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Tony F. Chan

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

BOUNDARY PROBE DOMAIN DECOMPOSITION PRECONDITIONERS FOR FOURTH ORDER PROBLEMS*

TONY F. CHAN†

Abstract. The boundary probing technique is a class of methods for the construction of efficient interface preconditioners in domain decomposition algorithms. The main idea is to capture the strong local coupling of the interface operator through subdomain solves with a few appropriately chosen "probing" boundary conditions. For second order elliptic problems, this technique has proven to be very successful and frequently performs better than other preconditioners. In this paper, we show how this technique can be extended to derive efficient domain decomposition preconditioners for fourth order problems. The main modifications are that the interface between subdomains now consists of two grid lines and a new set of probing boundary conditions is used. Numerical results for the biharmonic equation is presented.

Key words. Parallel algorithms, domain decomposition, fourth order partial differential equations, biharmonic equation, preconditioned conjugate gradient.

1. Introduction. In this paper, domain decomposition refers to a class of algorithms for solving boundary value problems for elliptic partial differential equations. The main idea is to decompose the original domain into smaller subdomains, solve the original problem on the subdomains, and somehow "patch" the subdomain solutions to form the solution to the original problem. In general, the above process has to be repeated through an iterative process until some convergence criteria is satisfied. There are two main approaches, characterized by the way the subdomains are constructed, namely overlapping and nonoverlapping. In this paper, we shall consider only the nonoverlapping approach.

There are several reasons why such a procedure would be useful. First, irregular domains can be decomposed into regular subdomains on which more efficient solvers can be used. Second, it is a natural way to design parallel algorithms for elliptic boundary value problems. Third, it allows large problems to be solved on computers with relatively small core memory. Finally, different mathematical models and different grid resolutions can be adaptively used in the different subdomains.

In the nonoverlapping approach of domain decomposition, the original problem is reduced to an equivalent one defined on the interfaces separating the subdomains. This problem (sometimes referred to as the Schur Complement system or capacitance system) is then solved iteratively, each iteration requiring a solve on each subdomain. To improve the rate of convergence, preconditioners (which can be thought of as easily invertible approximations to the interface operator) are often used. Due to its critical impact on the overall efficiency of the domain decomposition algorithm, the design and analysis of such preconditioners has been a main topic of research in this area. For a survey of this activity, see [4,9,6,2].

In this paper, we shall study in particular a class of preconditioners that we shall call *boundary probe preconditioners*. The main motivation behind these precondition-

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† Department of Mathematics, UCLA, Los Angeles, CA 90024. E-mail: chan@math.ucla.edu or na.tchan@na-net.stanford.edu. The research of the author is supported in part by The Department of Energy under contract DE-FG03-87ER25037, by the National Science Foundation under contract NSF-DMS87-14612 and by the Army Research Office under contract No. DAAL03-88-K-0085.

ers is the observation that for many elliptic problems, the reduced interface operator has strong spatial local coupling and weak global coupling. In the discrete case, the corresponding interface matrix has elements whose magnitude decay rapidly away from the main diagonal [7]. The boundary probe preconditioners are designed to capture the strong coupling (i.e. the main diagonals) via a few subdomain solves with appropriately chosen *probing* boundary conditions. They were first proposed in [3] for second order elliptic problems and have proven to be quite successful for a large class of problems [9], including convection diffusion problems [8]. Unlike some of the other domain decomposition preconditioners, its performance has also been shown to be relatively insensitive to the mesh size, the shapes of the subdomains and the variations in the coefficients of the differential equation.

The main purpose of this paper is to extend the boundary probe preconditioners to fourth order boundary value problems discretized with standard compact finite difference approximations. There are two main modifications needed. First, the interface between any two subdomains must now consist of two grid lines instead of one in order to competely decouple the subdomain problems. Second, slightly different probing boundary conditions must be used. These will be discussed in more details in Section 2. Results from numerical experiments for the biharmonic problem will be presented in Section 3. They show that the boundary probing technique does produce effective preconditioners.

We note that several preconditioners designed specifically for second order elliptic operators cannot be extended directly to fourth order problems, short of employing them in an inner iteration of an algorithm which solves second order problems at each step. For the biharmonic operator arising from the stream function formulation for the Stoke's problem in incompressible flows, some of these ideas are used in [10,11] to derive domain decomposition preconditioners through the alternate velocity-pressure formulation. On the other hand, the boundary probing technique makes it possible in a direct way to construct efficient domain decomposition preconditioners for problems with more complicated operators. For applications of the techniques developed here to the Navier Stoke's equations, see [1,5].

