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ON HOUSE SWAPPING WITH MONEY

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Abstract

We consider the "house-swapping" model of Shapley-Scarf (1974), except with a transferable resource. We consider two cases: First, in the TU case (where utility is transferable) we characterize the set of core points using linear programming. For the NTU case (where *money* but not utility is transferable), we present a "deferred acceptance procedure" which calculates a point in the core. The results here parallel work done by Shapley-Shubik (1972) and Crawford-Knoer (1981) for two-sided matching markets.

1. Introduction

Many times in economic theory, a single "unifying" paper comes along which ties together under one general model several seemingly unrelated results. In the area of two-sided matching markets (TSMMs), perhaps the best example is that of Quinzii (1984). Her idea was as follows. It is known that several "two-sided" games, such as those of Gale & Shapley (1962), Shapley & Shubik (1972), Crawford & Knoer (1981) and Gale & Demange (1985) are balanced, i.e., have nonempty cores. Can we present a class of balanced games containing all these models? Quinzii answers "yes", by presenting her class of "pairing models".

In an earlier part of the same paper, Quinzii uses a similar proof to show that a similar class of games with an indivisible good is balanced. These are the "house swapping" games, first defined [for the case of ordinal preferences] by Shapley and Scarf (1974). However, Quinzii's games also include those with a transferable resource [cardinal preferences], a much larger class of games.

In this paper, we analyze these house-swapping games with a transferable resource. The fact that Quinzii's "balancedness" proofs for these games and for two-sided matching markets are so similar suggests that our analysis might parallel work done with TSMMs. Indeed, this is the main theme of this paper.

First, we consider a house swapping game with transferable utility (TU). In a manner similar to Shapley & Shubik, linear programming is used to characterize the game's core points. Thus, the results of Curiel (1980) is extended. Also, the set of economic equilibrium prices is shown to be a sublattice. Again, a similar result applies in Shapley-Shubik.

Next, we consider a more general class of non-transferable (NTU) games, which can best be described as the "house swapping" analogue of Gale-Demange's two-sided matching market. First, we show that although economic equilibria always exist, the set of such prices is not necessarily a sublattice. We then provide an analogue of the "deferred acceptence procedure" used for TSMMs [Gale-Shapley, Crawford-Knoer, Kelso-Crawford (1984), Kamecke (1987), Quint (1987)] which finds a core point under certain conditions.

The paper is organized as follows. Section 2 presents background material, namely the Shapley-Scarf model. Section 3 discusses the TU model. Finally, Section 4 covers the NTU case.

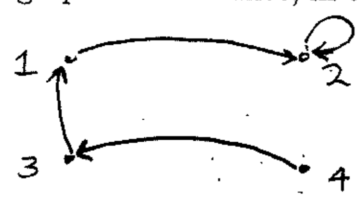
2. Background

Shapley and Scarf's house-swapping model is as follows. Suppose n traders in a market each originally own a house. The (initial) house of the ith trader is denoted as house i. Each trader also has a preference ordering, or ranking, over the entire set of n houses (including his own). Thus, in the four trader example presented below, Trader #1 prefers House #2 the most, House #3 second, his own house third, and Trader #4's last. Using the usual preference ordering notation, we write this as $2 \succ_1 3 \succ_1 1 \succ_1 4$.

T1: 2, 3, 1, 4
T2: 2, 4, 3, 1
T3: 1, 2, 3, 4
T4: 3, 2, 4, 1

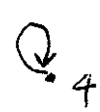
The object of course is to try and find a "sensible" assignment of houses to traders (presumably accomplished via house "swaps" between pairs of traders). To this end, let an assignment, or matching $\mu : N \longrightarrow N$ be any one-to-one mapping of traders to houses in which $\mu(i) = j$ means that trader i gets house j. We say that μ can be improved upon through S if $\exists$ assignment π with $\pi(S) = S$ and $\pi(i) \succ_i \mu(i)$ for all $i \in S$. An assignment μ is a core matching if there is no coalition S through which μ can be improved.

