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Abstract

A new approach for defining boundary conditions for the Navier-Stokes equations in the vorticity formulation is defined which is suitable for implementation with a deteministic vortex method. A numerical illustration of the efficiency of this boundary condition is presented in the case of the simulation of a flow past a cylinder.

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1.Introduction. The problem of boundary conditions for the Navier-Stokes equations arises when one wishes to deal with the vorticity formulations of the equations. Briefly speaking it consists in replacing one of the boundary conditions on the velocity (for instance the constraint on the tangential component) by a boundary condition on the vorticity. The role of the later is to ensure that one reconstructs an admissible vorticity field in the sense that the associated velocity field satisfies a posteriori the no slip condition. Afterwards it remains to subsitute this boundary condition in the numerical method used for solving the Navier-Stokes equations.

An attempt to solve this problem is the so-called Chorin's algorithm. This splitting based method consists in solving first the Euler equation then computing the slip of the velocity at the boundary which is further incorporated as a source term in the heat equation. This vorticity creation algorithm has a nice interpretation when discretizing the equations by finite differences methods. It is also suitable for random walk methods. However the following drawbacks of the method must be considered:

-the interpretation of this algorithm in terms of solving the proper boundary condition for the vorticity is not clear, which makes its analysis difficult ([3],[6]).

-to achieve its best accuracy it requires to follow the vorticity creation step by solving the heat equation with an homogeneous Neumann boundary condition.

When using a random walk method the second point merely amounts to reflecting particles which hit the boundary. This is however inconvenient when using deterministic vortex methods. In such methods (see [8]) the diffusion is dealt with by using an integral representation of the Laplace operator and using a numerical quadrature of this integral at the particles, which only move accordingly to the Euler part of the equations. Those methods have proved to be particularly efficient and accurate in absence of boundaries (see [5]) but they have difficulties in taking into account a given boundary condition. Moreover simply implementing Chorin's algorithm in this context probably leads to a very poor accuracy.

As for the exact boundary conditions giving an admissible vorticity field, C. Anderson ([1]) has derived one that involves a coupling between the kinetic and kinematic parts of the system. As an alternative this paper presents a method that:

-provides an exact boundary condition for the vorticity both in the Dirichlet an Neumann form.
-can be interprated as a vorticity creation algorithm overcoming the second drawback of Chorin's algorithm, making it particularly suitable when implemented along with a deterministic vortex method.
Furthermore the way this boundary condition is derived is mostly independent of the vorticity transport-diffusion equation which makes it easy to implement.

An outline of the paper is as follows: in section 2 we recall Chorin's algorithm; we sketch a proof of its consistency showing that it is of order 1. In section 3 we derive our boundary condition and prove its consistency. Finally in section 4 we present a vortex in cell code using this boundary condition and a deterministic resolution of the diffusion.

2. Consistency of Chorin's algorithm. In this section we present an analysis of the algorithm and for simplicity we consider only the Stokes equations (we indicate also briefly how to modify the argument to take into accont the nonlinear terms). We consider an exterior domain Ω with smooth boundary Γ whose exterior normal is denoted by ν; a vorticity formulation with Neumannn boundary conditions of the Stokes problem consists in the following set of equations:

$$\frac{\partial \omega}{\partial t} - \Delta \omega = 0 \qquad \text{in} \quad \Omega \tag{2.1}$$

$$\omega(0) = \omega_0 \qquad \text{in} \quad \Omega \tag{2.2}$$

$$\frac{\partial \omega}{\partial \nu} = g \qquad \text{on} \quad \Gamma, \tag{2.3}$$

where g is such that, solving

$$\begin{aligned} \text{curl} u &= \omega \qquad \text{in} \quad \Omega \\ u \cdot \nu &= 0 \qquad \text{on} \quad \Gamma \\ u &\to u_\infty \qquad \text{at infinity,} \end{aligned} \tag{2.4}$$

yields also $u.\tau = 0$ on Γ.
Sometimes we will consider the particular case of an half-space:

$$\Omega^\star = \{x = (x_1, x_2), x_2 < 0\}$$

but the crucial steps of the following analysis are mostly independent of this particular geometry. If $\tilde{\omega}$ denotes the even extension of ω out of $\Omega^\star$:

$$\tilde{\omega}(x_1, x_2) = \begin{cases} \omega(x_1, x_2) & \text{if } x_2 > 0; \\ \omega(x_1, -x_2) & \text{if } x_2 < 0. \end{cases}$$

then (2.1)–(2.3) can be rewritten as

$$\begin{aligned} &\frac{\partial \tilde{\omega}}{\partial t} - \Delta \tilde{\omega} = 2g(x_1) \otimes \delta(x_2) \qquad \text{in } \mathbf{R}^2 \\ &\tilde{\omega}(\cdot, 0) = \tilde{\omega}_0. \end{aligned} \tag{2.5}$$

Therefore a natural splitting of (2.1)–(2.3) or (2.5) consists in:

$$\begin{aligned} 1^{st} \text{ step}: \quad & \text{solve} \quad \frac{\partial \omega}{\partial t} - \Delta \omega = 0, \\ 2^{nd} \text{ step}: \quad & \text{solve} \quad \frac{\partial \omega}{\partial t} = 2g \otimes \delta. \end{aligned}$$

Let us now recall the design of Chorin's algorithm, still in the case of $\Omega^\star$. Given a time step Δt and the vorticity field $\tilde{\omega}^n$ at time t_n, $\tilde{\omega}^{n+1}$ is computed through the following steps:

a) if u^n is the solution of (2.4) with right hand side $\omega_n = \tilde{\omega}_{n|\Omega}$ we set

$$\tilde{\omega}^{n+\frac{1}{2}} = \tilde{\omega}^n + 2(u^n \cdot \tau)(x_1) \otimes \delta(x_2); \tag{2.6}$$

b) then $\tilde{\omega}^{n+1}$ is obtained as $\tilde{\omega}^h(\cdot, t_{n+1} - 0)$ where:

$$\begin{cases} \dfrac{\partial \tilde{\omega}^h}{\partial t} - \Delta \tilde{\omega}^h = 0 \\ \tilde{\omega}^h(\cdot, t_n) = \tilde{\omega}^{n+\frac{1}{2}}. \end{cases} \tag{2.7}$$

The step b) above also means that we are solving the heat equation in Ω with homogeneous Neumann boundary condition on Γ, which allows to write the algorithm no matter what the geometry is. However in the case of the geometry $\Omega^\star$ the following lemma is easy to prove.

