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Analysis of Domain Decomposition Preconditioners on L-Shaped and C-Shaped Regions

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We present an algebraic analysis of some domain decomposition preconditioners on irregular regions. We analyze a preconditioner proposed in [3] for the interface system and prove that, for all L-shaped regions and some C-shaped regions, it produces a convergence rate that is independent of the size of the discretization and the relative shape of the subdomains (aspect ratios). Specifically, we prove that the condition number of the preconditioned capacitance system is bounded by 2.16 for *all* L-shaped domains. We also give some results for other simple irregular geometries.

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1. Introduction

We consider the problem of solving an elliptic partial differential equation on a domain that is broken up into rectangular subregions. By using *domain decomposition* or *substructuring* techniques, the problem is reduced to separately solving approximate problems in the subdomains and updating the solution at the interfaces between two or more subregions. For the class of domain decomposition methods considered in this paper, the basic idea consists of the following: the differential operator is discretized on a grid imposed over the domain, which is partitioned into several subregions. Then, by applying block elimination to the discretized equations, a system is derived for the unknowns on the interfaces between subregions. This system is sometimes called the capacitance system. Forming the right hand side for the interface system requires the solution of independent elliptic problems on the subdomains. For certain constant coefficient problems on regular domains, fast direct methods can be applied to the solution of the interface system [3, 4]. Such is not the case, however, for more general operators or irregular domains. For efficiency reasons the system must then be solved by iterative methods, such as the preconditioned conjugate gradient method. Once the solution is known on the interfaces, one more elliptic problem must be solved on each subdomain with the computed values as boundary conditions.

In [3], an eigenvalue decomposition in terms of Fourier modes is given for the capacitance matrix for the case of the Poisson equation on a rectangle divided into two strips. This decomposition is described in section 2. In this paper, we are interested in the analysis of this decomposition, which we will call M_C, as a preconditioner on irregular domains and in particular, we want to study the dependency of the convergence rate on the gridsize *and* the shape of the domain. Many of the preconditioners, when applied to an L-shaped region, have convergence rates that are bounded independently of the gridsize. The bound, however, depends on the relative aspect ratios of the subdomains. For example, all of the preconditioners, except for M_C, are known to deteriorate when one of the subdomains becomes narrow. In section 3, we show that, if we use M_C as preconditioner for the capacitance matrix on any L-shaped region, the preconditioned matrix has a condition number that is bounded by 2.16, independently of gridsize *and aspect ratios.* Given an L-shaped region, there are two ways of separating it into two rectangular subregions. We prove, also in section 3, an interesting property of the preconditioner M_C, namely that the convergence rate is not affected by the way we choose to subdivide the domain. In section 4, we discuss the extention of some of the results in section 3 to C-shapes. In the proofs of sections 3 and 4, we often use a common operator, which describes the interaction between two perpendicular interior interfaces. This operator is analyzed in detail in the appendix.

2. The interface operator and its preconditioners

In order to illustrate the method, we will apply the process described above to a simple region Ω, which can be decomposed into two rectangles Ω_1 and Ω_2, with interface Γ_3, as shown in fig.1. Let the linear system

$$Au = f \tag{2.1}$$

represent the discretization of the differential operator on Ω. By ordering the variables in Ω_1 and Ω_2 first and then those in Γ_3, the system (2.1) can be written in block form as:

$$\begin{pmatrix} A_{11} & & A_{13} \\ & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \quad , \tag{2.2}$$

where the indexes for u and f corespond to gridpoints in Ω_1, Ω_2 and Γ_3, respectively. Based on

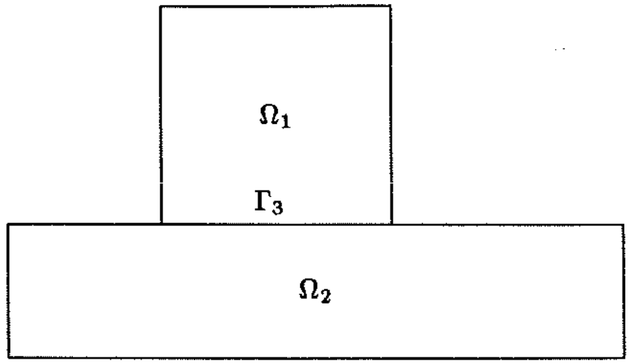

Figure 1: The domain Ω and its partition

the following block decomposition of the matrix in (2.2):

$$A = \begin{pmatrix} A_{11} & & \\ & A_{22} & \\ A_{13}^T & A_{23}^T & C \end{pmatrix} \begin{pmatrix} I & & A_{11}^{-1}A_{13} \\ & I & A_{22}^{-1}A_{23} \\ & & I \end{pmatrix} \quad , \tag{2.3}$$

where C is the Schur complement of A_{33} in A, i.e.

$$C = A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23} \quad , \tag{2.4}$$

the system (2.2) can be solved as follows:

Step 1: Compute

$$g = f_3 - A_{13}^T A_{11}^{-1} f_1 - A_{23}^T A_{22}^{-1} f_2 \tag{2.5}$$

and solve

$$Cu_3 = g \tag{2.6}$$

Step 2: Solve

$$A_{11}u_1 = f_1 - A_{13}u_3 \tag{2.7}$$

and

$$A_{22}u_2 = f_2 - A_{23}u_3 \tag{2.8}$$

The computation of g by (2.5) and u_1 and u_2 by (2.7) and (2.8), require the solution of independent problems on the subdomains. The matrix C given by (2.4), also called the capacitance matrix, is dense and expensive to compute. It is possible, however, to compute the action of C on a vector v at the cost of solving problems on the subdomains with boundary conditions on Γ given by v. Therefore, the interface system (2.6) is often solved by preconditioned conjugate gradients (PCG). Since each iteration involves solving problems on the subdomains, it is essential to keep the number of iterations low. For this reason, much effort has been devoted recently to the construction of good preconditioners for the capacitance matrix [6, 1, 7, 3, 4]. Many of the preconditioners proposed are spectrally equivalent to the exact boundary operator. They therefore yield convergence rates that are bounded independently of the gridsize. The method is particularly suited to problems for which the subproblems can be solved efficiently, for example, when the operator has separable coefficients. When the subdomain problems cannot be solved efficiently but they can be approximated by separable operators, it is possible to derive block preconditioners for the original system based on preconditioners for the interface system [8, 2, 5].

In [3], the case of a constant coefficient operator on a rectangular domain divided into two strips is analyzed. For this simple case, it is shown that, for many of the preconditioners proposed in the literature, while the condition number of the preconditioned system can be bounded independently of the gridsize h for a fixed domain, it can grow as a function of the aspect ratio of the subdomains. Roughly speaking, the aspect ratio of a rectangle is the ratio between its height and its width (note: for one of the preconditioners proposed in [1], the bound grows when only one of the subdomains becomes narrow). A fast direct solver for C based on Fourier analysis can be derived from the exact eigenvalue decomposition of the capacitance matrix. This operator takes aspect ratios into account and solves exactly the interface problem for the case of constant coefficients on a rectangle divided into two strips. It is therefore proposed in [3] to apply it as a preconditioner for interface systems on irregular regions or for variable coefficient operators. We will call this preconditioner M_C. For the case of a five point finite differences discretization of the Poisson equation:

$$-u_{xx} - u_{yy} = f \tag{2.9}$$

on a regular grid of size $h = \frac{1}{n+1}$, M_C has a decomposition of the form:

$$M_C = W_n \, \Lambda \, W_n^T \quad , \tag{2.10}$$

where Λ is a diagonal matrix and W_n is the matrix of sine modes of dimension n, whose elements are given by:

$$w_{ij} = \sqrt{\frac{2}{n+1}} \sin \frac{ij\pi}{n+1} \tag{2.11}$$

for $i, j = 1, \ldots, n$.