2. Formulation. In this case, we consider only the simplest case of a domain Ω split into two subdomains Ω_1 and Ω_2 sharing the interface Γ. Consider the problem $Lu = f$ on Ω where L is a fourth order partial differential operator with appropriate boundary conditions on u and its derivatives on $\partial\Omega$. We consider the use of a compact general 25-point discrete approximation of L on a finite difference grid, i.e. the discrete approximation at a point (i,j) involves only values at grid points (k,l) with $|i-k| \leq 2$ and $|j-l| \leq 2$. Further, we assume that the interface Γ consists of two adjacent grid lines, which we shall denote by Γ_1 and Γ_2. If we order the unknows for the internal points of the subdomains (which we denote by u_1 and u_2) first and those on the interface Γ (which we denote by u_3) last, then the discrete solution vector $u = (u_1, u_2, u_3)^T$ satisfies the linear system :

$$Au \equiv \begin{pmatrix} A_{11} & & A_{13} \\ & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad , \tag{2.1}$$

where the discrete vector $f = (f_1, f_2, f_3)^T$ contains the contribution of the right hand side f of the differential equation and of the boundary conditions. Note that in (2.1) the subdomain problems (the blocks A_{11} and A_{22}) are not coupled together.

System (2.1) can be solved by block Gaussian elimination which gives the equations for the interface variables u_3 :

$$Su_3 = \hat{f}_3 \quad , \tag{2.2}$$

with

$$S = A_{33} - A_{31}A_{11}^{-1}A_{13} - A_{32}A_{22}^{-1}A_{23}$$

and

$$\hat{f}_3 = f_3 - A_{31}A_{11}^{-1}f_1 - A_{32}A_{22}^{-1}f_2 \quad .$$

The matrix S is the Schur complement of A_{33} in the matrix A. It corresponds to the reduction of the operator L on Ω to an operator on the internal boundary Γ. Constructing the Schur complement would require the solution of n_Γ elliptic problems on each subdomain, where n_Γ is the number of internal points on Γ. Furthermore it is dense, so that factoring would be expensive.

Instead of solving the system (2.2) directly, iterative methods such as preconditioned conjugate gradient (PCG) can be applied in which only matrix vector product Sy are required. This product can be computed by one solve on each subdomain with boundary condition on Γ determined by y. Since each iteration is rather expensive, it is important to precondition this iteration with a good preconditioner in order to keep the number of iterations small.

We now consider using the boundary probing technique to construct efficient preconditioners for S. The main motivation for this approach is the observation that, for many elliptic operators, the reduced interfacial operator S exhibits a strong spatial local coupling and weak global coupling among the interfacial unknowns. To show this effect, consider the special case of the biharmonic operator $L \equiv \Delta^2$, on the unit square Ω with u and its normal derivatives given on $\partial\Omega$ and the usual 13-point second order central difference approximation on a uniform n by n grid. Let the interface Γ consist of the two vertical grid lines closes to $x = 0.5$, which we shall denote by w_1 and w_2. We shall order the unknowns first on one of the grid lines starting from $y = 0$ to $y = 1$ and then similarly on the other grid line. The Schur complement system corresponding to (2.2) can be written as a block 2 by 2 system:

$$S\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \equiv \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \hat{f}_{31} \\ \hat{f}_{32} \end{pmatrix}. \tag{2.3}$$

The blocks S_{11} and S_{22} account for the coupling of the unknowns on Γ_1 and Γ_2 respectively among themselves and the blocks S_{12} and S_{21} account for the coupling between the unknowns on the two interfaces. Figure 1 shows a plot of the elements of the matrix S for the case $n = 20$. The decay property can be seen clearly in the figure: the magnitude of the elements of the individual subblocks of S decays rapidly away from their respectively main diagonals, reflecting the strong local coupling and weak global coupling. Figure 2 shows the elements of the eighth row of S, showing that the elements of the main diagonal blocks S_{11} and S_{22} are negligible except for the 5 main diagonals and that the off diagonal blocks S_{12} and S_{21} have only 3 non-negligible main diagonals. This shows that the interfacial unknowns are most strongly coupled to its four nearest neighbors on its own grid line and to the three nearest neighbors on the adjacent grid line.

