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DOMAIN DECOMPOSITION INTERFACE PRECONDITIONERS FOR FOURTH ORDER ELLIPTIC PROBLEMS *

TONY F. CHAN†, WEINAN E‡ AND JIACHANG SUN§

Abstract: We present preconditioners for the interface system arising from solving fourth order elliptic equations with domain decomposition methods. These preconditioners are derived from a Fourier analysis of the interface operator . We show that the condition number of the interface Schur complement is of order $O(h^{-3})$, where h is the grid size. Precise estimates concerning the decay properties of the elements of the Schur complement are also obtained. Relationships between interface preconditioners for second order problems and fourth order problems are established. Analytical as well as numerical results are given to assess the performance of these preconditioners.

Keywords: *Domain Decomposition, biharmonic equation, Schur complement, interface preconditioner.*

1. Introduction. There are two key issues involved in the design of efficient domain decomposition methods for solving elliptic problems on domains decomposed into nonoverlapping subdomains. One is the coupling of the solutions on neighboring subdomains through interfaces and the other is the global coupling of non-neighboring subdomains. For second order elliptic problems, a vast amount of literature is available on how to handle these two kinds of couplings [9, 5, 11]. In particular for the first problem, a lot of insight has been gained by analyzing the following model problem: the interface operator for the Laplace equation on a rectangle which is decomposed into two rectangles. For example, the hierarchy of preconditioners constructed by Dryja [8], Golub and Mayers [10], Bjorstad and Widlund [3], and Chan [7] , can be viewed as successively more accurate approximations of the exact interface operator for the model problem. Such preconditioners can then be used as the basic building blocks in constructing preconditioners for the case when the domain is decomposed into more complicated geometries [4].

The situation is less satisfactory for fourth order problems which arises naturally in modeling the deformation of thin elastic plates and in the stream-function formulation of incompressible flows. For some fourth order problems, domain decomposition algorithms can be obtained by solving iteratively a system of second order problems and apply known domain decomposition techniques for the latter. Such is the case for incompressible flows which in velocity-pressure formulation, is described by a system of second order elliptic equations coupled with the incompressibility condition [12, 14]. On the other hand, not much is known about the interface operators obtained from solving directly the fourth order problems. For instance, it was not clear how to construct the analog of Dryja's preconditioner which was largely based on estimates of the condition numbers of the interface operators. To the best of our knowledge,

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there were even no precise rigorous estimates on the condition number of the interface operators for fourth order problems. The numerical results in [6] give an estimate of $O(h^{-2.7})$ where h is the grid size. We mention that [6] also contains an extension of the boundary probing technique to fourth order problems.

In this paper, we focus on analyzing the following model problem: the interface operators arising from solving the biharmonic equations on a rectangle which is decomposed into two rectangles. Using discrete Fourier analysis, we obtain the exact eigen-decomposition of the interface Schur complement (see Theorems 3.1 and 3.2). This allows us to read off precise information about the interface operators, including an estimate of their condition numbers which is shown to be $O(h^{-3})$, and the decay rate of their elements (see Corollary 3.5). We remark that such precise information on the decay rate of the elements is very important for designing and analyzing boundary probing type of preconditioners. Moreover, this also allows us to construct the analog of Golub-Mayers' and Chan's preconditioners (denoted by M_∞ and M respectively) for the fourth order problems. In addition, we give a simple recipe of converting interface preconditioners for second order problems to interface preconditioners for fourth order problems (see Theorem 3.6). In this way we obtain the fourth-order analog of most of the second-order preconditioners. We note that the analog of Golub-Mayers' and Chan's preconditioners constructed according to Theorem 3.6 (denoted by M_g and M_c respectively) are different from the ones we obtain from Theorems 3.1 and 3.2.

Some subtleties are involved for the fourth order problems with regard to the different types of boundary conditions. It is well-known that the biharmonic equation is not amendable to the method of separation of variables when the clamped boundary conditions are used (see Problem 1 of the next section), which happen to be of the most practical interest. For this reason, our analysis is carried out essentially for the simply supported boundary conditions only. We propose to use the same preconditioners for both problems. As has been pointed out [13], the elliptic operators with different boundary conditions are not necessarily spectrally equivalent. Indeed in our numerical experiments we observe a logarithmic growth of the condition number of the preconditioned interface operators for the clamped boundary conditions (see Figures 4 and 5). However, we also observe that only a finite number of eigenvalues of the preconditioned matrices grow. Therefore one naturally expects that, when a preconditioned conjugate gradient (hereafter abbreviated PCG) method is used, only a finite (bounded) number of iterations are needed for a given level of tolerance. We emphasize that while the *derivation* of the preconditioners are based on the model problem on a rectangular domain, the preconditioners themselves can be *applied* to more general fourth order problems on irregular domains by first obtaining an approximating biharmonic problem on a rectangular domain sharing the same interface with the irregular domain.

