4}[b+\sqrt{ac}(2-\sigma_j)]^2 - de}, \tag{3.3}$$

*with $\sigma_j \equiv 4\sin^2\left(\frac{j\pi h}{2}\right)$, and*

$$\gamma_j = \frac{\left((b+\sqrt{ac}(2-\sigma_j)) + \sqrt{(b+\sqrt{ac}(2-\sigma_j))^2 - 4de}\right)^2}{4de}.$$

*Proof.* First we will show that $C\tilde{w}_j = \lambda_j \tilde{w}_j$, where $\tilde{w}_j$ is the $j$th column of matrix $\tilde{W}$.

As pointed out in [4], [5], the term $-A_{31}A_{11}^{-1}A_{13}\tilde{w}_j$ can be computed by first solving the discrete equation (3.2) on $\Omega_1$ with homogeneous right-hand side and the boundary condition $u = \tilde{w}_j$ on $\Gamma$, and homogeneous boundary conditions elsewhere, and then taking the solution on the first row of grid point above $\Gamma$ multiplied by $d$. A similar observation applies to the term $-A_{32}A_{11}^{-1}A_{23}\tilde{w}_j$.

Let us now consider the term $-A_{31}A_{11}^{-1}A_{13}\tilde{w}_j$ first. As we mentioned above, this requires the solution of the discrete equation

$$au_{i-1,k} + bu_{i,k} + cu_{i+1,k} + du_{i,k+1} + eu_{i,k-1} = 0 \quad \text{on} \quad \Omega_1 \tag{3.4}$$

with boundary conditions

$$u_{i,k} = \tilde{w}_j \quad \text{on} \quad \Gamma \quad \text{and} \quad u_{i,k} = 0 \quad \text{on} \quad \partial\Omega_1/\Gamma. \tag{3.5}$$

Consider the solution vector of (3.4) of the form

$$u_{i,k} = d_k\sqrt{2h}(\sqrt{a/c})^i \sin j\pi i h, \tag{3.6}$$

where $0 \leq i \leq n+1$ and $0 \leq k \leq m_1 + 1$.

Substituting (3.6) into (3.4), we have

$$(dd_{k+1} + ed_{k-1})\sqrt{2h}\left(\sqrt{\frac{a}{c}}\right)^i \sin j\pi i h \tag{3.7}$$

$$+d_k\sqrt{2h}\left(\sqrt{\frac{a}{c}}\right)^i\left[b\sin j\pi i h + a\left(\sqrt{\frac{a}{c}}\right)^{-1}\sin j\pi(i-1)h + c\sqrt{\frac{a}{c}}\sin j\pi(i+1)h\right] = 0$$

with boundary conditions

$$d_0 = 1 \quad \text{and} \quad d_{m_1+1} = 0. \tag{3.8}$$

It follows that the $d_k$ satisfy the following difference equation

$$dd_{k+1} + [b+\sqrt{ac}(2-\sigma_j)]d_k + ed_{k-1} = 0 \tag{3.9}$$

with boundary conditions (3.8) and where $\sigma_j \equiv 4\sin^2\left(\frac{j\pi h}{2}\right)$. The roots of the corresponding characteristic polynomial for (3.9) are

$$r_{\pm} = \frac{-(b+\sqrt{ac}(2-\sigma_j)) \pm \sqrt{(b+\sqrt{ac}(2-\sigma_j))^2 - 4de}}{2d}.$$

The general solution of (3.7) is then given by

$$d_k = c_1 r_+^k + c_2 r_-^k. \tag{3.10}$$

Boundary condition (3.8) implies $c_1 + c_2 = 1$ and $c_1 r_+^{m_1+1} + c_2 r_-^{m_1+1} = 0$, which gives

$$c_1 = -\frac{r_-^{m_1+1}}{r_+^{m_1+1} - r_-^{m_1+1}}, \qquad c_2 = \frac{r_+^{m_1+1}}{r_+^{m_1+1} - r_-^{m_1+1}}.$$

Therefore we have

$$-A_{31}\, A_{11}^{-1}\, A_{13}\, \tilde{w}_j = d d_1 \tilde{w}_j, \tag{3.11}$$

where

$$d_1 = \left(\frac{r_- - r_+\gamma_j^{m_1+1}}{1-\gamma_j^{m_1+1}}\right), \qquad \gamma_j \equiv \frac{r_-}{r_+}. \tag{3.12}$$

Similarly,

$$-A_{32}\, A_{22}^{-1}\, A_{23}\, \tilde{w}_j = e d_1^* \tilde{w}_j, \tag{3.13}$$

where

$$d_1^* = \frac{r_-^* - r_+^*(\gamma_j^*)^{m_2+1}}{1-(\gamma_j^*)^{m_2+1}}, \qquad \gamma_j^* = \frac{r_-^*}{r_+^*},$$

$$r_{\pm}^* = \frac{-[b+\sqrt{ac}(2-\sigma_j)] \pm \sqrt{(b+\sqrt{ac}(2-\sigma_j))^2 - 4de}}{2e}.$$

One can easily see that $\gamma_j^* = \gamma_j$.