Shapley and Scarf proved that a core matching always exists. In fact, Gale ( ) gives an explicit procedure, called "top trading cycles" (TTCs) for finding such an assignment. In a nutshell, it runs as follows. Each trader's first preference is used to form a directed graph. For instance, in the example above, the graph is:



The graph's cycles determine part of the core assignment. In this case the cycle containing only "2" is the only such "top trading cycle". Set $\mu(2) = 2$, take "2" out of the model, and reiterate the process:

Now the top trading cycle is $\{1, 3\}$. Set $\mu(1) = 3$ and $\mu(3) = 1$, leaving us with



Obviously we set $\mu(4) = 4$. So the final outcome is $\mu = \{(1\ 3)(2\ 2)(3\ 1)(4\ 4)\}$. This is in fact in the core.

Interestingly enough, this procedure bears some resemblance to the deferred acceptence procedure (DAP) used to solve the marriage market of Gale-Shapley. In the TTC procedure, agents originally are matched to their favorite house. As the algorithm progresses, the agents (not in cycles) get matched to worse and worse houses. In the DAP, the men originally are matched with their favorite woman. Again, as the algorithm proceeds, the men (who are rejected) get matched to worse and worse women.

These similarities again suggest that there is a close relationship between two-sided matching markets and house-swapping models. We investigate these similarities further in the next section, as a house-swapping model with transferable utility is discussed.

3. The Transferable Utility Case

Before discussing the house-swapping model with TU, let us briefly review the assignment game of Shapley and Shubik (1972). This is a TU game in which there is a set $I = \{1, ..., n\}$ of buyers and a set $J = \{1, ..., n\}$ of sellers. The data for the model is an $n \times n$ matrix $\hat{C}$, whose entries $\hat{c}_{ij}$ represent the worth of a coalition consisting of only buyer i and seller j. These "one-buyer-one-seller" coalitions are the essential coalitions of the game. We define an assignment, or matching, as a 1-1 mapping $\mu : I \longrightarrow J$ and a maximal matching as a matching μ^* for which $\sum_{i \in I} \hat{c}_{i\mu^*(i)} \geq \sum_{i \in I} \hat{c}_{i\mu(i)}$ for all other assignments μ. The core is a triple (v, w, μ) satisfying $v_i + w_{\mu(i)} = \hat{c}_{i\mu(i)}\ \forall i$ [feasibility] and $v_i + w_j \geq \hat{c}_{ij}\ \forall i, j$ [stability]. The fundamental result of Shapley and Shubik states that the core is never empty, and that core triplets (v, w, μ) exist iff μ is a maximal matching.

We now describe the house-swapping market with TU. As in the Shapley-Scarf model, there are again n traders, but preferences are now expressed differently. Again, each trader in the market originally owns a house,[1] and the original house

[1] This actually covers the case where some traders originally have no house, through the allocation of "dummy houses" (which no one would desire) to those traders who are originally houseless.

of the ith trader is denoted as house i. The valuation that trader i has for house j is expressed as a constant in dollars, namely c_{ij}. Hence, as in Shapley-Shubik, we have a TU game, where "utility is identified with money". Furthermore, we assume $c_{ij} \geq 0$ for all i and j because it is always possible for trader i to "live out in the street" instead of using house j. Finally, since the $n \times n$ matrix C consisting of the c_{ij}'s will completely define the game, we refer to it as "house-swapping game C".

The characteristic function V for this game is then the following:

$$V(S) = \max_{\pi:\pi(S)=S} \sum_{i \in S} c_{i\pi(i)}$$

In other words, all that the members of S can do is to trade houses amongst themselves according to the π's. In particular, note that $V(\{i\}) = c_{ii}$.