We mention that a similar and more detailed development of the eigen-decomposition of the interface operators for the biharmonic equations and application to some fluid flow problems can be found in [16]. The idea of using Fourier eigen-decomposition to derive preconditioners (on the whole domain rather than on an interface) has been used in [2] to obtain a fast biharmonic solver.

This paper is organized as follows. In section 2, we present a class of discretization schemes for the biharmonic operator. In section 3, the Fourier analysis of the interface operator is carried out and our main results are presented . Finally, some numerical results are given in section 4. We will use $\kappa(C)$ to denote the condition number of a symmetric matrix C.

2. Discretizations. Consider the following biharmonic equation

$$\Delta^2 W = q \qquad \text{in} \quad \Omega, \tag{1}$$

where $\Omega = [0,1]\times[0,1]$. Let $\partial\Omega$ denote the boundary of Ω and $\partial\Omega_x$ and $\partial\Omega_y$ the part of Ω that are parallel to the x and the y axis respectively.

Three kinds of boundary conditions are usually imposed:

Problem 1. The clamped boundary conditions:

$$W = \frac{\partial W}{\partial n} = 0 \qquad \text{on} \quad \partial\Omega.$$

Problem 2. The simply supported boundary conditions:

$$W = \Delta W = 0 \qquad \text{on} \quad \partial\Omega.$$

Problem 3. The mixed boundary conditions:

$$W|_{\partial\Omega} = \Delta W|_{\partial\Omega_x} = \frac{\partial W}{\partial n}|_{\partial\Omega_y} = 0.$$

Note that Problem 1 is neither decomposable nor separable even for a rectangular domain. However, it is well known that Problems 2 and 3 are separable.

Problems 1-3 can be discretized by either a finite element method or a finite difference method [1]. In this paper , we will restrict ourselves to the family of difference schemes obtained by combining the standard 5-point scheme and the rotated (by 45 degrees) 5-point scheme for the Laplace operator :

$$\Delta_h^2 := h^{-4}\{t\begin{pmatrix}0 & 1 & 0\\ 1 & -4 & 1\\ 0 & 1 & 0\end{pmatrix} + (1-t)\begin{pmatrix}1 & 0 & 1\\ 0 & -4 & 0\\ 1 & 0 & 1\end{pmatrix}\}^2, \tag{2}$$

When $t = 0$, this scheme gives the standard 13 point scheme .

3. Fourier Analysis of the Interface Operator. Suppose that the rectangular domain Ω is decomposed into two subdomains Ω_1 and Ω_2, with interface Γ.

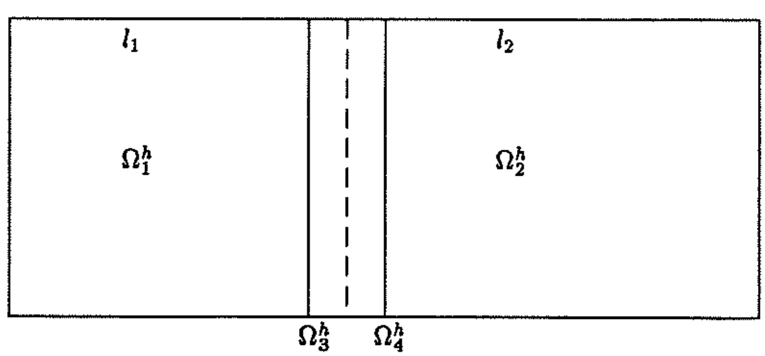

Figure 1: The domain Ω and its partition.