Finally, it can be verified directly:

$$A_{33}\tilde{w}_j = [b+\sqrt{ac}(2-\sigma_j)]\tilde{w}_j. \tag{3.14}$$

Thus (3.11), (3.13), and (3.14) together give $C\tilde{w}_j = \lambda_j \tilde{w}_j$, where

$$\lambda_j = [b+\sqrt{ac}(2-\sigma_j)] + d d_1 + e d_1^*, \tag{3.15}$$

which after some simplifications gives (3.3). □

**4. Convection-diffusion equations.** We now summarize results for a few special cases corresponding to various finite-difference discretizations of convection-diffusion equation. We use the standard five-point central and upwind differencing for the convection term. We also consider flows normal and tangential to the interface.

**4.1. Central differencing for $\Delta u + \eta(\partial u/\partial y) = 0$.** This is the normal flow case. The coefficients of the stencil are given by

$$a = 1, \quad b = -4, \quad c = 1, \quad d = 1 + \frac{\eta h}{2}, \quad e = 1 - \frac{\eta h}{2}.$$

The eigenvalues $\lambda_j$ depend on the value of $\eta h$. They have the following form:

$$\lambda_j = -\left[\frac{(1+\gamma_j^{m_1+1})}{(1-\gamma_j^{m_1+1})} + \frac{(1+\gamma_j^{m_2+1})}{(1-\gamma_j^{m_2+1})}\right]\sqrt{\sigma_j + \frac{\sigma_j{}^2}{4} + \left(\frac{\eta h}{2}\right)^2},$$

where

$$\gamma_j = \frac{(1 + \frac{\sigma_j}{2} - \sqrt{\sigma_j + \sigma_j^2/4 + (\eta h/2)^2} \quad)^2}{1 - (\eta h/2)^2}.$$

It is interesting to note that although $\gamma_j \to \infty$ as $|\frac{\eta h}{2}| \to 1$, $\lambda_j$ remain finite in the limit.

**4.2. Central differencing for $\Delta u + \delta(\partial u/\partial x) = 0$.** This is the tangential flow case. The coefficients of the stencil are given by

$$a = 1 - \frac{\delta h}{2}, \quad b = -4, \quad c = 1 + \frac{\delta h}{2}, \quad d = 1, \quad e = 1.$$

The eigenvalues $\lambda_j$ depend on the value of $\delta h$. They have the following form:

$$\lambda_j = -\left[\frac{(1+\gamma_j^{m_1+1})}{(1-\gamma_j^{m_1+1})} + \frac{(1+\gamma_j^{m_2+1})}{(1-\gamma_j^{m_2+1})}\right]\sqrt{(2-\mu)\mu\sigma_j + \mu^2\frac{\sigma_j{}^2}{4} + (2-\mu)^2 - 1},$$

where

$$\mu = \left(1 + \frac{\delta h}{2}\right)\sqrt{\frac{1 - \frac{\delta h}{2}}{1 + \frac{\delta h}{2}}},$$

and

$$\gamma_j = \frac{\left(\mu(2-\sigma_j) - 4 + \sqrt{(\mu(2-\sigma_j)-4)^2 - 4}\right)^2}{4}.$$

We note that in this case, the Fourier transform matrix $\bar{W}$ has a singularity as the cell Reynolds number $\delta h/2$ approaches 1 because the matrix $D$ in the Theorem 3.1 becomes singular. This difficulty can be removed by using upwind differencing.