A natural way to construct an interface preconditioner is therefore to capture efficiently the effects of these main diagonals of the individual sublocks of S. However, it would not be efficient to calculate all the elements of S in order to do this, for this would require $2n$ subdomains solves. Instead, we shall do this by probing the individual subblocks via a few matrix vector products $S_{ij}v_l$ with appropriately chosen probing vectors v_l's. As proposed in [3], a $2k+1$ diagonal approximation to subblock S_{ij} can be constructed by computing the action of S_{ij} on $2k+1$ "probing" vectors $v_l, l = 1, \ldots, 2k+1$ which for the cases $k = 0$ and $k = 1$ are given by:

$$\begin{aligned} k = 0: \quad & v_1 = (1,1,1,1,1,1,1,\ldots)^T \\ k = 1: \quad & v_1 = (1,0,0,1,0,0,1,\ldots)^T \\ & v_2 = (0,1,0,0,1,0,0,\ldots)^T \\ & v_3 = (0,0,1,0,0,1,0,\ldots)^T. \end{aligned}$$

The case $k = 0$ corresponds to a scaling of each row of the matrix S_{ij} by the sum of the elements of the row. For $k = 1$, if S_{ij} were indeed tridiagonal, all of its elements would be recovered in the vectors $S_{ij}v_l$, $l = 1, 2, 3$. Generalization to cases with $k > 1$ is straightforward [8].

Suppose we want to compute a preconditioner M_{kl} for S consisting of k-diagonal approximations for the diagonal blocks S_{11} and S_{22}, and l-diagonal approximations for the off-diagonal blocks S_{12} and S_{21}. Let V_k be a n by k matrix consisting of k probing vectors for any one of the subblocks described above. Then M_{kl} can be obtained by probing S by the columns of the following matrix:

$$\begin{pmatrix} V_k & V_l & 0 & 0 \\ 0 & 0 & V_k & V_l \end{pmatrix}.$$

This requires solving $2(k+l)$ subdomain problems with boundary conditions consisting of probing vectors from V_k or V_l on one grid line and zero on the other grid line. More efficient probing techniques, with fewer probing vectors and hence fewer subdomain solves for given values of k and l, can be constructed [5] but since our main concern in this paper is on the convergence rates of the preconditioned interface system, for simplicity we shall not present them here.

Finally, the block matrix M_{kl} can be permuted into a narrowly banded matrix by reordering the unknowns to preserve their physical proximity. For example, if we start from $y = 0$ and alternatively order the unknowns on the two grid lines, then M_{kl} is reordered into a banded matrix with bandwidth $2k-1$, assuming $l < k$ for simplicity. Therefore, the product $M_{kl}^{-1}w$ for a given interfacial vector w can be computed efficiently by banded Gaussian elimination.

3. Numerical Results. We now present some numerical results for the performance of the above boundary probing techniques on the biharmonic problem described in the last section. Figures 3a and 3b show the eigenvalue distribution of the unpreconditioned interface matrix S and the preconditioned matrix $M_{kl}^{-1}S$ for several values of (k,l). The figures show that the preconditioners produce a dramatic improvement in the conditioning of the interface operator. As a sample measure, the eigenvalues of S lie in the interval $(.02, 45)$ while those of the preconditioned system $M_{53}^{-1}S$ lie in $(0.3, 1.2)$. Moreover, many eigenvalues of the preconditioned system are clustered around unity. Figure 4 shows the condition number $M_{kl}^{-1}S$ in the spectral norm as a function of n for several values of k and l. These results show that not

only are the condition numbers of the preconditioned matrix much lower than S itself, but also that they grow at a slower rate (approximately $O(n)$ for the preconditioned cases versus $O(n^{2.8})$ for the unpreconditioned case). The plots also show that M_{53} is in some sense optimal because the more expensive M_{75} produces negligible improvement in the condition numbers. Finally, to show that the improvement in the condition number and the eigenvalue distribution of the preconditioned matrix does improve the performance in an iterative solution of the interfacial unknowns, we solve the interfacial system by the preconditioned conjugate gradient algorithm. Figure 5 shows the history of an iteration, with the norm of the residual plotted against the iteration step. It is clear that M_{53} produces a much faster convergence rate.

