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Abstract

We prove that for general second order elliptic partial differential equations, the Schwarz alternating procedure converges at a rate independent of the aspect ratio for L-shaped, T-shaped and C-shaped domains. Our results cover both continuous and discrete versions of the Schwarz algorithm. Moreover, they apply to the nonoverlapping Schur complement algorithms with the preconditioner proposed in Chan [4]. In particular, we show that the condition number of the preconditioned interface operator is bounded by 2.0 for all L-shaped and T-shaped domains. This improves similar geometry-independent convergence results for the Schur complement algorithms obtained previously by Chan and Resasco [5].

1 Introduction

Many domain decomposition algorithms have been proposed recently for solving elliptic boundary value problems. In these algorithms, the domain is decomposed into smaller subdomains and the solution is computed via an iterated sequence of solutions computed on the subdomains. There are two main approaches, characterized by the way the subdomains are constructed, namely overlapping (Schwarz) and nonoverlapping (Schur Complement) [10]. The convergence of the underlying iterative process is of course critical to the success of the methods and has therefore received a lot of study.

Some convergence theories are concerned with bounds on the convergence rate as the mesh size h (for discretized problems) tends to zero. For example, for many of the nonoverlapping algorithms it is possible to prove that the number of iterations required to achieve certain accuracy can be bounded by a constant independent of h (see [1,3,4,7,11]). Other kinds of convergence results are concerned with the geometrical aspects of the subdomains, such as dependence of the rate of convergence on the amount of overlapping of the subdomains in the overlapping type of algorithms [5,12,18]. Both kinds of results can provide useful information for the practical use of these methods.

This paper is concerned with convergence results of the geometric kind, for both continuous and the corresponding discretized problems. We shall only deal with the case of two overlapping subdomains although our theory can be easily generalized to more subdomains. We shall prove that, for several *classes* of geometrical shapes for the domain (L, T and C shapes), the convergence factor of the Schwarz overlapping method can be bounded above by a constant independent of the particular aspect ratio of the domain. In particular, we prove that for *all* L- and T-shaped domains with straight edges decomposed into two overlapping subdomains with maximum overlap, the Schwarz iteration for the Laplacian operator has a convergence factor bounded above by $\frac{1}{2}$. For C-shaped domains, the upper bound is $\frac{1}{\sqrt{2}}$. We shall also give similar bounds for convection-diffusion operators. Moreover, these results for the overlapping algorithms can be applied to certain nonoverlapping algorithms as well through an equivalence relationship between the two classes of methods [2,6]. These results improve on previous ones of this kind obtained in [5]. The methods of proof rely on maximum principle techniques as in [14,15].

The paper is organized as follows. In section 2, we review the convergence theory of the Schwarz iteration via the maximum principle and formulate three canonical problems on a rectangle with Dirichlet boundary value of 0 or 1 for L, T and C-shaped domains. Section 3 is devoted to study the convergence of the Schwarz algorithm for the Laplacian operator in the three canonical cases. Convection-diffusion operators are considered in section 4 and the discrete analog of the Schwarz method is studied in section 5. Finally we discuss in section 6 how our results for the overlapping Schwarz method can be applied to the non-overlapping Schur complement method.

2 Convergence of the Schwarz Iteration via Maximum Principle

Let Ω be a bounded, open domain in R^2. For simplicity we assume that Ω is smooth and connected. We then decompose Ω into two overlapping subdomains Ω_1 and Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2. \tag{1}$$

Denote by $\Gamma = \partial\Omega$, $\Gamma_1 = \partial\Omega_1$, $\Gamma_2 = \partial\Omega_2$, $\gamma_1 = \partial\Omega_1 \cap \Omega_2$, $\gamma_2 = \partial\Omega_2 \cap \Omega_1$, see Fig. 1.

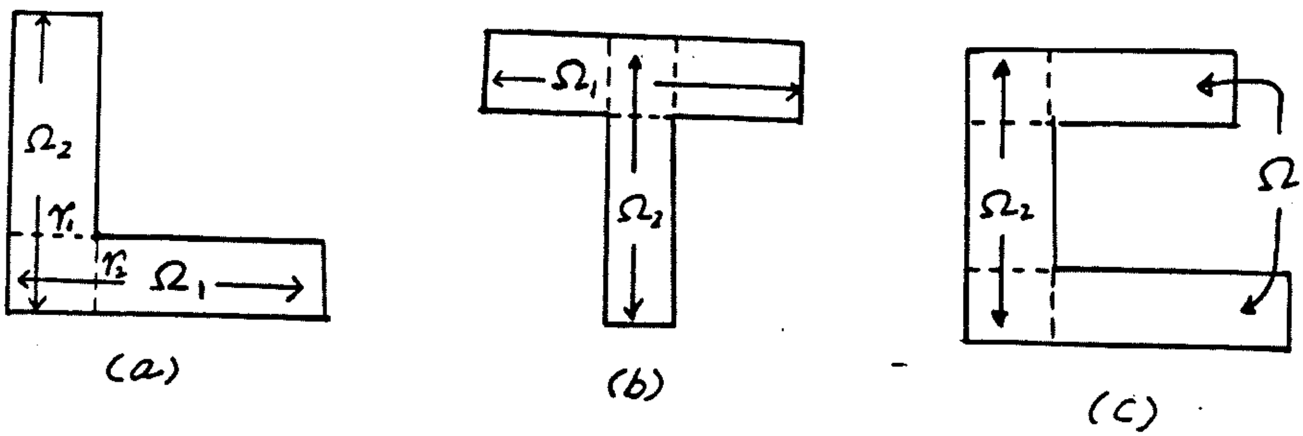


Figure 1

Consider the Poisson equation on Ω

$$-\Delta u = f \quad \text{in} \quad \Omega \;, \qquad u = 0 \quad \text{on } \partial\Omega \tag{2}$$

