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Fourier Analysis of Relaxed Incomplete Factorization Preconditioners*

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Abstract

We use Fourier analysis to study the behavior of a class of incomplete factorization preconditioners for elliptic problems, which blends the classical ILU and MILU preconditioners via a scalar relaxation parameter $\alpha \in [0,1]$. We obtain an expression for the eigenvalues of the preconditioned system from which we get information on both the condition number $K(\alpha)$ and the eigen-distribution of the preconditioned system. We derive an optimal value for α and show that $K(\alpha_{\text{opt}}) = O(h^{-1})$. The Fourier results agree extremely well with numerical results for the model Poisson problem. For example, they predict the sensitive behavior near $\alpha = 1$ (MILU). Finally, we showed that the relaxed methods are closely related (in fact *identical* for periodic problems) to the classical "modified" ILU (MILU) method.

1. Introduction.

The class of incomplete factorization methods (ILU, MILU) [6,8] has been used very successfully as preconditioners for linear systems of the form $Ax = b$, especially

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when A corresponds to discretizations of elliptic partial differential equations. The ILU method computes an approximate LU factorization M of A based on Gaussian Elimination in which fill-ins at the (i, j)-th element is dropped if $(A)_{ij} = 0$. In the MILU method, the dropped fill-ins are added back to the diagonal entry plus an additional term ch^2, where c is a constant and h is the mesh size. For matrices A arising from discretizations of second order elliptic problems, it can be proved [5,10,6,9] that the condition number $K(M^{-1}A)$ is bounded by $O(h^{-2})$ and $O(h^{-1})$ for ILU and MILU($c \neq 0$) respectively. Tests in [10] also showed that the number of PCG iterations for MILU varies very little with c. While these theoretical results may indicate a clear superiority for the MILU method, it has been observed in some practical problems that the MILU method (especially with $c = 0$) does not work as well as the ILU method [7]. This phenomenon led some researchers to consider a relaxed version of the two methods, which is similar to the MILU method except that only a fraction, say $\alpha \in [0, 1]$, of the dropped fill-ins are added back to the diagonal [2, 3, 7]. Thus, when $\alpha = 0$ we get the ILU method; and when $\alpha = 1$ we get the MILU method with $c = 0$. We shall denote this relaxed method by RILU(α)*. Numerical experiments in [1, 2, 3, 7] indicate that the number of iterations needed to reduce an initial residual by a fixed amount (when accelerated by the preconditioned conjugate gradient method) typically behaves as in Fig. 1 (for a more accurate plot, see [7]), which shows an extremely sensitive behavior near $\alpha = 1$, especially for 3-dimensional problems with badly behaved coefficients. This phenomenon has not been well-understood. In [7], it is attributed to "an early loss of orthogonality in the CG iteration, which spoils the convergence behavior". A value of $\alpha = 0.95$ is recommended as a safe cure.

*If A is symmetric positive definite, the corresponding preconditioner has been called the Relaxed Incomplete Cholesky (RIC) preconditioner in [2].

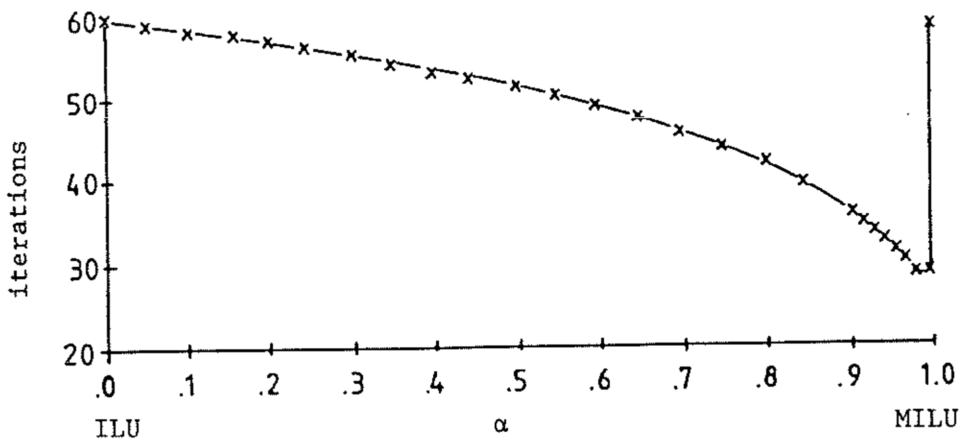

Fig. 1. Convergence Behavior of the RILU(α) method

The main purpose of this paper is to analyze (in Sec. 2) the convergence behavior of the RILU(α) method. The tool we shall use is Fourier analysis (following the framework in [4]), which is similar to the von Neumann stability analysis for finite difference schemes for time dependent problems. It is theoretically exact only for constant coefficient problems with periodic boundary conditions, but results in Chan-Elman [4] indicate that it can successfully predict the behavior of most classical iterative methods and preconditioners even for more general classes of problems with other boundary conditions. In particular, the results for the periodic problem with a mesh size h_p can be used to predict the convergence behavior of the corresponding Dirichlet boundary condition problem with mesh size $h_d = 2h_p$. In addition to deriving the condition number $K(M^{-1}A)$, the Fourier analysis also gives the eigenvalue distribution, which is extremely valuable in understanding the convergence behavior of the method. Moreover, one can easily derive optimal values for parameters of the method (such as α in the RILU(α) method).

For the RILU (α) method applied to the model Poisson problem, the Fourier analysis (Sec. 3) predicts an optimal value of $\alpha_{\text{opt}} = 1 - 8\sin^2 \pi h$ and $K(\alpha_{\text{opt}}) = \frac{1}{2\pi}h^{-1} + O(1)$. The estimate for $K(\alpha_{\text{opt}})$ agrees with the result derived earlier in [10] for the MILU method. The Fourier analysis also successfully predicts the behavior of the spectrum of $M^{-1}A$ as α varies.. In Sec. 4, we present some numerical results for the Dirichlet model problem which seem to verify these predictions. However, while there is excellent general agreement, there is still some discrepancy for the special case $\alpha = 1$. Finally in Sec. 5, we show that there is a close relationship between the RILU(α) method and the classical MILU(c) method.