Given integers n, m_1 and m_2, define

$$\lambda_j(n, m_1, m_2) = \left(\frac{1+\gamma_j^{m_1+1}}{1-\gamma_j^{m_1+1}} + \frac{1+\gamma_j^{m_2+1}}{1-\gamma_j^{m_2+1}} \right) \sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \tag{2.12}$$

where

$$\sigma_j = 4 \sin^2 \left(\frac{j}{(n+1)} \frac{\pi}{2} \right) \quad . \tag{2.13}$$

and

$$\gamma_j = \left(1 + \frac{\sigma_j}{2} - \sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \right)^2 \quad . \tag{2.14}$$

The eigenvalues of M_C are given by

$$\lambda_j = \lambda_j(n, m_1, m_2)$$

for $j = 1, \ldots, n$, where m_1 and m_2 are the number of grid points in the y-direction in Ω_1 and Ω_2 respectively.

The preconditioners proposed in [6] and [7] have the same eigenvectors as (2.10), but the eigenvalues are those of the square root of the one-dimensional discrete Laplacian, namely $\sqrt{\sigma_j}$ in [6] and $\sqrt{\sigma_j + \frac{\sigma_j^2}{4}}$ in [7]. For the case of the Poisson equation (2.9), it can be proved that one of the preconditioners given in [1] also has a decomposition of the form (2.10). The eigenvalues λ_j for this operator can be obtained by setting $m_2 = m_1$ in (2.12), i.e. $\lambda_j(n, m_1, m_1)$. This preconditioner is therefore exact for the case of a rectangle divided symmetrically into two identical rectangular subdomains.

3. L-shaped regions

In this section, we describe the interface operator and its preconditioners for an L-shaped domain, the simplest irregular shape that can be decomposed in rectangular subregions. Consider the Poisson equation on the region Ω of fig.2. It is clear that either interface, Γ_4 or Γ_5, will divide the domain into two rectangles. We might ask ourselves two questions: is a particular decomposition better than the other? And how does the convergence rate depend on the mesh size and the aspect ratios of the subdomains? We will show that for the particular preconditioner M_C we analyze, the two decompositions produce iteration matrices with the same convergence rate. We also give a bound for the condition number that is independent of the mesh size and the subdomain aspect ratios.

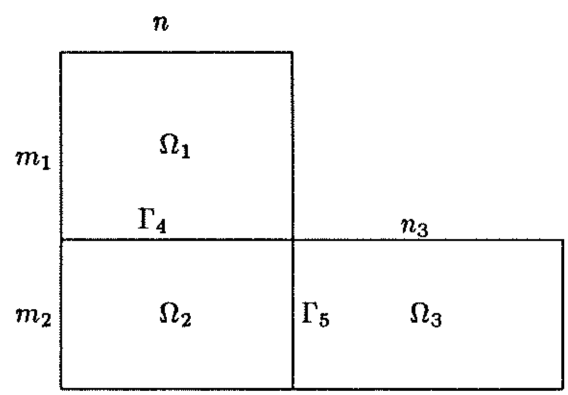

Figure 2: L-shaped domain

Let the linear system

$$Au = f \tag{3.1}$$

represent a standard second order five point discretization of the differential equation on a regular grid imposed on the domain Ω. Let us first consider the domain Ω as the union of two rectangles divided by the interface Γ_4. An interface system of the form

$$C_4 u_4 = g_4 \tag{3.2}$$

can be derived for the variables on Γ_4 by the process of block elimination, similarly to equations (2.5) to (2.8).

Similarly, we can consider the domain Ω as the union of two rectangles divided by the interface Γ_5 and an interface system of the form

$$C_5 u_5 = g_5 \quad , \tag{3.3}$$

can be derived for the variables on Γ_5.

On the other hand, by reordering the gridpoints on the subdomains first and then those on the interfaces Γ_4 and Γ_5, A can be written in block form as:

$$A = \begin{pmatrix} A_\Omega & P \\ P^T & A_\Gamma \end{pmatrix} \quad , \tag{3.4}$$

where

$$A_\Omega = \begin{pmatrix} A_{11} & & \\ & A_{22} & \\ & & A_{33} \end{pmatrix} \quad , \quad A_\Gamma = \begin{pmatrix} A_{44} & \\ & A_{55} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} A_{14} & \\ A_{24} & A_{25} \\ & A_{35} \end{pmatrix} \quad .$$

The matrix A of (3.4) can be decomposed as follows:

$$A = \begin{pmatrix} A_\Omega & \\ P^T & C_{45} \end{pmatrix} \begin{pmatrix} I & A_\Omega^{-1} P \\ & I \end{pmatrix} \quad , \tag{3.5}$$

where C_{45} is the Schur complement of A_Γ in A, i.e.,

$$C_{45} \equiv A_\Gamma - P^T A_\Omega^{-1} P = \begin{pmatrix} M_4 & -A_{24}^T A_{22}^{-1} A_{25} \\ -A_{25}^T A_{22}^{-1} A_{24} & M_5 \end{pmatrix} \quad , \tag{3.6}$$

with

$$M_4 = A_{44} - A_{14}^T A_{11}^{-1} A_{14} - A_{24}^T A_{22}^{-1} A_{24} \tag{3.7}$$

and

$$M_5 = A_{55} - A_{25}^T A_{22}^{-1} A_{25} - A_{35}^T A_{33}^{-1} A_{35}. \tag{3.8}$$

The matrix M_4 would be the capacitance matrix for Γ_4 if the domain Ω_3 were absent. Similarly, M_5 would be the capacitance matrix for Γ_5 if the domain Ω_1 were absent. In fact, they are nothing but the preconditioner M_C described in the previous section. Both M_4 and M_5 have eigenvalue decompositions of the form (2.10). According to the definition (2.12), the eigenvalues of M_4 are given by $\lambda_j(n, m_1, m_2)$ for $j = 1, \ldots, n$ and its eigenvectors, by W_n. The eigenvalues of M_5 are $\lambda_i(m_2, n, n_3)$ for $i = 1, \ldots, m_2$, with eigenvectors given by W_{m_2}.

The matrix C_4 of (3.2) is the Schur complement of A_{44} in A, but it can also be written as the Schur complement of M_4 in C_{45}. Similarly, C_5 is the Schur complement of A_{55} in A, but it can also be written as the Schur complement of M_5 in C_{45}. Therefore, we can derive the following expressions for C_4 and C_5:

Lemma 3.1. *The interface matrix for Γ_4 in Ω can be written as*

$$C_4 = M_4 - B^T M_5^{-1} B \quad , \tag{3.9}$$

where $B = A_{25}^T A_{22}^{-1} A_{24}$. Similarly, the interface matrix for Γ_5 in Ω can be written as

$$C_5 = M_5 - B M_4^{-1} B^T \quad . \tag{3.10}$$

∎

The preconditioner proposed in [3] for C_4 in (3.2) would correspond to $M_C = M_4$ and similarly, $M_C = M_5$ for C_5 in (3.3). Since M_C is positive definite, we can choose $\sqrt{M_C}$ as a symmetric preconditioner. Let us define the preconditioned matrices:

$$\hat{C}_4 = M_4^{-1/2} C_4 M_4^{-1/2} \quad \text{and} \quad \hat{C}_5 = M_5^{-1/2} C_5 M_5^{-1/2} \quad . \tag{3.11}$$

By (3.9), we have

$$\hat{C}_4 = I_n - \hat{B}^T \hat{B} \quad \text{and} \quad \hat{C}_5 = I_{m_2} - \hat{B} \hat{B}^T \quad , \tag{3.12}$$

where

$$\hat{B} = M_5^{-1/2} A_{25}^T A_{22}^{-1} A_{24} M_4^{-1/2} \quad . \tag{3.13}$$

If we choose Γ_4 as the interface, at each iteration subdomain problems will be solved on Ω_1 and $\Omega_2 \cup \Gamma_5 \cup \Omega_3$. Similarly, if we choose Γ_5 as the interface, at each iteration subdomain problems will be solved on $\Omega_1 \cup \Gamma_4 \cup \Omega_2$ and Ω_3. The work per iteration is therefore comparable for both ways of splitting the domain. We will next show that, by solving (3.2) with preconditioner M_4 and (3.3) with preconditioner M_5, both systems are also equivalent from the convergence point of view. Therefore, in a general case, there is no *a priori* reason to prefer one way of decomposing the domain over the other.