**4.3. Upwind differencing for $\Delta u + \eta(\partial u/\partial y) = 0$, $\eta > 0$.** The stencil coefficients are as follows

$$a = 1, \quad b = -4 - \eta h, \quad c = 1, \quad d = 1 + \eta h, \quad e = 1.$$

The eigenvalues $\lambda_j$ are:

$$\lambda_j = -\left[\frac{(1+\gamma_j^{m_1+1})}{(1-\gamma_j^{m_1+1})} + \frac{(1+\gamma_j^{m_2+1})}{(1-\gamma_j^{m_2+1})}\right]\sqrt{\sigma_j + \frac{\sigma_j^2}{4} + \sigma_j\eta h/2 + (\eta h)^2/4},$$

where $\gamma_j$ are given by

$$\gamma_j = \frac{\left[1 + \frac{\sigma_j}{2} + \eta h/2 - \sqrt{\sigma_j + \frac{\sigma_j{}^2}{4} + \eta h(\sigma_j/2 + \eta h/4)}\ \right]^2}{1+\eta h}.$$

An important feature about the upwind discretization is that the eigenvalues $\lambda_j$ are well defined for all values of the cell Reynolds number $\eta h$ ($\eta > 0$). This property is not shared by the central differencing discretization.

**4.4. Upwind differencing for $\Delta u + \delta(\partial u/\partial x) = 0, \delta > 0$.** The stencil coefficients are

$$a = 1, \quad b = -4 - \delta h, \quad c = 1 + \delta h, \quad d = 1, \quad e = 1.$$

The eigenvalues $\lambda_j$ are:

$$\lambda_j = -\left[\frac{(1+\gamma_j^{m_1+1})}{(1-\gamma_j^{m_1+1})} + \frac{(1+\gamma_j^{m_2+1})}{(1-\gamma_j^{m_2+1})}\right]\sqrt{\sigma_j + \frac{\sigma_j^2}{4} - \epsilon_j(2+\sigma_j-\epsilon_j)},$$

where the $\gamma_j$ and $\epsilon_j$ are given by

$$\gamma_j = \left[-1 - \frac{\sigma_j}{2} + \epsilon_j + \sqrt{\sigma_j + \frac{\sigma_j{}^2}{4} - \epsilon_j(2+\sigma_j-\epsilon_j)}\ \right]^2,$$

and

$$\epsilon_j = (1 - \frac{\sigma_j}{2})(\sqrt{1+\delta h} - 1) - \frac{\delta h}{2}.$$

Note that $\epsilon_j$ tends to zero as $\delta h$ goes to zero and the eigenvalues are well defined for all values of the cell Reynolds number $\delta h$ ($\delta > 0$).

**5. Effect of convection term on diffusion-based preconditioners.** Consider the equation $\Delta u + \eta(\partial u/\partial y) = 0$ discretized using both central and upwind differencing. Let $C(\eta)$ denote the interface matrix as a function of $\eta$. Consider preconditioning $C(\eta)$ by $C(0)$, the interface operator corresponding to the diffusion operator only. In Fig. 1, we plot the condition numbers $K(C^{-1}(0)C(\eta))$ (the ratio of the maximum eigenvalue to the minimum eigenvalue) versus the coefficient $\eta$ for $h = 0.02$, $m_1 = 50$, $m_2 = 100$. Note that the critical cell Reynolds number corresponds to $\delta = 100$ for the central differencing case, and the eigenvalues of $C^{-1}(0)C(\eta)$ are real in this case. We see that as $\delta$ approaches the critical value, the condition number can grow larger than 10.

Next we consider the equation $\Delta u + \delta(\partial u/\partial x) = 0$ with the same values for $h$, $m_1$, and $m_2$. In this case, the eigenvalues of $C^{-1}(0)C(\delta)$ are complex. Instead of a plot of the condition number, we show in Figs. 2 and 3 the distribution of the eigenvalues of $C^{-1}(0)C(\delta)$ for several values of $\delta$ for upwind and central differencing, respectively. These plots show how rapidly the spectrum spreads from unity as $\delta$ increases. The clustering around the value 1 can be easily seen, but even this effect weakens as $\delta$ increases. For many nonsymmetric iterative methods, such as conjugate-gradient–like methods and Chebychev methods, such information on the spectrum plays a crucial role in determining the convergence rates [10], [12]. We note that for $\delta > 65$, the formula for the eigendecomposition of the interface operator for the