where f is a given smooth function. The Schwarz alternating procedure [17] consists in solving successively the following problems: let u^0 be any initialization, say in $C_0(\bar{\Omega})$, we obtain $u^{2n+1}(n \geq 0)$ and $u^{2n}(n \geq 1)$ by solving respectively

$$-\Delta u^{2n+1} = f \quad \text{in } \Omega_1 \quad , \quad u^{2n+1} = u^{2n} \quad \text{on } \partial\Omega_1, \tag{3}$$

$$-\Delta u^{2n} = f \quad \text{in } \Omega_2 \quad , \quad u^{2n} = u^{2n-1} \quad \text{on } \partial\Omega_2, \tag{4}$$

and $u^{2n+1} \in C(\bar{\Omega}_1)$, $u^{2n} \in C(\bar{\Omega}_2)$, $u^{2n+1} = 0$ on $\Gamma_1 \cap \Gamma$, $u^{2n} = 0$ on $\Gamma_2 \cap \Gamma$. Also (3) and (4) require that u^{2n} and u^{2n+1} be defined on $\bar{\Omega}$. This can be done by extending $u^{2n+1} = u^{2n}$ on $\Omega \backslash \Omega_1$ and $u^{2n} = u^{2n-1}$ on $\Omega \backslash \Omega_2$. Thus we have u^{2n+1}, $u^{2n} \in C_0(\bar{\Omega})$.

The convergence of the Schwarz alternating procedure is given by

Proposition 1 There exist $k_1, k_2 \in (0,1)$, which depend only on (Ω_1, γ_2) and (Ω_2, γ_1), such that for all $n \geq 0$

$$\sup_{\bar{\Omega}_1} |u - u^{2n+1}| \leq {k_1}^n {k_2}^n \sup_{\gamma_1} |u - u^0|, \tag{5}$$

$$\sup_{\bar{\Omega}_2} |u - u^{2n}| \leq k_1^n k_2^{n-1} \sup_{\gamma_2} |u - u^0|. \tag{6}$$

Thus the effective rate of convergence is $\sqrt{k_1 k_2}$.

The following standard lemma provides estimates for constants k_1 and k_2. We will repeatedly use this Lemma in our convergence analysis throughout this paper.

Lemma 1. Let $w \in L^\infty(\Omega_1)$ be continuous on $\bar{\Omega}_1 - \{\overline{\partial\Omega_1 \cap \Omega} \cap \Omega\}$, satisfying

$$-\Delta w = 0 \text{ in } \Omega_1, \qquad w = 0 \quad \text{on } \partial\Omega_1 - \overline{\partial\Omega_1 \cap \Omega},$$
$$w = 1 \quad \text{on } \partial\Omega_1 \cap \Omega. \tag{7}$$

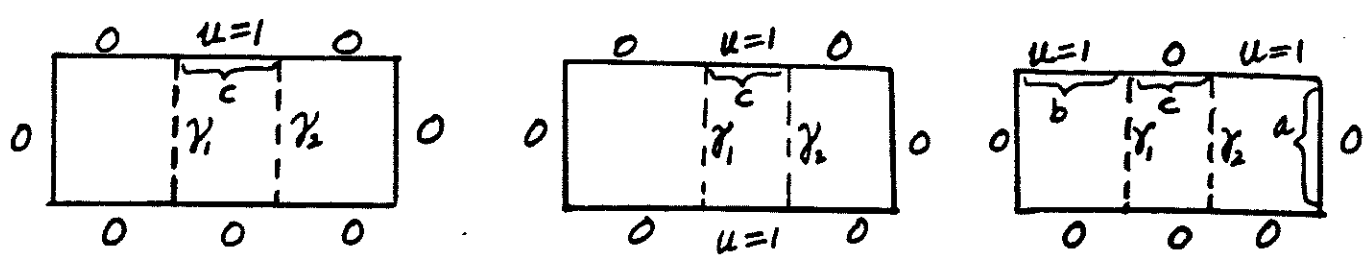
Then ,

$$k_1 = \sup\{|w(x)| \ : \ x \in \partial\Omega_2 \cap \bar{\Omega}\} \in (0,1). \tag{8}$$

A similar result applies to Ω_2, yielding $k_2 \in (0,1)$.

Proposition 1 and Lemma 1 are rather standard. We refer to [14,15] for proofs.

According to Proposition 1 and Lemma 1, the estimates for k_1 and k_2 can be obtained by considering the corresponding elliptic equation restricted to the three canonical rectangular domains with the boundary conditions described as in Figs. 2a, 2b and 2c. Then k_1 or k_2 is bounded from above by $\max_{\gamma_1,\gamma_2}\{w(x)\}$. For examples, we have $k_1 \le K_1$, $k_2 \le K_2$ for L and T-shaped domains, and $k_1 \le K_1$, $k_2 \le K_3$ for C-shaped domain. In the following section, we will give detailed estimates for k_1 and k_2 based on the analysis of these three canonical rectangles.



(a) : K_1 (b) : K_2 (c) : K_3

Figure 2

3 Laplace Operator

We first study the Poisson equation for the three canonical cases described in Fig. 2.

3.1 Case 1.

Let $v(x)$ be a solution of the boundary value problem defined in Fig. 3. By a standard comparison theorem and the maximum principle [9], the solution $w(x)|_{\gamma_1}$ corresponding to case 1 is bounded from above by the solution $v(x)|_{x_1=0}$.