If S is the grand coalition N consisting of all n traders, we have

$$V(N) = \max_{\pi:\pi(N)=N} \sum_{i \in N} c_{i\pi(i)}. \tag{3.1}$$

We call any element of the argmax in (3.1) a <u>maximal matching</u>, and denote one by π^*. Thus, evaluating $V(N)$ is equivalent to finding a maximal matching μ^* for the assignment game with "assignment matrix" $\hat{C} = \{c_{ij}\}_{i,j=1}^{n}$. This is not the only way in which this game is similar to Shapley-Shubik's.[2]

Next, the <u>core</u> of the game is defined as those vectors $u = (u_1, ..., u_n)$ satisfying

$$\sum_{i=1}^{n} u_i = V(N) \tag{3.2}$$

$$\sum_{i \in S} u_i \geq \max_{\pi:\pi(S)=S} \sum_{i \in S} c_{i\pi(i)} \ \forall S. \tag{3.3'}$$

These represent the usual feasibility and stability constraints for a TU game. Note that we can rewrite inequalities (3.3′) as

$$\sum_{i \in S} u_i \geq \sum_{i \in S} c_{i\pi(i)} \ \forall S, \pi : \pi(S) = S. \tag{3.3}$$

[2] Curiel (1980) refers to this game as the <u>permutation game</u>.

Theorem 3.1: (Curiel) The core of the house-swapping market with TU is nonempty.

Although this result is already known (Quinzii), this proof is of interest because it parallels the linear programming proof of core existence for Shapley and Shubik's market.

Proof: Consider the assignment linear program

$$\max \sum_{i,j=1}^{n} c_{ij} x_{ij} \qquad (P)$$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = 1$$

$$\sum_{i=1}^{n} x_{ij} = 1$$

$$x_{ij} \geq 0 \ \forall i, j.$$

We know this solves with any $x = x^*$, where

$$x^*_{ij} = 1 \longleftrightarrow \pi^*(i) = j$$

$$x^*_{ij} = 0 \longleftrightarrow \pi^*(i) \neq j$$

for some maximal matching π^*. Also, the optimal maximand of (P) is $V(N)$.

Now, take the dual of (P):

$$\min \sum_{i=1}^{n} v_i + \sum_{j=1}^{n} w_j \qquad (D)$$

$$\text{s.t. } v_i + w_j \geq c_{ij} \ \forall i, j. \qquad (D1)$$

Let $\{v_i^*\}_{i=1}^n$, $\{w_j^*\}_{j=1}^n$ be any optimal solution for (D). Define u by $u_i = v_i^* + w_i^*$ for all i.

Claim: Any u defined in this way is in the core.

Proof: To show feasibility (3.2), we have

$$\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} (v_i^* + w_i^*) = V(N),$$

where the last equality follows from the strong duality theorem of linear programming.[3] Finally, to show stability (3.3), consider any assignment π and any S with $\pi(S) = S$:

$$\sum_{i \in S} u_i = \sum_{i \in S} (v_i^* + w_i^*) = \sum_{i \in S} (v_i^* + w_{\pi(i)}^*) \geq \sum_{i \in S} c_{i\pi(i)}.$$

[The last inequality follows from the fact that constraints (D1) hold for (v, w).]

The next result states that *all* core vectors can be defined in this way.

Theorem 3.2: Let u be any core vector. Then u can be expressed as $v + w$, where (v, w) is an optimal solution for (D).

Proof: Let $u_1, ..., u_n$ be any core vector, and consider the assignment game defined by

$$\hat{C} = \begin{pmatrix} u_1 & c_{12} & \dots & c_{1n} \\ c_{21} & u_2 & \dots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \dots & u_n \end{pmatrix}$$

Claim: μ^* defined by $\mu^*(i) = i \ \forall i \in N$ is a maximal matching for assignment game $\hat{C}$ above.