If $n = m_2$, this fact is not surprising, considering that both interface systems have the same order and it is easy to see that $\hat{C}_4 = \hat{C}_5$. It is not obvious, however, whether one way of decomposing the domain should be prefered when $n \neq m_2$. As it turns out, even in this case the asymptotic convergence rate is the same for both systems, because the matrix C_{45} of (3.6) satisfies the following theorem:

Theorem 3.1. *Consider the following symmetric positive definite (SPD) system, written in block form:*

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad , \tag{3.14}$$

where the blocks A and B are square matrices. Also, define the Schur complement systems:

$$(A - BC^{-1}B^T)x = f - BC^{-1}g \tag{3.15}$$

and

$$(C - B^T A^{-1} B)y = g - B^T A^{-1} f \quad . \tag{3.16}$$

Consider the solution of (3.15) by the following fixed-point iteration, with splitting matrix given by A: given an initial guess x^0, define the i-th iterate as the solution to:

$$Ax^i = f - BC^{-1}g + BC^{-1}B^T x^{i-1} \tag{3.17}$$

for $i = 1, 2, \ldots$

Similarly, given y^0, define the i-th iterate of a fixed-point iteration for solving (3.16) with splitting matrix given by C, as:

$$Cy^i = g - B^T A^{-1} f + B^T A^{-1} B y^{i-1} \tag{3.18}$$

for $i = 1, 2, \ldots$

Then, the two iterations are convergent. Moreover, they are equivalent in the sense that for any given initial guess x^0 for (3.17), there exists an initial guess y^0 for (3.18), such that for all $i = 0, 1, \ldots$ we have:

$$y^i = q_y + P_y x^i \quad , \tag{3.19}$$

where $q_y = C^{-1}g$ and $P_y = -C^{-1}B^T$ and

$$\|e_x^{i+1}\|_A \leq \|e_y^i\|_C \leq \|e_x^i\|_A \quad , \tag{3.20}$$

where $e_x^i = x^i - x$, $e_y^i = y^i - y$ and $\|u\|_A$ denotes the A-norm of a vector u, i.e. $\sqrt{u^T A u}$.

Completely analogous results also hold for x^i and e_x^i, given an initial guess y^0 for (3.16).

Proof. Given x^0, define $y^0 = q_y + P_y x^0$. By induction, we can see that (3.19) satisfies (3.18) for every $i \geq 1$.

From the classical matrix iterative analysis for the convergence of block Gauss-Seidel iteration for SPD matrices, it can be shown that the two iterations converge. Also, since the matrix of (3.14) is SPD, so are the blocks A and C and their corresponding Schur complements. Therefore, $A^{-1/2}$ and $C^{-1/2}$ are well defined and

$$\|A^{-1/2}BC^{-1/2}\|_2 \leq 1 \quad . \tag{3.21}$$

We can also prove that

$$Ae_x^{i+1} = BC^{-1}B^T e_x^i$$

and

$$e_y^i = P_y e_x^i \quad .$$

Therefore, we have

$$A^{1/2}e_x^{i+1} = -(A^{-1/2}BC^{-1/2})C^{1/2}e_y^i \tag{3.22}$$

and

$$C^{1/2}e_y^i = -(C^{-1/2}B^T A^{-1/2})A^{1/2}e_x^i \quad . \tag{3.23}$$

Using (3.21), (3.22) and (3.23), we can prove (3.20).

∎

When an iterative method such as PCG is used, the rate of convergence depends on the condition number of the corresponding preconditioned matrix in (3.11). By applying the last theorem to C_{45} and by using (3.12), we can conclude the following:

Theorem 3.2. *Solving both systems (3.2) and (3.3) with preconditioners of the form M_C produce equivalent asymptotic convergence rates. Moreover, by (3.12), we have*

$$\mathcal{K}(\hat{C}_4) \leq \frac{1}{1 - \|\hat{B}^T\hat{B}\|_2} \tag{3.24}$$

and

$$\mathcal{K}(\hat{C}_5) \leq \frac{1}{1 - \|\hat{B}^T\hat{B}\|_2} \quad . \tag{3.25}$$

∎

Numerical computations show that the singular values β_i of $\hat{B}$ decrease very quickly with the index i. Therefore, in practice, only a few eigenvalues of $\hat{C}_4$ and $\hat{C}_5$ are different from 1, which leads to rapid convergence of the PCG method when applied to either matrix. For example, for the L-shaped region with corners at: $(0,0), (3,0), (3,0.25), (1,0.25), (1,1.25)$ and $(0,1.25)$, for $n = 31$ and 63, table 1 shows the singular values of $\hat{B}$ and the eigenvalues of $\hat{C}_5$, computed in single precision.

Our conclusion is that either way of decomposing an L-shaped region into two rectangles produces the same convergence rate, when preconditioner M_C is used. Moreover, we will be able to give an analytical bound on the condition number of the preconditioned capacitance matrix. This bound is derived from a bound on the norm of the operator $\hat{B}^T\hat{B}$.

But first, we will give an expression for the elements of a unitary transformation of $\hat{B}$. Let the elements of the matrix W_n be given by (2.11) and similarly, define the elements of W_{m_2} by replacing n by m_2 in (2.11).

The operator

$$Q_{NE} = A_{25}^T A_{22}^{-1} A_{24} \quad ,$$

which is part of the definition (3.13) of $\hat{B}$, is the operator that takes boundary values on the interface Γ_4, solving a Poisson problem on Ω_2 and then takes the values of the solution at the

$n = 31, m2 = 7$		$n = 63, m2 = 15$	
sv of $\hat{B}$	$\sigma(\hat{C}_5)$	sv of $\hat{B}$	$\sigma(\hat{C}_5)$
0.18204	0.96686	2.165E-01	0.95312
0.03868	0.99850	6.816E-02	0.99535
0.00514	0.99997	1.578E-02	0.99975
0.00045	0.99999	2.971E-03	0.99999
0.00002	1.00000	4.607E-04	0.99999
0.00000	1.00000	5.863E-05	1.00000
0.00000	1.00000	6.082E-06	1.00000
		5.093E-07	1.00000
		3.610E-08	1.00000
		...	...

Table 1: Eigenvalues of preconditioned capacitance system for an L-shaped region

gridpoints which are adjacent to Γ_5. It is possible to derive the elements of Q_{NE} when it is pre and post-multiplied, respectively, by the matrices W_{m_2} and W_n. The elements of

$$W_{m_2} Q_{NE} W_n$$

are given by

$$q_{ij} = \frac{2}{\sqrt{(m_2+1)(n+1)}} \frac{\sin\frac{i\pi}{m_2+1} \quad \sin\frac{j\pi}{n+1}}{\sigma_j^{(n)} + \sigma_i^{(m_2)}} \tag{3.26}$$

for $i = 1, \ldots, m_2$ and $j = 1, \ldots, n$. A proof of (3.26) can be found in the appendix (see lemma 5.2).

For any given integers n, m_1 and m_2, let $\lambda(n, m_1, m_2)$ be defined by (2.12), where γ_j is given by equation (2.14). By using (3.26), it is easy to prove the following lemma:

Lemma 3.2. *Let*

$$V = W_{m_2} \hat{B} W_n \quad . \tag{3.27}$$

Then, $\|V\|_2 = \|\hat{B}\|_2$ *and the elements of the matrix* V *are given by*

$$v_{ij} = \frac{2}{\sqrt{(n+1)(m_2+1)}} \frac{\sin\frac{i\pi}{m_2+1} \quad \sin\frac{jn\pi}{n+1}}{s_j^{(4)} s_i^{(5)} \left(\sigma_j^{(n)} + \sigma_i^{(m_2)}\right)} \tag{3.28}$$

for $i = 1, \ldots, m_2$ *and* $j = 1, \ldots, n$, *where* $s_j^{(4)} = \sqrt{|\lambda_j(n, m_1, m_2)|}$ *and* $s_i^{(5)} = \sqrt{|\lambda_i(m_2, n, n_3)|}$.