Lemma 2.1. *We have:* $(u^h \cdot \tau)(\cdot, t) \to 0$ *in* $\mathbf{L}^2(\Gamma)$ *as* $t \downarrow t_n$.

Let us just mention that this result follows from the continuity with respect to t of the solution of the heat equation in $\mathbf{H}^{-1}(\Omega)$. The interesting point here is that it is now possible to rewrite the step a) of the algorithm as:

$$\tilde{\omega}^{n+\frac{1}{2}} = \tilde{\omega}^n + 2 \int_{t_{n-1}}^{t_n} \frac{\partial}{\partial t}(u^h \cdot \tau)(x_1, t) \otimes \delta(x_2) dt \tag{2.8}$$

where $u^h(\cdot,t)$ denotes for $t \in]t_{n-1},t_n]$ the solution of (2.4) with $\omega = \omega^h(\cdot,t)$.

Therefore if we set

$$g^h(x,t) = \frac{\partial}{\partial t}(u^h \cdot \tau)(x,t), \quad x \in \Gamma,$$

and if we denote by $\bar{\omega}^h$ the solution of

$$\begin{aligned} \frac{\partial \bar{\omega}^h}{\partial t} - \Delta \bar{\omega}^h = 0 \qquad & \text{in} \quad \Omega \\ \frac{\partial \bar{\omega}^h}{\partial \nu} = g^h \qquad & \text{on} \quad \Gamma, \end{aligned} \tag{2.9}$$

then the error $\omega - \omega^h$ results in $\omega - \bar{\omega}^h$ plus an usual splitting error $O(\Delta t)$.

The next step consists in estimating $\omega - \bar{\omega}^h$ in terms of $g - g^h$ and vice-versa. Substracting (2.9) to (2.1),(2.3), multiplying by $\omega - \bar{\omega}^h$ and integrating over Ω give

$$\frac{d}{dt}\|\omega - \bar{\omega}^h\|_0^2 + \|\nabla(\omega - \bar{\omega}^h)\|_0^2 = \int_\Gamma (g - g^h)(x,t)(\omega - \bar{\omega}^h)(x,t) d\gamma(x) \tag{2.10}$$

where $\|\cdot\|_0$ means the $\mathbf{L}^2$ norm in Ω. Let us now set $\mathbf{G}(x) = 1/2\pi \log(|x|)$ and $\mathbf{K}(x) = \partial \mathbf{G}/\partial \tau(x)$, $x \in \Gamma$. The key step is then to prove the following integral identity

Lemma 2.2. *For $x \in \Gamma$ and $t \in]t_{n-1},t_n]$,*

$$(g - g^h)(x,t) = 2 \int_\Gamma \mathbf{K}(x-y) \frac{\partial}{\partial \tau}(\omega - \omega^h)(y,t) d\gamma(y).$$

Proof. Let $v = \frac{\partial}{\partial t}(u - u^h) - \Delta(u - u^h)$, for $t \in]t_{n-1},t_n]$, $x \in \bar{\Omega}$. From (2.1) and (2.7) we get $\mathrm{curl} v = 0$. Thus there exists ϕ such that $v = \mathrm{grad}\phi$. Since $\mathrm{div} v = 0$ we have also $\Delta\phi = 0$.

Moreover $u \cdot \nu = u^h \cdot \nu = 0$ so that:

$$\frac{\partial \phi}{\partial \nu} = v \cdot \nu = -\Delta(u - u^h) \cdot \nu = \mathrm{curl}(\omega - \omega^h) \cdot \nu = \frac{\partial}{\partial \tau}(\omega - \omega^h).$$

Therefore the following integral representation holds:

$$\phi(x) = 2 \int_\Gamma \mathbf{G}(x-y) \frac{\partial}{\partial \tau}(\omega - \omega^h)(y) d\gamma(y), \quad x \in \Gamma.$$

But

$$g - g^h = \frac{\partial \omega}{\partial \nu} - \frac{\partial}{\partial t}(u^h \cdot \tau) = v \cdot \tau = \frac{\partial \phi}{\partial \tau}$$

because

$$\frac{\partial}{\partial t}(u \cdot \tau) = 0 = \frac{\partial \omega^h}{\partial \nu} \qquad \text{on} \quad \Gamma.$$

This gives the desired result.