2. Fourier Analysis of the RILU(α) Method.

We shall restrict our attention to the model problem in 2D (our technique can be extended to more general problems):

$$-\Delta u = f$$

with periodic boundary conditions on the unit square $(0,1) \times (0,1)$ discretized by the standard 2nd order 5-point stencil on a uniform $n \times n$ mesh with size $h \equiv \frac{1}{n+1}$. The matrix A can then be represented by the stencil:

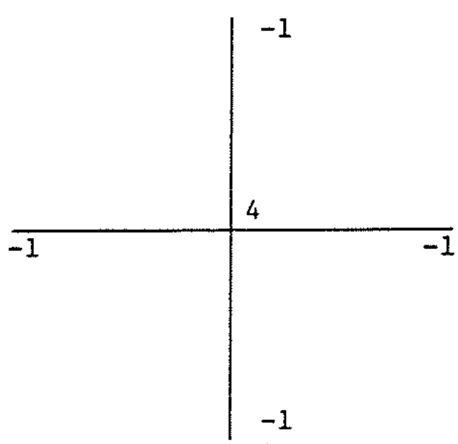

Fig. 2. Stencil for A

The RILU(α) method computes a preconditioner M of a form represented by the product of the following stencils:

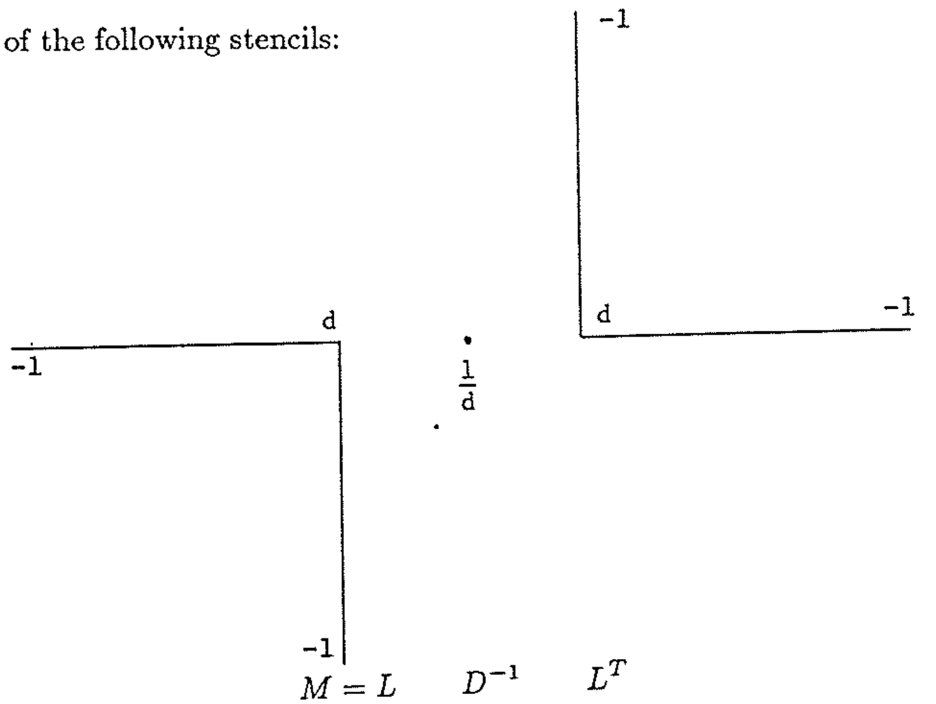

Fig. 3. Stencils for factors of RILU(α)

Note that due to the periodic boundary conditions, the matrix corresponding to L is not lower triangular but circulant (this is important for the Fourier analysis to follow). It can easily be verified that the product M is given by:

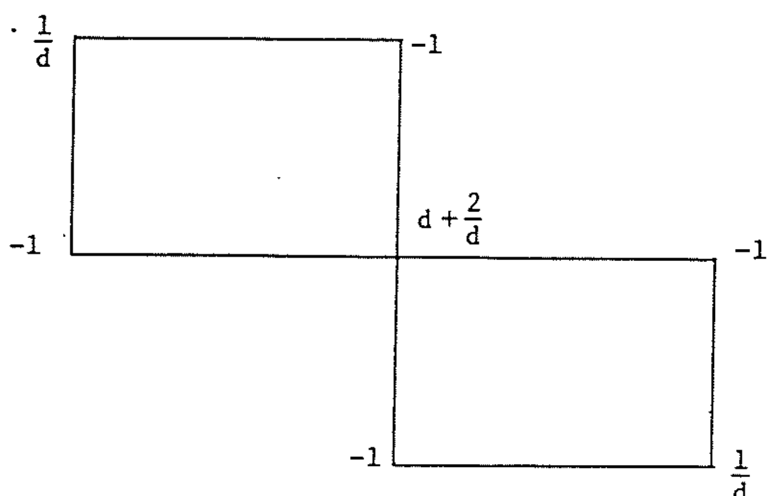

Fig. 4. Stencil for M of RILU(α)

The fill-ins correspond to the two $\frac{1}{d}$ terms, a fraction of which, $\alpha(\frac{2}{d})$, when added to the diagonal, $d + \frac{2}{d}$, should match the diagonal entry of A. This gives the following equation for d:

$$d + \frac{2}{d} + \frac{2\alpha}{d} = 4 \tag{1}$$

which gives

$$d = 2 + \sqrt{2(1-\alpha)}. \tag{2}$$

The Fourier analysis proceeds by computing the action of the operator $M^{-1}A$ on the eigenvectors represented by the Fourier modes whose value at the (j,k)-th grid point is given by:

$$\varphi_{st} = e^{ij\theta_s} e^{ik\phi_t} \tag{3}$$

where

$$i \equiv \sqrt{-1},\ \theta_s = 2\pi sh,\ \phi_t = 2\pi th,\ 1 \le s, t \le n. \tag{4}$$

(Following the framework in [4], we have excluded the indices $s = 0$ and $t = 0$, which correspond to φ_{st} being constant in one or both of the coordinates). For example, the Fourier transform $\lambda_{st}(A)$ of A is defined by:

$$A\varphi_{st} = \lambda_{st}(A)\varphi_{st}$$

and can be computed to be:

$$\begin{aligned} \lambda_{st}(A) &= 4 - e^{i\theta_s} - e^{-i\theta_s} - e^{i\phi_t} - e^{-i\phi_t} \\ &= 4(\sin^2\frac{\theta_s}{2} + \sin^2\frac{\phi_t}{2}). \end{aligned} \tag{5}$$