∎

As equations (3.24) and (3.25) suggest, in order to find a bound for the condition number of the preconditioned capacitance system, we need to bound the norm of $\hat{B}$, or V. Since we have an expression for the elements of V, we can bound $\|V\|_1$ and $\|V\|_\infty$ and then use the property:

$$\|V\|_2 \leq \sqrt{\|V\|_1 \|V\|_\infty} \quad .$$

The results are summarized in the next theorem. A proof can be found in appendix B:

Theorem 3.3. *Define the aspect ratio for the domain Ω_2 in fig.2 as $\alpha = \frac{m_2+1}{n+1}$. Then,*

a) $\|V\|_1 \leq \sqrt{\alpha}\, 0.733$ *and* $\|V\|_\infty \leq \sqrt{\frac{1}{\alpha}}\, 0.733$.

b) $\|\hat{B}^T\hat{B}\|_2 \leq \|\hat{B}\|_2^2 = \|V\|_2^2 \leq \|V\|_1\|V\|_\infty \leq 0.54$.

c) *For all gridsizes and all L-shaped regions,*

$$\mathcal{K}(\hat{C}_4) \leq 2.16 \quad \text{and} \quad \mathcal{K}(\hat{C}_5) \leq 2.16 \quad . \tag{3.29}$$

∎

In our experiments on L-shaped domains with many different aspect ratios, condition numbers larger than 1.2 have not been observed. The bound 0.54 in b), however, is fairly tight for $\|V\|_1\|V\|_\infty$, as was shown by numerical experiments with large values of n and m_2. Therefore, if a tighter bound is desired for the condition number, one would need to bound the 2-norm of $\hat{B}^T\hat{B}$ directly.

We would also like to discuss briefly how the parameter n_3 (or, respectively, m_1) affects the performance of preconditioner M_4 (M_5). Clearly, as n_3 tends to zero for large m_2, the domain Ω approaches the shape of a perfect rectangle. The preconditioner M_4 should reflect this by becoming the exact boundary operator. In other words, $\mathcal{K}(\hat{C}_4)$ should approach one. We can verify that this is the case as follows: v_{ij} in (3.28) depends on n_3 only through $\lambda_i(m_2, n, n_3)$ (defined in (2.12)). When the aspect ratio $\frac{n_3+1}{m_2+1}$ tends to zero (i.e. Ω_3 becomes thinner), $\lambda_i(m_2, n, n_3)$ tends to infinity and therefore v_{ij} tends to zero. However, we can see that this dependency is very weak, because $\lambda_j(m_2, n, n_3)$ tends rapidly to an asymptotic value independent of n_3 when such aspect ratio grows. Only the fact that

$$\lambda_j(m_2, n, n_3) \geq 2\sqrt{\sigma_j} \tag{3.30}$$

is used in the proof of theorem 3.3, which is true for all n_3. The discussion above implies that the performance of M_4 as a preconditioner for C_4 is fairly independent on how irregular the region is.

Incidentally, for the other preconditioners mentioned in this paper [6, 1, 7], the preconditioned capacitance matrix always has the form $X + \tilde{B}^T\tilde{B}$, for some operator $\tilde{B}$ to which the bounds (a) and (b) of theorem 3.3 can also be applied, as long as (3.30) holds. The bound given in (c), however, does not hold for other preconditioners, for which the norm of X may grow when the aspect ratio α of the domain Ω_2 decreases (see [3] for an example on a T-shaped region).

4. C-shaped regions

Some of the expressions and results of the previous section are more general than they appear and they can be used as basic components for more complicated regions that are unions of rectangles. For example, a C-shaped region can be subdivided as indicated in fig.3.

Similar to L-shaped domains, the region of fig.3 can be separated in three rectangles by either Γ_6 and Γ_7, or Γ_8 and Γ_9. By ordering the variables in $\Omega_i, i \leq 5$ first and then those on $\Gamma_j, 6 \leq j \leq 9$, the matrix A that represents the discrete differential operator on Ω can be written in block form as in (3.4), where

$$A_\Omega = \begin{pmatrix} A_{11} & & \\ & \ddots & \\ & & A_{55} \end{pmatrix} \quad , \quad A_\Gamma = \begin{pmatrix} A_{66} & & \\ & \ddots & \\ & & A_{99} \end{pmatrix} \quad , \tag{4.1}$$

and

$$P = \begin{pmatrix} A_{16} & & A_{18} & \\ A_{26} & A_{27} & & \\ & \ddots & & \\ & & & A_{59} \end{pmatrix} \quad .$$

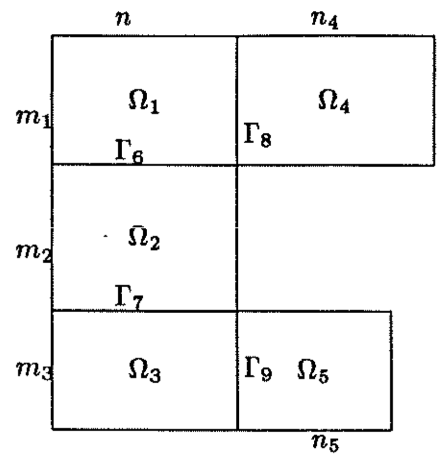

Figure 3: C-shaped domain

A system

$$C_{67}\begin{pmatrix} u_6 \\ u_7 \end{pmatrix} = g_{67} \tag{4.2}$$

can be derived by block elimination for the interfaces Γ_6 and Γ_7, where C_{67} is the Schur complement in A of the blocks A_{66} and A_{77}. In [4], a multistrip operator M_{67} is described, which solves, exactly, the problem on a rectangle divided into three strips (Ω_1, Ω_2 and Ω_3). We will analyze M_{67} as a preconditioner for C_{67}. The operator M_{67} has the following block structure:

$$M_{67} = \begin{pmatrix} H_6 & S \\ S & H_7 \end{pmatrix} \quad , \tag{4.3}$$

where

$$H_6 = A_{66} - A_{16}^T A_{11}^{-1} A_{16} - A_{26}^T A_{22}^{-1} A_{26} \quad ,$$
$$H_7 = A_{77} - A_{37}^T A_{33}^{-1} A_{37} - A_{27}^T A_{22}^{-1} A_{27}$$

and

$$S = -A_{26}^T A_{22}^{-1} A_{27} \quad .$$

The blocks H_6, H_7 and S have eigenvalue decompositions of the form (2.10), with eigenvalues given by $\lambda_j(n, m_1, m_2), \lambda_j(n, m_2, m_3)$ and

$$\delta_j(n, m_2) = -2\sqrt{\sigma_j + \frac{\sigma_j^2}{4}}\left(\frac{\gamma_j^{\frac{m_2+1}{2}}}{1-\gamma_j^{m_2+1}}\right) \tag{4.4}$$

respectively, for $j = 1, \ldots, n$. (See lemma 5.1 in appendix A).

Similarly, a system

$$C_{89}\begin{pmatrix} u_8 \\ u_9 \end{pmatrix} = g_{89} \tag{4.5}$$

can be derived for the interfaces Γ_8 and Γ_9, where C_{89} is the Schur complement in A of the blocks A_{88} and A_{99}. The system (4.5) can be preconditioned by a block diagonal preconditioner M_{89}, with diagonal blocks M_8 and M_9. M_8 is the exact interface system for Γ_8 with respect to the subdomains

Ω_1 and Ω_4, and M_9 is the exact interface system for Γ_9 with respect to the subdomains Ω_3 and Ω_5. Both M_8 and M_9 have decompositions of the form (2.10).