If we denote by e the difference $\omega - \bar{\omega}^h$ and by R the splitting error $\bar{\omega}^h - \omega^h$, the right hand side of (2.10) can be rewritten as twice

$$\int_\Gamma \int_\Gamma \mathbf{K}(x-y)\frac{\partial e}{\partial \tau}(x)e(y)d\gamma(x)d\gamma(y) + \int_\Gamma \int_\Gamma \mathbf{K}(x-y)\frac{\partial R}{\partial \tau}(x)e(y)d\gamma(x)d\gamma(y)$$
$$= -\int_\Gamma \int_\Gamma \mathbf{G}(x-y)\frac{\partial e}{\partial \tau}(x)\frac{\partial e}{\partial \tau}(y)d\gamma(x)d\gamma(y) + \int_\Gamma \int_\Gamma \mathbf{K}(x-y)\frac{\partial R}{\partial \tau}(x)e(y)d\gamma(x)d\gamma(y).$$

Next we observe that the first integral has a distinguished sign:

Lemma 2.3. *For any a in $\mathbf{L}^2(\Gamma)$ we have*

$$\int_\Gamma \int_\Gamma \mathbf{G}(x-y)a(x)a(y)d\gamma(x)d\gamma(y) \geq 0.$$

Proof. If ψ is the solution of

$$\Delta\psi = 0 \quad \text{in } \Omega, \quad \frac{\partial \psi}{\partial \nu} = a \quad \text{on } \Gamma, \quad \psi(x) = O(\frac{1}{|x|}) \quad \text{at infinity,}$$

then:

$$2\int_\Gamma \int_\Gamma \mathbf{G}(x-y)a(x)a(y)d\gamma(x)d\gamma(y) = \int_\Gamma \psi(x)\frac{\partial \psi}{\partial \nu}(x)d\gamma(x)$$
$$= \int_\Omega |\nabla\psi(x)|^2 dx \geq 0,$$

which proves the desired result.

As a consequence, (2.10) now yields

$$\frac{d}{dt}\|e(\cdot,t)\|_0^2 + \|\nabla e(\cdot,t)\|_0^2 \leq 2\Big|\int_\Gamma \int_\Gamma \mathbf{K}(x-y)\frac{\partial R}{\partial \tau}(x)e(y)d\gamma(x)d\gamma(y)\Big|$$
$$\leq C\|e(\cdot,t)\|_{\mathbf{H}^{1/2}(\Gamma)}\|R\|_{\mathbf{H}^{1/2}(\Gamma)}$$
$$\leq C\|e(\cdot,t)\|_{\mathbf{H}^1(\Omega)}\|R\|_{\mathbf{H}^{1/2}(\Gamma)}$$
$$\leq \|e(\cdot,t)\|^2_{\mathbf{H}^1(\Omega)} + C'\|R\|^2_{\mathbf{H}^1(\Omega)},$$

which gives

$$\frac{d}{dt}\|e(\cdot,t)\|_0^2 \leq \|e(\cdot,t)\|_0^2 + C'\|R(\cdot,t)\|^2_{\mathbf{H}^1(\Omega)}.$$

Therefore

$$\|e(\cdot,t)\|_0^2 + \int_0^t \|\nabla e(\cdot,s)\|_0^2 ds \leq C(T)\int_0^t \|R(\cdot,s)\|^2_{\mathbf{H}^1(\Omega)}ds, \quad t \in [0,T].$$

This finally proves that the error e in $\mathbf{L}^\infty\big(0,T;\mathbf{L}^2(\Omega)\big) \cap \mathbf{L}^2\big(0,T;\mathbf{H}^1(\Omega)\big)$ reduces to the splitting error R in the same space, resulting in an order one algorithm.

Let us now briefly describe how to modify the above argument in the case of the full Navier-Stokes system. In this case a convection step must follow the diffusion step. Up to the additional splitting error introduced in this process (as analyzed in [2] in the case of the whole space), the consistency of this algorithm will result from the consistency of the algorithm consisting in replacing (2.6),(2.7) by

$$\tilde{\omega}^{n+\frac{1}{2}} = \tilde{\omega}^n + 2(u^n \cdot \tau)(x_1) \otimes \delta(x_2);$$
$$\begin{cases} \dfrac{\partial \tilde{\omega}^h}{\partial t} + \nabla(\tilde{u}^h \otimes \tilde{\omega}^h) - \Delta \tilde{\omega}^h = 0 \\ \tilde{\omega}^h(\cdot, t_n) = \tilde{\omega}^{n+\frac{1}{2}}, \end{cases}$$

where $\tilde{u}^h = (\tilde{u}_1^h, \tilde{u}_2^h)$ is the following extension of u^h in $\mathbf{R}^2$:

$$(\tilde{u}_1^h, \tilde{u}_2^h)(x_1, x_2) = \begin{cases} (u_1^h, u_2^h)(x_1, x_2), & \text{if } x_2 > 0; \\ (u_1^h, -u_2^h)(x_1, -x_2), & \text{if } x_2 < 0. \end{cases}$$

In this situation Lemma 2.1 remains valid and we use it to rewrite (2.8) in a slightly different way: let us denote by $X(s; x, t)$ (resp $X^h(s; x, t)$) the characteristic at time s associated to u (resp u^h) which was on Γ at the point $(x, 0)$ at time t. Then, since $u^h \cdot \nu = 0$ on Γ, X^h remains on Γ for all time and it is possible to write

$$\begin{aligned} \tilde{\omega}^{n+\frac{1}{2}} &= \tilde{\omega}^n + 2\int_{t_{n-1}}^{t_n} \frac{\partial}{\partial t}\{(u^h \cdot \tau)\big(X^h(t; x_1, t_n), t\big)\} \otimes \delta(x_2) dt \\ &= \tilde{\omega}^n + 2\int_{t_{n-1}}^{t_n} \{\frac{\partial}{\partial t}(u^h \cdot \tau)\big(X^h(t; x_1, t_n), t\big) + \\ &\qquad + (u^h \cdot \nabla)(u^h \cdot \tau)\big(X^h(t; x_1, t_n), t\big)\} \otimes \delta(x_2) dt. \end{aligned} \tag{2.11}$$

Setting $g^h(x, t) = \frac{\partial}{\partial t}\{(u^h \cdot \tau)\big(X^h(t; x_1, t_n), t\big)\}$ and observing that $x = X(t; x, t_n)$ (because $u = 0$ on Γ), it is easy to prove with the same techniques as for Lemma 2.2:

$$(g - g^h)(x, t) = g^h\big(X(t; x, t_n), t\big) - g^h\big(X^h(t; x, t_n), t\big) + \\ + 2\int_\Gamma \mathbf{K}(x - y) \frac{\partial}{\partial \tau}(\omega - \omega^h)(y, t) d\gamma(y).$$

This identity along with the enstrophy inequality (2.10), which is still valid for the Navier-Stokes system, makes it possible to end the proof by following the same lines as in the linear case.

