UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

The Least Action Principle and the Related Concept of Generalized Flows for Incompressible Perfect Fluids

Yann Brenier

April 1988

CAM Report 88-17

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

"It is a most beautiful and awe-inspiring fact that all the fundamental laws of Classical Physics can be understood in terms of one mathematical construct called the Action" (Pierre Ramond, Field Theory a modern primer).

1. Introduction.

Let us consider the motion of a perfect incompressible fluid inside a closed bounded domain X in the Euclidian space $\mathbf{R}^d$. The fluid is not necessarily homogeneous, which means that the initial mass density $\rho_0(x)$ may depend on the space variable x. If the external forces derive from a potential U(t,x) (where t is the time variable), then the motion is described in terms of the density field $\rho(t,x)$, the velocity field v(t,x) and the pressure field p(t,x), by the following equations :

(1.1) $\quad \mathrm{div}\, v = 0$ (incompressibility condition),

(1.2) $\quad \partial_t\rho + v.\mathrm{grad}\rho = 0$ (conservation of mass),

(1.3) $\quad \rho\{\partial_t v + v.\mathrm{grad} v + \mathrm{grad} U\} + \mathrm{grad} p = 0$ (Newton's law).

The initial and boundary conditions are :

(1.4) $\quad v(t{=}0,x) = v_0(x), \quad \rho(t{=}0,x) = \rho_0(x),$

(1.5) $\quad v.n = 0$ along ∂X where n = outward normal (impermeability condition).

The pressure field is the Lagrange multiplier associated with the incompressibility condition (1.1) and needs neither boundary nor initial conditions. This system can be reformulated in terms of the flow map g(t,x) defined on RxX, valued in X and solution to the ordinary differential system :

(1.6) $\quad g(0,x) = x, \ \partial_t g(t,x) = v(t,g(t,x)).$

From (1.2-3-4), one gets :

(1.7) $\quad \rho(t,g(t,x)) = \rho(t{=}0,x) = \rho_0(x),$

(1.8) $\quad \rho_0(x)\{\partial_t^2 g(t,x) + \mathrm{grad}U(t,g(t,x))\} + \mathrm{grad}p(t,g(t,x)) = 0,$

(1.9) $\quad g(0,x) = x, \ \partial_t g(0,x) = v_0(x).$

If the velocity and the density fields are smooth enough, as well as X and its boundary, then the incompressibility and impermeability conditions (1.1-5) exactly mean that, for each time t, g(t,.) is a smooth diffeomorphism from X into itself that preserves both orientation and volume. In other words :

(1.10) $\quad$ for any t, g(t,.) belongs to G where G is the set of all smooth orientation and volume preserving mappings from X into X :

(1.11) $G = \{ \gamma : X \to X$, diffeomorphism s.t. $\det D\gamma(x) \equiv 1 \}$ ($D\gamma(x)$ denotes the Jacobian matrix of γ at x).

Then the motion is entirely described by equations (1.8-9) and condition (1.10-11). Behind these equations, as has been known for a long time [Arnol'd], there is the Least Action Principle. Here the Action is the sum of the kinetic energy and the potential energy and is defined at each time t by :

(1.12) $$A(g,t) = \int_X \rho_0(x)\{1/2\|\partial_t g(t,x)\|^2 - U(t,g(t,x))\}dx .$$

The Least Action Principle says that if $t_1-t_0>0$ is not too large, then :

(1.13) $$\int_{t_0}^{t_1} A(g,t)dt \le \int_{t_0}^{t_1} A(\gamma,t)dt$$

holds for any flow map γ such that :

(1.14) $\gamma(t,.)\in G$, for $t_0\le t\le t_1$, $\gamma(t_0,.)=g(t_0,.)$, $\gamma(t_1,.)=g(t_1,.)$.

In other words, the Action integrated from t_0 to t_1 is minimal for g.
In this paper, instead of considering the initial value problem (1.8...11), we concentrate on the related minimization problem : given $t_0 < t_1$, g_0 and g_1 in G, find a flow map g such that :

(1.15) $g(t,.)\in G$, for $t_0\le t\le t_1$, $g(t_0,.)=g_0$, $g(t_1,.)=g_1$,

(1.16) and $\int_{t_0}^{t_1} A(g,t)dt$ is minimal.

This can be considered as a "shortest path problem" on G, the manifold of smooth orientation and volume preserving mappings from X into itself. Since G is a group under the composition rule, it is not a restriction to substitute the identity map for g_0 and $h= g_1 \circ g_0^{-1}$ for g_1. In a similar way, one can substitute the time interval [0,T] for $[t_0,t_1]$, where T is a given strictly positive number. h will be called the "final configuration" and T the "final time". Thus, let us consider :

The shortest path problem

- Given T>0 and $h\in G$, find a flow map $t \to g(t,.)\in G$, $0\le t\le T$, that reaches the final configuration h at time T and minimizes the Action :

(1.17) $$g\to \int_0^T \int_X \rho_0(x)\{1/2\|\partial_t g(t,x)\|^2 - U(t,g(t,x))\}dx\, dt$$

where $\rho_0 \ge 0$ and U are given.

From contributions by various authors [Ebin & Marsden], one can solve this problem when h is very smooth (h belongs to some high order Sobolev space, say) and lies in a small neighborhood (related to some very strong topology) of the identity map. However, for arbitrary data, this minimization problem seems highly difficult since the quantity to be minimized does not involve any spatial derivatives of the flow map, while the incompressibility constraint is expressed in terms of the Jacobian determinant. Therefore, strong convergence of minimizing sequences cannot be obtained by classical ways. Moreover, the appropriate strong topologies for G are totally unrelated to the metrics induced by the kinetic energy. The goal of this paper is to overcome these difficulties in two steps :

i) enlarge the framework with an adequate concept of generalized flows, in the spirit of L.C. Young's ideas on the Calculus of Variations [Young], and prove the existence of a generalized solution to the "shortest path problem" ;

ii) check that classical solutions cannot be missed in this new framework.

Before defining our favorite concept of generalized flow, let us review several possible generalizations of the shortest path problem. The first natural idea would be to substitute for the set G a broader set of volume preserving mappings, for example :

(1.19) $S = \{ \gamma : X \rightarrow X$, if Y is a measurable subset of X, then $\gamma^{-1}(Y)$ is measurable and $\mathrm{meas}(\gamma^{-1}(Y)) = \mathrm{meas}(Y)\}$.

This definition is very classical in ergodic theory [Arnol'd & Avez]. An equivalent definition is :

(1.19) $S = \{ \gamma : X \rightarrow X$, if f is a continuous function on X, then $f\circ\gamma$ is measurable and $\int_X f(\gamma(x))dx = \int_X f(x)dx\}$.

These definitions make sense for any measure space (X,dx) when X is a compact set and dx a positive Borel (Radon) measure. When X is a closed bounded domain in $\mathbf{R}^d$, G is obviously a subset of S, since for any f in C(X) and γ in G, $f\circ\gamma$ is continuous and :

(1.20) $\int_X f(x)dx = \int_X f(\gamma(y))\det D\gamma(y)dy = \int_X f(\gamma(y))dy$ (change of variable : $x=\gamma(y)$).

However, S contains many more mappings than G. The one dimensional case X=[-1,1] is striking since, then, G has a single element (the identity map) while S contains various mappings such as $\gamma(x)=x+1$ if $x<0$, $x-1$ if $x>0$, which is discontinuous, $\gamma(x)=-x$ which is orientation reversing, or $\gamma(x)=2x+1$ if $x<0$, $1-2x$ if $x>0$, which is not one-to-one, So, G and S seem very different from each other. However the gap between G and S is a specific property of the one-dimensional case and it is a reasonable **conjecture** to state that **S actually is the strong L^2 closure of** G **if $d\geq2$**. We believe that such a result is probably available somewhere in the litterature but, since it is not strictly necessary for our discussion, we will not try to prove it. In the shortest path problem, the quantity to be minimized looks like a L^2 norm with respect to the space variables. Therefore it makes sense to substitute for G what we believe to be its (strong) L^2 closure, that is S. Then, one gets :

The first generalization of the shortest path problem

• Given T>0 and $h \in S$, find a measurable mapping $(t,x) \in [0,T] \times X \to g(t,x) \in X$, s.t. :

(1.21) $g(0,x) = x$, $g(T,x) = h(x)$ ("g reaches h at time T"),

(1.22) $\int_X f(g(t,x))dx = \int_X f(y)dy$ for any $t \in [0,T]$ and $f \in C(X)$ (incompressibility),

(1.23) $g \to \int_0^T \int_X \rho_0(x)\{1/2\|\partial_t g(t,x)\|^2 - U(t,g(t,x))\}dx\, dt$ is minimal •

This problem seems easier to solve than the original one, since the classical incompressibility condition :

(1.24) $\det Dg(t,x) \equiv 1$,

that involves in a very non-linear way the space derivatives of the flow map is now replaced by condition (1.22) which makes sense even when the flow map is not continuous ! However it is still a non-linear constraint. At this point of the discussion, it is highly questionable whether or not such a generalization is justified from a physical point of view. For example, the important property for the flow map to be, at least, one-to-one and orientation preserving is completely missed by this new framework. A reasonable answer to this question would be to prove, at least, that, whenever there is a classical solution to the original shortest path problem, it is automatically the solution to the generalized problem. Otherwise, the new framework would be worthless. Nevertheless, this is not our main concern at the moment, since it is even not clear that the generalized shortest path problem always has a solution ! Indeed, as has been mentioned earlier, the action to be minimized does not involve any spatial derivative of the flow map and, therefore, there is no control of the amount of oscillation (in the space variables) that can be produced by the minimizing sequences. In some cases, for example :

(1.25) $X=[-1,1]$ and $h(x)=-x$, or more generally : $X=[-1,1]^d$ and $h(x)=(-x_1,x_2,\ldots,x_d)$,