It can be easily shown that C_Γ, the Schur complement of the blocks A_Γ in A, can be written in block form as:

$$C_\Gamma = \left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{M_{67}}} & Q_{SE} & 0 \\ & & 0 & Q_{NE} \\ \hline Q_{SE}^T & 0 & M_8 & 0 \\ 0 & Q_{NE}^T & 0 & M_9 \end{array}\right] ,$$

where $Q_{SE} = A_{16}^T A_{11}^{-1} A_{18}$ and $Q_{NE} = A_{37}^T A_{33}^{-1} A_{39}$. Again, by applying theorem 3.1, we have that both ways of dividing the domain are equivalent, in the sense that initial residuals can be found such that the same number of iterations are necessary when PCG is applied to C_{67} with preconditioner M_{67} than when PCG is applied to C_{89} with preconditioner M_{89}. The preconditioned interface operator for Γ_6 and Γ_7,

$$\hat{C}_{67} \equiv M_{67}^{-1/2} C_{67} M_{67}^{-1/2} ,$$

can be written in the form

$$\hat{C}_{67} = I - \hat{B}^T \hat{B} , \tag{4.6}$$

where $\hat{B} \in R^{(m_1+m_3)\times 2n}$ and

$$\hat{B} = \begin{pmatrix} M_8 & 0 \\ 0 & M_9 \end{pmatrix}^{-1/2} \begin{pmatrix} Q_{SE}^T & 0 \\ 0 & Q_{NE}^T \end{pmatrix} M_{67}^{-1/2} .$$

Similarly, the preconditioned interface operator for for Γ_8 and Γ_9,

$$\hat{C}_{89} \equiv \begin{pmatrix} M_8 & 0 \\ 0 & M_9 \end{pmatrix}^{-1/2} C_{89} \begin{pmatrix} M_8 & 0 \\ 0 & M_9 \end{pmatrix}^{-1/2} ,$$

can be written in the form

$$\hat{C}_{89} = I - \hat{B}\hat{B}^T . \tag{4.7}$$

The condition numbers of $\hat{C}_{67}$ and $\hat{C}_{89}$ are bounded by

$$\mathcal{K}(\hat{C}_{67}) \leq \frac{1}{1 - \|\hat{B}^T \hat{B}\|_2} \tag{4.8}$$

and

$$\mathcal{K}(\hat{C}_{89}) \leq \frac{1}{1 - \|\hat{B}^T \hat{B}\|_2} . \tag{4.9}$$

Define V as the following unitary tranformation of $\hat{B}$:

$$V = \begin{pmatrix} W_{m_1} & 0 \\ 0 & W_{m_3} \end{pmatrix} \hat{B} \begin{pmatrix} W_n & 0 \\ 0 & W_n \end{pmatrix} .$$

Then $\|V\| = \|\hat{B}\|$. The matrix V can be written as a block two by two matrix

$$V = \begin{pmatrix} V_{66} & V_{67} \\ V_{76} & V_{77} \end{pmatrix} , \tag{4.10}$$

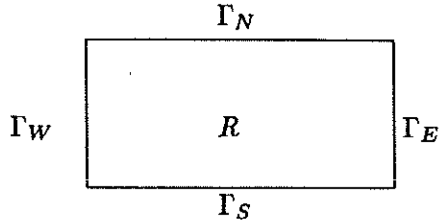

Figure 4: Interaction between interfaces

whose block elements have expressions similar to the matrix V for L-shaped regions, namely,

$$\begin{aligned} V_{66} &= W_{m_1} M_8^{-1/2} Q_{SE}^T W_n R_6 \\ V_{67} &= W_{m_1} M_8^{-1/2} Q_{SE}^T W_n R_- \\ V_{76} &= W_{m_3} M_9^{-1/2} Q_{NE}^T W_n R_- \\ V_{77} &= W_{m_3} M_9^{-1/2} Q_{NE}^T W_n R_7 \end{aligned} \quad . \tag{4.11}$$

where R_6, R_7 and R_- are diagonal matrices such that:

$$\begin{pmatrix} R_6 & R_- \\ R_- & R_7 \end{pmatrix} = \begin{pmatrix} W_n & 0 \\ 0 & W_n \end{pmatrix} M_{67}^{-1/2} \begin{pmatrix} W_n & 0 \\ 0 & W_n \end{pmatrix} \quad .$$

For the case when $m_1 = m_3 \leq m_2$, a simple expression can be found for R_6, R_7 and R_-, namely $R_6 = R_7 = R_+$, with the diagonal elements of $R_\pm$ given by

$$r_j^\pm = \frac{1}{2}\left(\frac{1}{\sqrt{\lambda_j - |\delta_j|}} \pm \frac{1}{\sqrt{\lambda_j + |\delta_j|}}\right) \quad , \tag{4.12}$$

where λ_j is $\lambda_j(n, m_1, m_2)$, given by (2.12) and δ_j is $\delta_j(n, m_2)$, given by (4.4). Arguments similar to those in theorem 3.3 can be applied to give the following:

Theorem 4.1. *Consider a C-shaped region like fig.3, where $m_1 = m_3 \leq m_2$ and α is the aspect ratio for the domain Ω_1 or Ω_3 in the picture, i.e. $\alpha = \frac{m_1+1}{n+1}$. Then,*

a) $\|V\|_1 \leq \sqrt{\alpha}\, 0.7877$ *and* $\|V\|_\infty \leq \frac{1}{\sqrt{\alpha}}\, 0.7877$.

b) $\|V^T V\|_2 \leq \|V\|_2^2 = \|\hat{B}\|_2^2 \leq \|\hat{B}\|_1 \|\hat{B}\|_\infty \leq 0.62$.

c) $\mathcal{K}(\hat{C}_{67}) \leq 2.63$ *and* $\mathcal{K}(\hat{C}_{89}) \leq 2.63$ *for all gridsizes and all C-shaped regions such that* $m_1 = m_3 \leq m_2$.

Proof. In appendix B.

∎

5. Appendix A: The interaction between interior edges

In this appendix, we define the operators that represent the interaction between two interfaces of a given subdomain. Consider the rectangular region R of fig.4, with edges Γ_N, Γ_E, Γ_S and Γ_W. This region R represents a generic rectangular subdomain in the domain Ω. Let n_1 be the number of gridpoints in Γ_N (or Γ_S) and n_2, the number of gridpoints in Γ_E (or Γ_W). The corner points are not included in the edges. They may or may not be interior to Ω.

For the case of constant coefficients operators, it is possible to describe, in terms of Fourier modes, an operator Q which takes boundary values on one of the edges and computes the solution on the gridpoints adjacent to the same or other edge.

Let A be the matrix which represents the discretization of the differential operator in Ω. If the interface Γ_k, where $k = N, S, E$ or W, is interior to Ω, then we can define P_k, the submatrix of A that represents the coupling between gridpoints of R and gridpoints on Γ_k. Also define A_R, the diagonal block corresponding to the interior points of R, in other words, A_R is the restriction of the differential operator to the region R. We can now define the operator Q_{kl} which represents the interaction between the edges Γ_k and Γ_l as:

$$Q_{kl} = P_k^T A_R^{-1} P_l \tag{5.1}$$

For constant coefficients operators, when Γ_k and Γ_l are parallel, the operator Q_{kl} is diagonalizable by Fourier modes. For example, for the case of the Poisson equation we can prove the following lemma. Here, for any given n, W_n is the matrix of sine modes of dimension n, with elements given by

$$w_{ij} = \sqrt{\frac{2}{n+1}} \sin \frac{ij\pi}{n+1} \quad . \tag{5.2}$$

for $i, j = 1, \ldots, n$.