Similarly, the Fourier transform $\lambda_{st}(M)$ of M is:

$$\begin{aligned} \lambda_{st}(M) &= [d - (e^{-i\theta_s} + e^{-i\phi_t})]\frac{1}{d}[d - (e^{i\theta_s} + e^{i\phi_t})] \\ &= 4(\sin^2\frac{\theta_s}{2} + \sin^2\frac{\phi_t}{2}) + \frac{2}{d}(\cos(\theta_s - \phi_t) - \alpha). \end{aligned} \tag{6}$$

Thus the Fourier transform μ_{st} of $M^{-1}A$ (i.e. the eigenvalues of $M^{-1}A$) is given by

$$\mu_{st}(\alpha) = \frac{4(\sin^2\frac{\theta_s}{2} + \sin^2\frac{\phi_t}{2})}{4(\sin^2\frac{\theta_s}{2} + \sin^2\frac{\phi_t}{2}) + (\frac{2}{2+\sqrt{2(1-\alpha)}})(\cos(\theta_s - \phi_t) - \alpha)}. \tag{7}$$

The condition number $K(\alpha)$ is defined by:

$$K(\alpha) = \frac{\max_{s,t} \mu_{st}(\alpha)}{\min_{s,t} \mu_{st}(\alpha)} \tag{8}$$

From (7), $K(\alpha)$ can be easily computed numerically for a given mesh size h. In Figure 5, $K(\alpha)$ is plotted versus α for the case $n = 40$. The sensitive behavior near $\alpha = 1$ is evident.

In Figs. 6a-6e, the eigenvalue distribution for 5 representative values of α are plotted in a 3D perspective. In Fig. 7, the same eigenvalues are plotted in a linear fashion. (Similar plots for the Dirichlet model problem are given in [2]). In Fig. 8, the maximum and minimum eigenvalues of $M^{-1}A$ are plotted as a function of α. We can make several observations from these plots:

1. In all cases, there is a clustering of eigenvalues around 1.

2. For $\alpha \approx 0$ (ILU) most of the eigenvalues are ≤ 1, while for $\alpha \approx 1$ (MILU) most are ≥ 1.

3. The 3D plots show clearly the transition from ILU to MILU. The two "humps" near $(s,t) = (1,n)$ and $(n,1)$ shoot up steeply when α rises past the optimal value of ≈ 0.96. These account for the large eigenvalues of $M^{-1}A$. The small eigenvalues of A occur near $(s,t) = (1,1)$, (n,n) and rise smoothly from ≈ 0 to ≈ 1 as α goes from 0 to 1. Thus the sensitive behavior in Figs. 1 and 5 can be traced to the sharp rise of the large eigenvalues of $M^{-1}A$ as α approaches 1.

4. The condition number $K(\alpha)$ is substantially lower at the optimal value of $\alpha(\approx 0.96)$ than at either $\alpha = 0$ or $\alpha = 1$.

3. Optimal Relaxation Parameter.

In this section, we shall derive the optimal value for α which minimizes the condition number $K(\alpha)$. To do this, we first find the minimum and maximum eigenvalues as a function of α.

First we show that the minimum eigenvalue occurs at $(s,t) = (1,1)$. This follows from the inequalities:

$$\begin{aligned}\mu_{s,t} &\geq \frac{g_{s,t}}{g_{s,t} + 2(1-\alpha)/(2+\sqrt{2(1-\alpha)})} \\ &\geq \frac{g_{1,1}}{g_{1,1} + 2(1-\alpha)/(2+\sqrt{2(1-\alpha)})} \\ &= \mu_{1,1}\end{aligned}$$

where

$$g_{s,t} \equiv 4(\sin^2 \frac{\theta_s}{2} + \sin^2 \frac{\phi_t}{2}).$$

To simplify the algebra, we define

$$\varepsilon^2 \equiv 2(1-\alpha),$$

with $\varepsilon \in [0, \sqrt{2}]$.

Thus, we obtain from (7) that:

$$\mu_{\min}(\varepsilon) = \mu_{11}(\varepsilon) = \frac{\beta}{\beta + \frac{\varepsilon^2}{8(2+\varepsilon)}}, \tag{10}$$

where

$$\beta \equiv \sin^2 \pi h. \tag{11}$$

Next we shall find the maximum eigenvalue. We shall treat the variables θ_s and ϕ_t as continuous variables θ and ϕ, which allows us to use differential calculus to find the maximum. This is justified in the sense that any maximum found this way is achievable in the discrete case for small enough h.

Due to the symmetries

$$\mu(\theta,\phi) = \mu(\phi,\theta)$$

and

$$\mu(\theta,\phi) = \mu(2\pi-\phi,\ 2\pi-\theta),$$

we can restrict our search for the maximum to the triangular region T defined by:

$$T = \{(\theta,\phi) | \phi \geq 0, \quad \theta+\phi \leq 2\pi, \quad \theta \geq \phi\}.$$

By differentiating $\mu(\theta,\phi)$ in (7) with respect to the variable $\theta+\phi$ while keeping $\theta-\phi$ fixed, it can be verified that μ is strictly decreasing or strictly increasing along any $\theta-\phi=$ constant line in T. Therefore, the maximum of μ must occur on the lines $\phi=0$ or $\theta+\phi=2\pi$.

To obtain the maximum eigenvalue on the line $\theta+\phi=2\pi$, we eliminate the variable ϕ and obtain:

$$\mu(\theta) = \frac{8\sin^2\frac{\theta}{2}}{8\sin^2\frac{\theta}{2} + \left(\frac{2}{2+\varepsilon}\right)(\cos 2\theta - 1 + \frac{\varepsilon^2}{2})} \tag{12}$$

Noting that

$$\cos 2\theta - 1 = -8\sin^2\frac{\theta}{2}(1-\sin^2\frac{\theta}{2}) \tag{13}$$

and defining

$$x \equiv \sin^2\frac{\theta}{2}, \tag{14}$$

we can simplify (12) to obtain

$$\mu(x) = \frac{(2+\varepsilon)x}{\varepsilon x + 2x^2 + \frac{\varepsilon^2}{8}}. \tag{15}$$

We find $\mu'(x) = 0$ when

$$\varepsilon x + 2x^2 + \frac{\varepsilon^2}{8} = x(\varepsilon + 4x)$$

which gives the value of x where $\mu_{\max}$ occurs:

$$x_{\max} = \frac{\varepsilon}{4}. \tag{16}$$

Substituting the value of $x_{\max}$ into (15), we get

$$\mu^{(1)}_{\max}(\varepsilon) = \frac{1}{2} + \frac{1}{\varepsilon}. \tag{17}$$

For the maximum on the line $\phi = 0$, it can be easily verified that it occurs at $\theta = \pi$ with

$$\mu^{(2)}_{\max}(\varepsilon) = \frac{4}{4 + \frac{2}{2+\varepsilon}(\frac{\varepsilon}{2}^2 - 2)}.$$

A simple calculation show that

$$\mu^{(1)}_{\max}(\varepsilon) \geq \mu^{(2)}_{\max}(\varepsilon)$$

whenever $\varepsilon \leq 2$. Since $\varepsilon \leq \sqrt{2}$, it follows that $\mu_{\max} = \mu^{(1)}_{\max}$.