Also suppose that a uniform mesh with size h is used on Ω, with n interior grid points in the vertical y direction. The simplest way of decoupling the two subdomains is to introduce two computational grid interfaces Ω_3^h and Ω_4^h near the physical interface Γ as shown in Figure 1. Assume that

$$l_1 = m_1 h, \; l_2 = m_2 h,$$

where m_1, m_2 denote the number of grid points along the horizontal x direction of the two subdomains, as shown in Fig. 1. If we order the unknowns in such a way that the interior points in the sub-domains appear first and those on the two interfaces appear last, then the discrete solution vector $u = (u_1, u_2, u_3, u_4)$, where u_i denote the set of unknowns on Ω_i^h, satisfies the linear system

$$Au = b$$

which can be expressed in block form as:

$$(3) \qquad \begin{pmatrix} A_{11} & 0 & A_{13} & A_{14} \\ 0 & A_{22} & A_{23} & A_{24} \\ A_{13}^T & A_{23}^T & A_{33} & A_{34} \\ A_{14}^T & A_{24}^T & A_{34} & A_{44} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

Here the matrix A and the vector b depend on the specific discretization method we choose and the boundary conditions, etc., but they can always be expressed in the above form. The Schur complement (corresponding to the reduced interface operator) on the interfaces is given by:

$$(4) \qquad C = \begin{pmatrix} A_{33} & A_{34} \\ A_{34}^T & A_{44} \end{pmatrix} - \begin{pmatrix} A_{13} & A_{14} \\ A_{23} & A_{24} \end{pmatrix}^T \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & A_{22}^{-1} \end{pmatrix} \begin{pmatrix} A_{13} & A_{14} \\ A_{23} & A_{24} \end{pmatrix}.$$

We are concerned with the problem of solving efficiently the reduced equations:

$$(5) \qquad C \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} g_3 \\ g_4 \end{pmatrix}.$$

Because C is dense and expensive to form explicitly, the PCG methods are usually preferred in this case. Since C is an ill-conditioned matrix (as we shall show later), and for fourth order problems it is much more so than the corresponding ones for the second order case, it is imperative to use an efficient preconditioner.

We now proceed to compute the exact eigen-decomposition of the matrix C for problems 2 and 3, using discrete Fourier analysis. The basic idea is the same as the one used in [7] , although the actual computations are much more complicated.

Denote by w_j, $j = 1, \ldots, n$ the eigen-vectors of the one dimensional discrete Laplace operator:

$$w_j = \sqrt{2h}(\sin j\pi h, \sin 2j\pi h, \ldots, \sin nj\pi h)^T,$$

and let

$$W = [w_1, w_2, \ldots, w_n]$$

be the matrix form by these eigenvectors. We shall diagonalize C by diagonalizing each of its four individual blocks by a similarity transformation using W. As in [7], this will involve the computation of terms such as $A_{13}^T A_{11}^{-1} A_{13} w_j$, which corresponds to solving the biharmonic equation on Ω_1 with appropriately chosen boundary conditions related to w_j. Therefore, we first need a general solution for the discrete biharmonic equations on the subdomains. We get this by using the method of separation of variables.

Substituting the expression

$$V(ih, kh) = d_k \ (w_j)_i = d_k \sqrt{2h} \sin ij\pi h$$

into the discrete model

$$\Delta_h^2 V = 0,$$

we get the following fourth order difference equation for d_k :

$$b_0 d_{k+2} + b_1 d_{k+1} + b_2 d_k + b_1 d_{k-1} + b_0 d_{k-2} = 0 \tag{6}$$

where

$$\begin{aligned} b_0 &= 2a_{22} \cos 2j\pi h + 2a_{21} \cos j\pi h + a_{20}, \\ b_1 &= 2a_{21} \cos 2j\pi h + 2a_{11} \cos j\pi h + a_{10}, \\ b_2 &= 2a_{20} \cos 2j\pi h + 2a_{10} \cos j\pi h + a_{00}, \end{aligned}$$

and the $a_{i,j}$'s were defined in the previous section.

We will denote by $\{d_k\}$ the solution of (6) with the boundary condition

$$d_0 = 0, d_1 = 1, d_m = 0, d_{m-1} = d_{m+1};$$

for Problem 2, and

$$d_0 = 0, d_1 = 1, d_m = 0, d_{m-1} = -d_{m+1};$$

for Problem 3. These boundary conditions are needed for the computation of terms such as $-A_{13}^T A_{11}^{-1} A_{13} w_j$. Similarly we denote by $\{\tilde{d}_k\}$ the solution of (6) with the boundary condition

$$\tilde{d}_0 = 1, \tilde{d}_1 = 0, \tilde{d}_m = 0, \tilde{d}_{m-1} = \tilde{d}_{m+1}$$