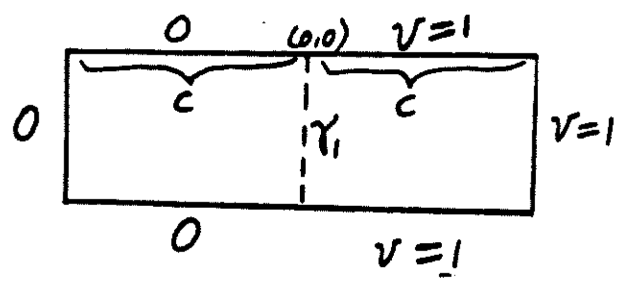


Figure 3

Note that $1 - v(-x_1, x_2)$ also satisfies the same boundary value problem defined in Fig. 3. By the uniqueness, we have $v(x_1, x_2) = 1 - v(-x_1, x_2)$, which implies that

$$v(0, x_2) = \frac{1}{2}.$$

Thus we obtain

$$max_{\gamma_1} w(x) \leq max_{\gamma_1} v(x) = \frac{1}{2} \quad \text{independent of the geometry.} \qquad (9)$$

Similarly we could prove that $max_{\gamma_2} w(x) \leq \frac{1}{2}$ independent of the geometry.

3.2 Case 2.

Arguing exactly as in case 1 and using the symmetry property of the domain for the corresponding solution $v(x)$, we conclude that $max_{\gamma_1} w(x)$ and $max_{\gamma_2} w(x)$ are bounded from above by 0.5 independent of the geometry.

3.3 Case 3.

Consider the boundary value problem defined in Fig. 2c. If $c \geq c_0 > 0$ and a is bounded, then the strong maximum principle [9] implies that

$$\max_{\gamma_1, \gamma_2} w(x) \leq k(c_0) < 1.$$

However, $k(c_0) \to 1$ as $c_0 \to 0$. To see this, observe that solution $w(x)$ can be expressed as the sum of $v_1(x)$ and $v_2(x)$ defined in Fig. 4 below.

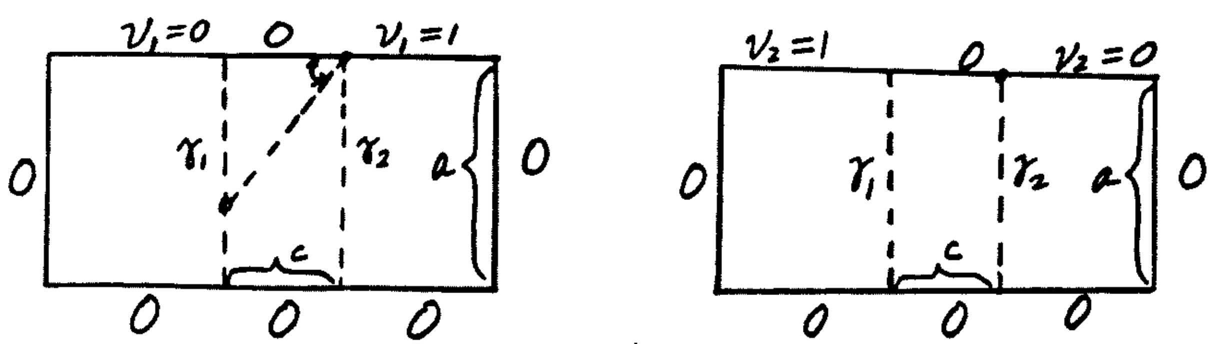


Figure 4

For $v_2(x)$, using the symmetry argument, we get $max_{\gamma_1} v_2(x) \leq \frac{1}{2}$. For $v_1(x)$, note that $\frac{1}{\pi} arg(x)$ is a supersolution for the corresponding boundary value problem, since $\frac{1}{\pi} arg(x)$ is the imaginary part of an analytic function $ln(x)$. Thus we obtain

$$max_{\gamma_1} w(x) = max_{\gamma_1}(v_1(x) + v_2(x))$$

$$\leq \frac{1}{2} + \frac{1}{\pi} tan^{-1}(\frac{a}{c}) \rightarrow 1 \quad \text{as} \quad \frac{a}{c} \rightarrow \infty.$$

3.4 Convergence Rates for L, T and C Shapes

Suppose that we have decomposed respectively the L, T and C-shaped domains into the unions of two overlapping subdomains as in Fig. 1. Combining the results of cases 1, 2 and 3 with Lemma 1, we come to the following conclusions:

(1) For the L and T-shaped domains, k_1 and k_2 are bounded from above by 0.5 independent of the geometry.

(2) For the C-shaped domain, k_1 is bounded from above by 0.5 independent of the geometry. But k_2 may approach 1 as certain aspect ratio of the domain goes to infinity.

We summarize these results in the following theorem.

THEOREM 1. For all L, T and C-shaped domains, the Schwarz alternating algorithm converges independent of the geometry. More precisely, for the L and T-shaped domains, the rate of convergence is given by

$$\sup_{\Omega} \mid u^n - u \mid \; \leq \; (\frac{1}{2})^n sup|u^0 - u|. \tag{10}$$

For the C-shaped domain, the rate of convergence is given by

$$\sup_{\Omega} \mid u^n - u \mid \; \leq \; (\frac{1}{\sqrt{2}})^n sup|u^0 - u|. \tag{11}$$

Remark: As we see from the derivation of k_1 and k_2, these rates of convergence are sharp in the sense that there exist boundary value problems with boundary values 0 and 1 for which the above inequalities actually become equalities.