Therefore, from (10) and (17) we get

$$K(\varepsilon) = \frac{(\frac{1}{2} + \frac{1}{\varepsilon})}{(\frac{\beta}{\beta + \frac{\varepsilon^2}{8(2+\varepsilon)}})} = \frac{8\beta(2+\varepsilon) + \varepsilon^2}{16\beta\varepsilon}.$$

To find the minimum value of $K(\varepsilon)$, we find that $K'(\varepsilon) = 0$ when

$$(8\beta + 2\varepsilon)\varepsilon = 8\beta(2+\varepsilon) + \varepsilon^2$$

which gives

$$\varepsilon_{\text{opt}} = 4\sqrt{\beta} \equiv 4\sin\pi h \tag{18}$$

or equivalently

$$\alpha_{\text{opt}} = 1 - 8\sin^2 \pi h. \tag{19}$$

After some manipulations, we obtain

$$K(\alpha_{\text{opt}}) = \frac{4(1 + \sin \pi h)}{8 \sin \pi h}. \tag{20}$$

Therefore

$$K(\alpha_{\text{opt}}) = \frac{1}{2\pi} h^{-1} + O(1). \tag{21}$$

REMARKS.

1. Notice that $\alpha_{\text{opt}} = 1 - O(h^2)$. For $h = \frac{1}{41}$, $\alpha_{\text{opt}} \approx 0.953$, which is the value recommended in [7]. For smaller h, the value 0.95 will be an underestimate of the optimal value.

2. The optimal blending gives

$$K(\alpha_{\text{opt}}) = O(h^{-1})$$

which is similar to that for the classical MILU method with $c \neq 0$ [5].

3. From (17), we see that the value of $\mu_{\max}$ grows like $\frac{1}{\varepsilon}$ near $\varepsilon = 0$ (i.e. $\alpha = 1$) but is bounded below by $\frac{1}{2} + \frac{1}{\sqrt{2}}$ for any ε. From (10), we see that $\mu_{\min}$ grows from $O(h^2)$ at $\varepsilon = \sqrt{2}$ (i.e. $\alpha = 0$) to 1 at $\varepsilon = 0$ (i.e. $\alpha = 1$).

4. Due to the sharp rise in $K(\alpha)$ as $\alpha \to 1$, it is better in practice to underestimate α_{opt} than to overestimate it.

5. From (17), it may appear that $\mu_{\max}$ becomes unbounded as $\varepsilon \to 0$. However, this results from the assumption that in (16), $x_{\max}$ can take on any value in

(0, 1). However, for finite h, x must lie in $[\sin^2 \pi h, 1)$. Therefore,

$$\mu_{\max}(\varepsilon = 0) = \frac{1}{\sin^2 \pi h}. \tag{22}$$

Since $\mu_{\min}(\varepsilon = 0) = 1$, this implies that:

$$K(\varepsilon = 0) = \frac{1}{\sin^2 \pi h} = O(h^{-2}). \tag{23}$$

This agrees with the Fourier results in [4] for MILU with $c = 0$, although for the model problem with Dirichlet boundary conditions, it has been observed experimentally that, for MILU, even with $c = 0$ the condition number grow like $O(h^{-1})$. We note that the theory in [5] cannot predict this behavior either*. In [4], a plausible explanation is given, based on the fact that the $O(h^{-2})$ behavior in the Fourier case results from a very delicate cancellation in the denominator in (7) at $c = 0$, which may not occur in the Dirichlet model problem. The Fourier results, together with the experimental results reported in [7] suggest that for practical problems the $O(h^{-2})$ behavior can indeed occur (unlike for the model problem) and therefore it is not a good idea to use $c = 0$ in MILU (i.e. $\alpha = 1$ in RILU) for more general problems.

4. Comparison with Dirichlet Model Problem.

We have carried out some numerical experiments to verify how well the Fourier results predict the behavior of the RILU preconditioner as applied to the model Poisson problem with *Dirichlet* boundary conditions.

To this end, we consider the model Dirichlet problem with the forcing function:

$$f(x, y) = x(x-1)y(y-1)e^{xy}.$$

*A more detailed analysis for the model problem carried out by Beauwens [11] does explain this phenomena.

We solve the discrete problem by the conjugate gradient method preconditioned by the Dirichlet RILU method (with non-constant diagonals) using zero as initial guess. We stop the iteration when the 2-norm of the residual has been reduced by a factor of 10^{-7}. The results are given in Table 1.

Table 1. Model Dirichlet problem: Number of PCG iterations

	Grid Size $n_d \times n_d$			
α	15×15	31×31	63×63	127×127
0.0	10	19	37	74
0.3	9	17	34	67
0.6	8	15	29	58
0.8	8	13	25	50
0.9	8	12	21	42
0.95	8	12	19	36
0.98	8	12	17	30
0.99	8	12	17	26
1.0	9	13	20	30

Table 2. Number of PCG iterations with α_{opt}

Grid	α_{opt}	PCG iters
15×15	0.923	8
31×31	0.981	12
63×63	0.995	17
127×127	0.9988	25

In Table 2, we give the values of

$$\alpha_{\text{opt}} = 1 - 8\sin^2(\frac{\pi h_d}{2})$$

(where $h_d = \frac{1}{n_d+1}$) and the corresponding number of PCG iterations using the optimal values.

By comparing Tables 1 and 2, we see that:

1. Due to the discrete nature of the number of PCG iterations, it is difficult to locate precisely where the optimal value of α is from Table 1. Nonetheless, the Fourier α_{opt} is consistent with the data in Table 1. We note that Axelsson and Lindskog [2] suggested that $\alpha_{\text{opt}} = 1 - O(h)$ on the basis of their experimental results. However, on careful examination, the α_{opt} obtained from our Fourier analysis is also consistent with the numerical results in [2], to the resolution allowed.