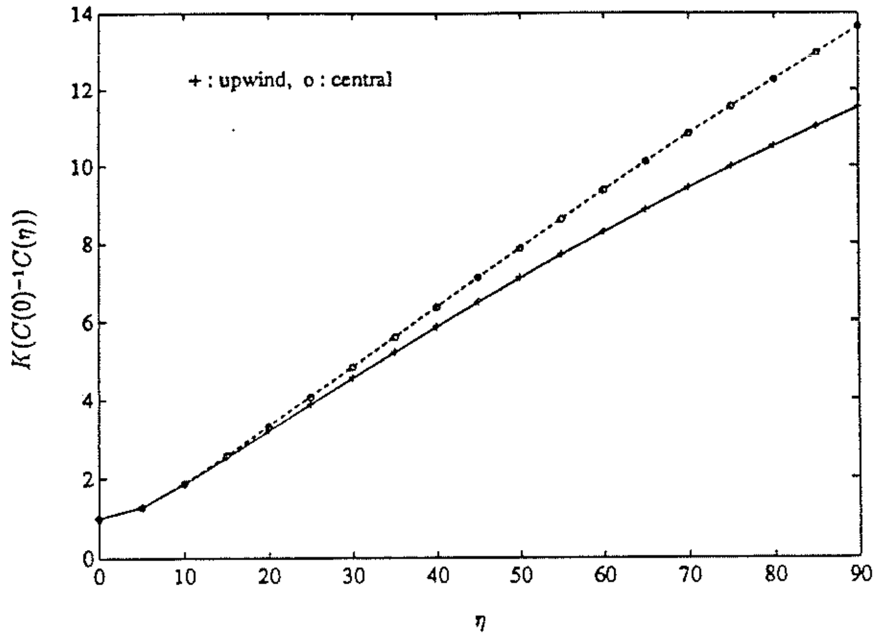


FIG. 1. $K(C(0)^{-1}c(\eta))$ *versus* $\eta$ *for* $\Delta u + \eta(\partial u/\partial y) = 0$.

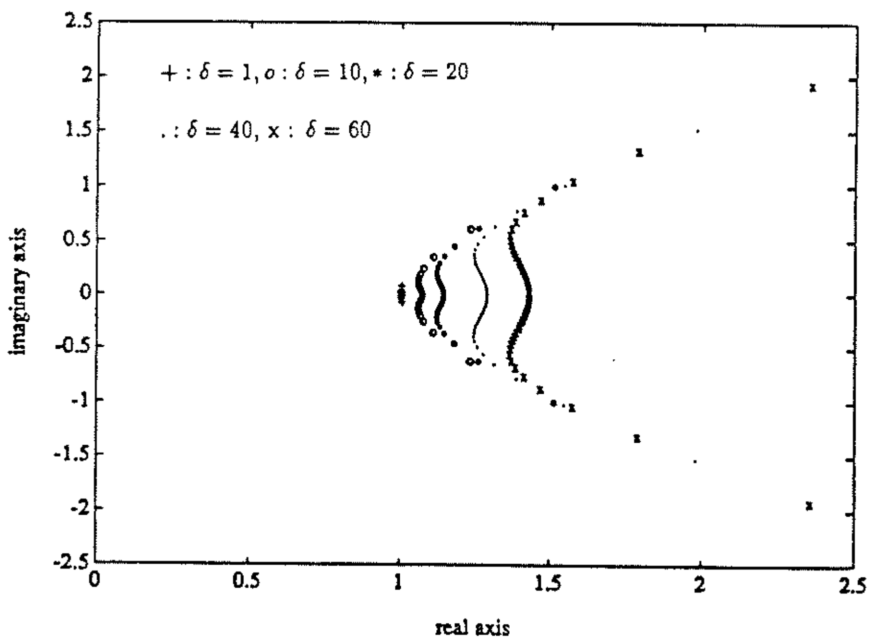


FIG. 2. *Eigenvalue distributions for upwind difference for* $\Delta u + \delta(\partial u/\partial x) = 0$.