The above analysis does not depend solely on the symmetry of a rectangular domain with vertical interface. Similar results apply to problems with angled interfaces. Let's consider the angled L-shaped domain for example. Decompose the domain into the union of two subdomains as in Fig. 5.

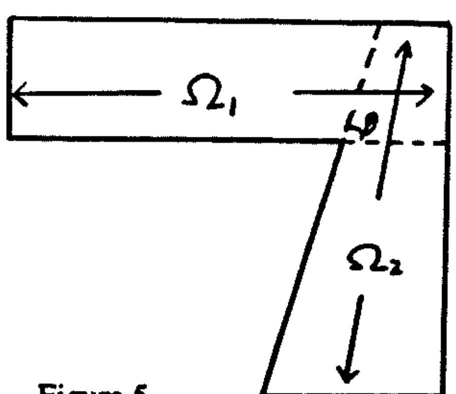


Figure 5

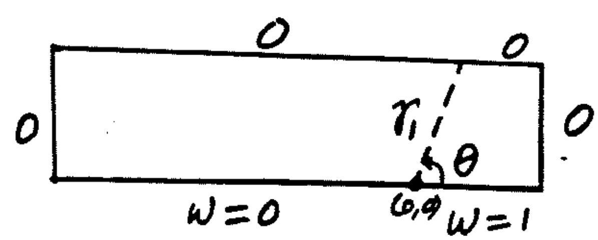


Figure 6

Consider the solution $w(x)$ of the boundary value problem defined by Fig. 6. Obviously $v(x) = \frac{1}{\pi} arg(x)$ is a supersolution for the above boundary value problem. On the other hand, the maximum principle gives

$$k_1 = max_{\gamma_1} w(x) \leq max_{\gamma_1} v(x) = \frac{\pi - \theta}{\pi} \tag{12}$$

independent of the aspect ratio of the domain.

Similarly we have $k_2 \leq \frac{\pi-\theta}{\pi}$ independent of the aspect ratio of the domain. Therefore the rate of convergence $\sqrt{k_1 k_2} \rightarrow 1$ as $\theta \rightarrow 0$ and $\sqrt{k_1 k_2} \rightarrow 0$ as $\theta \rightarrow \pi$.

We would like to point out that our analysis relies only on the maximum principle of elliptic differential equations over regular domains (see [9], for instance, for some classical results of these types). Therefore it applies to domains of more general shapes than the rectangular domains considered here. In particular, it includes the domain of the following shape:

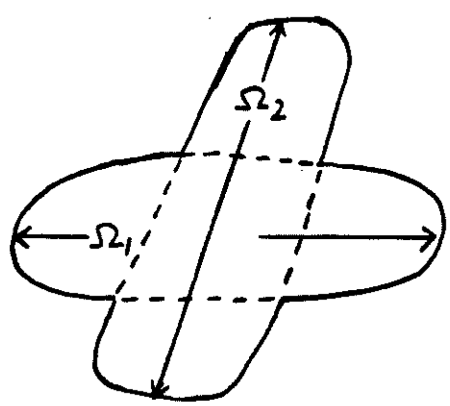


Figure 7.

The rate of convergence in this case is determined by the angle between the interface and the boundary. We refer to [15] for more discussions about convergence of the Schwarz alternating method for more general domains.

Moreover, our results in this section apply to three dimensional domains. For instance, the rate of convergence for a three dimensional L-shaped domain is determined by the solution $v(x)$ to the canonical boundary value problem defined in Fig. 8:

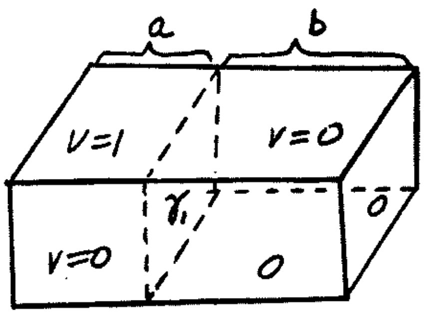


Figure 8

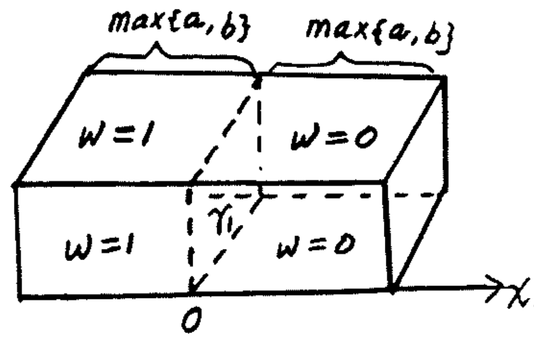


Figure 9

As in the two dimensional case, we have the rate of convergence k_1 bounded by $sup_{\gamma_1} v(x)$. By the comparison theorem, the solution w to the boundary value problem defined in Figure 9 is a supersolution of v. But the symmetry implies that $w(x_1, x_2, x_3) = 1 - w(-x_1, x_2, x_3)$. Therefore, we obtain

$$sup_{\gamma_1} v(x) \leq sup_{x_2,x_3} w(0, x_2, x_3) = 1/2.$$

That is the rate of convergence is bounded by 1/2 .

4 Convection-Diffusion Operators

We now turn to study the convection-diffusion operator:

$$-\Delta u + \mathbf{b}(x) \cdot \nabla u. \tag{13}$$

As before, the convergence study for L, T and C-shaped domains can be reduced to consider the three canonical boundary value problems defined in Figure 2. For simplicity, we just consider the case defined in Figure 10.