2. The number of PCG iterations using α_{opt} grows like $O(h_d^{-\frac{1}{2}})$, which is consistent with the well-known asymptotic behavior of the number of PCG iterations growing like $\sqrt{K(\alpha_{\text{opt}})}$ with $K(\alpha_{\text{opt}}) = O(h_d^{-1})$.

The above results show that the Fourier analysis does predict successfully α_{opt} and $K(\alpha_{\text{opt}})$ for the Dirichlet problem.

5. Relation Between RILU(α) and MILU(c).

In this section, we show that the RILU(α) method can really be viewed as an MILU(c) method in which the parameter c, which in the original method is constrained to be independent of h, is now allowed to vary with h.

The MILU(c) method has the same stencils as in Figs. 3 and 4. The only differences are that

(1) the dropped terms (rather than a fraction of them) are added to the diagonal

(2) and in addition a term ch^2 is added to the diagonal, where c is a constant independent of h.

Thus the diagonal d for MILU(c) satisfy the relationship (see [4]):

$$d + \frac{2}{d} + \frac{2}{d} = 4 + ch^2 \tag{24}$$

which gives

$$d = 2 + \frac{ch^2}{2} + \frac{1}{2}\sqrt{8ch^2 + (ch^2)^2}. \tag{25}$$

The RILU(α) method is *identical* (for the periodic problem) to the MILU(c) method if their diagonals d are the same, i.e. if

$$2 + \sqrt{2(1-\alpha)} = 2 + \frac{ch^2}{2} + \frac{1}{2}\sqrt{8ch^2 + (ch^2)^2} \tag{26}$$

or

$$\varepsilon = \frac{ch^2}{2} + \frac{1}{2}\sqrt{8ch^2 + (ch^2)^2}, \tag{27}$$

or equivalently,

$$c = (\frac{\varepsilon^2}{2+\varepsilon})h^{-2}. \tag{28}$$

Therefore, there is a $1-1$ correspondence between the two methods. (This is strictly true only for the periodic case, but the asymptotic values of the corresponding Dirichlet preconditioners should also be equal). Of course, if $\varepsilon \neq O(h)$, then (28) implies that $c \neq O(1)$, which violates the constraint on c in the original MILU method. However, as we can see from (18), $\varepsilon_{\text{opt}} = O(h)$ and therefore the corresponding $c_{\text{opt}} = O(1)$. In fact, from (18) and (28), we have

$$c_{\text{opt}} = \Big(\frac{8\sin^2\pi h}{1 + 2\sin\pi h}\Big)h^{-2}, \tag{29}$$

giving

$$\lim_{h \to 0} c_{\text{opt}} = 8\pi^2 \tag{30}$$

Which agrees with the result derived in [4] for MILU(c).

In a sense, the RILU(α) method is slightly more general than the MILU(c) method in that the corresponding value of c is not constrained to be $O(1)$. In fact, the results in this paper *confirm* that the addition of the term ch^2 with $c = 0(1)$ is indeed optimal and one cannot improve on the $O(h^{-1})$ bound for K within the class of RILU(α) or MILU(c) methods.

Finally, concerning the use of the two methods in practice, it seems better to use MILU(c) because the behavior is much less sensitive to c than that of RILU(α) to α, as long as c is bounded away from zero. One gets the RILU(α) methods by allowing c to grow with h as in (28). In some sense, c is a local scaled parameter around the region $\alpha = \alpha_{\text{opt}}$ in Fig. 5.

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Figure 5. Condition Number of RILU(alpha) vs alpha, n=40