(that are not classical, since the final configuration h is orientation reversing) oscillations occur and (see section 6) solutions must be looked for in a much larger class of flowmaps. This kind of trouble is typical in the Calculus of Variations and this is why L.C. Young [Young] introduced illuminating probabilistic concepts , known as "Young's measures", to describe the behaviour of oscillating minimizing sequences. These techniques have been popularized and extended to other fields by Tartar [Tartar] and used in Fluid Mechanics, for both compressible [DiPerna] and incompressible [DiPerna & Majda] flows, through the concepts of "measure-valued solutions". Unfortunately this last concept is essentialy Eulerian and not Lagrangian, in the sense that it is based on the description of the motion in terms of velocity fields (through equations (1.1...4)) rather than flowmaps (through (1.6...11)). Therefore, it does not seem suitable for the shortest path problem here considered and a different (but consistent) concept will be introduced in this paper. Before describing what we believe to be the right probabilistic concept for our problem, let us first consider an intermediate generalization that turns out to be inadequate but shows interesting features. Indeed, it is a natural probabilistic idea to replace the concept of volume preserving mappings by the one of "doubly stochastic probability measures" on the product space XxX. This is classical in the litterature devoted to the Monge-Kantorovitch problem [Rachev] and has been recently used by the author [Brenier] to define the rearrangement of vector fields. A doubly stochastic probability measure on XxX is a positive Borel measure $\mu(dx,dy)$ such that

(1.26) $\int_{XxX} f(x)\mu(dx,dy) = \int_{XxX} f(y)\mu(dx,dy) = \int_X f(x)dx$, for each f in C(X).

To any volume preserving mapping h in S, one can associate a unique doubly stochastic probability measure μ_h defined by :

(1.27) $\mu_h(dx,dy) = \delta(y-h(x))dx$, i.e. $\int_{XxX} f(x,y)\mu_h(dx,dy) = \int_X f(x,h(x))dx$, for f in C(XxX).

Indeed, if h belongs to S, then, for each f in C(X):

$$\int_{XxX} f(x)\mu_h(dx,dy) = \int_X f(x)dx,$$

$$\int_{XxX} f(y)\mu_h(dx,dy) = \int_X f(h(x))dx = \int_X f(x)dx \text{ (since h is volume preserving).}$$

By (1.27), S can be identified as a subset of P, the set of all doubly stochastic probability measures on XxX. In the same way as S was conjectured to be the strong L^2 closure of G, we believe that P is the weak-* closure of S. Once again, since such a result is probably available somewhere in the litterature and not strictly necessary for our discussion, there will be no attempt to prove it here. It is now tempting to solve the shortest path problem in the class of time parameterized families of doubly stochastic probability measures on XxX:

(1.28) $t \in [0,T] \rightarrow \mu(t\,;dx,dy) \in \text{Prob}(XxX)$.

$\mu(t\,;dx,dy)$ can be interpreted as the probability for a particle to go from x at time 0 to y at time t. It is now easy to translate the initial and final conditions (1.21), as well as the incompressibility condition (1.22) in terms of μ :

(1.29) $\mu(0\,;dx,dy) = \delta(y-x)dx$, $\mu(T\,;dx,dy) = \delta(y-h(x)dx$

(1.30) $\int_{X \times X} f(y)\mu(t\,;dx,dy) = \int_X f(x)dx$ for each f in C(X) and t in [0,T].

Unfortunately, at this point, it becomes extremely hard to define the Action in terms of μ and this is why this approach is given up in this paper. The main trouble is the lack of dynamics in the description of a generalized flow as a one parameter family of doubly stochastic probability measures (similar problems have been discussed in the case of hyperbolic systems of conservation laws [DiPerna]). Indeed, the Action involves the velocity of the particles and it is therefore necessary to consider not only the probability that a particle issued from x at time 0 reaches y at time t, but also the probability that it goes from x_0 at time t_0 to x_1 at time t_1 when t_1 and t_0 are infinitesimaly close etc...In this paper, a richer concept is used that fully takes into account the dynamics of the particles : to each path $t \in [0,T] \rightarrow z(t) \in X$ one associates the probability that it is followed by some material particle. This defines generalized flows as probability measures on the set Ω of all possible paths. Obviously this kind of idea is rather common in both statistical and quantum physics, but, surprisingly, seems ignored in continuum mechanics. It is also closely related to the concept of path integrals [Reed & Simon] as well as the construction of the Wiener integral. The nicest feature of the new framework is the fact that i) initial, final and incompressibility conditions are (w-* continuous) linear constraints on the set Prob(Ω), ii) the Action turns out to be nearly a linear functional. Therefore, one obtains what we could call (following the terminology used for the Monge-Kantorovich problem [Rachev]) a "continuous linear programming problem", for which it is rather easy to prove the existence of an optimal generalized flow. Then, it is not difficult to check that classical solutions to the Euler equations cannot be missed in our framework : in any case, under some natural restrictions on the time scale, the corresponding flow is the unique solution to the generalized shortest path problem. Moreover, there are examples of optimal generalized flows that are not deterministic and can be explicitly computed. Finally, one can *formally* derive from any optimal generalized flow a corresponding "measure-valued solution" to the Euler equations in the sense of DiPerna and Majda.

The paper is organized as follows :

section 2 : a probabilistic concept of generalized flows ;

section 3 : the generalized shortest path problem ;

section 4 : a complete existence result in the case of the d-dimensional torus ;

section 5 : classical solutions and generalized solutions ;

section 6 : explicit examples of non deterministic generalized solutions to the Euler equations ;

appendix : a formal link with the measure-valued solutions in the sense of DiPerna and Majda.

2. A probabilistic concept of generalized flows.

Here X is a compact set in R^d and dx a probability measure on X (X can be a manifold and dx can be different from the Lebesgue measure). T>0 is a fixed time. Sometimes, Q will denote the set $[0,T]xX$ and τ a generic finite subset $\{t_1,\ldots,t_n\}$ of [0,T].
The product space : $\Omega = X^{[0,T]}$, which is the set of all paths z : $t \in [0,T] \rightarrow z(t) \in X$, is compact for the product topology. A function F defined on Ω can be viewed as a "path functional" and, therefore, the notation $F[t \rightarrow z(t)]$ will be sometimes used instead of F(z). Given a finite subset $\tau=\{t_1,\ldots,t_n\}$ of [0,T] and a continuous function f on X^n, $F(z) = f(z(t_1),\ldots,z(t_n))$ defines a function on Ω which is always continuous for the product topology [Reed & Simon]. The function(al)s of this kind will be called "continuous functionals of finite type". By Stone-Weierstrass' theorem, the set $C_{fin}(\Omega)$ of all such functionals is a dense subspace of the space $C(\Omega)$ of all continuous functionals on Ω. $C(\Omega)$ is a Banach space for the sup-norm and the dual space $C(\Omega)'$ is exactly the set of all positive Borel (Radon) measures on Ω. Our first result is elementary but essential for our purposes :

Proposition 2.1.

•Let g be an incompressible flow on X that reaches a "final configuration" $h \in S$ at time T (in the sense of (1.21-22)).Then there exists a unique probability measure q on Ω defined by :

(2.1) $\int_\Omega F(z)q(dz) = \int_X F[t \rightarrow g(t,x)]dx$, for each F in $C_{fin}(\Omega)$.

q satisfies the following properties :

(2.2) $\int_\Omega f(z(t_0))q(dz) = \int_X f(x)dx$, for each f in C(X) and t_0 in [0,T],

(2.3) $\int_\Omega f(z(0),z(T))q(dz) = \int_{XxX} f(x,y)\eta(dx,dy)$, for each f in C(XxX),

where $\eta(dx,dy) = \delta(y-h(x))dx$ defines a doubly stochastic probability measure on XxX, i.e.

(2.4) $\int_{XxX} f(x)\eta(dx,dy) = \int_{XxX} f(y)\eta(dx,dy) = \int_X f(x)dx$, for each f in C(X)•

Proof

•If F belongs to $C_{fin}(\Omega)$, it can be written as :
$F(z) = f(z(t_1),\ldots,z(t_n))$ for some finite set $\{t_1,\ldots,t_n\}$ and some continuous function f on X^n.
Thus formula (2.1), that can be rewritten as :

(2.5) $\int_\Omega f(z(t_1),\ldots,z(t_n))q(dz) = \int_X f(g(t_1,x),\ldots,g(t_n,x))dx,$

defines a unique positive linear functional on $C_{fin}(\Omega)$ and therefore, by density, a unique positive Borel measure q on Ω. Since dx is a probability measure, q is also a probability measure. Property (2.2) is a straightforward consequence of the incompressibility condition (1.22) and (2.3) immediately follows from conditions (1.21) and definition (2.5). Since h is a volume preserving mapping, the correponding measure η is necessarily a doubly stochastic probability measure, as has been seen in section 1. Proposition 2.1 suggests the following :

Definition 2.2.

•Any probability measure q on Ω is called a "generalized flow". If (2.2) holds, we say that q is incompressible. Any doubly stochastic probability measure η on XxX is called a "(generalized) final configuration". If q satisfies (2.3), we say that q reaches the final configuration η at time T•

There is no doubt that generalized flows and final configurations can be approximated (in the weak-* sense) by classical flows (at least when X is a nice d- dimensional domain, with $d \geq 2$). Since this result is not, strictly speaking, necessary to our discussion, no proof will be provided here.
As an obvious consequence of Proposition 2.1 and Definition 2.2, one finally gets:

Proposition 2.3.

•For the weak-* star topology, on C(Ω)', the set of all "generalized incompressible flows" is compact. The same is true for the set of all generalized flows that reaches a given final configuration at a given time T, and for the set of all such flows that are incompressible•

3. The generalized shortest path problem.

In order to keep notations as simple as possible, integrals with respect to t (resp. x, resp. x and y, resp. z) are implicitely performed over [0,T] (resp. X, resp. XxX, resp. Ω), the initial density ρ_0 is simply denoted by ρ.