Lemma 5.1. *Consider the Poisson equation on a domain Ω which contains the rectangular region R. Let Q_{NS} be the operator that represents the coupling between interfaces Γ_N and Γ_S, defined as in (5.1). Then,*

$$W_{n_1} Q_{NS} W_{n_1} = D_{NS} \quad ,$$

where the matrix D_{NS} is diagonal, with diagonal entries given by

$$d_{jj} = \sqrt{\gamma_j^{n_2}} \left(\frac{1-\gamma_j}{1-\gamma_j^{n_2+1}} \right) \quad , \tag{5.3}$$

where

$$\gamma_j = \left(1 + \frac{\sigma_j^{(1)}}{2} - \sqrt{\sigma_j^{(1)} + \frac{\left(\sigma_j^{(1)}\right)^2}{4}} \right)^2 \tag{5.4}$$

and

$$\sigma_j^{(1)} = 4 \sin^2 \frac{j\pi}{2(n_1+1)} \quad . \tag{5.5}$$

A similar expression can be found for Q_{EW}.

Also,

$$W_{n_1} Q_{NN} W_{n_1} = D_{NN} \quad ,$$

where the matrix D_{NN} is diagonal, with diagonal entries given by

$$d_{jj} = -\sqrt{\gamma_j} \left(\frac{1-\gamma_j^{n_2}}{1-\gamma_j^{n_2+1}} \right) \quad . \tag{5.6}$$

Similar expressions can be derived for Q_{SS}, Q_{EE} and Q_{WW}.

Proof. Proofs for formulas (5.3) and (5.6) can be found in [3] and [4]. Here we give a different – more general – proof using direct (or tensor) products. The matrices P_N and P_S can be written as:

$$P_N = e_1^{(2)} \otimes I_1 \tag{5.7}$$

$$P_S = e^{(2)}_{n_2} \otimes I_1 \tag{5.8}$$

where I_l, for $l = 1, 2$, is the identity matrix of dimension n_l and $e^{(l)}_i$ is the i-th column of I_l. The matrix A_R is the discrete Laplacian operator on the region R and it has the following block tridiagonal form:

$$A_R = \begin{pmatrix} T & I_1 & & \\ I_1 & T & & \\ & & \ddots & I_1 \\ & & I_1 & T \end{pmatrix} \quad , \tag{5.9}$$

where $T = \text{tridiag}(1, -4, 1)$. It is easy to prove that

$$W_{n_1} T W_{n_1} = D_T \quad ,$$

where $D_T = \text{diag}(-2 - \sigma^{(1)}_j)$. Then we have

$$W_{n_1} Q_{NS} W_{n_1} = (e^{(2)}_1 \otimes I_1) \begin{pmatrix} D_T & I_1 & & \\ I_1 & D_T & & \\ & & \ddots & I_1 \\ & & I_1 & D_T \end{pmatrix}^{-1} (e^{(2)}_{n_2} \otimes I_1) \tag{5.10}$$

By reordering the equations in (5.10), we have:

$$W_{n_1} Q_{NS} W_{n_1} = (I_1 \otimes e^{(2)}_1)^T \begin{pmatrix} T_1 & & & \\ & T_2 & & \\ & & \ddots & \\ & & & T_{n_1} \end{pmatrix}^{-1} (I_1 \otimes e^{(2)}_{n_2}) \quad ,$$

where $T_j = \text{tridiag}(1, -2 - \sigma^{(1)}_j, 1)$. Therefore, $W_{n_1} Q_{NS} W_{n_1}$ is diagonal and its diagonal elements are given by

$$e^{(2)^T}_1 T_j^{-1} e^{(2)}_{n_2} \quad ,$$

which can be proved to be given by (5.3).

Similarly, we can prove that $W_{n_1} Q_{NN} W_{n_1}$ is diagonal and its diagonal elements are given by

$$e^{(2)^T}_{n_2} T_j^{-1} e^{(2)}_{n_2} \quad ,$$

which can be proved to be given by (5.6).

∎

Operators like Q_{NE}, on the other hand, which represent the interaction between perpendicular edges, are not diagonalizable by Fourier modes. Moreover, they are, in general not square, but n_1 by n_2 rectangular matrices. It is possible, however, to describe the elements of the matrices $W_{n_1} Q_{NE} W_{n_2}$ and $W_{n_1} Q_{NW} W_{n_2}$ for constant coefficients cases.

Lemma 5.2. *Consider the Poisson equation on a domain Ω which contains the rectangular region R. The elements of*

$$\hat{Q}_{NE} = W_{n_1} Q_{NE} W_{n_2}$$

are given by

$$q_{ij}^{NE} = \frac{2}{\sqrt{(n_1+1)(n_2+1)}} \frac{\sin\frac{i\pi}{n_1+1} \quad \sin\frac{j\pi}{n_2+1}}{\sigma_i^{(1)} + \sigma_j^{(2)}} \tag{5.11}$$

for $i = 1, \ldots, n_1$ *and* $j = 1, \ldots, n_2$, *where* $\sigma_j^{(l)} = 4\sin^2\frac{j\pi}{2(n_l+1)}$, *for* $l = 1, 2$. *Similarly, the elements of the matrix*

$$\hat{Q}_{NW} = W_{n_1} Q_{NW} W_{n_2}$$

are given by

$$q_{ij}^{NW} = \frac{2}{\sqrt{(n_1+1)(n_2+1)}} \frac{\sin\frac{n_1 i\pi}{n_1+1} \quad \sin\frac{j\pi}{n_2+1}}{\sigma_i^{(1)} + \sigma_j^{(2)}} \tag{5.12}$$

Proof. The eigenvalue decomposition of the matrix A_R is well known and it is given by

$$A_R = (W_{n_2} \otimes W_{n_1}) \quad \Lambda \quad (W_{n_2} \otimes W_{n_1}) \tag{5.13}$$

where Λ is the $n_1 n_2 \times n_1 n_2$ diagonal matrix whose diagonal elements are

$$\lambda_J = -\sigma_i^{(1)} - \sigma_j^{(2)} \quad ,$$

with $J = (j-1)n_1 + i$, for $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2$. Also, we have

$$P_W = I_2 \otimes e_1^{(1)} \quad . \tag{5.14}$$

By replacing equations (5.13) to (5.14) in (5.1) and then applying the following two properties of tensor products:

$$\begin{aligned} &i) \quad (X \otimes Y)^T = X^T \otimes Y^T \text{and} \\ &ii) \quad (X_1 \otimes Y_1)(X_2 \otimes Y_2) = (X_1 X_2) \otimes (Y_1 Y_2) \quad , \end{aligned}$$

we have:

$$Q_{NW} = \left((e_1^{(2)T} W_2) \otimes W_1\right) \Lambda^{-1} \left(W_2 \otimes (W_1 e_1^{(1)})\right) \tag{5.15}$$

and therefore,

$$\hat{Q}_{NW} = \left((e_1^{(2)T} W_2) \otimes I_1\right) \Lambda^{-1} \left(I_2 \otimes (W_1 e_1^{(1)})\right) \cdot \tag{5.16}$$

Then we can see that the j-th column of (5.16) is given by

$$\sqrt{\frac{2}{n_2+1}} \sin\frac{j\pi}{n_2+1} \left(\sigma_j^{(2)} I_1 + \operatorname{diag}(\sigma_i^{(1)})\right)^{-1} W_1 \, e_{n_1}^{(1)} \quad ,$$

from which (5.11) follows.

Similarly, (5.12) can be derived by using

$$P_E = I_2 \otimes e_{n_1}^{(1)} \tag{5.17}$$

instead of (5.7).

∎

6. Appendix B: Proof of theorems 3.3 and 4.1

Proof of Theorem 3.3

Proof.