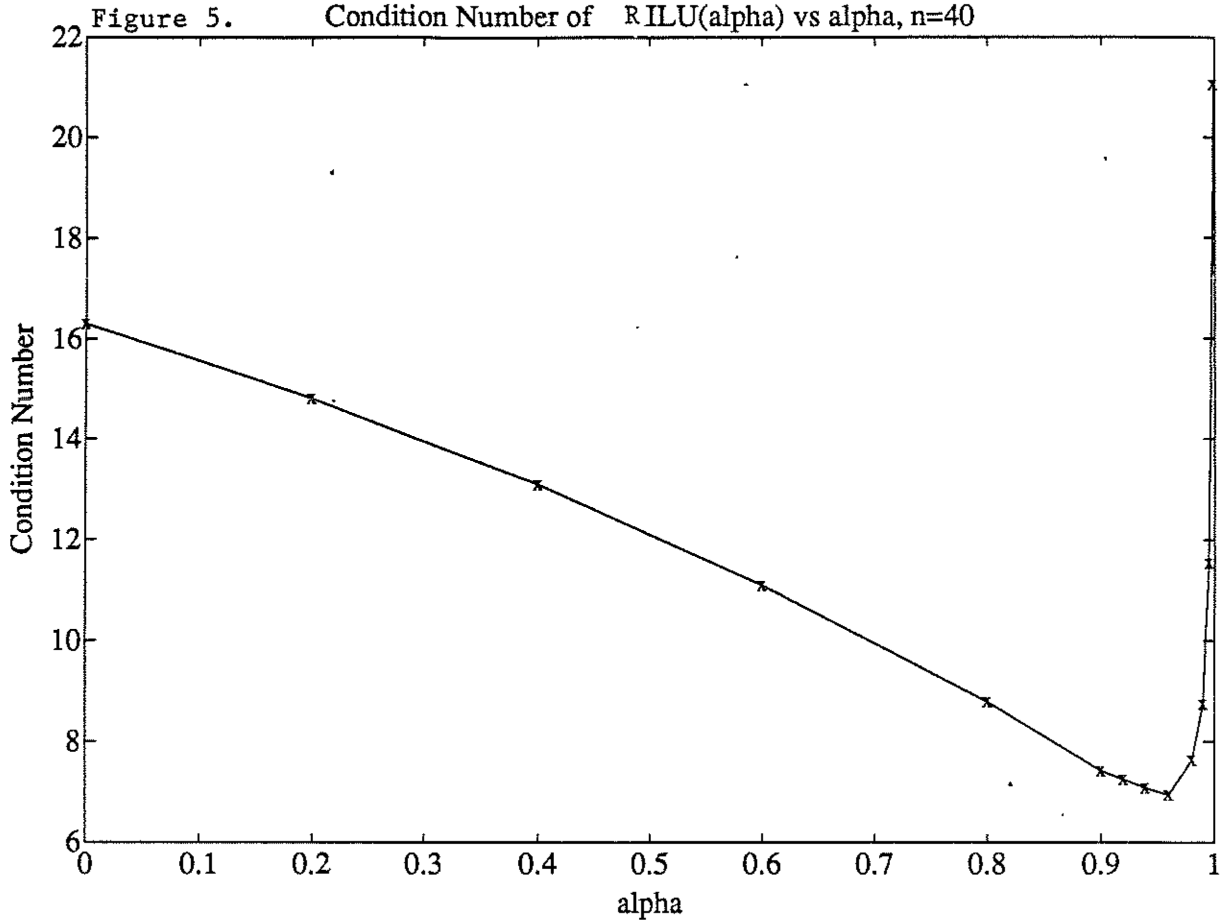

Figure 6a. Fourier Eigenvalues for RILU, alpha=0

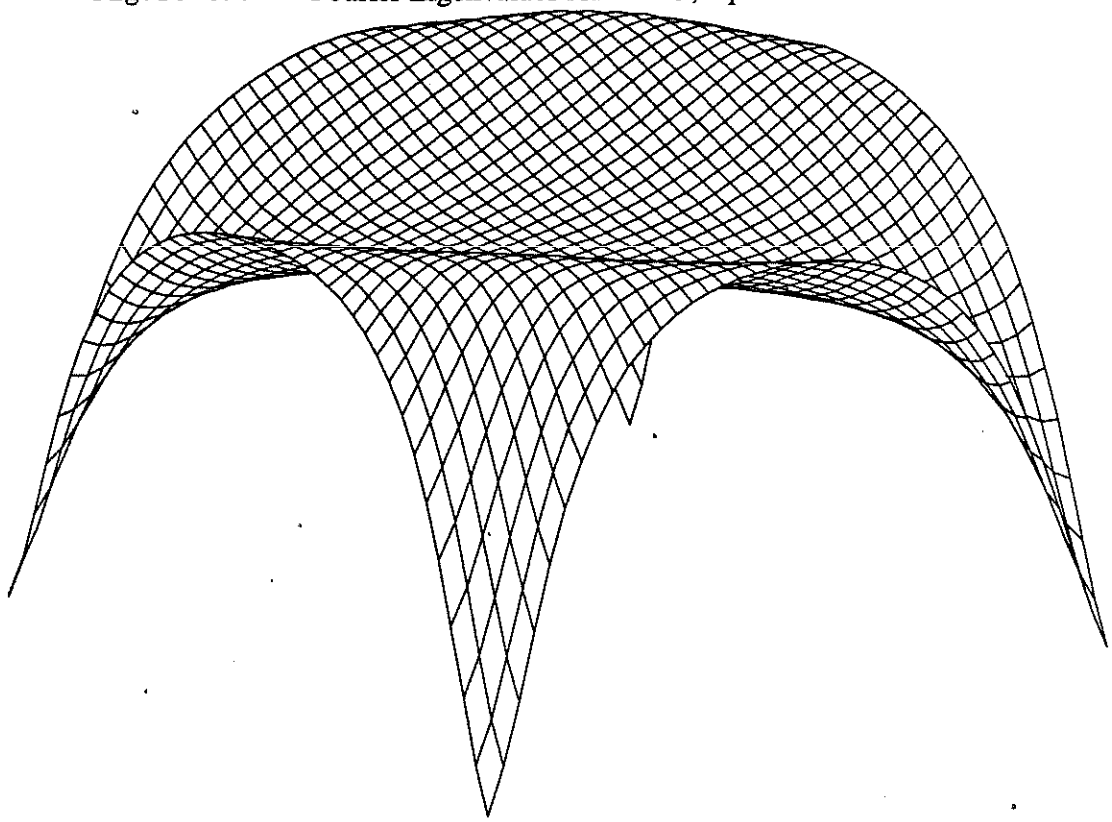

Figure 6b. Fourier Eigenvalues for RILU, alpha=0.6

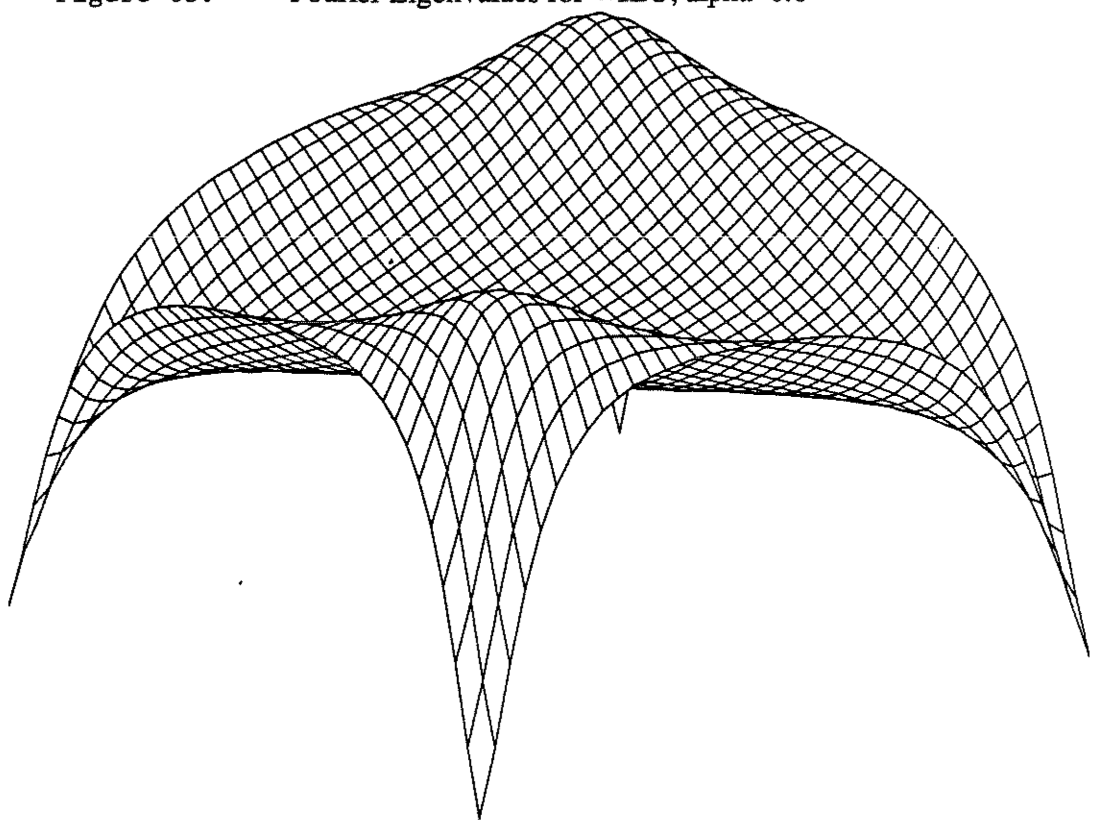