In the previous section, it has been established that one can associate to any classical flow g a unique generalized flow q defined as a probability measure on Ω by (2.1). A formal substitution into (2.1) of $F(z) = \int \|z'(t)\|^2 dt$ leads to :

(3.1) $\int\int \|z'(t)\|^2 dt\ q(dz) = \int\int \|\partial_t g(t,x)\|^2 dt\ dx.$

In the same way, by substituting for F into (2.1) :

(3.2) $a(z) = \rho(z(0))\{1/2\int \|z'(t)\|^2 dt - \int U(t,z(t))dt\},$

one gets the formal identity :

(3.3) $\int \rho(z(0))\{1/2\int \|z'(t)\|^2 dt - \int U(t,z(t))dt\}q(dz) = \int\int \rho(x)\{1/2\ \|\partial_t g(t,x)\|^2 - U(t,g(t,x))\}\ dt\ dx.$

These formulae provide formal but simple definitions of the kinetic energy and the Action as *linear* functionals on the set of all generalized flows. The goal of this section is to get rigorous definitions and, then, to show that the shortest path problem, set in the class of all generalized incompressible flows, always has a solution provided that the final configuration can be reached by at least one generalized incompressible flow having a finite kinetic energy . To state these results, we need :

Assumption 3.1.

• ρ is a lower semi-continuous mapping from X into $[0,+\infty[$ and belongs to $L^r(X)$ for some r, $1 \leq r \leq +\infty$; U is in $L^s([0,T]xX)$ if $r>1$, and in C([0,T]xX) if $r=1$, where $1/r+1/s=1$ •

Then, we have one of our main results :

Theorem 3.2.

•For any generalized incompressible flow q, the generalized kinetic energy :

(3.4) $E(q) = \int \rho(z(0)) \int 1/2 \|z'(t)\|^2 dt\ q(dz)$ is well defined in $[0,+\infty]$.

If E(q) is finite, then one can define the generalized Action :

(3.5) $A(q) = \int \rho(z(0))\{\int 1/2\|z'(t)\|^2 dt - \int U(t,z(t))dt\}\ q(dz).$

•For any generalized final configuration η , if there is one generalized incompressible flow that can reach η at time T with a finite kinetic energy, then there is such a flow that minimizes the Action•

The proof is based on several technical results :

Proposition 3.3.

•For any z in Ω, let us define :

(3.6)
$e(z) = 0$ if $\rho(z(0)) = 0$,
$e(z) = \rho(z(0))\int 1/2\|z'(t)\|^2 dt$ if z is in the Sobolev space $H^1[0,T]$ and $e(z) = +\infty$ otherwise.

Then e is a lower semi-continuous mapping from Ω into $[0,+\infty]$ and the upper integral :

(3.7) $E(q) = \int^* e(z)\ q(dz)$

defines a (weak-*) lower semi-continuous mapping from the set of all probability measures on Ω, Prob(Ω), into $[0,+\infty]$. Moreover :

(3.8) $E(q) = \sup\ \{ \int e_\tau(z)\ q(dz)\ ;\ \tau$ finite subset of $[0,T]\}$, where :

(3.9) $e_\tau(z) = \rho(z(0))\Sigma\ 1/2\|z(t_k)-z(t_{k-1})\|^2\ (t_k-t_{k-1})^{-1}$ for $\tau=\{t_1,\ldots,t_n\}$, $0\leq t_1\leq\ldots\leq t_n\leq T$,

defines a lower semi-continuous mapping from Ω into $[0,+\infty[$•

Proposition 3.4.

•For any generalized incompressible flow q such that E(q) is finite,

(3.10) $V(z) = \rho(z(0))\int U(t,z(t))dt$

is well defined as a q-integrable function on Ω and :

(3.11) $\int |V(z)|q(dz) \leq \|\rho\|_{L^r}\ \|U\|_{L^s}.$

(3.12) If $\rho = cst = 1$, then $\int V(z)q(dz) = \int\int U(t,x)dt\ dx$•

Proposition 3.5.

•If (q_m) is a sequence of generalized incompressible flows, with uniformly bounded kinetic energy, that converges toward q in the weak-* sense, then :

$$E(q) \leq \liminf E(q_m) \text{ and } \int V(z)q(dz) = \lim \int V(z)q_m(dz)\bullet$$

Proof of proposition 3.3.

• According to classical results [Bourbaki, Integration] on the integration of lower semi-continuous (l.s.c.) functions valued in $[0,+\infty]$, it is enough to prove that i) the family (e_τ), where $\tau=\{t_1,\dots,t_n\}$ $0\leq t_1<\dots<t_n\leq T$ are arbitrary finite subsets of $[0,T]$, is a filtering increasing family of positive l.s.c. functions defined on Ω, ii) e can be written as the supremum of the e_τ.

i) ρ is a positive l.s.c. mapping from X into $[0,+\infty[$ and, thus, $z\to\rho(z(0))$ is also positive and l.s.c. on Ω for the product topology ; $z\to\Sigma\,\|z(t_k)-z(t_{k-1})\|^2\,(t_k-t_{k-1})^{-1}$ is a positive "functional of finite type", continuous on Ω. Therefore, as a product of a l.s.c. function by a continuous function that are both valued in $[0,+\infty[$, each e_τ is also l.s.c and valued in $[0,+\infty[$. The fact that (e_τ) (which is not a countable family !) is filtering increasing follows from the Cauchy-Schwarz inequality. Indeed, for any sequence $0\leq t_1<\dots<t_n\leq T$, we have :

$$\|z(t_n)-z(t_1)\|^2\,(t_n-t_1)^{-1}\ \leq \Sigma_{k=2,n}\,\|z(t_k)-z(t_{k-1})\|^2\,(t_k-t_{k-1})^{-1}$$

Therefore, given two finite sets τ_1, τ_2 in $[0,T]$, by setting $\tau_3=\tau_1\cup\tau_2$, one easily deduces : $e_{\tau_3}\geq\max(e_{\tau_1},e_{\tau_2})$. This precisely means that (e_τ) is filtering increasing.

ii) Let us now prove that $e=e^\#$, where $e^\#=\sup e_\tau$ in two steps :

a) if z belongs to $H^1[0,T]$, then $e^\#(z)\leq e(z)<+\infty$;

b) if $e^\#(z)<+\infty$, then :

either $\rho(z(0))=0$ and $e^\#(z)=e(z)$ or z belongs to $H^1[0,T]$ and $e(z)\leq e^\#(z)$.

The first statement is another trivial consequence of the Cauchy-Schwarz inequality, since, for any z in $H^1[0,T]$ and any set $\tau=\{t_1,\dots,t_n\}$, $0\leq t_1<\dots<t_n\leq T$, one has :

$\Sigma_{k=2,n}\,\|z(t_k)-z(t_{k-1})\|^2\,(t_k-t_{k-1})^{-1}\ \leq\int\|z'(t)\|^2dt$,

and, thus, $e_\tau(z)\leq e(z)$.

Let us now prove the second statement. Let z be a path such that $e^\#(z)<+\infty$. If $\rho(z(0))=0$, then both $e(z)$ and $e^\#(z)$ are equal to zero. Let us consider the case when $\rho(z(0))>0$. For each pair $t_1<t_2$, we get for $\tau=\{t_1,t_2\}$: $e_\tau(z)=\rho(z(0))\|z(t_2)-z(t_1)\|^2\,(t_2-t_1)^{-1}\ \leq e^\#(z)<+\infty$. Thus, z is necessarily Hölder continuous from $[0,T]$ into X, and it makes sense to consider its derivative z' in the sense of distributions. For any C^∞ compactly supported mapping ζ from $[0,T]$ into R^d, $\langle z',\zeta\rangle=-\int\zeta'(t).z(t)dt$. Since ζ,ζ' and z are continuous, for any $\varepsilon>0$, one can find a finite set $\tau=\{t_1,\dots,t_n\}$, $0=t_1<\dots<t_n=T$, such that :

$$|-\langle z',\zeta\rangle\ -\Sigma_{k=1,n-1}\,(\zeta(t_{k+1})-\zeta(t_k))\,z(t_k)\,|\leq\varepsilon$$

$$|\,\Sigma_{k=2,n}\,\|\zeta(t_k)\|^2(t_k-t_{k-1})\ -\int\|\zeta(t)\|^2dt\,|\leq\varepsilon.$$

Since ζ vanishes at 0 and T, one gets :

$$\Sigma_{k=1,n-1}\,(\zeta(t_{k+1})-\zeta(t_k))\,z(t_k)=-\Sigma_{k=2,n}\,(z(t_k)-z(t_{k-1}))\,\zeta(t_k)$$

$\leq (\Sigma_{k=2,n} \|z(t_k)-z(t_{k-1})\|^2 (t_k-t_{k-1})^{-1})^{1/2} (\Sigma_{k=2,n} \|\zeta(t_k)\|^2 (t_k-t_{k-1}))^{1/2}$
$\leq (\rho(z(0))^{-1} e^{\#}(z))^{1/2} (\int \|\zeta(t)\|^2 dt + \varepsilon)^{1/2}$

Thus :

$| \langle z',\zeta \rangle | \leq \varepsilon + (\rho(z(0))^{-1} e^{\#}(z))^{1/2} (\int \|\zeta(t)\|^2 dt + \varepsilon)^{1/2}$

Since ε is arbitrary, it follows that z belongs to $H^1[0,T]$ and $\int \|z'(t)\|^2 dt \leq \rho(z(0))^{-1} e^{\#}(z)$. Thus $e(z) \leq e^{\#}(z)$, which proves statement b) and, therefore, achieves the proof of Proposition 3.3•

Proof of propositions 3.4 and 3.5.