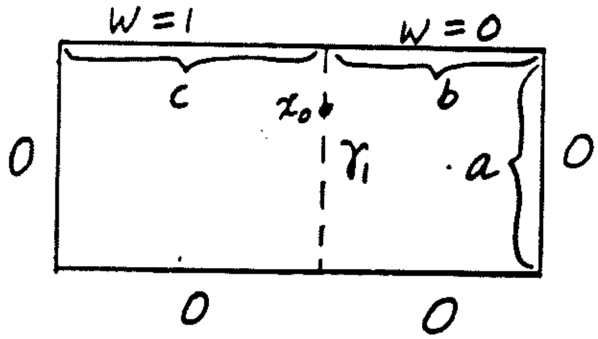


Figure 10

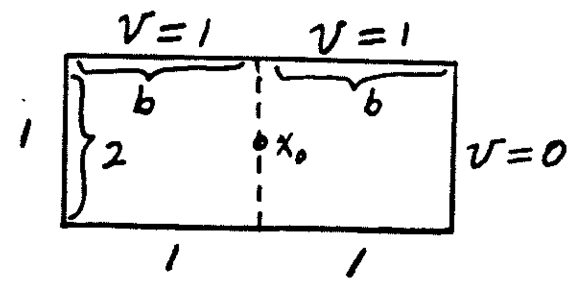


Figure 11

THEOREM 2. Let $w(x)$ be the solution to the canonical boundary value problem described in Fig. 10. Suppose there exist finite constants B and M such that $\|\mathbf{b}(\cdot)\|_\infty \le B$ and a or b or c is bounded from above by M, then $sup_{\gamma_1} w(x) \le K(B, M) < 1$ independent of all other parameters.

Proof. We will prove the theorem case by case.

Case 1. a is bounded. Let $v(x)$ be the solution to the corresponding boundary value problem with $b = \infty$ and $c = \infty$. The existence of such solution v is not entirely trivial and follows for instance from elliptic estimates and maximum principle when letting b and c go to ∞; the uniqueness is not in fact needed in the argument below. Next, since $\mathbf{b}(x)$ and a are bounded, the strong maximum principle implies that

$$0 \leq v(x)|_{\gamma_1} \leq K(B,M) < 1.$$

This fact (intuitively obvious) requires some justification: for points of γ_1 at any positive distance from 0, it is enough to invoke the strong maximum principle (see [9] for instance). Next, as x_0 goes to 0, one just observes that v converges to $1/2$ uniformly in $b(\cdot)$ (and in a). This follows from a tedious local argument. And these two facts prove the above claim.

Using the comparison theorem, we obtain

$$w(x)|_{\gamma_1} \leq v(x)|_{\gamma_1} \leq K(B,M) < 1.$$

Case 2. b is bounded. Let x_0 be a point on γ_1. If $|x_0| \leq 2.0$, then we can show by a similar argument as in case 1 that

$$sup_{\gamma_1} w(x) \leq K(B,2) < 1 \quad \text{for} \quad |x_0| \leq 2.0.$$

For the case $|x_0| \geq 2.0$, consider the corresponding elliptic equation with the boundary condition defined in Fig. 11. By the maximum principle and the comparison theorem, $w(x_0) \leq v(x_0)$. Now make a change of variables to translate x_0 to the origin. We get

$$\Delta \tilde{v} - \tilde{\mathbf{b}} \cdot \nabla \tilde{v} = 0.$$

Since $|\tilde{\mathbf{b}}(x)| \leq M$, the strong maximum principle gives

$$w(x_0) \leq \tilde{v}(0) \leq K(B, M) < 1.$$

Combining the above results, we conclude that

$$sup_{\gamma_1} w(x) \leq K(B, M) < 1. \tag{14}$$

Case 3. c is bounded. This can be shown in a similar way as in case 2. This completes the proof of the theorem. □

Next we study the case when $\mathbf{b}(x)$ is an arbitrary constant vector.

$$-\Delta u - \alpha u_{x_1} - \beta u_{x_2} = 0, \tag{15}$$

where α and β are assumed to be constants.

We have the following result.

THEOREM 3. Let w be the solution of (15) satisfying the boundary condition described in Figure 10. Then, except for the case when $\beta \geq 0$ and $\alpha < 0$, we have

$$\sup_{\gamma_1, \gamma_2} w(x) \leq K < 1,$$

independent of the geometry.

Proof. By the comparison theorem, it is enough to consider and estimate along the semi-axis $\gamma_1 = \{(0, x_2) : x_2 < 0\}$ a solution w of (15) in the lower half space satisfying $0 \leq w \leq 1$, $w(x_1, 0) = 1$ if $x_1 < 0$, $w(x_1, 0) = 0$ if $x_1 > 0$. In fact, the uniqueness of such a w is not necessary for the proof below and the existence follows upon solving the same problem in a box

$\{-R < x_1 < R, -R < x_2 < 0\}$, imposing boundary conditions of the form: $w(x_1, 0) = w(x_1, -R) = 1$ if $x_1 < 0$, $w(x_1, 0) = w(x_1, -R) = 0$ if $x_1 > 0$, $w(-R, x_2) = 1$, $w(R, x_2) = \phi(x_2)$ for some function ϕ satisfying $0 \leq \phi \leq 1$, and letting R go to ∞, using elliptic estimates.