Theorem 3.3: Let the *aspect ratio* for the domain Ω_2 in fig.2 be defined as $\alpha = \frac{m_2+1}{n+1}$. Then,

a) $\|\tilde{V}\|_1 \leq \sqrt{\alpha}\, 0.733$ and $\|\tilde{V}\|_\infty \leq \sqrt{\frac{1}{\alpha}}\, 0.733$.

b) $\|V^T V\|_2 \leq \|V\|_2^2 = \|\tilde{V}\|_2^2 \leq \|\tilde{V}\|_1 \|\tilde{V}\|_\infty \leq 0.54$.

c) For all gridsizes and all L-shaped regions,

$$\mathcal{K}(\hat{C}_4) \leq 2.16 \quad \text{and} \quad \mathcal{K}(\hat{C}_5) \leq 2.16 \quad . \tag{6.1}$$

Proof. (b) follows from (a). (c) follows from (b) and from equations (3.24) and (3.25). In order to prove (a), we will first need to prove some lemmas that give bounds for the column and row sums of the absolute values of (3.28), the elements v_{ij} of the matrix $\tilde{V}$. The eigenvalues (2.12) of M_4 and M_5 can be bounded by

$$\lambda_j(n, m_1, m_2) \geq 4 \sin \frac{j\pi}{2(n+1)} \tag{6.2}$$

for all $j = 1, \ldots, n$ and

$$\lambda_i(m_2, n, n_3) \geq 4 \sin \frac{i\pi}{2(m_2+1)} \tag{6.3}$$

for all $i = 1, \ldots, m_2$. It is easy to show that

$$|v_{ij}| \leq \frac{1}{2\sqrt{\alpha}} \frac{f(x_i, y_j)}{n+1} \quad , \tag{6.4}$$

where the function f is defined by

$$f(x, y) = \frac{\sqrt{\sin x\frac{\pi}{2}}\ \cos x\frac{\pi}{2}\ \sqrt{\sin y\frac{\pi}{2}}\ \cos y\frac{\pi}{2}}{\sin^2 x\frac{\pi}{2} + \sin^2 y\frac{\pi}{2}} \quad , \tag{6.5}$$

$x_i = \frac{i}{m_2+1}$ and $y_j = \frac{j}{n+1}$. Similarly, we have

$$|v_{ij}| \leq \frac{\sqrt{\alpha}}{2} \frac{f(x_i, y_j)}{m_2+1} \quad . \tag{6.6}$$

The column and row sums of $|v_{ij}|$ can be then bounded by expressions that involve the integrals of f with respect to x or with respect to y. The following lemma gives an expression for the integral of f with respect to x, for a fixed y. Since $f(x, y) = f(y, x)$, an analogous result holds for the integral of f with respect to y, for a fixed x.

Lemma 6.1. *Given $y \in (0,1)$ and $a, b \in (0,1)$ such that $a \leq y < b$,*

$$\int_a^b f(x, y)\, dx = \frac{2}{\sqrt{2\pi}} \cos y\frac{\pi}{2} \left(\pi - g(\sin b\frac{\pi}{2}, \sin y\frac{\pi}{2}) + g(\sin a\frac{\pi}{2}, \sin y\frac{\pi}{2})\right) \quad , \tag{6.7}$$

where

$$g(z, w) = \frac{1}{2} \log \frac{z + \sqrt{2zw} + w}{z - \sqrt{2zw} + w} + \arctan \frac{\sqrt{2zw}}{z - w} \quad . \tag{6.8}$$

Proof. By replacing $z = \sin x\frac{\pi}{2}$ and $w = \sin y\frac{\pi}{2}$ in (6.5) and defining

$$F(z,w) = \frac{\sqrt{zw}}{z^2 + w^2} \quad , \tag{6.9}$$

we get

$$\begin{aligned}\int_a^b f(x,y)\,dx &= \frac{2}{\pi}\cos y\frac{\pi}{2}\int_{\sin a\frac{\pi}{2}}^{\sin b\frac{\pi}{2}} F(z,w)dz \\ &= \frac{2}{\sqrt{2}\pi}\cos y\frac{\pi}{2}\left(-\frac{1}{2}\log\frac{z+\sqrt{2zw}+w}{z-\sqrt{2zw}+w} + \arctan\frac{\sqrt{2zw}}{w-z}\right)\Bigg|_{\sin a\frac{\pi}{2}}^{\sin b\frac{\pi}{2}} \\ &= \frac{2}{\sqrt{2}\pi}\cos y\frac{\pi}{2}\left(\pi - g(\sin b\frac{\pi}{2}, \sin y\frac{\pi}{2}) + g(\sin a\frac{\pi}{2}, \sin y\frac{\pi}{2})\right) \quad .\end{aligned}$$

∎

We will also need to describe the behiavor of $f(x,y)$, for a fixed $y \in (0,1)$, in the interval $x \in (0,1)$. We can easily see that, when $x, y \in (0,1)$, $f(x,y) > 0$. In the next lemma we prove that $f(.,y)$ has one and only one relative maximum in $(0,1)$.

Lemma 6.2. *Given $y \in (0,1)$, there exists a unique $x^*(y) \in (0,1)$ such that:*

$$\max_{0<x<1} f(x,y) = f(x^*,y) \quad ,$$

$f(.,y)$ is monotonically increasing on the interval $(0,x^*)$ and $f(.,y)$ is monotonically decreasing on $(x^*,1)$. Moreover, $f(x^*,y)$ is bounded by

$$f(x^*,y) \le \frac{3}{4\sqrt[4]{3}}\cot y\frac{\pi}{2} \quad . \tag{6.10}$$

Proof. The partial derivative of f with respect to x is given by:

$$\frac{\partial f}{\partial x} = \xi(x,y)(\sin^2 x\frac{\pi}{2} - z_-)(\sin^2 x\frac{\pi}{2} - z_+) \quad ,$$

where $\xi(x,y) > 0$ for all $x, y \in (0,1)$ and

$$z_\pm = \frac{3}{2}(1 + \sin^2 y\frac{\pi}{2}) \pm \sqrt{\frac{9}{4}(1+\sin^2 y\frac{\pi}{2})^2 - \sin^2 y\frac{\pi}{2}} \quad .$$

It can be shown that

$$0 < z_- \le \frac{\sin^2 y\frac{\pi}{2}}{3} < 1 \tag{6.11}$$

and $z_+ > 1$. Therefore, $\frac{\partial f}{\partial x} > 0$ for $x < x^*$ and $\frac{\partial f}{\partial x} < 0$ for $x > x^*$, where x^* is the unique solution in $(0,1)$ to

$$\sin^2 x^*\frac{\pi}{2} = z_- \quad . \tag{6.12}$$

Therefore, f has a unique maximum in $(0,1)$ at x^*. Moreover, since for all $x \in (0,1)$,

$$f(x,y) \le F(\sin x\frac{\pi}{2}, \sin y\frac{\pi}{2}) \quad ,$$

where F is defined by (6.9), we have

$$f(x^*, y) \leq \max_{0<z<1} F(z, \sin y\frac{\pi}{2}) = \frac{3}{4\sqrt[4]{3}} \cot y\frac{\pi}{2} \quad .$$

∎

We can now prove (a) in theorem 3.3:

We will only prove the inequality $\|\tilde{V}\|_1 \leq \sqrt{\alpha}\ 0.733$. The proof of $\|\tilde{V}\|_\infty \leq \sqrt{\frac{1}{\alpha}}\ 0.733$ is completely analogous, by using (6.4) instead of (6.6).