•Since ρ is a positive integrable l.s.c. function on X, for each integer $n>0$, there exists a continuous approximation ρ_n such that $0 \leq \rho_n \leq \rho$ and $\| \rho_n - \rho \|_{L^1} \leq 1/n$.

Since U is in $L^s([0,T]xX)$ if $r>1$, and in $C([0,T]xX)$ if $r=1$, where $1/r+1/s=1$, there is, for each integer $m>0$, a Lipschitz continuous approximation U_m such that : $\| U_m - U \|_{L^s} \leq 1/m$.

To show that $V(z) = \rho(z(0)) \int U(t,z(t))dt$ can be well defined as a q-integrable function for any generalized flow q having a finite kinetic energy $E(q) = \int^* e(z)\, q(dz) < +\infty$, let us introduce, for any integers $n,m,k>0$, the following approximations to V (where $T=1$ is set for simplicity) :

(3.13) $\quad V_{nmk}(z) = \rho_n(z(0))\, k^{-1} \Sigma_{j=1,k} U_m(j/k,z(j/k)),$

(3.14) $\quad V_{nm}(z) = 0$ if $\rho_n(z(0))=0$, and $\rho_n(z(0)) \int U_m(t,z(t))dt$ if z belongs to H^1.

Notice that V_{nm} is defined q-almost everywhere, since $E(q)$ is finite, which implies q-almost surely either $\rho(z(0))=0$ (and therefore $\rho_n(z(0))=0$) or $z \in H^1$. Also notice that, in the special case when ρ is identically equal to 1, one can take ρ_n identically equal to 1, and, then, since q is assumed to be incompressible :

(3.15) $\quad \int V_{nmk}(z) q(dz) = k^{-1} \Sigma_{j=1,k} \int U_m(j/k,x)dx$ (if $\rho=1$).

By integrating in z over Ω, we get for any n,n',m,k :

$\int |V_{nmk}(z) - V_{n'mk}(z)|\, q(dz) \leq \sup |U_m| \int |\rho_n(z(0)) - \rho_{n'}(z(0))|\, q(dz)$

$= \sup |U_m| \, \| \rho_n - \rho_{n'} \|_{L^1}$ (since q is incompressible).

By using Hölder inequalities (twice), we also get for any n,m,m',k :

$\int |V_{nmk}(z) - V_{nm'k}(z)| q(dz) \leq (\int \rho_n(z(0))^r q(dz))^{1/r} (k^{-1} \Sigma_{j=1,k} \int |U_m(j/k,z(j/k)) - U_{m'}(j/k,z(j/k))|^s q(dz))^{1/s}$

$= (\int \rho_n(x)^r dx)^{1/r} (k^{-1} \Sigma_{j=1,k} \int |U_m(j/k,x) - U_{m'}(j/k,x)|^s dx)^{1/s}$ (since q is incompressible).

It follows that :

(3.16) $\| V_{nmk}-V_{nm'k} \|_{L^1} \leq \| \rho \|_{L^r} (\| U_m-U_{m'} \|_{L^s} + \delta(k)(c_m+c_{m'}))$

where c_m depends only on U_m and $\delta(k)$ tends to 0 when k tends to$+\infty$.

In the same way, one obtains :

(3.17) $\| V_{nmk}\|_{L^1} \leq \| \rho \|_{L^r} (\| U_m\|_{L^s} + \delta(k)\, c_m)$.

In the special case $\rho=1$, one also gets :

(3.18) $|\int V_{nmk}(z)q(dz) - \int\int U_m(t,x)dt\, dx \,| \leq k^{-1}c_m$.

Let us now consider a path z such that $\rho(z(0))>0$. It belongs to $H^1[0,T]$ q-almost surely and therefore :

$|V_{nmk}(z)-V_{nm}(z)| \leq \rho_n(z(0))\, k^{-1}c_m \int(1+ |z'(t)|\,)dt \leq 2\rho_n(z(0))\, k^{-1}c_m \;\; (\int (1+ |z'(t)|^2)dt \,)^{1/2}$.

If $\rho(z(0))=0$, then $\rho_n(z(0))=0$, and the right-hand side is anyway equal to zero. Thus, after integrating with respect to z over Ω, one gets (by using the Schwarz inequality):

$\int |V_{nmk}(z)-V_{nm}(z)|q(dz) \leq 2k^{-1}c_m \;\; (\int \rho_n(z(0))q(dz) \int \rho_n(z(0)) \int (1+ |z'(t)|^2)dt \;\, q(dz))^{1/2}$

Since q is incompressible, one has :

$\int \rho_n(z(0))q(dz) = \int \rho_n(x)dx \leq \int \rho(x)dx$,

and it follows that :

(3.19) $\| V_{nmk}-V_{nm} \|_{L^1} \leq 1/k\, cc_m$ (where c depends only on E(q) and $\| \rho \|_{L^1}$).

These estimates show that, for any integer i>0, one can choose $m=m_i$, then $n=n_i$, $k=k_i$, in such a way that $W_i=V_{n_i m_i k_i}$ satisfies :

(3.20) $\| V_{n_i m_i}-W_i \|_{L^1} \leq 1/i$,

(3.21) $\| W_{i'}-W_i \|_{L^1} \leq 1/i + 1/i'$,

(3.22) $\| W_i \|_{L^1} \leq \| \rho \|_{L^r} (\| U \|_{L^s} +1/i)$,

and:

(3.23) $| \int W_i(z)q(dz) - \int\int U(t,x)dt\, dx \,| \;\; \leq 1/i \quad$ if $\rho=1$.

It follows that (W_i) is a Cauchy sequence in L^1 and has a unique limit V, that can be formally defined by (3.10) : $\quad V(z) = \rho(z(0))\int U(t,z(t))dt$. Moreover :

(3.24) $\| V \|_{L^1} \leq \| \rho \|_{L^r} \| U \|_{L^s}$

and:

(3.25) $\int V(z)q(dz) = \int\int U(t,x)dt\, dx$ if $\rho=1$.

This achieves the proof of proposition 3.4.

To prove proposition 3.5, let us consider a sequence (q_m) of generalized incompressible flows that weak-* converges toward q and satisfies : sup $E(q_m) < +\infty$. Necessarily, q belongs to the set of all generalized incompressible flows since this set is weak-* close (Proposition 2.3). Moreover, the generalized kinetic energy E is weak-* l.s.c. Therefore, $E(q) \leq \liminf E(q_m) < +\infty$, which proves the first part of proposition 3.5. To prove the remainder, let us first remark that all the estimates that have been previously obtained for the Cauchy sequence (W_i) are uniform in q, as long as E(q) stays uniformly bounded. It follows that the $L^1(\Omega,q_m)$ norm of $V-W_i$ uniformly goes to 0 when i tends to $+\infty$. Since each W_i is a continuous functional (of finite type) and q_m converges toward q in the weak-* sense, we get, for a fixed i :

$\int W_i(z)q(dz) = \lim \int W_i(z)q_m(dz)$.

Then,the fact that V is uniformly (with respect to m) approximated by W_i shows that V is q-integrable and :

$\int V(z)q(dz) = \lim \int V(z)q_m(dz)$.

This achieves the proof of proposition 3.5•

Proof of Theorem 3.2.

•Let us call P_η the set of all generalized incompressible flows q that reach η at time T with a finite kinetic energy $E(q)<+\infty$. By proposition 3.4, for each q in P_η, $V(z) = \rho(z(0))\int U(t,z(t))dt$ defines a q-integrable function and, by (3.11), $\int |V(z)|\, q(dz)$ is uniformly bounded by a constant C that depends only on ρ and U. Therefore, one gets for each q in P_η:

$-\infty < -C \leq A(q) = \int (V(z)+e(z))\, q(dz) \leq E(q) + C < +\infty$.

Thus, there certainly exists a minimizing sequence (q_m) in P_η such that :

$\lim A(q_m) = A_{opt} = \mathrm{Inf}\{A(q),\ q \text{ in } P_\eta\}$ and : $C' = \sup A(q_m) < +\infty$.

By proposition (2.3), it is not a restriction to suppose that (q_m) converges toward a generalized incompressible flow q that reaches η at time T. Moreover since : $\sup E(q_m) \leq C+C' < +\infty$, proposition 3.5 can be used. Thus : $E(q) \leq \liminf E(q_m)$ and $\int V(z)\, q(dz) = \lim \int V(z)\, q_m(dz)$. It follows that q has a finite kinetic energy and $A(q) \leq \lim A(q_m) = A_{opt}$, which shows that q is optimal and achieves the proof•

4. A complete existence result in the case of the d-dimensional torus.

By theorem 3.2, we know that, for a given $T>0$ and a given final configuration η, there exists an optimal generalized incompressible flow that minimizes the Action, ***provided*** that η can be reached at time T by at least one generalized incompressible flow with finite kinetic energy. Thus our main concern is now to exhibit such a flow. A complete answer can be obtained in the case of the d-dimensional torus $X=R^d/Z^d$ (that is the d-dimensional unit cube with periodic boundary conditions) or, slightly more generally, in the case when X and dx satisfy :

Property 4.1.

•There exists a constant C and a measurable mapping $(t,x,y)\rightarrow\gamma(t,x,y)$ from $[0,1]\times X\times X$ into X such that :

(4.1) $\gamma(0,x,y)=x,\ \gamma(1,x,y)=y$;

(4.2) $\int\|\partial_t\gamma\,(t,x,y)\|^2dt \leq C$, for almost every x,y in X ;

(4.3) $\iint f(\gamma(t,x,y))dxdy =\int f(x)dx$ for any f in C(X) and any t in [0,1]•

This property is satisfied by $X= R^d/Z^d$ and dx=Lebesgue measure, once one defines $t\rightarrow\gamma(t,x,y)$ to be the geodesical path between x and y, which is uniquely defined for almost every pair of points on the torus. Indeed, (4.1-2) are obvious and, from the straightforward translation invariance property :

(4.4) $\gamma(t,x,y)=x+ \gamma(t,0,y-x)$,

one easily deduces for any f in C(X) :

$\iint f(\gamma(t,x,y))dxdy = \iint f(x+ \gamma(t,0,y-x))dxdy = \iint f(x+ \gamma(t,0,y'))dxdy' = \iint f(x)dxdy' =\int f(x)dx.$

Thanks to property 4.1, we can prove our main result :

Theorem 4.2.