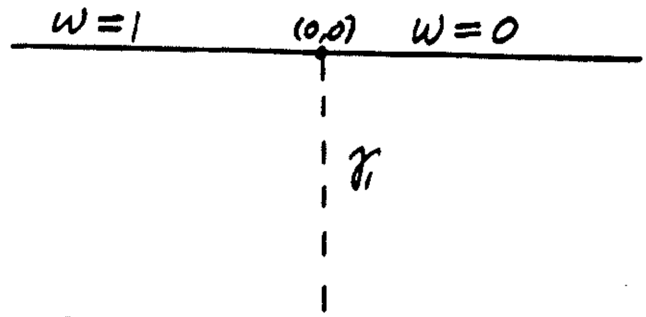


Figure 12

Case 1. $\alpha = 0$. We choose $\phi = 0$ in the above construction and we observe that $1 - w(-x_1, x_2)$ satisfies the same boundary value problem. By the uniqueness, we have

$$w(x_1, x_2) = 1 - w(-x_1, x_2),$$

which implies that

$$w(0, x_2) = \frac{1}{2}.$$

Therefore, $sup_{\gamma_1} w(x) \leq \frac{1}{2}$.

Case 2. $\alpha \geq 0$. We again choose $\phi = 0$ and we observe (using the maximum principle) that $w(x_1, x_2)$ is a nonincreasing function with respect to x_1. Consequently, we obtain

$$-\Delta w - \beta w_{x_2} = \alpha w_{x_1} \le 0. \tag{16}$$

Let $v(x)$ be the solution of

$$-\Delta v - \beta v_{x_2} = 0,$$

with the same boundary condition. Then equation (16) implies that $v(x)$ is a supersolution of $w(x)$. As a result, we have

$$sup_{\gamma_1} w(x) \le sup_{\gamma_1} v(x) \le \frac{1}{2},$$

by the result of case 1.

Case 3. $\beta < 0$. We take $\phi(x_2) = exp(-\beta x_2)$. Note that $w = exp(-\beta x_2)$ is a special solution for equation (15), satisfying $exp(-\beta x_2)|_{x_2=0} = 1$. Therefore we get

$$w(x_1, x_2) \le exp(-\beta x_2) \quad \text{for} \quad x_2 < 0.$$

On the other hand, since $\lim_{x_2 \to 0^-} w(0, x_2) = \frac{1}{2}$, there exists a $\delta_0 > 0$ such that

$$|w(0, x_2) - \frac{1}{2}| < \frac{1}{4} \quad \text{for} \quad 0 \ge x_2 \ge -\delta_0. \tag{17}$$

Combining these two results we conclude for $\beta < 0$

$$sup_{\gamma_1} w(x) \le max\{\frac{3}{4}, exp(\beta\delta_0)\} < 1,$$

independent of the geometry. This completes the proof of theorem 3. □

We would like to point out that the result of theorem 3 is sharp. The Schwarz method corresponding to the exceptional case $\beta \ge 0$ and $\alpha < 0$

does not converge independent of the geometry in general. In the following, we will show that $sup_{\gamma_1} w(x)$ approaches to 1 with probability one as the aspect ratio goes to infinity.

We first scale the independent variable x by λ. We get

$$-\Delta w_\lambda - \lambda\alpha(w_\lambda)_{x_1} - \lambda\beta(w_\lambda)_{x_2} = 0. \tag{18}$$

To study $w(0, x_2)$ as $x_2 \to -\infty$ is then equivalent to study $w_\lambda(0,-1)$ as $\lambda \to \infty$. Let (B_t^1, B_t^2) be the two dimensional Brownian motions. The stochastic particle path for equation (18) starting at point (0,-1) is governed by the following stochastic differential equation [8]:

$$\begin{aligned} X_t^1 &= 0 + \lambda\alpha t + B_t^1 \\ X_t^2 &= -1 + \lambda\beta t + B_t^2 \end{aligned} \tag{19}$$

The first exit time τ when (X_t^1, X_t^2) reaches the boundary is given by

$$\begin{aligned} 0 < \tau &= inf\{t \geq 0 : X_t^2 = 0\} \\ &= inf\{t \geq 0 : B_t^2 = 1 - \lambda\beta t\} < \infty, \end{aligned} \tag{20}$$

since $\lambda\beta \geq 0$. Denote by $P\{w_\lambda(0,-1) = 1\}$ the probability of $w_\lambda(0,-1) = 1$. Then we have

$$\begin{aligned} P\{w_\lambda(0,-1) = 1\} &= P\{X_\tau^1 < 0\} \\ &= P\{B_\tau^1 < \lambda|\alpha|\tau\} \to 1 \quad \text{as} \quad \lambda \to \infty. \end{aligned} \tag{21}$$

Therefore we conclude that $sup_{\gamma_1} w(x) = 1$ as the aspect ratio of the domain goes to infinity.

Remark. The geometry-independent convergence results in this section can be easily extended to the corresponding three dimensional problems since the argument is dimension independent.

5 Discretized Problems

Consider the Poisson equation on a canonical rectangular domain

$$-\Delta u = 0, \quad u|_{\partial\Omega} = g. \tag{22}$$

Let $\Delta x_1 = \Delta x_2 = h$, $n+1 = \frac{a}{h}$, $m+1 = \frac{b}{h}$, where a and b are the width and length of the rectangle.

Denote by $u_{i,j}$ the approximation of $u(ih, jh)$. Discretizing the Poisson equation with standard five-point central difference, we obtain the following discrete scheme:

$$-\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} - \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = 0, \tag{23}$$

for $1 \leq i \leq n, 1 \leq j \leq m$, with boundary condition $u_{i,j}|_{\Omega} = g_{i,j}$.