for problem 2, and

$$\tilde{d}_0 = 1, \tilde{d}_1 = 0, \tilde{d}_m = 0, \tilde{d}_{m-1} = -\tilde{d}_{m+1}$$

for problem 3. These are needed for the computation of terms such as $-A_{14}^T A_{11}^{-1} A_{14} w_j$. Explicit expressions of d_k and $\tilde{d}_k$ will be given later. Once we have these solutions, it is easy to see that the following relations hold,

$$\begin{aligned} A_{33} w_j &= b_2 w_j, \quad -A_{13}^T A_{11}^{-1} A_{13} w_j = (b_0 d_3(m_1) + b_1 d_2(m_1)) w_j, \\ & \quad -A_{32}^T A_{22}^{-1} A_{23} w_j = b_0 \tilde{d}_2(m_2) w_j, \end{aligned}$$

$$\begin{aligned} A_{34} w_j &= b_1 w_j, \quad -A_{31}^T A_{11}^{-1} A_{14} w_j = b_0 d_2(m_1) w_j, \\ & \quad -A_{32}^T A_{22}^{-1} A_{24} w_j = (b_0 \tilde{d}_3(m_2) + b_1 \tilde{d}_2(m_2)) w_j, \end{aligned}$$

$$A_{44}w_j = b_2 w_j, \quad -A_{24}^T A_{22}^{-1} A_{24} w_j = (b_0 \tilde{d}_3(m_2) + b_1 d_2(m_2)) w_j,$$
$$-A_{41}^T A_{11}^{-1} A_{14} w_j = b_0 \tilde{d}_2(m_1) w_j,$$

Here the argument m is given explicitly to distinguish the two subdomains which may have a different number of grid points in the x direction.

We summarize these results by stating the following

THEOREM 3.1. *For Problems 2 and 3, the interface Schur complement C has the following diagonalized form:*

$$C = \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12} & \Lambda_{22} \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \tag{7}$$

where

$$\Lambda_{11} = diag(\lambda_{11,j}), \; \Lambda_{12} = diag(\lambda_{12,j}), \; \Lambda_{22} = diag(\lambda_{22,j})$$

and

$$\begin{aligned} \lambda_{11,j} &= b_2 + b_0 d_3(m_1) + b_1 d_2(m_1) + b_0 \tilde{d}_2(m_2) \\ \lambda_{12,j} &= b_1 + b_0 \tilde{d}_3(m_2) + b_1 \tilde{d}_2(m_2) + b_0 d_2(m_1) \\ \lambda_{22,j} &= b_2 + b_0 d_3(m_2) + b_1 d_2(m_2) + b_0 \tilde{d}_2(m_1). \end{aligned}$$

We now proceed to derive explicit expressions for d_k and $\tilde{d}_k$. The characteristic roots η for the difference scheme (6) satisfy the following relation

$$b_0(\eta + \eta^{-1})^2 + b_1(\eta + \eta^{-1}) + b_2 - 2b_0 = 0$$

or

$$\eta + \eta^{-1} = \frac{1}{2b_0}\{-b_1 \pm \sqrt{b_1^2 + 8b_0^2 - 4b_2 b_0}\}. \tag{8}$$

In principle, the solutions $d_k(m)$ and $\tilde{d}_k(m)$ can be obtained for any discretization of the biharmonic equation which gives rise to a constant coefficient difference equation of the form (6) . However, this is typically a very tedious computation. A simplification occurs for the class of difference schemes in (2). In this case $b_1^2 + 8b_0^2 = 4b_2 b_0$, and equation (8) has two double roots r and r^{-1}, where

$$r = \frac{-b_1 - \sqrt{b_1^2 - 4b_0^2}}{2b_0} \quad (0 < r < 1).$$

A straight-forward computation yields the following solutions. For Problem 2, we have:

$$d_k(m) = \frac{k(r^m + r^{-m})(r^{k-m} - r^{m-k}) + 2(m-k)(r^k - r^{-k})}{(r^m + r^{-m})(r^{1-m} - r^{m-1}) + 2(m-1)(r - r^{-1})},$$

$$\tilde{d}_k(m) = \frac{(1-k)(r^{k-m} - r^{m-k})(r^{m-1} + r^{1-m}) + 2(m-k)(r^{1-k} - r^{k-1})}{(r^{-m} - r^m)(r^{1-m} + r^{m-1}) \; + \; 2m(r - r^{-1})}$$

and for Problem 3 we have:

$$d_k(m) = \frac{k(r+r^{-1})(r^m - r^{-m})(r^{k-m} - r^{m-k}) + 2m(m-k)(r-r^{-1})(r^k - r^{-k})}{(r+r^{-1})(r^m - r^{-m})(r^{1-m} - r^{m-1}) + 2m(m-1)(r-r^{-1})^2},$$

$$\tilde{d}_k(m) = \frac{(1-k)(r+r^{-1})(r^{k-m} - r^{m-k})(r^{m-1} - r^{1-m}) + 2(m-k)(m-1)(r^{k-1} - r^{1-k})^2}{(r+r^{-1})(r^m - r^{-m})(r^{1-m} - r^{m-1}) \ + \ 2m(m-1)(r-r^{-1})^2}.$$

Although these expressions can be used in a practical computation, they are too complicated to analyze. In order to gain further insight, we make the simplification by taking the limit as m_1 and m_2 go to infinity (the case of an infinite strip) . We then have

$$d_k = kr^{k-1}, \qquad \tilde{d}_k = (1-k)r^k. \tag{9}$$

The results in Theorem 3.1 now take the form of

THEOREM 3.2. *For the discretization scheme (2) on an infinite strip ($m_1, m_2 = +\infty$), we have:*

$$\lambda_{11,j} = \sigma_j^{1/2}(4 + \sigma_j(1-4t))^{1/2}(2 + (1-2t)\sigma_j),$$

$$\lambda_{12,j} = \sigma_j^{1/2}(4 + \sigma_j(1-4t))^{1/2}(-2 + 2t\sigma_j),$$

and

$$\lambda_{22,j} = \lambda_{11,j},$$

where $\sigma_j = 4\sin^2(j\pi h/2)$.

If we let $K = W\text{diag}(\sigma_j)W$, (i.e. the discrete one dimensional Laplace operator), then the result of Theorem 3.2 for $t = 0$ can be re-stated as

$$C = \begin{pmatrix} (2I+K)K^{1/2}(4I+K)^{1/2} & -2K^{1/2}(4I+K)^{1/2} \\ -2K^{1/2}(4I+K)^{1/2} & (2I+K)K^{1/2}(4I+K)^{1/2} \end{pmatrix}.$$

The next theorem shows how to reduce C to diagonal form.

THEOREM 3.3. *For Problems 2 and 3,*

$$C = \tilde{W}\Lambda\tilde{W} \tag{10}$$

where

$$\tilde{W} = \frac{1}{\sqrt{2}}\begin{pmatrix} W & W \\ W & -W \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$$

$$\Lambda_1 = diag(\lambda_{1,j}), \ \Lambda_2 = diag(\lambda_{2,j})$$

$$\lambda_{1,j} = \lambda_{11,j} + \lambda_{12,j}, \qquad \lambda_{2,j} = \lambda_{11,j} - \lambda_{12,j}.$$

In particular for discretization (2) on an infinite strip, we have:

$$\begin{aligned}\lambda_{1,j} &= \sigma_j^{3/2}\{4+\sigma_j(1-4t)\}^{1/2}, \\ \lambda_{2,j} &= \sigma_j^{1/2}\{4+\sigma_j(1-4t)\}^{3/2}.\end{aligned}$$

Since σ_j varies from $O(h^2)$ to $O(1)$, the eigenvalues of C vary from $O(h^3)$ to $O(1)$ and therefore we have:

COROLLARY 3.4.

$$\kappa(C) = O(h^{-3}) \tag{11}$$

where $\kappa(C)$ is the condition number of C.

Note that the corresponding condition number for second order operators is $O(h^{-1})$.

In [6], the decay properties for the entries of C were explored to construct boundary probing preconditioners. Now, by using Theorem 3.3, it is easy to quantify the decay phenomenon.

COROLLARY 3.5. *Denote the blocks of C by*

$$C = \begin{pmatrix} C_{33} & C_{34} \\ C_{34}^T & C_{44} \end{pmatrix}$$

and the Schur Complements of C by

$$\begin{aligned}C_{33}^* &= C_{33} - C_{43}^T C_{44}^{-1} C_{43} \\ C_{44}^* &= C_{44} - C_{34}^T C_{33}^{-1} C_{34}.\end{aligned}$$

Let $c_{i,j}$ be the entries of either one of the submatrices C_{33}, C_{34}, C_{44} and $c_{i,j}^*$ be the entries of either one of the matrices C_{33}^*, C_{44}^*, then we have:

$$|c_{ij}| = O(|i-j|^{-2})$$

and

$$|c_{ij}^*| = O(|i-j|^{-4}).$$

We shall only sketch the proof here since it basically follows the one in [10], where the decay properties of the interface operator for second order problems are obtained. From Theorem 3.3, the elements $c_{i,j}$ can be written as a sum of three terms — two from the W terms and one from the diagonal terms. Approximating this sum by an integral and integrating by parts twice gives the result for $c_{i,j}$. Similarly one obtains the other result.