3. An explicit boundary condition. For simplicity let us begin with the case of a bounded domain. Let us consider again the elliptic sysytem giving the velocity in terms of the vorticity

$$\begin{aligned} \text{curl} u &= \omega \quad \text{in} \quad \Omega \\ \text{div} u &= 0 \quad \text{in} \quad \Omega \\ u \cdot \nu &= 0 \quad \text{on} \quad \Gamma. \end{aligned} \tag{3.1}$$

Due to the second equation above the rotational of the first one gives $-\Delta u = \mathrm{curl}\omega$. Now if we consider the elliptic system

$$\begin{cases} -\Delta u = \mathrm{curl}\omega & \text{in} \quad \Omega \\ \qquad u = 0 & \text{on} \quad \Gamma, \end{cases} \tag{3.2}$$

we get a well-posed problem in which the no slip condition on Γ is taken into account explicitely. Obviously the systems (3.1) and (3.2) are not equivalent (in particular the solution of (3.2) is not necessarily divergence free). However this becomes the case if one of the conditions $\mathrm{curl}u = \omega$ or $\mathrm{div}u = 0$ is constrained on Γ. This can be done through the boundary condition needed for solving the transport-diffusion equation on ω.

A possible system including this boundary condition in the Neumann form is as follows

$$\frac{\partial \omega}{\partial t} + \nabla(u \otimes \omega) - \Delta\omega = 0 \qquad \text{in} \quad \Omega \tag{3.3}$$

$$\omega(\cdot, 0) = \omega_0 \qquad \text{in} \quad \Omega \tag{3.4}$$

$$-\Delta u = \mathrm{curl}\omega \qquad \text{in} \quad \Omega \tag{3.5}$$

$$u = 0 \qquad \text{on} \quad \Gamma \tag{3.6}$$

$$\frac{\partial \omega}{\partial \nu} = \frac{\partial}{\partial \nu}(\mathrm{curl}u) - \frac{1}{|\Gamma|}\int_\Gamma \frac{\partial}{\partial \nu}(\mathrm{curl}u)d\gamma \qquad \text{on} \quad \Gamma. \tag{3.7}$$

The role of the constant $-\frac{1}{|\Gamma|}\int_\Gamma \frac{\partial}{\partial \nu}(\mathrm{curl}u)d\gamma$ appearing in the right hand side of (3.7) is to enforce the correct circulation. More precisely we know that if (u, ω) is the exact solution of the Navier-Stokes equations then:

$$\int_\Omega \omega dx = \int_\Gamma u \cdot \tau d\gamma = 0.$$

Moreover, from the transport-diffusion equation (3.3) we easily get:

$$\frac{d}{dt}\int_\Omega \omega dx = \int_\Gamma \frac{\partial \omega}{\partial \nu} d\gamma = 0, \tag{3.8}$$

which means that in order to be consistent a Neumann boundary condition on ω must satisfy $\int_\Gamma \frac{\partial \omega}{\partial \nu} d\gamma = 0$. This is certainly true with (3.7). It is worth noticing that (3.8) is the only information we need to extract from (3.3) to construct our boundary condition. Let us now prove the consistency of this boundary condition.

Theorem. *Assuming that ω_0 and Γ are smooth enough, the system (3.3)–(3.7) has a unique solution, which is the solution of the Navier-Stokes equations.*

Proof. Let us first consider the solution u^{NS} of the Navier-Stokes equations and $\omega^{NS} = \mathrm{curl}u^{NS}$. Then of course (ω^{NS}, u^{NS}) is solution of (3.3),(3.4) and (3.6). We have also that

$$\begin{aligned}\mathrm{curl}\omega^{NS} &= \mathrm{curl}(\mathrm{curl}u^{NS}) \\ &= -\Delta u^{NS} + \mathrm{grad}(\mathrm{div}u^{NS}) \\ &= -\Delta u^{NS},\end{aligned}$$

which gives (3.5). Finally

$$\int_\Gamma \frac{\partial}{\partial \nu}(\mathrm{curl}u^{NS})d\gamma = \int_\Gamma \frac{\partial \omega^{NS}}{\partial \nu}d\gamma = \frac{d}{dt}\int_\Omega \omega^{NS}dx = 0,$$

so that (3.7) is trivially satisfied.

Reciprocially let (ω, u) be a solution of (3.3)–(3.7). Taking the rotational of (3.5) gives along with (3.7):

$$\begin{cases} -\Delta(\mathrm{curl}u - \omega) = 0 & \text{in } \Omega \\ \dfrac{\partial}{\partial \nu}(\mathrm{curl}u - \omega) = constant & \text{on } \Gamma. \end{cases}$$

This proves first that the above constant is 0 and then that $\mathrm{curl}u - \omega$ is constant in Ω. But integrating $\mathrm{curl}u$ gives, due to (3.6)

$$\int_\Omega \mathrm{curl}u dx = \int_\Gamma u \cdot \tau d\gamma = 0.$$

On the other hand integrating (3.3) and using (3.7) yield

$$\frac{d}{dt}\int_\Omega \omega dx = -\int_\Gamma \frac{\partial \omega}{\partial \nu}d\gamma = 0$$

so that

$$\int_\Omega \omega dx = \int_\Omega \omega_0 dx = 0.$$

Therefore

$$\omega = \mathrm{curl}u \quad \text{in } \Omega.$$

It remains to prove that $\mathrm{div}u = 0$. Since $\omega = \mathrm{curl}u$ we get

$$\mathrm{curl}\omega = \mathrm{curl}(\mathrm{curl}u) = \mathrm{grad}(\mathrm{div}u) - \Delta u.$$