Define matrix A to be

$$A = \begin{pmatrix} D & -I & 0 & \cdots & 0 & 0 & 0 \\ -I & D & -I & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & -I & D & -I \\ 0 & 0 & 0 & \cdots & 0 & -I & D \end{pmatrix}_{mn \times mn},$$

where I is the identity matrix and D is defined by

$$D = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & -1 & 4 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 4 \end{pmatrix}_{n\times n}.$$

Ordering $u_{i,j}$ such that

$$\mathbf{u} = (u_{1,1}, \cdots, u_{n,1}, \cdots, u_{n,m})^T,$$

then the discrete scheme can be expressed in the matrix form

$$A\mathbf{u} = \mathbf{f},$$

where $\mathbf{f}$ is given by the boundary condition.

LEMMA 2. Matrix A is an M-matrix ($A^{-1} \geq 0$). Moreover, we have $A^{-1} > 0$.

Lemma 2 is a well-known fact. Its proof can be found in most text books on linear algebra, see e.g. [19].

An immediate consequence of Lemma 2 is the maximum principle for the discrete elliptic problem.

LEMMA 3. For $1 \leq i \leq n$ and $1 \leq j \leq m$ we have

$$\min_{\partial\Omega} u_{k,l} \leq u_{i,j} \leq \max_{\partial\Omega} u_{k,l}.$$

The comparison theorem for the discrete problem follows from Lemma 3.

For elliptic equations with convection terms

$$-\triangle u - \mathbf{b}(x) \cdot \nabla u = 0,$$

the corresponding matrix A still satisfies $A^{-1} > 0$, provided that $\frac{h}{2}\|\mathbf{b}(\cdot)\|_\infty < 1$.

Remark. The convergence of the Schwarz alternating algorithm for discrete problems was also investigated by Miller in [16]. Our interest here is to study how the rate of convergence depends on the geometry.

We now study the convergence of the Schwarz method for the discrete problem. For the purpose of illustration, we consider an L-shaped domain decomposed into two overlapping subdomains, as in Fig.1. Since the maximum principle is still true for the discrete problem, the discrete analog of Proposition 1 holds. Moreover, the rates of convergence can be estimated in a similar way as in Lemma 1.

LEMMA 4. Let $w_{i,j}$ be the numerical solution to

$$-\frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{h^2} - \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{h^2} = 0, \qquad (24)$$

with boundary conditions

$$w_{i,j} = 0 \ \text{ on } \ \partial\Omega_1 \backslash (\overline{\Omega_1 \cap \Omega}), \quad \text{and} \quad w_{i,j} = 1 \ \text{ on } \ \partial\Omega_1 \cap \Omega. \qquad (25)$$

Then ,

$$k_1 = sup\{|w_{i,j}| \ : \ (ih, jh) \in \partial\Omega_2 \cap \bar{\Omega}\} \in (0,1). \qquad (26)$$

Similar result applies to k_2.

Lemma 4 follows directly from the discrete maximum principle. We omit the proof.

We now turn to estimate on the convergence rates for L, T and C-shaped domains. By Lemma 4, we need only to estimate the discrete solution $w_{i,j}$ on γ_1 subject to the boundary conditions described in Fig. 2.

By the comparison theorem, it is enough to consider the discrete solution $v_{i,j}$ corresponding to the boundary condition described in Fig. 3. Due to the symmetry of the domain, we have $v_{i,j} = 1 - v_{-i,j}$ by the uniqueness of the discrete solutions. Thus we have $v_{0,j} = 1/2$ and $k_1 \leq 1/2$. Similarly, $k_2 \leq 1/2$.

The other cases for the Laplace equation follow in the same way as in the continuous case. We thus have proved the following theorem:

THEOREM 4. The results in Theorem 1 is still true for the corresponding discrete problem.

More generally, we could handle angled interfaces. First of all, when γ has a right angle with the boundary, the previous result gives $w_{i,j}|_\gamma \leq \frac{1}{2}$. Consider the boundary value problem defined in Fig. 6. In the discrete case, we can not use $\frac{1}{\pi}arg(x)$ as a supersolution anymore, since it is not a discrete harmonic function. But we could still make use of the symmetry of the domain to obtain an upper bound for the rate of convergence.

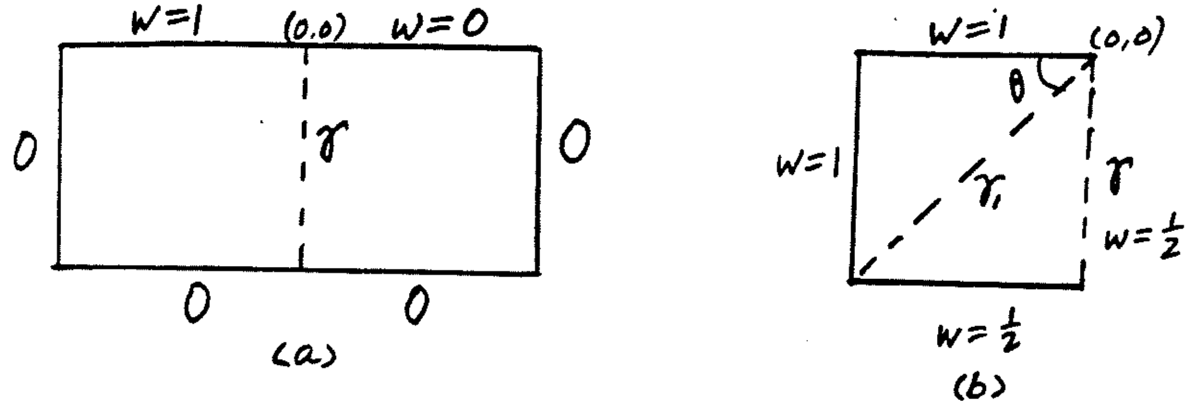


Figure 13

For the boundary value problem in Fig.13a, we know that $w_\gamma = 1/2$. Now consider the boundary value problem defined in Fig.13b. Due to the symmetry, we again have

$$w(r,\theta) = \frac{3}{2} - w(r, \frac{\pi}{2} - \theta),$$

which gives $w(r, \frac{\pi}{4}) = \frac{3}{4}$. The above observation also applies to the discrete problem. Thus we obtain

$$0 \le w_{i,j} \le \frac{3}{4} \quad \text{for} \quad (ih,\ jh) \in \gamma_1.$$

By the maximum principle, we conclude that

$$0 \le w_{i,j} \le \frac{3}{4} \quad \text{for} \quad \frac{\pi}{4} \le \theta \le \pi.$$