We have also performed experiments with the biharmonic operator on nonrectangular domains, such as the L-shaped domain resulting from cutting away a quarter of the unit square. The results are completely similar. More recently, we have successfully applied the above boundary probing technique to the driven cavity problem of incompressible flow [1,5]. However, these results are only preliminary evidence that the boundary probing technique can be applied successfully to fourth order problems. Much further work needs to be done, especially concerning the decay rates of the elements of S and the properties of the preconditioners derived from it.

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Fig. 1 Plot of elements of S, n=20

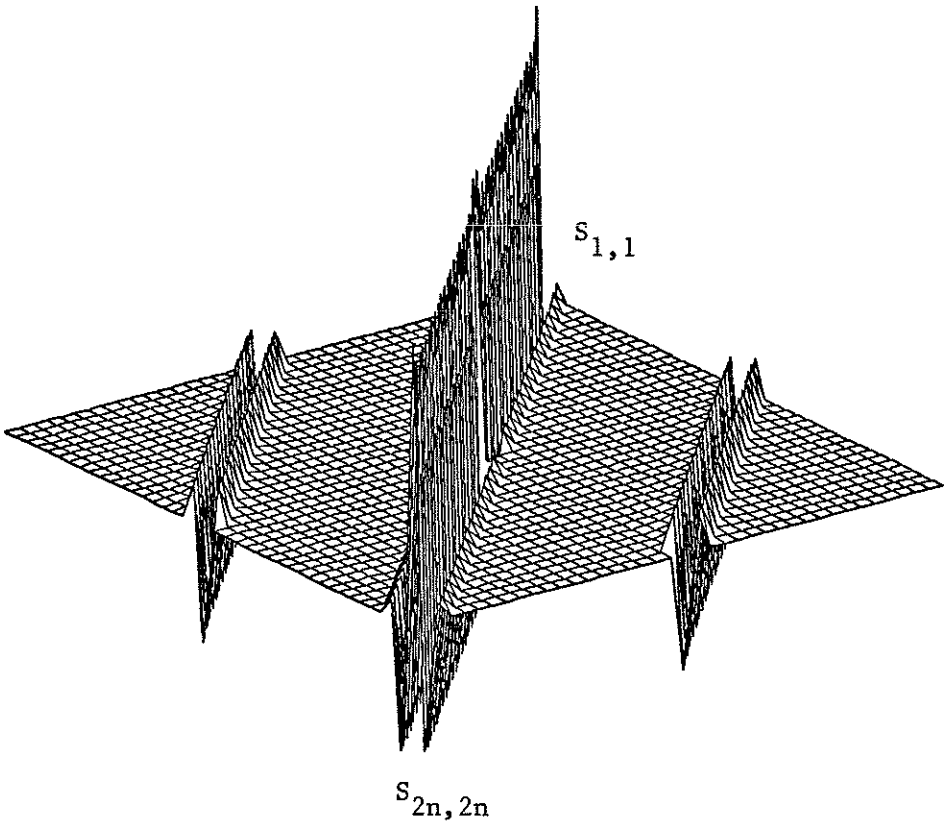

Fig. 2 elements of the Eighth row of S, n=20

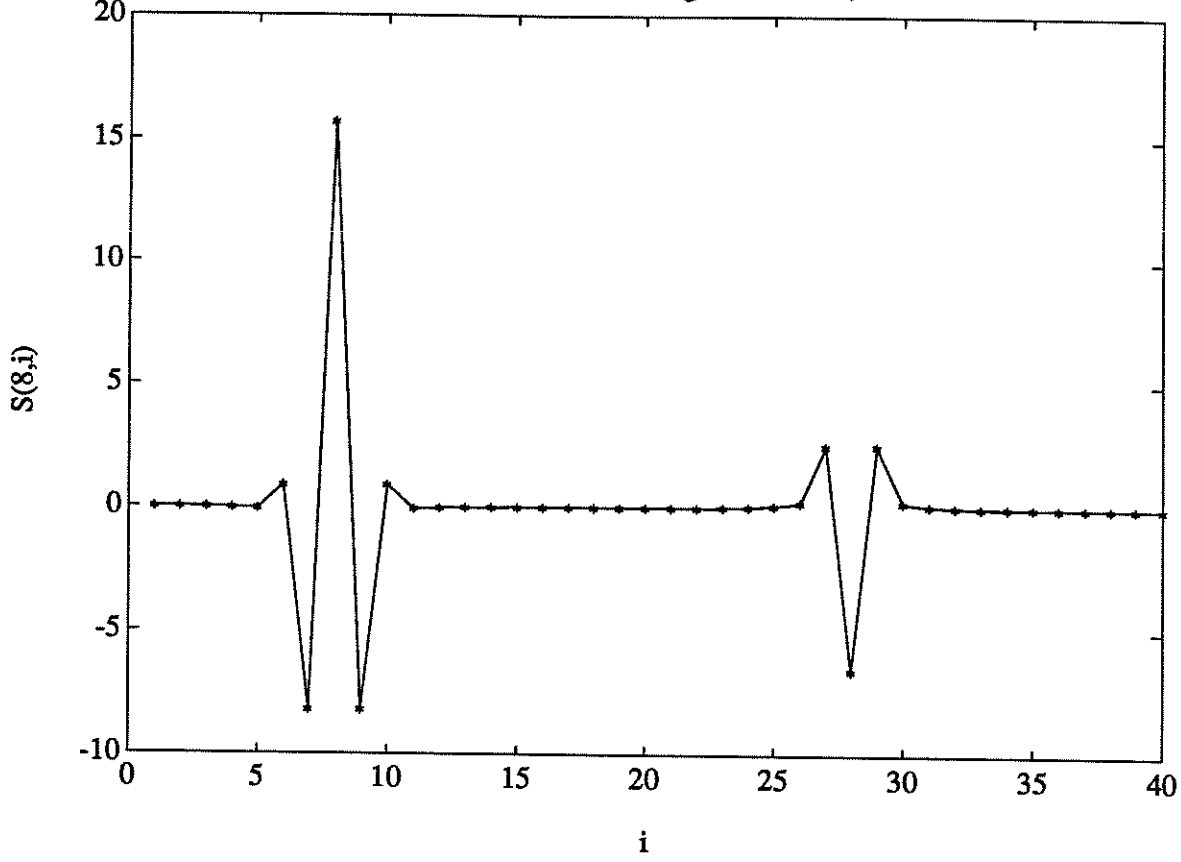