By (6.6), we have

$$\|\tilde{V}\|_1 \equiv \max_{1\leq j\leq n} \sum_{i=1}^{m_2} |v_{ij}| \leq \frac{\sqrt{\alpha}}{2} \frac{1}{m_2+1} \max_{1\leq j\leq n} \sum_{i=1}^{m_2} f(x_i, y_j) \quad . \tag{6.13}$$

Let $h = \frac{1}{m_2+1} = x_1$. Since $f > 0$ for $x, y \in (0,1)$ and, by lemma 6.2, f is monotonic in the intervals $(0, x^*(y_j))$ and $(x^*(y_j), 1)$, it is easy to see, by using graphical arguments, that

$$h\sum_{i=1}^{m_2} f(x_i, y_j) \leq \int_h^1 f(x, y_j)dx + hf(x^*(y_j), y_j) \quad , \tag{6.14}$$

when $h \leq x^*(y_j)$. On the other hand, when $h > x^*(y_j)$, all the values of $x_i, i = 1, \ldots, m_2$ are on the interval $(x^*, 1)$, where f is monotonically decreasing. Then, we have

$$h\sum_{i=1}^{m_2} f(x_i, y_j) = hf(h, y_j) + h\sum_{i=2}^{m_2} f(x_i, y_j) \leq \int_h^1 f(x, y_j)\mathrm{d}x + hf(h, y_j) \quad . \tag{6.15}$$

Let us first assume that $h \leq x^*(y_j)$. By (6.11) and (6.12), we can prove that $x^*(y) \leq y$ for all $y \in (0,1)$. Then, by (6.7), we have:

$$\int_h^1 f(x, y_j)\ dx \leq \frac{2}{\sqrt{2\pi}} \left(\pi + g(\sin h\frac{\pi}{2}, \sin y_j\frac{\pi}{2})\right) \tag{6.16}$$

because $g(z, w) \geq 0$ for $w < z$. Define the function $G(h, \beta) = g(\sin h\frac{\pi}{2}, \sin\beta h\frac{\pi}{2})$. Then, the right hand side of (6.16) can be written as:

$$\frac{2}{\sqrt{2\pi}} (\pi + G(h, \beta)) \quad , \tag{6.17}$$

with $\beta = \frac{y_j}{h} \geq 1$. By differentiating G with respect to β we can see that $\frac{\partial G}{\partial \beta} < 0$ for all $h \in (0,1)$ and $\beta \in [1, +\infty)$. Therefore, G decreases with β, i.e.

$$G(h, \beta) \leq G(h, 1) = \lim_{w\to z^+} g(z, w) = \frac{1}{2}\log\frac{2+\sqrt{2}}{2-\sqrt{2}} - \frac{\pi}{2} \quad . \tag{6.18}$$

for all $h \in (0,1)$ and $\beta \in [1, +\infty)$. We can then bound (6.16) by

$$\int_h^1 f(x, y_j)\, dx \le \frac{1}{\sqrt{2}\pi}\left(\pi + \log\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) \quad . \tag{6.19}$$

On the other hand, by (6.10), we have

$$hf(x^*(y_j), y_j) \le \frac{3}{4\sqrt[4]{3}} h \cot \beta h \frac{\pi}{2} \quad . \tag{6.20}$$

The right hand side of (6.20) can also be proven to decrease with β and h and therefore we have

$$hf(x^*(y_j), y_j) \le \frac{3}{4\sqrt[4]{3}} h \cot h\frac{\pi}{2} \le \frac{3}{2\sqrt[4]{3}\pi} \quad . \tag{6.21}$$

By replacing (6.19) and (6.21) in (6.14), we have

$$h\sum_{i=1}^{m_2} f(x_i, y_j) \le \frac{1}{\sqrt{2}\pi}\left(\pi + \log\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) + \frac{3}{2\sqrt[4]{3}\pi} = 1.4666 \tag{6.22}$$

and therefore, by (6.13), we have

$$\|\tilde{V}\|_1 \le \sqrt{\alpha}\; 0.7333 \tag{6.23}$$

when $h \le x^*(y_j)$.

When $h > x^*(y_j)$, by (6.15) we have

$$h\sum_{i=1}^{m_2} f(x_i, y_j) \le \int_{x^*(y_j)}^1 f(x, y_j)\mathrm{dx} + hf(h, y_j) \quad . \tag{6.24}$$

By (6.7),

$$\begin{aligned}\int_{x^*(y_j)}^1 f(x, y_j)\mathrm{dx} &\le \frac{2}{\sqrt{2}\pi}\cos y_j\frac{\pi}{2}\left(\pi + g(\sin x^*\frac{\pi}{2}, \sin y_j\frac{\pi}{2})\right)\\ &\le \frac{2}{\sqrt{2}\pi}\left(\pi + G(x^*, \frac{y_j}{x^*})\right)\end{aligned}$$

and, by (6.18), we have

$$\int_{x^*(y_j)}^1 f(x, y_j)\mathrm{dx} \le \frac{1}{\sqrt{2}\pi}\left(\pi + \log\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) \quad . \tag{6.25}$$

Since $f(h, y_j) \le f(x^*(y_j), y_j)$, by (6.21) we have

$$hf(h, y_j) \le \frac{3}{2\sqrt[4]{3}\pi} \tag{6.26}$$

By replacing (6.25) and (6.26) in (6.24), we have

$$h\sum_{i=1}^{m_2} f(x_i, y_j) \leq 1.4666$$

and therefore, by (6.13), we have

$$\|\tilde{V}\|_1 \leq \sqrt{\alpha}0.7333 \tag{6.27}$$

when $h > x^*(y_j)$. By (6.23) and (6.27), we proved that (6.23) holds for all $h < 1$.

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Proof of Theorem 4.1

Proof. Define the function

$$f(x) = \frac{1 + x - \sqrt{x}}{1 - x} \quad .$$

We can easily prove that

$$f(x) \geq 0.866 \qquad \text{for all} \quad x \in [0, 1) \quad . \tag{6.28}$$

By (4.4) and (2.12), we have

$$\lambda_j(n, m_1, m_2) - |\delta_j(n, m_2)| = \left(\frac{1 + \gamma_j^{m_1+1}}{1 - \gamma_j^{m_1+1}} + \frac{1 + \gamma_j^{m_2+1}}{1 - \gamma_j^{m_2+1}} - 2\frac{\gamma_j^{\frac{m_2+1}{2}}}{1 - \gamma_j^{m_2+1}} \right) \sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \quad . \tag{6.29}$$

Since $\gamma_j < 1$ and $m_1 \leq m_2$, we have

$$\begin{aligned} \lambda_j(n, m_1, m_2) - |\delta_j(n, m_2)| &\geq 2 \left(\frac{1 + \gamma_j^{m_2+1}}{1 - \gamma_j^{m_2+1}} - \frac{\gamma_j^{\frac{m_2+1}{2}}}{1 - \gamma_j^{m_2+1}} \right) \sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \\ &= 2\sqrt{\sigma_j + \frac{\sigma_j^2}{4}} f(\gamma_j^{m_2+1}) \\ &\geq 1.73\sqrt{\sigma_j + \frac{\sigma_j^2}{4}} \end{aligned} \tag{6.30}$$

Expressions for the elements of Q_{SE} and Q_{NE} of (4.11) are given in appendix A. We can easily verify that the elements of both matrices have the same absolute values. Also, both M_8 and M_9 have eigenvalues that are bounded from below by $4\sin\frac{j\pi}{2(m_1+1)}$.

By (4.10), (4.12) and (6.30), we can see that $\|V\|_1$ is bounded by

$$\|V\|_1 \leq \frac{1}{\sqrt{0.866}} \left(\frac{\sqrt{\alpha}}{2} \frac{1}{m_1 + 1} \max_{1 \leq j \leq n} \sum_{i=1}^{m_1} \frac{\sqrt{\sin x_i \frac{\pi}{2}} \; \cos x_i \frac{\pi}{2} \; \sqrt{\sin y_j \frac{\pi}{2}} \; \cos y_j \frac{\pi}{2}}{\sin^2 x_i \frac{\pi}{2} + \sin^2 y_j \frac{\pi}{2}} \right) \quad , \tag{6.31}$$

where $x_i = i/(m_1 + 1)$ and $y_j = j/(n + 1)$, for $i = 1, \ldots m_1$ and $j = 1, \ldots n$. The proof of theorem 3.3 applies now to the expression in parenthesis.

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