Therefore the Schwarz method converges independent of the geometry if the angle between the interface and the boundary lies in the interval $[\frac{\pi}{4}, \pi]$.

For convection-diffusion equations

$$-\Delta u - \mathbf{b}(x) \cdot \nabla u = 0,$$

if $\|\mathbf{b}(\cdot)\|_\infty \leq B < \infty$ and a or b or c is bounded, then the Schwarz method for the discrete problem converges independent of the geometry. The proof is essentially the same as in the continuous case since we have the strong maximum principle for $\mathbf{b}(x)$ bounded (recall $A^{-1} > 0$).

Finally we remark that the results in this section can be easily extended to three dimensional problems.

6 Results for the Schur Complement Method

Besides the class of the Schwarz overlapping domain decomposition methods discussed so far in this paper, there is another class of widely used domain decomposition methods which takes the nonoverlapping approach. In this section, we shall make use of a recently discovered relationship between these two classes of methods [2,6] to extend the geometry-independent convergence results for the overlapping approach derived earlier in this paper to equivalent convergence results for the corresponding nonoverlapping methods. For simplicity, we shall restrict our discussion to L-shaped domains as in Fig. 1a.

In the nonoverlapping approach, the problem (continuous or discrete) on the domain Ω is first reduced to an equivalent problem

$$Cx = b \quad , \tag{27}$$

defined on an interface (e.g. γ_1 in Fig. 1a) which divides Ω into two nonoverlapping subdomains. This reduced problem on the interface is then solved by iterative methods, often preconditioned by an easily invertible operator M which approximates C and accelerated by conjugate gradient methods. The action of C on an iterate on the interface can be evaluated via independent subdomain solves. The convergence rate of this preconditioned iteration depends on the condition number $K(M^{-1}C)$.

Many preconditioners for C have been proposed in the literature (see [1,3,4,7,11]). We shall consider a particular one proposed in [4], which we denote by M_C, and is defined to be the exact reduction of the Laplace operator $-\Delta$ on Ω_2 onto the interface γ_1. It was shown in [4] that for rectangular Ω_2's with γ_1 parallel to one of the sides of the rectangle, M_C is efficiently invertible via FFTs.

Define the splitting

$$C = M_C - N \quad . \tag{28}$$

Now consider the Schwarz overlapping method as an iterative method for refining an initial guess $u^0_{\gamma_1}$ to obtain u_{γ_1}. In [2,6], the following result is proved:

THEOREM 5 The iterates in the Schwarz method are precisely those generated via

$$M_C v^{i+1} = N v^i + b \tag{29}$$

with

$$v^0 = u^0_{\gamma_1}. \tag{30}$$

In other words, the Schwarz method can be viewed as a stationary iterative method for the reduced interface problem with a splitting given in (28).

Through the connection between the Schwarz and the nonoverlapped method as expressed in theorem 5, we can extend the geometry-independent convergence results in theorem 1 to obtain an upper bound for $K(M_C^{-1}C)$ governing the convergence of the nonoverlapped method.

THEOREM 6 For the operator $-\Delta$ on all L-shaped domains, we have

$$K(M_C^{-1}C) \leq 2.$$

Proof. From theorem 1, we have

$$\|M_C^{-1}N\|_\infty \leq \frac{1}{2}.$$

Since

$$|\lambda(M_C^{-1}N)| \leq \rho(M_C^{-1}N) \leq \|M_C^{-1}N\|_\infty,$$

where $\lambda(M_C^{-1}N)$ denotes any eigenvalue of $M_C^{-1}N$, $\rho(M_C^{-1}N)$ is the spectral norm of $M_C^{-1}N$, we have

$$|\lambda(M_C^{-1}N)| \leq \frac{1}{2}.$$

Moreover, it was proved in [5] that C, M_C and N are all symmetric positive definite matrices and therefore it follows that the eigenvalues of $M_C^{-1}N$ are real and positive. Thus we have

$$0 \leq \lambda(M_C^{-1}N) \leq \frac{1}{2}.$$

Combining this with the relation

$$M_C^{-1}C = I - M_C^{-1}N,$$

we obtain

$$\lambda_{max}(M_C^{-1}C) \leq 1, \quad \lambda_{max}(M_C^{-1}C) \geq \frac{1}{2},$$

and thus

$$K(M_C^{-1}C) \leq 2.$$

Remark: The equivalence between the Schwarz and the nonoverlapped methods extends to T and C-shaped domains as well. Using a similar derivation as in theorem 6, we can show that

$$K(M_C^{-1}C) \leq 2 \quad for \quad all \quad T-shaped \quad domains$$

and

$$K(M_C^{-1}C) \leq \frac{1}{1-1/\sqrt{2}} \quad for \quad all \quad C-shaped \quad domains.$$

This improves slightly on similar results obtained earlier in [5] via Fourier analysis.

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