Combined with (3.5) this implies

$$\text{grad}(\text{div}u) = 0 \qquad \text{in} \quad \Omega,$$

and therefore

$$\text{div}u = constant \qquad \text{in} \quad \Omega.$$

But

$$\int_\Omega \text{div}u dx = \int_\Gamma u \cdot \nu d\gamma = 0,$$

so that

$$\text{div}u = 0 \qquad \text{in} \quad \Omega.$$

We have thus proved that u is solution of the original elliptic problem (3.1). Since in addition u satisfies $u \cdot \tau = 0$ on Γ, this means that u is solution of the original Navier-Stokes system. •

Let us now describe the first time step of a naive implementation of the proposed boundary condition. Let Δt be a time step; let $(\omega^{\Delta t}, u^{\Delta t})$ be the solution at time Δt of (3.3)–(3.6) with boundary condition $\frac{\partial \omega_0}{\partial \nu}$. Let us now construct the boundary condition needed at time Δt to proceed. Accordingly to (3.7) we compute

$$\frac{\partial \omega}{\partial \nu}\Big|_{t=\Delta t} = \frac{\partial}{\partial \nu}(\text{curl}u^{\Delta t}) - I$$

where $I = 1/|\Gamma| \int_\Gamma \frac{\partial}{\partial \nu}(\text{curl}u^{\Delta t}) d\gamma$. But we can write

$$\begin{aligned}\frac{\partial}{\partial \nu}(\text{curl}u^{\Delta t}) &= \text{curl}(\text{curl}u^{\Delta t}) \cdot \tau \\ &= (\text{grad}(\text{div}u^{\Delta t}) - \Delta u^{\Delta t}) \cdot \tau.\end{aligned}$$

Since $u^{\Delta t}$ is computed from $\omega^{\Delta t}$ trough (3.5),(3.6) we have

$$\begin{aligned}-\Delta u^{\Delta t} \cdot \tau &= \text{curl}\omega^{\Delta t} \cdot \tau \\ &= \frac{\partial \omega^{\Delta t}}{\partial \nu} = \frac{\partial \omega_0}{\partial \nu}.\end{aligned}$$

Finally

$$\frac{\partial}{\partial \nu}(\text{curl}u^{\Delta t}) = \frac{\partial}{\partial \tau}(\text{div}u^{\Delta t}) + \frac{\partial \omega_0}{\partial \nu}.$$

Therefore $I = 0$ and the boundary condition reduces to

$$\frac{\partial \omega}{\partial \nu}\Big|_{t=\Delta t} = \frac{\partial}{\partial \tau}(\text{div}u^{\Delta t}) + \frac{\partial \omega_0}{\partial \nu}. \tag{3.9}$$

Equation (3.9) describes how to correct the vorticity created from one time step to the next. It means in particular that if $\mathrm{div} u^{\Delta t} = 0$ then $\omega^{\Delta t}$ is actually an admissible vorticity field and no correction is needed at the boundary.

Here is what we consider as the main advantage of this approach: (3.9) gives a way to define a vorticity creation algorithm even if the boundary condition used between two creation steps is not known, provided it gives the correct global circulation.

Let us now discuss the case of an exterior domain. For simplicity we assume that $u_\infty = 0$. In this case (3.7) is still a natural boundary dition. However the equivalence between (3.3)–(3.7) and the Navier-Stokes equations is not clear because solving the system (3.2) does not tell anything about the behaviour at infinity of u. For the solution of (3.2) to be such that $u \longrightarrow 0$ at ∞, ω must satisfy a compatibility condition which does not reduce to $\int_\Omega \omega = 0$.

Nevertheless in practical situations this problem is naturally overcome since the computational domain Ω has to be enclosed within an artificial boundary Γ_∞. On Γ_∞ artificial boundary conditions must supplement both (3.6) and (3.7) (for instance $u = u_\infty$, $\frac{\partial \omega}{\partial \nu} = 0$ on Γ_∞). The resulting system is then clearly an approximation, up to a truncature error, of the Navier-Stokes equations.

4. A deterministic Vortex-In-Cell code for a flow past a cylinder. We now come to the description of a Vortex In Cell code which uses the boundary condition as defined in section 3. Ω denotes the exterior domain $\{x \in \mathbf{R}^2; |x| > 1\}$ and Γ is the circle of radius 1. The viscosity is denoted by σ and Re is the Reynolds number.

First of all let us summarize the main features of the method by distinguishing between the inside and the boundary part.

Inside Ω:

a) Vorticity is carried by particles with positions X_p and weights α_p; the weights α_p take together the local value of the vorticity ω_p and the volume of the particle w_p.

b) The transport-diffusion equation (3.3) is dealt with by a purely deterministic method: particles are convected along the velocity field u; their weigths are modified, to take into account the diffusion, following an approach introduced in [8].

c) In order to solve (3.5),(3.6), we use a Vortex In Cell approach, following [4]: we define an assignment scheme from the particles to a polar grid, leading to values (ω_i, J_i) which respectively

represent the vorticity and the volume of the cell at the grid point i.

On the boundary Γ:

d) The divergence of u is evaluated on Γ. Then vorticity is created accordingly to (3.9) and distributed among source particles that are located in the immediate neighborhood of Γ.

Let us now give a more detailed description of the algorithm. Steps *a*) and *b*) mean that we write

$$\omega(x,t) \approx \sum_p \alpha_p(t)\delta\big(x - X_p(t)\big); \quad \alpha_p = \omega_p w_p.$$

The initial distribution $x_p = X_p(0)$ is defined in a rectangular grid with grid size h surrounding Γ. The width of this region is defined from a standard evaluation of the thickness ρ of the boundary layer in which most of the vorticity is expected to be found, namely:

$$x_p \in \left\{ x \in \Omega; |x| \leq 5\sqrt{\frac{2\pi}{Re}} \right\}.$$

The parameter h must be kept proportional to ρ so that the initial number of particles does not increase with the Reynolds number; in our experiments the ratio ρ/h is about 10.