•Assume that $X=R^d/Z^d$ (or property 4.1 holds); ρ is a lower semi-continuous mapping from X into $[0,+\infty[$ and belongs to $L^r(X)$ for some r, $1\leq r\leq+\infty$; U is in $L^s([0,T]\times X)$ if $r>1$, and in $C([0,T]\times X)$ if $r=1$, where $1/r+1/s=1$. Then, for *any* final configuration η (i.e. *any* doubly stochastic probability measure on $X\times X$) and any final time $T>0$, there exists a generalized incompressible flow that reaches η at time T with a finite kinetic energy and minimizes the Action :
$A(q)=\int\rho(z(0))\{\int 1/2\|z'(t)\|^2dt-\int U(t,z(t))dt\}q(dz)$•

This result is a straightforward corollary of theorem 3.2 and :

Proposition 4.3.

•If $X=R^d/Z^d$, then for any final configuration η and any final time $T>0$, there exists a generalized incompressible flow q that reaches η at time T with a finite kinetic energy :
$E(q)=\int\rho(z(0))\int 1/2\|z'(t)\|^2 dt\, q(dz) \leq 2C/T \int \rho(x)dx$
where C is the constant considered in property 4.1•

Proof of proposition 4.3.

•Let us explicitely define q by :
(4.5) $\int F(z)q(dz) = \iiint F[\, t\to G(t,x,x',y)]\, \eta(dx,dy)dx'$, for each F in $C_{fin}(\Omega)$,
where :
(4.6) $G(t,x,x',y) = \gamma(2t/T,x,x')$ if $0\leq t\leq T/2$, $\gamma(2-2t/T,x',y)$ if $T/2\leq t\leq T$.
This intuitively means that a particle issued from x at time 0 can reach at time T/2 any point x' in X with the same uniform probability and then reaches y at time T according to the probability law $\eta(dx,dy)$. During the intermediate times, $0\leq t\leq T/2$ and $T/2\leq t\leq T$, each particle follows a geodesic on the torus.To prove proposition 4.3, it is sufficient to check that i) q is a generalized incompressible flow, ii) q reaches η at time T , iii) q has a finite kinetic energy.
i) G is a measurable mapping from $[0,1]\times X\times X$ into X and (4.5) defines a generalized flow on X. To check that q is incompressible, let us fix t in [0,T] and f in C(X). For $0\leq t\leq T/2$, one has :
$\int f(z(t))q(dz) = \iiint f(\gamma(2t/T,x,x'))\, \eta(dx,dy)dx'$ (by definitions (4.5-6))
$= \iint f(\gamma(2t/T,x,x'))dxdx'$ (since η is doubly stochastic)
$= \int f(x)dx$ (by property (4.3)). One gets the same result when $T/2\leq t\leq T$, for the same reasons.
ii) For any f in $C(X\times X)$, one has :
$\int f(z(0),z(T))q(dz) = \iiint f(G(0,x,x',y),G(T,x,x',y))\, \eta(dx,dy)dx'$ (by definition (4.5))
$= \iiint f(x,y)\, \eta(dx,dy)dx'$ (by definition (4.6) and property (4.1)) $= \iint f(x,y)\eta(dx,dy)$.
Thus, q reaches η at time T.
iii) By definition of q, one immediately gets :
$E(q)=\iiint \rho(G(0,x,x',y))[\int 1/2\|\partial_t G(t,x,x',y)\|^2 dt]\, \eta(dx,dy)dx'$ (definition (4.5))
$= 1/T \iiint \rho(x)[\int \|\partial_t\gamma(\tau,x,x')\|^2 d\tau + \int \|\partial_t\gamma(\tau,x',y)\|^2 d\tau\,]\, \eta(dx,dy)dx'$ (by definition (4.6) ; here the $d\tau$-integrals are performed over the unit interval [0,1])
$\leq 2C/T \iiint \rho(x)\, \eta(dx,dy)dx'$ (by definition of C in property 4.1)
$= 2C/T \int \rho(x)dx$ (since η is doubly stochastic).
This achieves the proof of proposition 4.3•

5. Classical solutions and generalized solutions.

In this section, it is shown that, under a natural restriction on the time scale, any classical solution to the Euler equations satisfies the generalized Least Action principle and is the unique solution to the corresponding generalized shortest path problem. This is a consequence of the following result :

Theorem 5.1.

• Let q be a generalized incompressible flow and let η be the final configuration that it reaches at time T. Assume that :

(5.1) ρ is bounded away from zero by some constant $\alpha > 0$;

U is Lipschitz continuous on $Q = [0,T] \times X$ and there is a constant C such that :

(5.2) $D^2U(t,x) \leq C$ (in the sense of distributions and symmetric matrices);

there exists a Lipschitz continuous "pressure field" $p(t,x)$ on Q such that :

(5.3) $D^2p(t,x) \leq R$ holds (in the same sense as above) for some real constant R;

q-almost surely, a path z in Ω belongs to the Sobolev space $H^1[0,T]$ and satisfies the ordinary differential equation :

(5.4) $\rho(z(0))\{z''(t) + \mathrm{grad}U(t,z(t))\} + \mathrm{grad}p(t,z(t)) = 0$, in an appropriate sense ;

the kinetic energy E(q) is finite.

Then q satisfies the generalized Least Action Principle in the sense that it minimizes the Action $A(q)=\int\rho(z(0))\{\int 1/2\|z'(t)\|^2dt-\int U(t,z(t))dt\}q(dz)$ among all generalized incompressible flows that reach the same final configuration η at time T, provided that inequality:

(5.5) $(R\alpha^{-1}+ C)T^2 \leq \pi^2$ holds.

If this inequality is strict, then q is the unique minimizer and has the following "deterministic" property : q-almost surely, two paths z and $z^{\#}$ that satisfy $z(0)= z^{\#}(0)$ and $z(T)= z^{\#}(T)$ are equal. Conversely there are cases when either q is not the unique minimizer or q is not "deterministic" and

(5.6) $(R\alpha^{-1}+ C)T^2 = \pi^2$•

Before proving theorem 5.1, let us introduce some notations and prove some technical results. For any path z in $H^1[0,T]$, let us define :

(5.7) $B(z) =\int \{\rho(z(0))[1/2\|z'(t)\|^2 -U(t,z(t))] - p(t,z(t))\}dt$,

and, for any x and y in X :

(5.8) $b(x,y) = \mathrm{Inf}\,\{B(z), z\in\Omega$, z belongs to $H^1[0,T]$, $z(0)=x$ and $z(T)=y\}$.

Then ,we have :

Lemma 5.2.

•Under the same assumptions as in theorem 5.1, if inequality (5.5) holds, then q-almost every path z satisfies :

(5.9) $B(z) = b(z(0),z(T))$.

If the inequality is strict, then, for any path $z^\#$ that belongs to $H^1[0,T]$,

(5.10) $B(z) = B(z^\#)$, $z(0) = z^\#(0)$, $z(T) = z^\#(T)$ imply : $z = z^\#$•

Proof of lemma 5.2.

•Let us first remark that, by definitions (5.7-8) of B and b, we obviously have :

(5.11) $B(z) \geq b(z(0),z(T))$, for *any* path z that belongs to $H^1[0,T]$.

Lemma 5.2 says that the corresponding *equality* holds q-almost everywhere on Ω as soon as inequality (5.5) is satisfied. Moreover, it says that, q-almost surely, there is only one z such that $B(z) = b(z(0),z(T))$ when z(0) and z(T) are prescribed and inequality (5.5) is strict.

Let us pick up an arbitrary path z in Ω. By assumption, q-almost surely, z is in $H^1[0,T]$ and is a solution to the ordinary differential equation (5.4), which exactly means that the first variation of $B(z^\#)$ (defined by (5.7)) vanishes at $z^\# = z$, when $z^\#$ is an arbitrary path in $H^1[0,T]$ such that $z^\#(0)=z(0)$ and $z^\#(T)=z(T)$. It does not a priori mean that $B(z^\#)$ *is minimal* at $z^\# = z$. This is why we need inequality (5.5) to make sure that z actually is a minimum point (and not a maximum or a saddle-point). The proof is an elementary application of the Poincaré inequality. Let us first introduce some notations :

(5.12) $\beta = \rho(z(0))$, $K = \beta C + R$ (C and R are defined in (5.2-3)),

(5.13) $\phi(t,x) = K/2\|x\|^2 - \beta U(t,x) - p(t,x) = \beta(1/2\, C\|x\|^2 - U(t,x)) + 1/2\, R\, \|x\|^2 - p(t,x)$.

Since both p and U are Lipschitz continuous on Q and satisfy (5.2-3), it follows that $\phi(t,x)$ is Lipschitz continuous and convex in x.Thus, one gets :

(5.14) $\phi(t,z^\#(t)) \geq \phi(t,z(t)) + w(t).(z^\#(t)-z(t))$, for every paths $z^\#$, z in $H^1[0,T]$,

and any measurable curve $t \in [0,T] \rightarrow w(t) \in \mathbf{R}^d$ such that :

(5.15) $w(t) \in \partial\phi(t,z(t))$ a.e.on $0 \leq t \leq T$, where $\partial\phi(t,.)$ is the subdifferential of $\phi(t,.)$

[Ekeland & Temam]. Equation (5.4) can be rewritten more accurately as a "multivocal" o.d.e. :

(5.16) $\beta z''(t) + Kz(t) \in \partial\phi(t,z(t))$, a.e. on $0 \leq t \leq T$,

which means :

(5.17) $\int \{-\beta z'(t).[z^{\#\prime}(t)-z'(t)] + [Kz(t)-w(t)].[z^\#(t)-z(t)]\}\,dt = 0$,

for any path $z^\#$ in $H^1[0,T]$ such that $z^\#(0)=z(0)$, $z^\#(T)=z(T)$ and some curve w that satisfies (5.15).