Proof: Consider any other matching μ, and define $I = \{i : \mu(i) = i\}$ and $I^c = \{i : \mu(i) \neq i\}$. We trivially have $\sum_{i \in I} \hat{c}_{i\mu^*(i)} = \sum_{i \in I} u_i = \sum_{i \in I} \hat{c}_{i\mu(i)}$. Furthermore, $\sum_{i \in I^c} \hat{c}_{i\mu^*(i)} = \sum_{i \in I^c} u_i \geq \sum_{i \in I^c} \hat{c}_{i\mu(i)}$ because u in the core satisfies (3.3) with $S = I^c$. These two facts taken together prove the Claim.

[3] Dantzig, p. 129.

Now let $(\hat{v}, \hat{w})$ be any core vector of Shapley-Shubik game $\hat{C}$. Since $\mu^* : \mu^*(i) = i$ is maximal, we have $\hat{v}_i + \hat{w}_i = \hat{c}_{ii} = u_i$. Next, $(\hat{v}, \hat{w})$ satisfies constraints (D1) because its in the core of $\hat{C}$. Finally, its optimal for (D) because of the strong duality theorem and the fact that $\sum \hat{v}_i + \sum \hat{w}_j = \sum_{i=1}^{n} u_i = V(N)$. Q.E.D.

We interpret Theorems 3.1 and 3.2 as follows. Basically, we can view the house swapping market as an assignment game in which Trader i is playing the role of *both* Buyer i and Seller i. Hence, it is not surprising that in core allocations, Trader i's welfare is the sum of the welfare he derives from being a "buyer" (v_i) and from being a "seller" (w_i).

Next, we define an <u>equilibrium</u> as a set of prices $p = (p_1, ..., p_n)$ on the houses and an assignment π with

$$\pi(i) \in \arg\max_j (c_{ij} + p_i - p_j) \ \forall i.$$

<u>Lemma</u>: Equilibria (p, π) exist iff π is a maximal matching. Furthermore, if μ is a maximal matching, the set of equilibrium prices is precisely the set of w's contained in optimal solutions of linear program (D) above (see p. 6).

We note here that the first part of the Lemma can be deduced from core-equilibrium equivalence results of Wako (1986). However, we again give an alternative proof.

<u>Proof</u>: First we show that if (p, π) is an equilibrium, then π is a maximal matching. Suppose not, i.e., (p, π) is an equilibrium but $\sum c_{i\pi(i)} < \sum c_{i\pi^*(i)}$ for some π^*. Since (p, π) is an equilibrium,

$$c_{i\pi(i)} + p_i - p_{\pi(i)} \geq c_{i\pi^*(i)} + p_i - p_{\pi^*(i)} \ \forall i, \text{ which implies} \qquad (3.4)$$

$$\sum_{i \in N} c_{i\pi(i)} + p_i - p_{\pi(i)} \geq \sum_{i \in N} c_{i\pi^*(i)} + p_i - p_{\pi^*(i)}.$$

But then, all the "p" terms cancel, yielding $\sum c_{i\pi(i)} \geq \sum c_{i\pi^*(i)}$. This is of course a contradiction.

Now, in order to prove the entire Theorem, we need to show that, given maximal matching π^*, (p, π^*) is an equilibrium iff (v, p) solves linear program (D) for some vector v. To this end, let (v^*, p^*) be a solution to (D). By the complementary slackness theorem of linear programming[4], $\pi^*(i) = j \iff x^*_{ij} = 1 \implies v^*_i + p^*_j = c_{ij}$. Since (v^*, p^*) satisfies (D1), we thus have $0 = v^*_i + p^*_{\pi^*(i)} - c_{i\pi^*(i)} \leq v^*_i + p^*_j - c_{ij}\ \forall i, j$, or,