Figure 6c. Fourier Eigenvalues for RILU, alpha=0.96

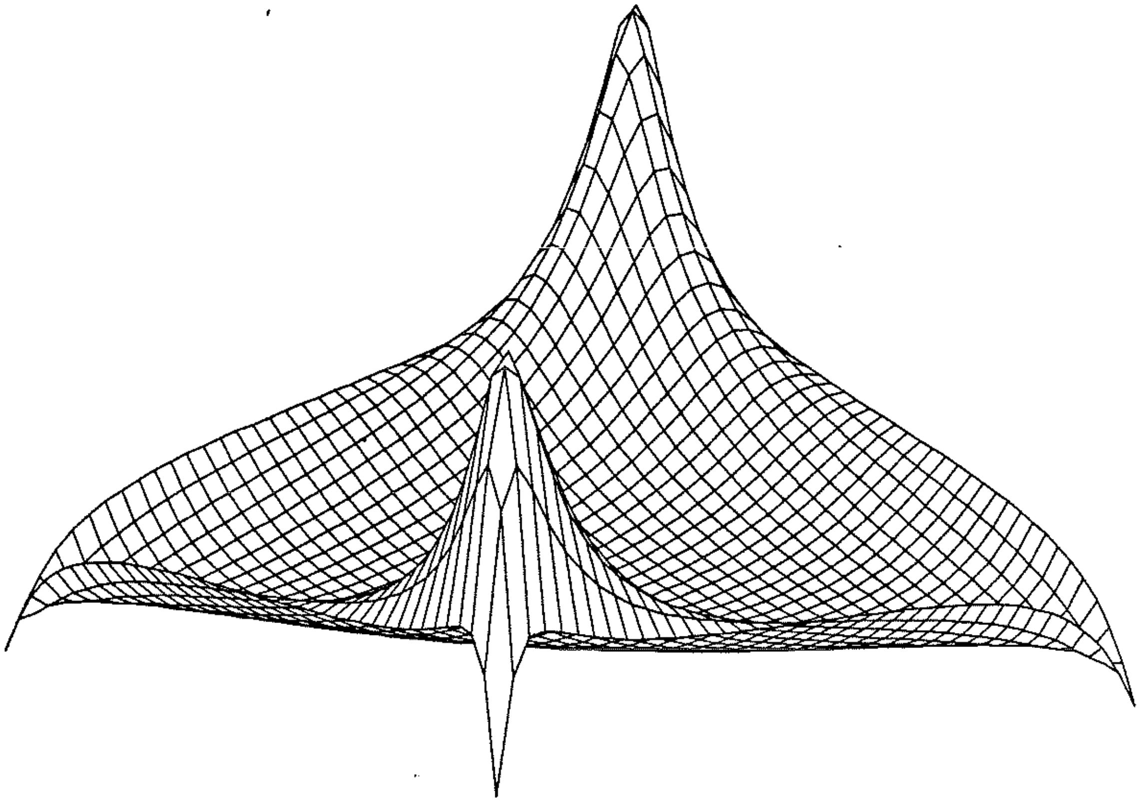

Figure 6d. Fourier Eigenvalues for RILU, alpha=0.995

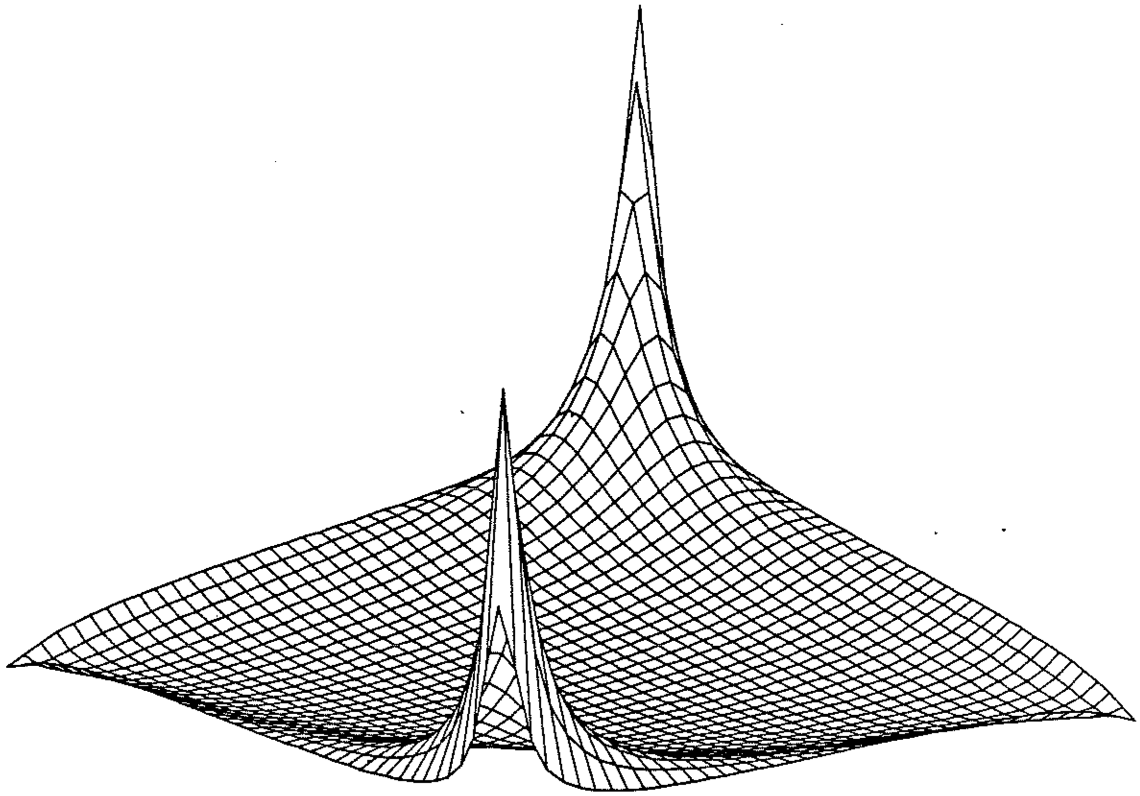

Figure 6e. Fourier Eigenvalues for R ILU, alpha=1