Because of the convexity of ϕ, it follows that :

$$\int \{-\beta z'(t).[z^{\#\prime}(t)-z'(t)] + Kz(t).[z^\#(t)-z(t)]\}\,dt \leq \int [\phi(t,z^\#(t)-\phi(t,z(t))]\,dt,$$

that is :

$$\int \{ [\beta/2\|z'(t)\|^2 - K/2\,\|z(t)\|^2 + \phi(t,z(t))] - [\beta/2\|z^{\#\prime}(t)\|^2 - K/2\,\|z^\#(t)\|^2 + \phi(t,z^\#(t))] \}\,dt$$
$$\leq \int \{ -\beta/2\|z^{\#\prime}(t)-z'(t)\|^2 + K/2\,\|z^\#(t)-z(t)\|^2 \}\,dt .$$

The left hand side is precisely $B(z)-B(z^{\#})$ (by definition (5.7-12-13) of K, β, o and B) and the right hand side can be estimated with the help of Poincaré's inequality :

$\int \|z^{\#}(t)-z(t)\|^2\,dt \leq (T/\pi)^2 \int \|z^{\#\prime}(t)-z'(t)\|^2 dt$, since $z^{\#}(0)-z(0)=z^{\#}(T)-z(T)=0$.

Thus, on gets :

$B(z)-B(z^{\#}) \leq 1/2\ ((T/\pi)^2K-\beta) \int \|z^{\#\prime}(t)-z'(t)\|^2 dt$.

By definition (5.12-13) of K and β, $(T/\pi)^2K-\beta$ is precisely negative when inequality (5.5) holds. Therefore $B(z)-B(z^{\#}) \leq 0$, which proves that z minimizes B among all paths $z^{\#}$ in $H^1[0,T]$ such that $z^{\#}(0)=z(0)$, $z^{\#}(T)=z(T)$. Moreover, if inequality (5.5) is strict, then z is the unique minimizer. This achieves the proof of lemma 5.2.

The proof of theorem 5.1. uses the following result :

Lemma 5.3.

• Let q be a generalized incompressible flow that reaches the final configuration η at time T with finite kinetic energy. Assume that $B(z)=b(z(0),z(T))$ holds q-almost everywhere on Ω (where b and B are defined by (5.7-8). Then q is optimal.

Proof of lemma 5.3.

•Since ρ is bounded away from zero and the kinetic energy E(q) is finite, $\iint \|z'(t)\|^2 dt\ q(dz)$ is also finite. It follows that q-almost every path z belongs to $H^1[0,T]$ and, by proposition 3.4,

(5.18) $V(z) = \rho(z(0))\int U(t,z(t))dt$

(5.19) $P(z) = \int p(t,z(t))dt$

define q-integrable functions on Ω. Moreover, (by taking $\rho=1$ in proposition 3.4) one gets :

(5.20) $\int P(z)q(dz) = \int\int p(t,x)dt\ dx$.

These properties also hold for any other generalized incompressible flow $q^{\#}$ that has a finite kinetic energy. For such a flow, let us compute the integral of B(z) (defined by (5.7)). We have $B(z)=e(z)-V(z)-P(z)$ where $e(z) = \rho(z(0))\int 1/2\|z'(t)\|^2 dt$. We get :

$\int B(z)q^{\#}(dz) = \int (e(z)-V(z))q^{\#}(dz) - \int P(z)q^{\#}(dz)$ (since e, V and P are $q^{\#}$-integrable).

$= \int (e(z)-V(z))q^{\#}(dz) - \int\int p(t,x)dt\ dx$ (by (5.20))

$= A(q^{\#}) - \int\int p(t,x)dt\ dx$ (by definition (3.5) of the Action). We deduce :

$A(q^{\#}) - A(q) = \int B(z)q^{\#}(dz) - \int B(z)q(dz)$, since the pressure term is the same for q and $q^{\#}$. To prove that q is optimal, it is therefore enough to show $\int B(z)q^{\#}(dz) \geq \int B(z)q(dz)$. To do that, let us consider the auxiliary function $z \in \Omega \rightarrow b(z(0),z(T))$. We know that $B(z) \geq b(z(0),z(T))$ holds everywhere on Ω (by (5.11)), while $B(z)=b(z(0),z(T))$ is true q-*almost* everywhere, by assumption. Thus, $z \in \Omega \rightarrow b(z(0),z(T))$ is q-integrable and therefore b(x,y) is $\eta(dx,dy)$ integrable, since q reaches η at time T where η is defined by : $\int f(z(0),z(T))q(dz) = \int f(x,y)\eta(dx,dy)$ for each f in $C(X \times X)$. It follows that $z \in \Omega \rightarrow b(z(0),z(T))$ is also $q^{\#}$-integrable, since $q^{\#}$ also reaches η at time T. Thus :

$\int b(z(0),z(T))q^{\#}(dz) = \int b(z(0),z(T))q(dz) = \int\int b(x,y)\eta(dx,dy)$,

and consequently :

$\int B(z)q^{\#}(dz) \geq \int b(z(0),z(T))q^{\#}(dz) = \int b(z(0),z(T))q(dz) = \int B(z)q(dz)$,

which achieves the proof of lemma 5.3•

Proof of Theorem 5.1.

•The first statement of theorem 5.1 immediately follows from lemma 5.1 and 5.2 : under assumptions (5.1...5), the generalized flow q is necessarily optimal. Let us now consider the case when inequality (5.5) is strict. From lemma 5.2, it is easy to deduce that q is deterministic. Indeed, let us consider two paths z and $z^{\#}$ that satisfy $z(0)=z^{\#}(0)$ and $z(T)=z^{\#}(T)$. We obviously have $b(z^{\#}(0),z^{\#}(T))=b(z(0),z(T))$ and, q-almost surely, z and $z^{\#}$ belong to $H^1[0,T]$.Thus, by lemma 5.2 (first satement), we deduce $B(z)=B(z^{\#})$. By lemma 5.2 (second statement) again, it follows that $z=z^{\#}$, which precisely shows that q is "deterministic".

Let us now prove that q is the unique optimal flow that reaches η at time T. Let us consider another generalized incompressible flow $q^{\#}$ that has a finite kinetic energy, reaches η at time T and has the same Action as q. We have seen earlier that : $A(q^{\#}) - A(q) = \int B(z)q^{\#}(dz) - \int B(z)q(dz)$
$= \int B(z)q^{\#}(dz) - \int b(z(0),z(T))q(dz) = \int \{B(z)-b(z(0),z(T))\}q^{\#}(dz)$.

Since $B(z)-b(z(0),z(T))\geq 0$ and $A(q^{\#})-A(q)=0$, it follows that : $B(z)=b(z(0),z(T))$ holds $q^{\#}$-almost everywhere on Ω.We know that the same property also holds q-almost everywhere on Ω. From the last statement of lemma 5.2 it follows that, for q-almost every z and $q^{\#}$-almost every $z^{\#}$, $z(0)=z^{\#}(0)$, $z(T)=z^{\#}(T)$ implies $z=z^{\#}$. Since q and $q^{\#}$ reach the same final configuration, we conclude that they must be equal.

This achieves the proof of theorem 5.1, except for the last statement about the case when equality (5.6) holds. There is a trivial example when (5.6) holds for which the minimizer is not unique : let us consider the two-dimensional motion inside the unit disk of an homogeneous fluid ($\rho(x)$=cst=1) without external forces. To reach at time $T=\pi$ the "mirror" final configuration $\eta(dx,dy) = \delta(y-h(x))dx$, where $h(x_1,x_2)=(-x_1,-x_2)$, there are two different classical solutions to the Euler equations corresponding to two rotational flows with opposite constant angular velocities :

$g_\varepsilon(t, x_1,x_2) = (x_1\cos(\varepsilon t)+ x_2\sin(\varepsilon t), - x_1\sin(\varepsilon t)+ x_2\cos(\varepsilon t))$, $0\leq t\leq\pi$, $\varepsilon=\pm 1$.

In both cases the pressure field is $p(t,x)=1/2\|x\|^2$ and thus R=1. Since $\alpha=1$ and C=0, equality (5.6) becomes $RT^2=\pi^2$ and therefore is trivially satisfied.

There are also cases when (5.6) holds for which solutions to the shortest path problem are not deterministic. This will be discussed in section 6.

6. Explicit examples of non deterministic generalized solutions to the Euler equations.

In this section, explicit examples of solutions to the generalized shortest path problem are obtained in the case of homogeneous ($\rho=1$) fluids without external forces ($U=0$). These solutions are of the form considered in section 5 and theorem 5.1 : the generalized incompressibility condition is satisfied, their kinetic energy is finite and there exists a smooth pressure field p such that each trajectory z satisfies the dynamical equation : $z''=-\mathrm{grad}p$. Therefore, they satisfy the least action principle in the sense that they are solutions to the generalized shortest path problem, provided that the final time T is not too large. Since the trajectories are not deterministic in general, these solutions can be considered as *probabilistic* generalized solutions to the Euler equations. *In each case*, the corresponding velocity field is a stationnary measure-valued solution in the sense of DiPerna and Majda.The first step to get these explicit solutions is :

Proposition 6.1.