$$p^*_{\pi^*(i)} - c_{i\pi^*(i)} \leq p^*_j - c_{ij}\ \forall i, j. \tag{3.5}$$

However, this implies (3.4), meaning that (p^*, π^*) is an equilibrium. For the converse, suppose (p, π^*) is an equilibrium. This implies (3.5). Let $v^*_i = c_{i\pi^*(i)} - p^*_{\mu^*(i)}\ \forall i$. Then we claim (v^*, p^*) solves (D). Substituting back into (3.5), we get $-v^*_i \leq p^*_j - c_{ij}\ \forall i, j$, which is (D1). Furthermore, since $v^*_i + p^*_{\pi^*(i)} = c_{i\pi^*(i)}\ \forall i$, we know $\sum v^*_i + \sum p^*_i = \sum c_{i\mu^*(i)} = V(N)$. Thus (v^*, p^*) is optimal for (D) by the strong duality theorem. Q.E.D.

Definition: Let a and b be any two points in $\Re^n$, define the join of a and b by y where $y_i = \max(a_i, b_i)$, and the meet of a and b by z where $z_i = \min(a_i, b_i)$. Then a sublattice in $\Re^n$ is any subset L of $\Re^n$ for which

$$a, b \in L \implies \text{join}(a, b) \in L \text{ and } \text{meet}(a, b) \in L.$$

Corollary: The set of equilibrium prices for a house-swapping game with TU is a sublattice.

We note here that a similar result holds for the model of Shapley and Shubik.[5]

[4] Dantzig, pp. 135-6.
[5] Shapley and Shubik, p. 120.

Proof: From the previous Theorem, the set of equilibrium prices is the set of w's contained in optimal solutions of (D). Since in any optimal solution (v^*, w^*), $v_i^* + w_{\pi^*(i)}^* = c_{i\pi^*(i)}$ for some maximal matching π^*, our set is the set of w's for which

$$c_{i\pi^*(i)} - w_{\pi^*(i)} + w_j \geq c_{ij} \ \forall i, j.$$

Due to a theorem of Veinott's (1987), this is in fact a sublattice.

4. The NTU Case

We next move to the NTU case, in which instead of splitting up an amount of *utility*, a coalition (in additon to swapping houses) makes *monetary* transfers. The utility that a trader derives [given that he gets a particular house] is then an increasing, but not necessarily linear, function of the amount of money he obtains. Thus, the model will be analogous to Demange and Gale's (1985) model for the two-sided matching case, and, later to Crawford & Knoer's (1981) when we "discretize" it.

The model is as follows. Let $N = \{1, ..., n\}$ be the set of traders in the market, and again assume trader i initially owns house i. Let $u_{ij}(x) : \Re \longrightarrow \Re$ be the utility to trader i if he ends up with house j and a net monetary transfer to him of x. Assume each $u_{ij}(x)$ is a strictly increasing onto function. Finally, we assume that each trader would only desire to own one house.

In order to calculate the characteristic function $V(S)$, first define the inverse functions $f_{ij}(u_i)$ of the u_{ij}'s. Given that he is to receive house j, $f_{ij}(u_i)$ is the amount of money that trader i needs in order to raise his utility level to u_i. Using this notation, we have

$$V(S) = \{u \in \Re^S : \exists \pi \text{ with } \pi(S) = S \text{ and } \sum_{i \in S} f_{i\pi(i)}(u_i) \leq 0\} \tag{4.1}$$

In other words, the set of feasible vectors for a coalition S are those in which its members swap houses according to some π and require a net nonpositive input of money from the outside world.

Finally, the <u>core</u> of the game is the set of vectors $u = (u_1, ..., u_n)$ such that

$$u \in V(N) \tag{4.2}$$

$$\nexists S,\ w \in V(S) \text{ such that } w_i > u_i\ \forall i \in S. \tag{4.3}$$

Having stated these definitions, we should note that the house-swapping game with TU is just the special case of this game, with $u_{ij}(x) = c_{ij} + x$ and $f_{ij}(u_i) = u_i - c_{ij}\ \forall i, j$.