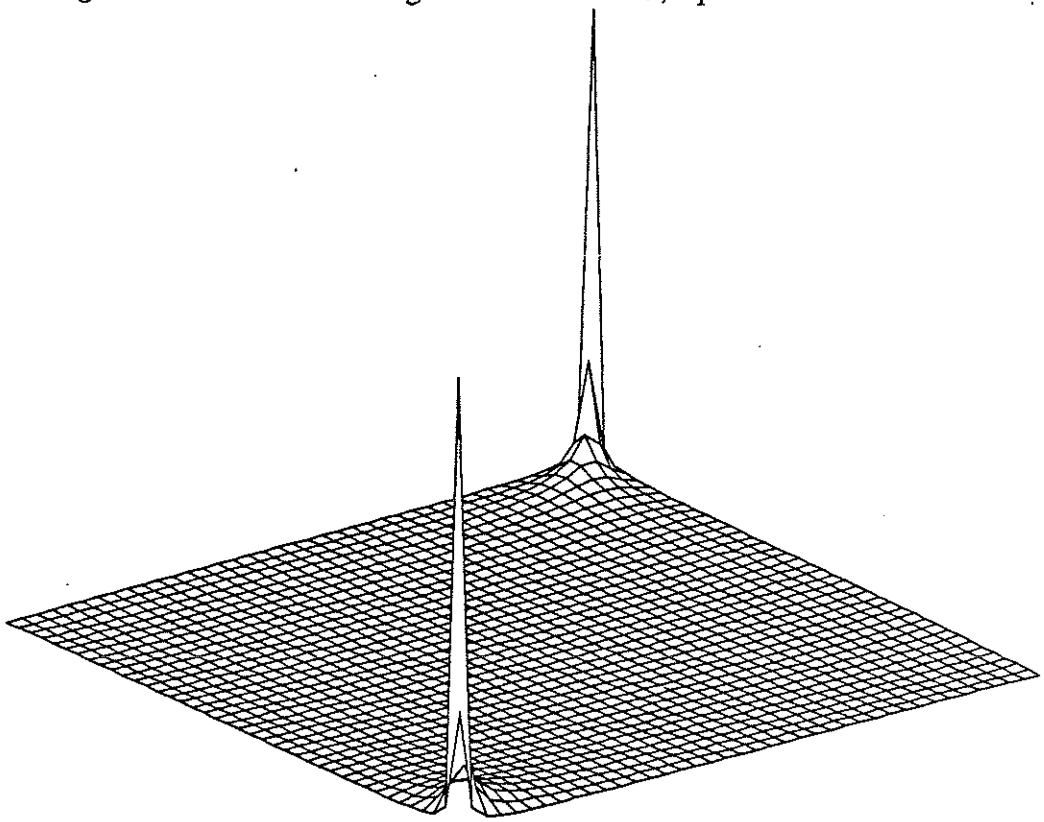

Figure 7. Distribution of Fourier Eigenvalues for RILU

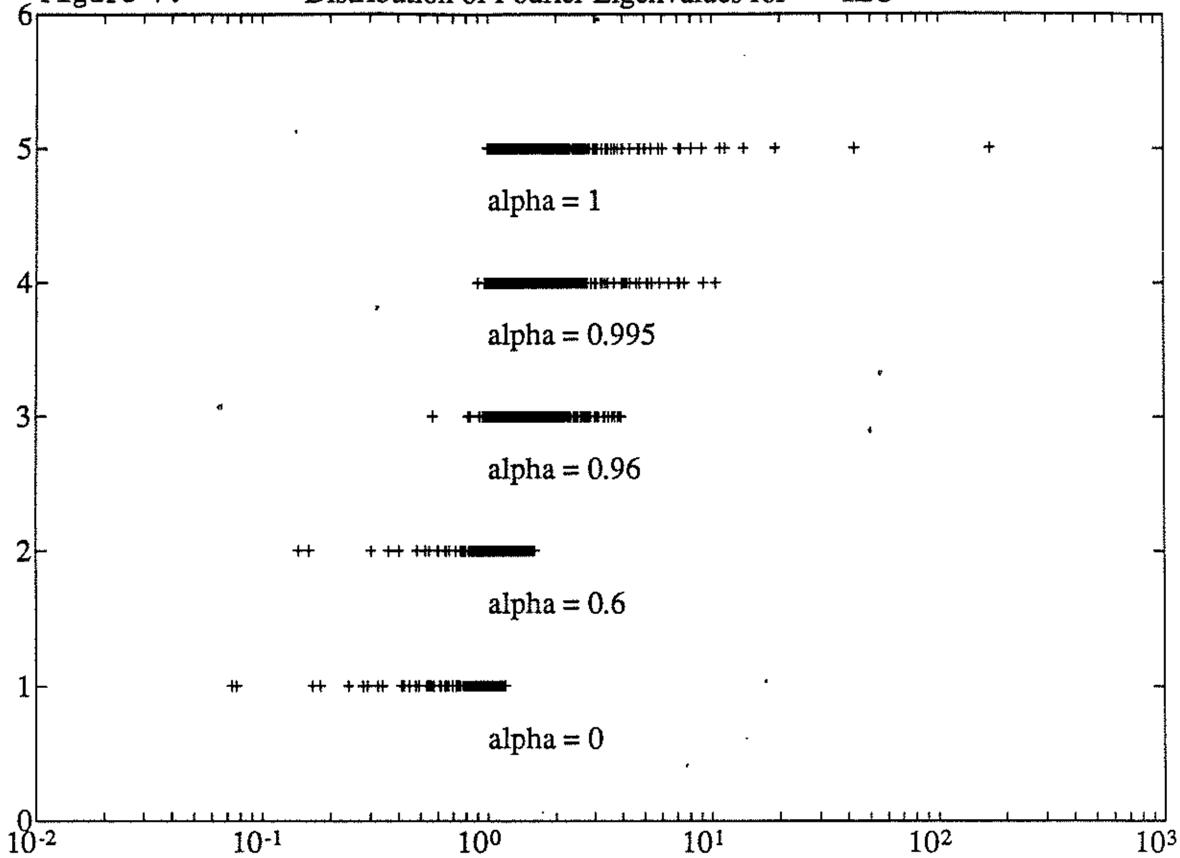

Figure 8. Max and Min Fourier E.V. vs alpha for R ILU, n=40