•Let p be a "pressure" field given in $W^{2,\infty}(\mathbf{R}^d)$ and G be the global flow map :

(6.1) $(t,x,v)\in \mathbf{R} \times \mathbf{R}^d\times \mathbf{R}^d \rightarrow (X(t,x,v),V(t,x,v))\in \mathbf{R}^d\times \mathbf{R}^d$,

corresponding to the ordinary differential system in $\mathbf{R}^{2d}$: $x'=v$, $v'=-\mathrm{grad}p(x)$.

Assume that, for some real c, $X=\{x\in \mathbf{R}^d ; p(x) \leq c\}$ defines a smooth compact subset of $\mathbf{R}^d$ of Lebesgue measure equal to 1.Then $K=\{(x,v); p(x)+1/2\|v\|^2 \leq c\}$ defines a compact subset of $\mathbf{R}^d\mathrm{x}\mathbf{R}^d$, invariant under G, and its x-projection is X.

Let $\mu(dx,dv)$ be a probability measure on K, **invariant under G**, i.e.

(6.2) $\iint f(X(t,x,v),V(t,x,v))\mu(dx,dy) = \iint f(x,v)\mu(dx,dv)$, for any time t and f in C(K).

Then $\mu(dx,dv)$ always is a stationnary measure-valued solution to the Euler equations [DiPerna and Majda]. Given T>0, the generalized flow q defined by :

(6.3) $\int F(z)q(dz) =\iint F\,[t\rightarrow X(t,x,v)]\ \mu(dx,dv)$, for each F in $C_{fin}(\Omega)$, $\Omega=X^{[0,T]}$,

is incompressible on X, if and only if :

(6.4) for any f in C(X), $\iint f(x)\mu(dx,dv) =\int_X f(x)dx$.

Moreover, if $D^2p(x) \leq \pi^2/T^2$ holds a.e. on X, then q solves the shortest path problem for the final configuration η defined by :

(6.5) $\iint_{X\mathrm{x}X} f(x,y)\eta(dx,dy) = \iint f(x,X(T,x,v))\mu(dx,dv)$ for any f in $C(X\mathrm{x}X)$•

Remark.

•It is very natural to consider $\mu(dx,dv)$ as the (time independent) velocity field associated with q.

$\int f(z(t),z'(t))q(dz) = \iint f(X(t,x,v),V(t,x,v))\mu(dx,dv)$ (by definition (6.3)) $= \iint f(x,v)\mu(dx,dv)$ (by property (6.2))•

Proof of proposition 6.1.

•Since p belongs to $W^{2,\infty}(R^d)$, $(x,v)\to(v,-\mathrm{grad}p(x))$ defines a Lipschitz continuous vector field on $R^d x R^d$ and the flow map G is globally defined on $R \times R^d \times R^d$. Since $H(x,v) = p(x)+1/2\|v\|^2$ is constant along each trajectory, K is invariant under G ; it is also easy to check that K is compact and X is the x-projection of K.

Let us first prove that $\mu(dx,dv)$ is a time independent solution to the Euler equations in the sense of DiPerna and Majda. From the invariance property (6.2), one deduces for any time t and any smooth bounded function f defined on X :

(6.6) $\iint f(X(t,x,v))\mu(dx,dv) = \iint f(x)\mu(dx,dv)$,

(6.7) $\iint V(t,x,v)f(X(t,x,v))\mu(dx,dv) = \iint vf(x)\mu(dx,dv)$.

Since $(X(t,x,v),V(t,x,v))$ is solution to $x'=v$, $v'=-\mathrm{grad}p(x)$, by expanding these equalities with respect to t about 0, we deduce :

(6.8) $\iint v.\mathrm{grad}f(x)\mu(dx,dv) = 0$, $\iint[v(v.\mathrm{grad}f(x))-\mathrm{grad}p(x)f(x)]\mu(dx,dv) = 0$,

which means that μ is a time independent measure valued solution to the Euler equations [DiPerna & Majda]. Let us now consider the generalized flow q defined by (6.3). This flow is well defined on X since for μ-almost every (x,v) in $R^d x R^d$, the curve $t\to X(t,x,v)$ is valued in X (indeed G is invariant under G and X is the x-projection of K).

The incompressibility condition means $\int f(z(t))q(dz) = \int f(x)dx$, for any t and f in C(X). Since, from definition (6.3) and property (6.2) $\int f(z(t))q(dz) = \iint f(X(t,x,v))\mu(dx,dy) = \iint f(x)\mu(dx,dv)$, it follows that q is a generalized incompressible flow on X if and only if condition (6.4) holds. The kinetic energy of q, which is exactly equal to $1/2\iint\|v\|^2\mu(dx,dv)$, is finite. Finally, since q-almost every path z in Ω is of the form $z(t) = X(t,x,v)$ and is solution to the ordinary differential equation : $z''=-\mathrm{grad}p(z)$, it follows from theorem 4.1 that q is a solution to the generalized shortest path problem for the final configuration η given by (6.5), which achieves the proof•

Among the invariant positive measures for the Hamiltonian system $x'=v$, $v'=-\mathrm{grad}p$, there is obviously dxdv (by Liouville's theorem) and also any measure μ of the form :

(6.9) $\mu(dx,dv) = \lambda(H(x,v))dxdv$, $H(x,v) = 1/2\|v\|^2 + p(x)$,

where λ is an arbitrary positive function (indeed, H is constant along the trajectories).

For these measures, that are uniform on each level set of H, we have :

Proposition 6.2.

•In the same conditions as in proposition 6.1, let μ be a probability measure of the form (6.9). Then, although μ always is a measure valued solution to the Euler equations in the sense of DiPerna and Majda, the corresponding generalized flow q defined by (6.3) is incompressible (and, therefore, solves the generalized shortest path problem) only in the following cases :
i) $\lambda(t) = Cst.\ \delta(t-c)$, when d=2 ; ii) $\lambda(t) = Cst.\ (c-t)^{-1/2}$, when d=1 ; iii) and never when d=3•

Proof of proposition 6.2.

•If μ is of the form (6.9), and more generally if μ is a measure depending only on x and $\|v\|$ (which means that, at each point x, the velocity is isotropically distributed), then it is automatically a measure solution in the sense of DiPerna and Majda.

Indeed, if $\mu(dx,dv) = \sigma(x,\|v\|)dxdv$, one gets :

$\iint v_i f(x)\mu(dx,dv) = \iint v_i\, \sigma(x,\|v\|)f(x)dxdv = 0,$

$\iint v_i\, v_j f(x)\mu(dx,dv) = \iint v_i\, v_j\, \sigma(x,\|v\|)f(x)dxdv = \delta_{ij} \int \phi(x)f(x)dx.$ where :

$\phi(x)=1/d \int \|v\|^2\, \sigma(x,\|v\|)dv.$

Then, it is easy to check that μ is a measure valued solution to the Euler equations.

Conversely, the requirement that the associated generalized flow q is incompressible (and, therefore, satisfies the generalized least action principle, in the sense that it is a solution to the shortest path problem when T is not too large) is considerably more restrictive. According to proposition 6.1, this happens if and only if property (6.4) holds.

Since $\mu(dx,dv) = \lambda(1/2\|v\|^2+p(x))dxdv$, this exactly means :

$\int \lambda(1/2\|v\|^2+p(x))dv = 1$ if x belongs to X, 0 otherwise, and, therefore, is equivalent to :

$\int \lambda(1/2\|v\|^2+s)dv = 1$ if $s\leq c$, 0 if $s>c$, since $X=\{x\in \mathbf{R}^d,\ p(x) \leq c\}$. Finally, by using polar coordinates, we see that λ must satisfy the following classical Volterra equation :

(6.10) $C_d \int_{r>0} \lambda(r + s)r^{d/2-1}dr = 1$ if $s\leq c$, 0 if $s>c$,

where C_d is a numerical constant depending on the space dimension. This equation as no positive solution when d=3. The unique solution is (up to a multiplicative constant depending on d) :

(6.11) $\lambda(r)=\delta(r-c)$ when d=2

(then λ is not a function and (6.9) is more a convenient notation than a rigorous definition).

(6.12) $\lambda(r)= (c-r)^{-1/2}$ if $r<c$, 0 if $r\geq c$, when d=1.

This achieves the proof of proposition 6.2•

This result allows us to construct a large familly of generalized solutions to the Euler equations in dimension 2 and...1 ! Let us consider the simplest examples, when :

(6.13) $p(x)=1/2\|x\|^2$, $X=\{x\in \mathbf{R}^d,\ \|x\| \leq 1\}$, d=1 or 2.

Then the flow map G is trivially given by :

Then the flow map G is trivially given by :

(6.14) $X(t,x,v)=x\cos t+v\sin t,\ V(t,x,v)=-x\sin t+v\cos t.$

Following proposition 6.2, after some elementary computations, one finds out a two-dimensional generalized solution q defined by :

(6.15) $\int F(z)q(dz) =\pi^{-2}\iint F\,[t\to x\cos t+v\sin t]\ \delta(\|x\|^2+\|v\|^2 -1)dxdv$, for each F in $C_{fin}(\Omega)$,

that is :

(6.17) $\int F(z)q(dz) = \pi^{-2}\int_{\|x\|<1}\int_{0<\theta<\pi} F\,[t\to x\cos t+(1- \|x\|^2)^{1/2}(\cos\theta,\sin\theta)\sin t]d\theta\ dx$

The corresponding one-dimensional solution is obtained in a similar way :

(6.18) $\int F(z)q(dz) = (2\pi)^{-1}\int_{|x|<1}\int_{0<\theta<\pi} F\,[t\to x\cos t+ (1-x^2)^{1/2}\cos\theta\sin t]d\theta\ dx.$

When $T=\pi$, in both cases the final configuration η is deterministic and given by :

(6.19) $\eta(dx,dy) = \delta(y-h(x))dx$, where $h(x)=-x$.