<u>Theorem</u>: The core of the house-swapping game with NTU is nonempty.

<u>Proof</u>: Quinzii (1984).

<u>Definition</u>: An <u>equilibrium</u> is a set of prices $p = (p_1, ...p_n)$ on the houses together with an assignment π satisfying

$$\mu(i) \in \arg\max_j u_{ij}(p_i - p_j)\ \forall i \tag{4.6}$$

<u>Theorem</u>: The set of equilibria for the house-swapping game with NTU is nonempty.

<u>Proof</u>: Quinzii (1984); Gale (1984).

The two concepts of core and equilibrium are related in that any competitive equilibrium corresponds to an element in the core (Wako 1987). In fact, our "algorithm to find a core point" actually computes an equilibrium (for the "discretized game").

Thus, it is of interest to study the set of equilibrium prices. In particular, we might first wish to find out if (as in the TU case) it is again always a sublattice. The answer, as the following example demonstrates, is "no".

Example: (with $n = 3$.)

$$u_{11}(x) = u_{22}(x) = u_{33}(x) = x$$

$$u_{12}(x) = x^3$$

$$u_{21}(x) = x + (x)^+$$

$$u_{13}(x) = x + ((x)^+)^2$$

$$u_{31}(x) = 2x + 1$$

$$u_{23}(x) = \frac{x}{2} - 1$$

$$u_{32}(x) = x - 2$$

Let $p^1 = (0, 0, 0)$, let $p^2 = (\frac{1}{2}, \frac{-1}{4}, 0)$, and π be defined by $\pi(1) = 3$, $\pi(2) = 2$, and $\pi(3) = 1$. Then (p^1, π) and (p^2, π) are both equilibria.

However, now consider the meet of p^1 and p^2, namely $p^3 = (0, \frac{-1}{4}, 0)$. Now suppose (p^3, π^3) is to be an equilibrium for some π^3. Equation (4.6) for $i = 1$ gives that $\pi^3(1) = 2$; while, for $i = 2$, we must have $\pi^3(2) = 2$. Hence, there can be no assignment for which (4.6) holds with p^3. Hence, the set of equilibrium prices is not a sublattice.

There is one other characteristic of the Demange and Gale two-sided matching model which we would also like to extend to the house-swapping model with NTU. This is the fact that there exist numerous algorithms which calculate core points using modifications of Gale & Shapley's "deferred acceptance procedure". Not only do these procedures provide a method for finding a core-point, but they also have economic interpretations as bargaining procedures whose final outcomes are core points. Examples include the algorithms of Crawford-Knoer (1982), Kamecke (1987), and Quint (1988).

We now proceed to define a similar procedure for the house swapping game with NTU. However, in order for such a procedure to work, we need to make the following separability assumption:

$$u_{ij}(x) = c_{ij} + u_i(x)\ \forall i, j. \tag{4.7}$$

In other words, for each trader i, the effect of gaining income does not depend on, or is *separate* from, which house he receives. We can also restate (4.7) as the following:

$$u_{ij}(x_1) > u_{ik}(x_2) \Longrightarrow u_{ij}(x_1 + \alpha) > u_{ik}(x_2 + \alpha)\ \forall i, j, k, x_1, x_2, \text{ and } \alpha > 0.$$

To interpret, if Trader i prefers a particular house-money bundle over another, then, sweetening both bundles by \$$\alpha$ will not change his preference. The mathematical importance of this assumption will become apparent later.

To avoid technical difficulties, we follow Crawford and Knoer's lead and consider the *discretized* market in which only *integral* salaries are allowed. This adds the requirement "and $f_{i\pi(i)}(u_i)$ is an integer for each $i \in S$" to the definition of $V(S)$ (p.10). The definition of the core remains the same.[6]

Again, in keeping with Crawford and Knoer, define the $n \times n$ matrix $P^t = \{p_{ij}^t\}$, the matrix of integer monetary offers that traders i are permitted to make for house j at time t. Time is measured discretely.