In addition particles are generated upstream in such a way that the incoming flow maintain a grid resolution which is roughly constant in time.

The particle positions are defined from the velocity through

$$\begin{cases} \dfrac{dX_p}{dt} = u\big(X_p(t),t\big) \\ X_p(0) = x_p, \end{cases}$$

while the volumes of the particles w_p follow the law

$$\begin{cases} \dfrac{dw_p}{dt} = \mathrm{div} u\big(X_p(t),t\big) w_p \\ w_p(0) = h^2. \end{cases}$$

Finally the diffusion is dealt with by modifying the weights α_p according to the following equation:

$$\frac{d\alpha_p}{dt} = 2\frac{\sigma}{\eta^2} \sum_{p'} (w_p \alpha_{p'} - w_{p'}\alpha_p) \Lambda_\eta (X_p - X_{p'}). \tag{4.1}$$

The function Λ_η above is defined by $\Lambda_\eta(x) = \eta^{-2}\Lambda(x/\eta)$, where Λ is a radially symmetric function satisfying

$$\int_{\mathbf{R}^2} x_i^2 \Lambda(x) dx = 1, \quad i = 1,2;$$

in our simulations we have choosen a positive function rapidly decaying at infinity:

$$\Lambda(r) = \frac{16}{\pi^2(1+r^8)}.$$

The resulting approximation of the diffusion operator can then easily be proved to be of order 2, assuming that there are enough particles in any boxe of size η. This led us to choose for η a value slightly larger that $\sqrt{\sigma}$. Moreover, in order to have a fast evaluation of the right hand side of (4.1) we have used a truncature of Λ at $r = 2$.

In step c) a polar grid $(\delta r, \delta\theta)$ is introduced in the domain $1 \leq r \leq 6$. The vorticity is assigned from the particles onto the grid in the following way: if x a current point of Ω we denote by ξ the doublet (r,θ) such that $x = (r\cos\theta, r\sin\theta)$; associated to the grid in the ξ-space, let ϕ be a **TSC** basis function (that is ϕ is zero out of a nine points box); then define

$$\begin{cases} J_i = \sum_p w_p \phi(\xi_p - \xi_i) \\ \omega_i = \frac{1}{J_i} \sum_p \alpha_p \phi(\xi_p - \xi_i). \end{cases}$$

The above assignment scheme in particular enjoyes the following stability property

$$\sum_i J_i |\omega_i|^2 \leq \sum_p w_p |\omega_p|^2.$$

Next, curlω is computed on the grid by finite differences and (3.5),(3.6) is solved by a second order finite difference method.

It remains now to describe the step d) involving the boundary Γ. As pointed out in the introduction, the approach leading to (4.1) has proved to be very efficient in absence of boundaries. Moreover it enjoyes nice mathematical properties (see [7], [8]). Unfortunately so far it has not been possible to formulate in this context a simple way to deal with a given boundary condition. In the approach developped in section 3 this is not a drawback since the formula (3.9) makes it possible to introduce vorticity such as to correct the effect of (4.1) at the boundary. At the end of step c) div$u$ is evaluated at the boundary nodes of the polar grid, and vorticity is created at those points with a strength $\delta\theta \frac{\partial}{\partial\tau}(\text{div}u)$.

Several sheet of particles (2 in our simulations) are also considered in the neighborhood of Γ. They coincide there with the polar grid and are used to prevent a possible distortion of the particle distribution that might occur in this region. They do not move and their volume is computed as

the difference between the ideal volume of the corresponding cell (that is $r\delta r\delta\theta$) and the volume J_i computed from the particles. As a consequence, even if particles X_p are locally missing in the neighborhood of Γ (so that $J_i \approx 0$), those source particles allow to resolve the diffusion in this critical region. We believe that he process just described gives a natural way to couple in the same domain particle methods and more conventional methods (e.g. finite differences) and we plan to further investigate those techniques in the future.

Finally let us mention that actually, rather than modifying the volumes of the particles, which would be necessary because $\mathrm{div} u \neq 0$, we have prefered to construct a second velocity field by solving (3.1). This divergence free velocity is used to convect the particles, so that their volume remain constant. Moreover, the value of this velocity field at the boundary gives a natural control of the accuracy of the method. In the resulting algorithm the step c) consists now in solving 3 elliptic systems on the polar grid (2 for (3.2) and 1 for (3.1)).

The results presented concern a simulation for a Reynolds number $Re = 800$. In this case the width ρ of the boundary layer is .443. The parameters used for the discretization are:

$$\Delta t = .01 \quad ; \quad h = .048 \quad ; \quad \delta r = .055 \quad ; \quad \delta\theta = .065 \quad .$$

The resultant initial number of particles is 1778, while the polar grid involves 90×96 points. Between time 0 and time 5.5 about 1000 particles had to be generated upstream.

Figure 1 represents the velocity at the boundary, as computed from (3.1), in both the $\mathbf{L}^\infty$ and $\mathbf{L}^2$ norm on Γ. The results show that the accuracy of the method improves when time goes on in the early stage of the flow. As a comparison, the maximum of the velocity inside Ω is rougly constant, equal to 1.7 (with $u_\infty = 1.$).