Moreover the pressure field identically satisfies $D^2p(x) = \pi^2/T^2\, I$ (where I is the identity). Since $T=\pi$ exactly corresponds to the limit case of theorem 5.1 (when equality (5.6) holds), it follows that the generalized flows defined by (6.17) and (6.18) are solutions to the shortest path problem. Notice that they are not deterministic, although the final configuration is deterministic ! Thus, the last statement of theorem 5.1 is now entirely justified.

In dimension 2, the final configuration $h(x)=-x$ is classical (it is a smooth volume and orientation preserving map) and the shortest path problem has two trivial classical solutions, already considered in section 4, the two rotational flows with opposite constant angular velocities :

(6.20) $g_\varepsilon(t, x_1,x_2)=(x_1 \cos(\varepsilon t)+ x_2 \sin(\varepsilon t), - x_1 \sin(\varepsilon t)+ x_2 \cos(\varepsilon t)),\ 0\le t\le\pi,\ \varepsilon=\pm1.$

Thus (6.17) defines another, highly non classical, solution to the same (generalized) shortest path problem ! It can easily be checked that the three different solutions have the same kinetic energy and, more surprisingly, the same pressure field $p(x)=1/2\|x\|^2$. In constrast with the two classical flows, the probabilistic one has a zero mean velocity field, while the pressure is in exact balance with the "inertial tensor" :

(6.21) $E[v_i] = 0,\ E[v_i v_j] = \delta_{ij}\,(1/2-p(x))$, where E denotes the expected value.

In the one-dimensional case, the final configuration is still deterministic but not classical (indeed $h(x)=-x$ is volume preserving but orientation reversing). Therefore there is no classical solution to the shortest path problem. Furthermore, one can prove :

Proposition 6.3.

•Let $X=[-1,+1]$. For the final configuration $h(x)=-x$, and the final time $T=\pi$, the generalized shortest path problem has a unique solution q defined by (6.18)•

Proof of proposition 6.3.

•We already know that (6.18) defines a solution q to the shortest path problem. Let us prove that this solution is unique. Because q-almost every path z is solution to $z''=-z$ and therefore satisfies : $\int \|z'(t)\|^2 dt = \int \|z(t)\|^2 dt$ (where the integral is performed over $[0,\pi]$), it follows that :
$E(q)= \int\int 1/2\ \|z'(t)\|^2 dt\ q(dz) = \int\int 1/2\ \|z(t)\|^2 dt\ q(dz)$
$= \int\int 1/2\ \|x\|^2 dt\ dx = \pi \int 1/2\ \|x\|^2 dx$ (since q is incompressible).
Therefore, any optimal flow q* must satisfy :
$\int\int 1/2\ \|z'(t)\|^2 dt\ q^*(dz) = E(q) = \pi \int 1/2\ \|x\|^2 dx = \int\int 1/2\ \|x\|^2 dt\ dx$
$= \int\int 1/2\ \|z(t)\|^2 dt\ q^*(dz)$ (indeed, q* *also* is incompressible).
Since the final configuration at time $T=\pi$ is $h(x)=-x$, we get $z(\pi)=-z(0)$ for q*-almost every path z. By Poincaré inequality $z(\pi)=-z(0)$ implies $\int \|z'(t)\|^2 dt \geq \int \|z(t)\|^2 dt$ (where the integral is performed over $[0,\pi]$) and the corresponding equality holds if and only if z is of the form $z(t)=x\cos t + v\sin t$, for some real numbers x and v. Because z maps $[0,\pi]$ into $[-1,+1]$, (x,v) must belong to the unit disk. Then, it is not hard to deduce that there exists a probability measure $\mu(dx,dv)$ supported by the unit disk such that q* can be defined by :

(6.22) $\int F(z) q^*(dz) = \int\int F\,[t \to x\cos t + v\sin t]\,\mu(dx,dv)$, for each F in $C_{fin}(\Omega)$.

Since q* must be incompressible, $\mu(dx,dv)$ must satisfy :

(6.23) $\int f(x\cos t+v\sin t)\ \mu(dx,dv) = \int f(x)dx$, for any f in $C[-1,+1]$ and t in $[0,\pi]$

(here dx denotes the Lebesgue measure multiplied by 1/2, in order to be a probability measure on $[-1,+1]$). In particular (6.23) holds for any f of the form $f(x) = \exp(ix\xi)$. Then, it can be easily seen that the Fourier transform of $\mu(dx,dv)$ (which is a bounded continuous function defined on $\mathbf{R}^2$) is completely determined (notice that this assertion would be wrong in the 2-dimensional case when X is the unit disk!). Thus, μ is unique and, consequently, q* also is unique. This achieves the proof of proposition 6.3•

Appendix. A formal link with the measure-valued solutions in the sense of DiPerna and Majda.

In this appendix, it is shown that one can **formally** associate to any optimal generalized flow a generalized velocity field that (formally) solves the Euler equations in the sense of DiPerna and Majda. For simplicity, it is assumed that X is the closure of a smooth bounded open set in R^d, the fluid is homogeneous ($\rho=1$) and there is no external forces ($U=0$).

Let q be an optimal generalized incompressible flow. Let us introduce the probability measure $\mu(dt,dx,dv)$ on $[0,T]\times X\times R^d$, **formally defined** by :

$$\iiint f(t,x,v)\mu(dt,dx,dv) = T^{-1}\iint f(t,z(t),z'(t))dt\ q(dz),$$

for any suitable f defined on $[0,T]\times X\times R^d$ (it is not easy, for several reasons, to justify this definition). μ can be considered as a generalized velocity field, has a finite kinetic energy :

$$\iint 1/2\ \|v\|^2\mu(dt,dx,dv) = E(q) < +\infty,$$

and is solution to the Euler equations in the following generalized sense [cf. DiPerna & Majda] :

$$\iiint [v.w(x)f'(t)+v(Dw(x).v)f(t)]\mu(dt,dx,dv) = 0,$$

for any smooth real function f compactly supported in $]0,T[$ and any smooth vector field w on X such that : div w = 0 inside X, w.n = 0 along ∂X (n = outward normal).

Let us give a formal proof of this claim.

Because of the properties of w, there is a smooth classical volume preserving flow map γ :

$$(t,x)\in R\times X \rightarrow \gamma(t,x)\in X, \text{ such that : } \gamma(0,x)=x,\ \partial_t\gamma(t,x) = w(\gamma(t,x)).$$

For ε small, let us consider the *modified* generalized flow q_ε defined by :

$$\int_\Omega F(z)q_\varepsilon(dz) = \int_\Omega F\,[t\rightarrow\gamma(\varepsilon f(t),z(t))]q(dz), \quad \text{for each F in } C_{fin}(\Omega).$$

It is easy to check that q_ε is incompressible. Indeed, for each f in C(X) and t in [0,T]:

$$\int f(z(t))q_\varepsilon(dz) = \int f(\gamma(\varepsilon f(t),z(t)))q(dz) = \int f(\gamma(\varepsilon f(t),x))dx \text{ (since q is incompressible)}$$

$$= \int f(x)dx \text{ (since } \gamma \text{ is volume preserving).}$$

Let us now compare the kinetic energies of q and q_ε . A formal computation leads to :

$$\gamma(\varepsilon f(t),z(t)) = z(t) + \varepsilon f(t)w(z(t)) + 0(\varepsilon^2),$$

$$\partial_t\,[\gamma(\varepsilon f(t),z(t))] = z'(t) + \varepsilon\partial_t\,[f(t)w(z(t))] + 0(\varepsilon^2).$$

It follows that :

$$E(q_\varepsilon)-E(q) = \iint [1/2\ \|\partial_t\,[\gamma(\varepsilon f(t),z(t))]\|^2 - 1/2\ \|z'(t)\|^2]dt\ q(dz)$$

$$= \varepsilon \iint z'(t).\partial_t\,[f(t)w(z(t))]dt\ q(dz) + 0(\varepsilon^2)$$

$$= \varepsilon \iint [z'(t).w(z(t))f'(t) + z'(t)(Dw(z(t)).z'(t))]dt\ q(dz) + 0(\varepsilon^2)$$

$$= \varepsilon \iiint [v.w(x)f'(t) + v(Dw(x).v)f(t)]\ \mu(dt,dx,dv) + 0(\varepsilon^2),$$

which shows that $\iiint [v.w(x)f'(t) + v(Dw(x).v)f(t)]\ \mu(dt,dx,dv)$ must vanish for each suitable w.

References

V.I.Arnol'd, Méthodes mathématiques de la mécanique classique, Mir, Moscou 1976.

V.I.Arnol'd, A.Avez, Problèmes ergodiques de la mécanique classique, Gauthier-Villars, Paris 1967.

N.Bourbaki, Eléments de mathématique, Integration (chapitre 4), Hermann, Paris 1965.

R.DiPerna, Measure-valued solutions of conservation laws, Arch. Rat. Mech. Anal. 8 (1985).

R.DiPerna, A.Majda, Oscillations and concentrations in weak solutions of the incompressible fluid equations, Comm. Math. Physics 108 (1987) 667-689.

D.Ebin, J.Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Annals of Math. 92 (1970) 102-163.

I.Ekeland, R.Temam, Analyse convexe et problèmes variationnels, Dunod, Paris 1974.

S.T.Rachev, The Monge-Kantorovitch mass transference problem, Theory of Probability and its Applications, XXIX, 4 (1985) 647-676.

M.Reed, B.Simon, Methods of modern mathematical Physics, II, Academic Press, New York 1978.

L.Tartar, The compensated compactness method applied to systems of conservation laws, in Systems of nonlinear PDE, J.Ball ed., NATO ASI Series, C.Reidel Publishing Col 1983.

L.C.Young, Lectures on the Calculus of variations and optimal control theory, W.B.Saunders co, Philadelphia-London 1969.

Y.Brenier, Décomposition polaire et réarrangement monotone des champs de vecteurs, C.R.Acad.Sci, série I, 305 (1987) 805-808.