The procedure is then as follows:

1) Initially $t = 0$, $p_{ij}^0 = 0$, and $p_i^0 = 0\ \forall i, j$.

2) Let $t = t + 1$. Each trader i makes an offer for his favorite house, i.e., he makes an offer for house j, where $j \in \arg\max u_{ij}(p_i^{t-1} - p_{ij}^{t-1})$. [He may break ties however he likes.] For each j, let $K^t(j)$ be the set of i's who make offers for his house.

3) Each trader j who has one or more offers for his house now rejects all but a maximal one, which he tentatively holds on to. [He may break ties however he likes.] He sets the new price on his home as $p_j = \max_{i \in K^t(j)} p_{ij}^{t-1}$.

[6] However, in this discretized case, there *is* a difference between core and strict core (see Crawford and Knoer for details). In this paper only the notion of core is used, not strict core.

4) Offers not yet rejected remain in force. If in this iteration trader j rejected i's offer for his house, set $p_{ij}^t = p_{ij}^{t-1} + 1$. Otherwise, set $p_{ij}^t = p_{ij}^{t-1}$.

5) If there have been no rejections in iteration t, the process stops and all outstanding offers are accepted. This defines a matching π^* of houses to traders, a vector of prices p^* for the houses, and a utility vector u^* where $u_i = u_{i\pi^*(i)}(p_i^* - p_{\pi^*(i)}^*)$. Otherwise, go back to 2).

Theorem 4.1: The procedure outlined above results in finite time in a discrete core allocation in the discrete market for which it is defined.

We prove the Theorem via a series of Lemmata:

Lemma 1: If Trader j has at least one offer for house j at time t, then, at all subsequent times $t' > t$ he still has at least one offer.

Proof: Suppose Trader j has an offer at time t, say, from Trader i. Thus, $u_{ij}(p_i^{t-1} - p_{ij}^{t-1}) \geq u_{ik}(p_i^{t-1} - p_{ik}^{t-1})\ \forall k$. Note that changes in p_i cannot "wreck" this inequality because of (4.7). Thus, the only way he could not have an offer from i during the next iteration is if $p_{ij}^t > p_{ij}^{t-1}$. In this case, j has rejected i's offer and thus must have another offer which he didn't reject. This unrejected offer, say from l, will remain in effect next iteration because no p_{lk} will have changed. Q.E.D.

Lemma 2: After a finite number of periods, no rejections are issued, every trader has exactly one offer for his house, and the process stops.

Proof: Suppose not, i.e., the process goes on indefinitely. Since someone gets rejected on every iteration, $\exists i, j$ such that $\lim_{t\to\infty} p_{ij}^t = \infty$. In other words, the set $IJ^\infty \stackrel{\text{def}}{=} \{(i,j) : \lim_{t\to\infty} p_{ij}^t = \infty\}$ is nonempty. Furthermore, since $\max_i p_{ij}^t - 1 \leq p_j^t \leq \max_i p_{ij}^t$, we have that $I^\infty \stackrel{\text{def}}{=} \{i : \lim_{t\to\infty} p_i^t = \infty\}$ is nonempty. Consider $I^F \stackrel{\text{def}}{=} \{i : \lim_{t\to\infty} p_i^t < \infty\}$. For each $i \in I^F$, let B_i be a least upper bound for

$\lim_{t\to\infty} p_i^t$. Next, for each such i, let R_i be a number such that, for all $r \geq R_i$,

$$u_{ij}(B_i - r) < u_{ii}(0) \text{ for all } j \in I^\infty. \tag{4.8}$$

Next, let $R = \max_{i \in I^F} R_i$. Let T_1 be a time at which $p_{ij}^{T_1} \geq R \; \forall (i,j) \in IJ^\infty$. Then clearly, for time $t > T_1$, no $i \in I^F$ will ever make an offer to a j where $(i,j) \in IJ^\infty$ [because, by inequality (4.8), i would make an offer to himself rather than j].