Figure 2 shows the ratio between the divergence of u, as computed from (3.2), and the maximum vorticity of in Ω (which is about 30). This somehow indicates how far the vorticy field is from and admissible one and thus provides an other control of the accuracy of the method. The second curve shows the evolution of $\left|\frac{\omega_{max}+\omega_{min}}{\omega_{max}-\omega_{min}}\right|$ which gives a measure of the symmetry of the solution.
The divergence remains more or less constant, at .01, while the symmetry deteriorates a little bit, indicating a possible sensitivity of the treatment of the diffusion to round-off errors.
Finally the representation of the velocity field (computed from (3.1)) at time 5.5 in figure 3 shows the recirculation zone behind the cylinder.

5. Conclusion. The purpose of this paper was to describe a new approach for deriving vorticity boundary conditions both in the Dirichlet and Neumann form. Those boundary conditions are simple to implement because they are mostly decoupled from the kinematic part of the problem. The Neumann boundary condition leads to a simple vorticity creation algorithm which can be used together with a deterministic vortex method. Numerical simulations involving also a recently designed Vortex In Cell approach for the computation of the velocity has shown to give a qualitative agreement with the experiments. In the future we plan to further investigate the capabilities of the method, in particular by using a fourth order particle method. It would also be useful to implement this method with more conventional dicretization methods for the Navier-Stokes equations in the vorticity form, such as finite differences or spectral methods.

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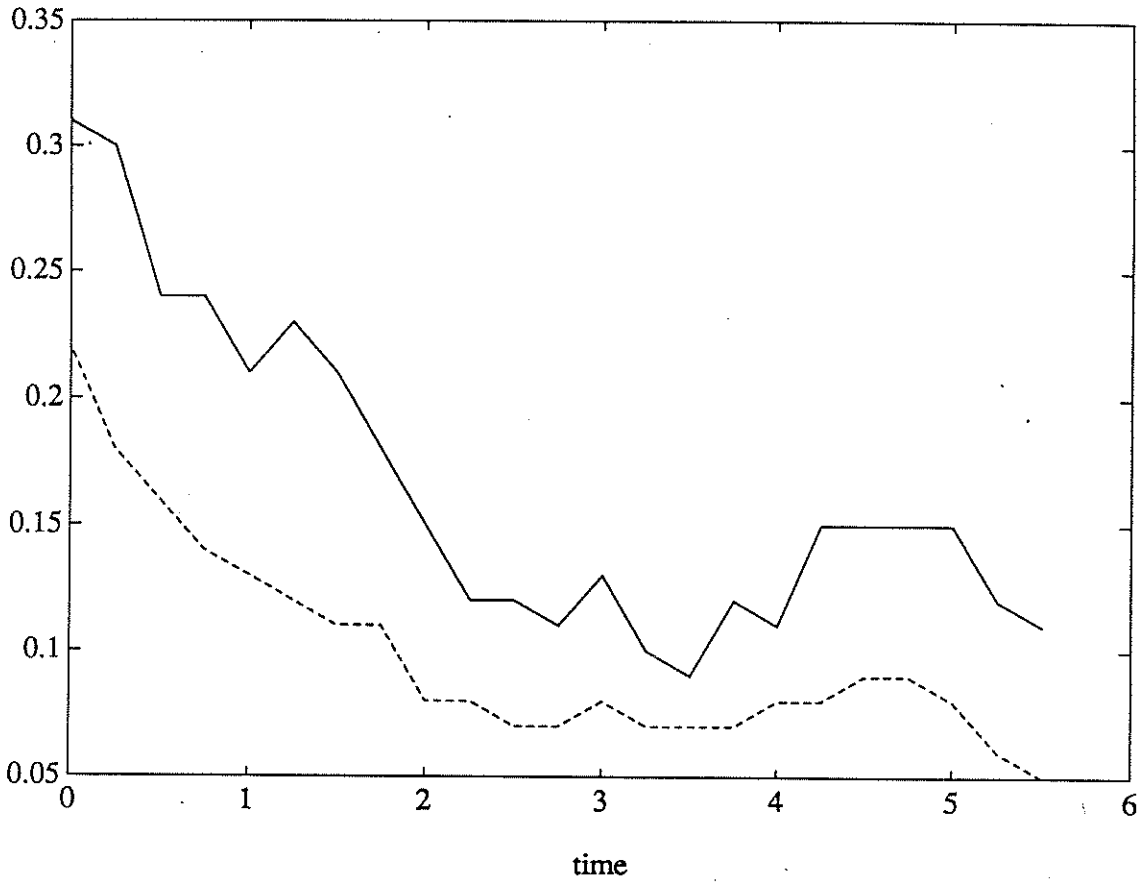

Figure 1. $\mathbf{L^2}$ (dotted line) and $\mathbf{L^\infty}$ norm of the velocity at the boundary versus time.