Fig.3a Eigenvalue Distribution of S, n=20

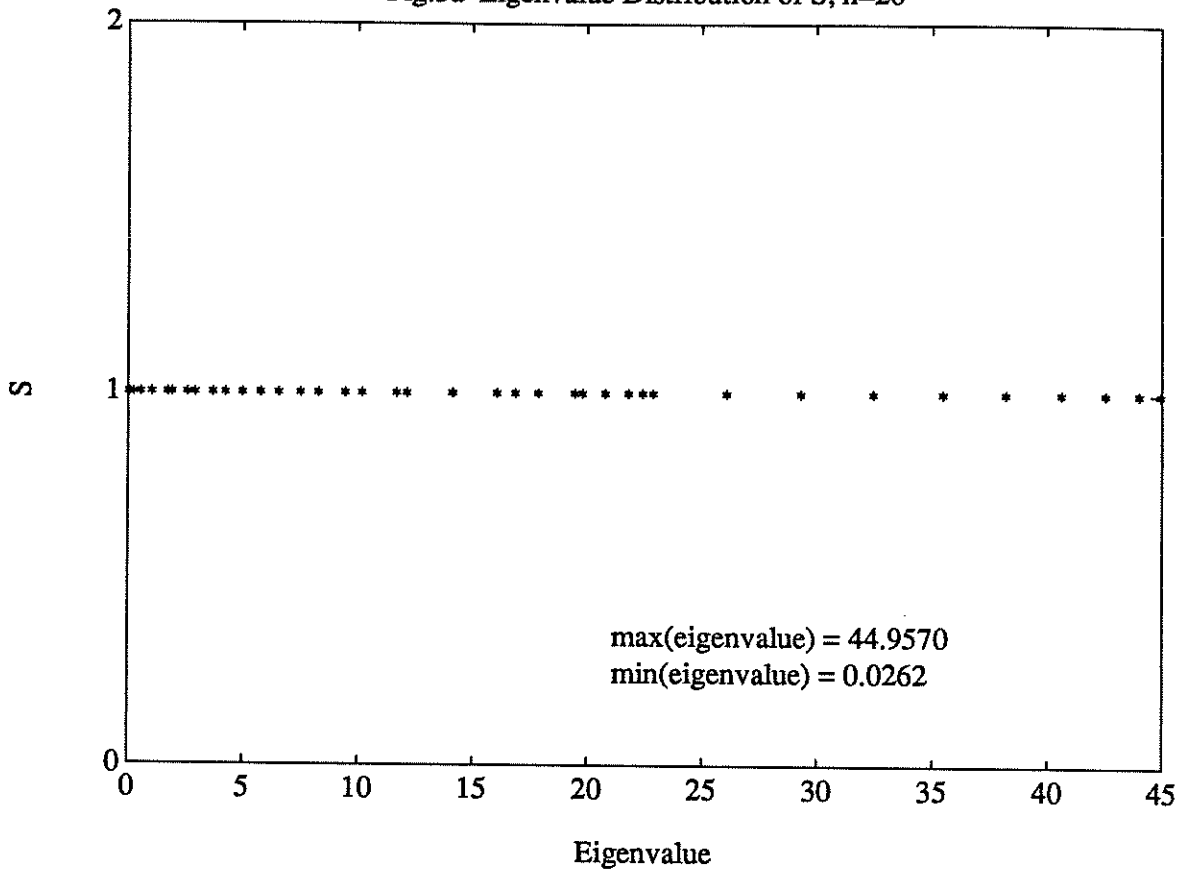

Fig.3b Eigenvalue Distribution of inv(m)*S, n=20

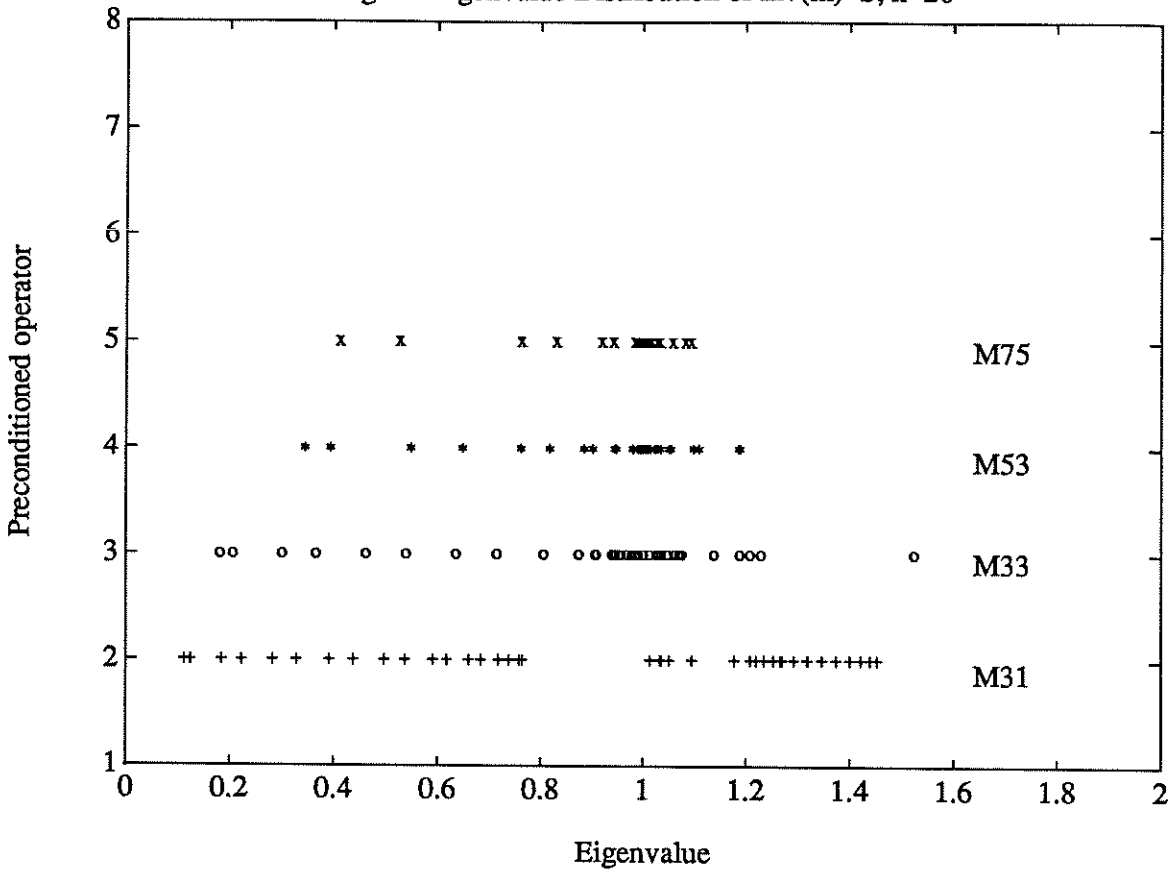

Fig. 4 Condition number of inv(m)*S vs n

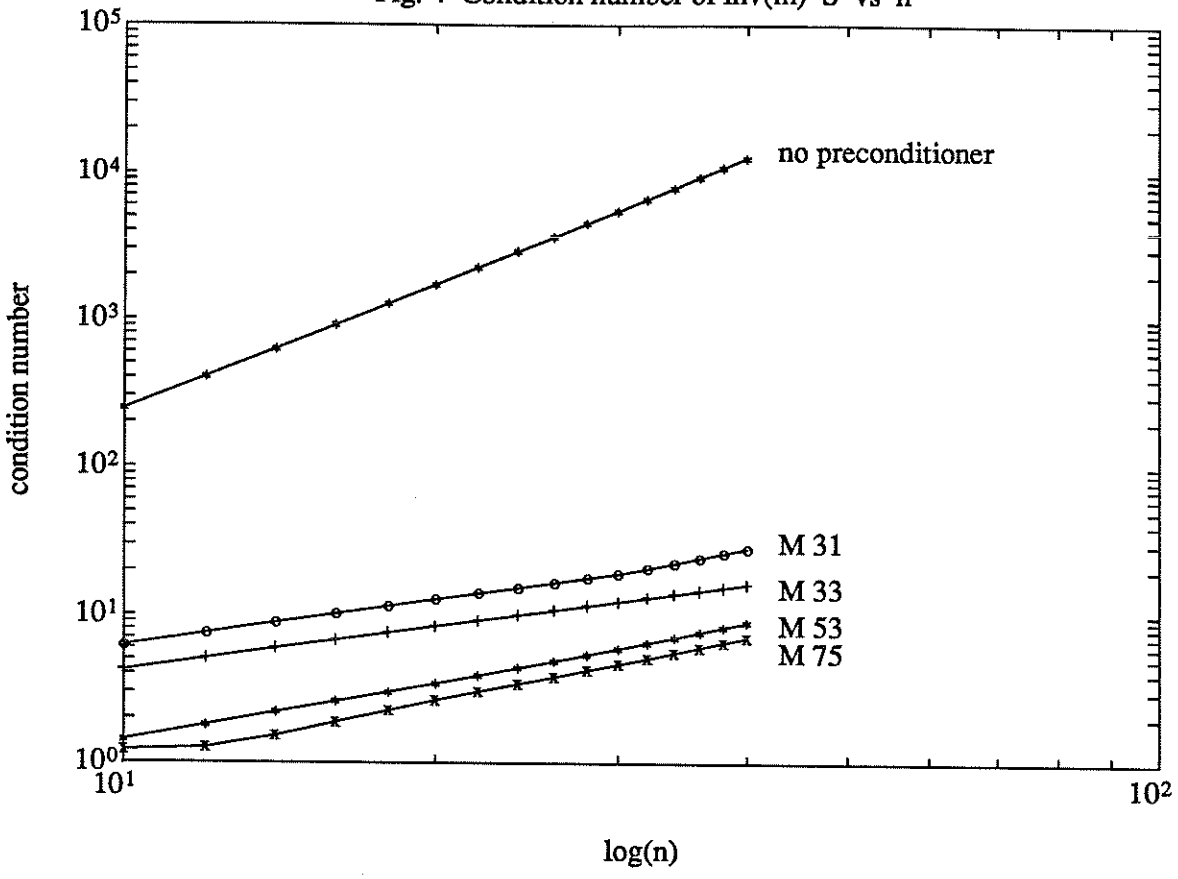

Fig. 5 Convergence history of PCG iteration