So, since after time T_1, $i \in I^F$ only makes offers to js where $(i,j) \notin IJ^\infty$, it must be that i can only get rejected a finite number of times. Thus, there must be a time T_2 after which $i \in I^F$ never gets rejected.

Next, consider whether or not, after $t = T_2$, $i \in I^F$ could ever make an offer for house j where $j \in I^\infty$. From the last paragraph, we know i's offer would have to be accepted. However, since $p_j^t \to \infty$, j will get an infinite number of subsequent offers for his house, the best of which will be infinitely high. Thus, i must eventually be rejected because the highest offer he could make is R_i.

So, the upshot of the previous argument is that after time $t = T_2$, no $i \in I^F$ can ever A) get rejected, or B) make an offer for a $j \in I^\infty$.

Next, consider $t > T_2$. Lemma 1 implies that since the algorithm hasn't terminated, there is a trader $\tilde{i}_1$ who has *never* received an offer. Thus, $\tilde{i}_1 \in I^F$, with $p_{\tilde{i}_1}^t = 0$. Suppose $\tilde{i}_1$ makes an offer to $\tilde{i}_2$ during period t. By B), $\tilde{i}_2 \in I^F$ also. Note $\tilde{i}_2 \neq \tilde{i}_1$, because by hypothesis $\tilde{i}_1$ never gets an offer. Next, suppose $\tilde{i}_2$ makes an offer to $\tilde{i}_3$ at time t. Then $\tilde{i}_3 \neq \tilde{i}_2$, because, otherwise, both $\tilde{i}_1$ and $\tilde{i}_2$ would be making an offer to $\tilde{i}_3$, resulting in one of them getting rejected next iteration. This would violate A) above. Also, $\tilde{i}_3 \neq \tilde{i}_1$ because $\tilde{i}_1$ never gets an offer. Finally, $\tilde{i}_3 \in I^F$ by B). Next, let $\tilde{i}_4$ be the trader that Trader 3 makes an offer to...

Continuing in this fashion, we see that for each k, $\tilde{i}_k \neq \tilde{i}_1$ because $\tilde{i}_1$ can never get an offer, and $\tilde{i}_k \neq \tilde{i}_l$, $l = 2, ..., k-1$ because that would mean a rejection for a member of I^F on the next iteration. Since there are only a finite number of agents, we get a contradiction.

Lemma 3: The process terminates at a discrete core allocation for the discrete market in which it is defined.

Proof: Suppose not, i.e., suppose the algorithm terminates at $(\pi^*, p_1^*, ..., p_n^*)$, with $p_i^* = p_{i\pi^*(i)}^t$. Then, since this is not in the core, there must be a coalition S, a matching $\pi : \pi(S) = S$, and integral prices p_i, $i \in S$ such that

$$u_{i\pi(i)}(p_i - p_{\pi(i)}) > u_{i\pi^*(i)}(p_i^* - p_{\pi^*(i)}^*) \; \forall i \in S. \tag{4.9}$$

We rewrite this as:

$$u_{i\pi(i)}(p_i^* - [p_i^* - p_i + p_{\pi(i)}]) > u_{i\pi^*(i)}(p_i^* - p_{\pi^*(i)}^*) \; \forall i \in S.$$

If $\pi(i) \neq \pi^*(i)$, this implies that

$$p_{\pi(i)}^* > p_i^* - p_i + p_{\pi(i)} \; \forall i \in S \tag{4.10}$$

because otherwise i would have made an offer (on the last iteration) to $\pi(i)$ and *not* to $\pi^*(i)$. On the other hand, if $\pi(i) = \pi^*(i)$, (4.10) holds directly from (4.9).

Summing up inequalities (4.10) over $i \in S$ yields a contradiction.

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