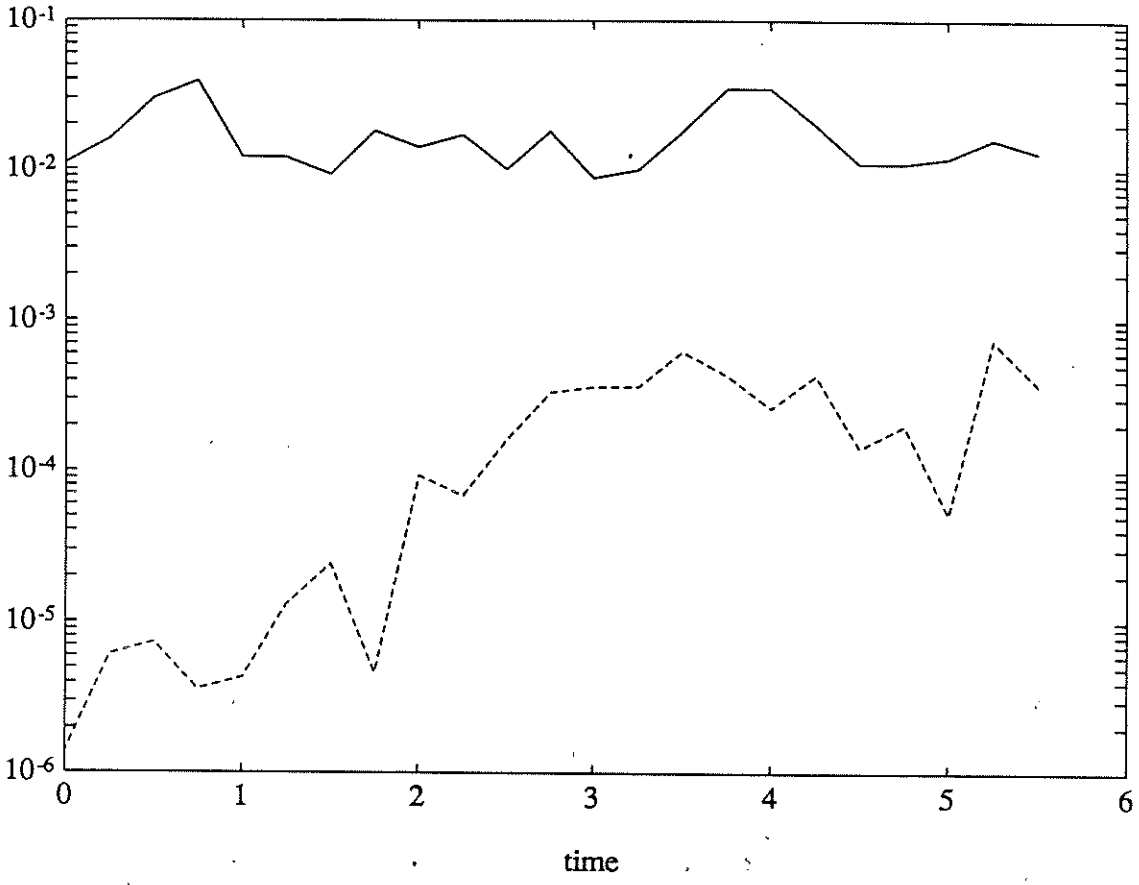

Figure 2. $\left|\frac{\omega_{max}+\omega_{min}}{\omega_{max}-\omega_{min}}\right|$ (dotted line) and $\frac{\mathrm{div}_u}{\omega_{max}}$ versus time.

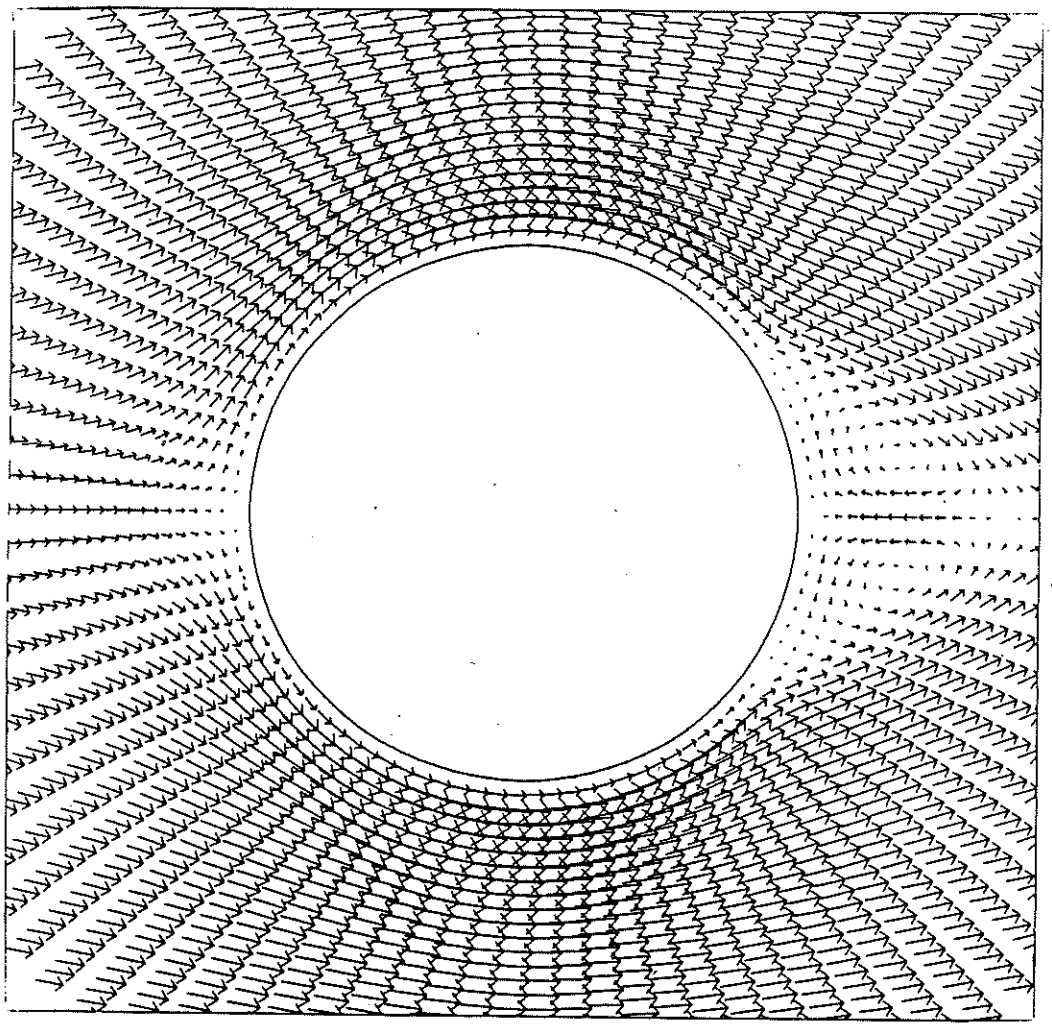

Figure 3. Velocity field at time 5.5.