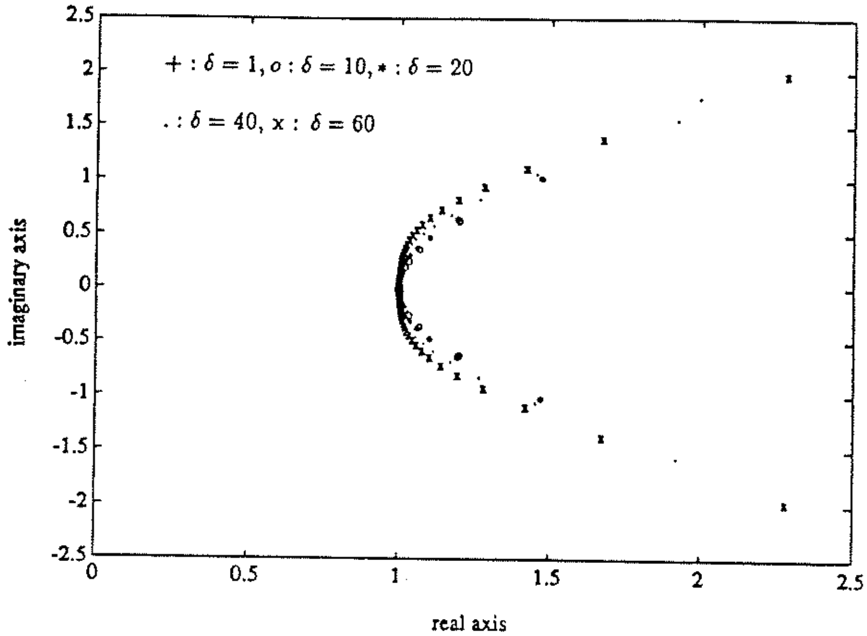


FIG. 3. *Eigenvalue distributions for central difference for* $\triangle u + \delta(\partial u/\partial x) = 0$.

central differencing case becomes unstable and hence the eigenvalues are not plotted in Figs. 2 and 3.

These calculations show that the effectiveness of the diffusion-based preconditioners (as reflected in either the condition number or the spread of the spectrum of the preconditioned interface system) can deteriorate appreciably as the size of the cell Reynolds number increases to $O(1)$.

**Acknowledgment.** The authors would like to thank Professor David Keyes for many helpful suggestions and for catching some errors in an earlier version of the paper.

## REFERENCES


[1] P. E. BJØRSTAD AND O. B. WIDLUND, *Iterative methods for the solution of elliptic problems on regions partitioned into substructures*, SIAM J. Numer. Anal., 23 (1986), pp. 1097–1120.

[2] J. H. BRAMBLE, J. E. PASCIAK, AND A. H. SCHATZ, *The construction of preconditioners for elliptic problems by substructuring*, I, Math. Comp., 47 (1986), pp. 103–134.

[3] X. C. CAI, *Some domain decomposition algorithms for nonself-adjoint elliptic and parabolic partial differential equations*, Ph.D. thesis, Courant Institute, New York University, New York, NY, 1989.

[4] T. F. CHAN, *Analysis of preconditioners for domain decomposition*, SIAM J. Numer. Anal., 24 (1987), pp. 382–390.

[5] T. F. CHAN AND D. C. RESASCO, *A framework for the analysis and construction of domain decomposition preconditioners*, in Proceedings of the First International Symposium on Domain Decomposition Methods for Partial Differential Equations, 1987, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1988, pp. 217–230.

[6] ———, *Analysis of domain decomposition preconditioners on irregular regions*, in Proceedings of the Sixth IMACS International Symposium on Computer Methods for Partial Differential

Equations, R. Vichnevetsky and R. S. Stepleman, eds., June 23–26, 1987, Lehigh University, Bethelem, PA, pp. 317–322.

[7] ———, *A domain-decomposed fast Poisson solver on a rectangle*, SIAM J. Sci. Statist. Comput., 8 (1987), pp. s14–s26.

[8] T. F. CHAN AND D. E. KEYES, *Interface preconditionings for domain-decomposed convection-diffusion operators*, in Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, R. Glowinski, J. Périaux, and O. B. Widlund, eds., Society for Industrial and Applied Mathematics, Philadelphia, PA, 1990, pp. 245–262.

[9] M. DRYJA, *A capacitance matrix method for Dirichlet problem on polygonal region*, Numer. Math., 39 (1982), pp. 51–64.

[10] R. W. FREUND AND N. M. NACHTIGAL, QMR: *a quasi-minimal residual method for non-Hermitian linear systems*, Tech. Report 90.51, Research Institute for Advanced Computer Science, NASA Ames Research Center, Moffett Field, CA, December, 1990.

[11] G. H. GOLUB AND D. MAYERS, *The use of pre-conditioning over irregular regions*, in Computing Methods in Applied Sciences and Engineering VI, R. Glowinski and J. L. Lions, eds., North-Holland, 1984, pp. 3–14.

[12] Y. SAAD AND M. H. SCHULTZ, GMRES: *A generalized minimal residual algorithm for solving nonsymmetric linear systems*, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 856–869.

[13] B. F. SMITH, *An optimal domain decomposition preconditioner for the finite element solution of linear elasticity problems*, Tech. Report 482, Courant Institute, 1989; to appear in Proceedings of Copper Mountain Conference on Iterative Methods, April 1990.