Returning to our original problem of constructing preconditioners for the interface Schur complements, we can readily interpret our results in Theorems 3.1 and 3.2 as ways of achieving such a goal. For this purpose, let us denote by M and M_∞ the matrices obtained in Theorem 3.1 and 3.2 respectively. It is clear from their construction that M and M_∞ are the respective analogs of Chan's and Golub-Mayers' preconditioners for the fourth order problems.

The next theo[199zrem is motivated by the desire of seeking the analog of Dryja's preconditioner for the fourth order problems, and the symbolic expression of C obtained above (stated after Theorem 3.2). This theorem gives a procedure for converting a general interface preconditioner for second order problems into an interface preconditioner for fourth order problems.

THEOREM 3.6. *If M_2 is an interface preconditioner for a second order elliptic operator, with $\kappa(M_2^{-1}C_2) = O(1)$, then*

$$M_4 = \begin{pmatrix} M_2(1+M_2^2) & -M_2(1-M_2^2) \\ -M_2(1-M_2^2) & M_2(1+M_2^2) \end{pmatrix} \tag{12}$$

is an interface preconditioner for the biharmonic operator satisfying

$$\kappa(M_4^{-1}C_4) = O(1),$$

where C_2 and C_4 are the Schur complements for the second order operator and the biharmonic operator on a rectangular domain, respectively.

Proof. Let

$$M_2 = W\Lambda_{(2)}W.$$

By assumption M_2 is spectrally equivalent to C_2. Furthermore it is well-known [8] that C_2 is spectrally equivalent to $K^{1/2}$. Thus $\Lambda_{(2)}$ is spectrally equivalent to $K^{1/2}$. From the definition of M_4, we have

$$\begin{aligned} M_4 &= \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \begin{pmatrix} \Lambda_{(2)}(I+\Lambda_{(2)}^2) & -\Lambda_{(2)}(I-\Lambda_{(2)}^2) \\ -\Lambda_{(2)}(I-\Lambda_{(2)}^2) & \Lambda_{(2)}(I+\Lambda_{(2)}^2) \end{pmatrix} \begin{pmatrix} W & 0 \\ 0 & W \end{pmatrix} \\ &= \tilde{W} \begin{pmatrix} \Lambda_{(2)}^3 & 0 \\ 0 & \Lambda_{(2)} \end{pmatrix} \tilde{W}, \end{aligned}$$

and

$$CM_4^{-1} = \tilde{W} \begin{pmatrix} \Lambda_1\Lambda_{(2)}^{-3} & 0 \\ 0 & \Lambda_2\Lambda_{(2)}^{-1} \end{pmatrix} \tilde{W}.$$

From the formulas for Λ_1 and Λ_2 in Theorem 3.3, it can be easily verified that they are spectrally equivalent to $K^{3/2}$ and $K^{1/2}$ respectively, which are in turn spectrally equivalent to $\Lambda_{(2)}^3$ and $\Lambda_{(2)}$. □

Using Theorem 3.6 we can construct, for fourth order problems, the analogues of the preconditioners of Dryja [8], Golub-Mayers[10] and Chan [7], which are known for second order problems. For notational purpose, we will denote these preconditioners by M_d, M_g, M_c respectively. Recall that the matrices constructed in Theorems 3.1 and 3.2 are denoted by M and M_∞ respectively.

Although we have derived the exact eigen-decomposition for the interface operator C for Problems 2 and 3, the procedure is not applicable to Problem 1 because it is not separable. Instead, we propose to use the matrices C constructed in Theorems 3.1 and 3.2 as *preconditioners* for the interface operator arising from Problem 1 and more general fourth order problems. The hope is that the interface operator of Problems 2 and 3 are good approximations for the interface operator of Problem 1 since the only difference is in the boundary condition on part of the boundary. Since at this point we do not have theoretical results on the performance of this preconditioner, we present the results of some preliminary numerical experiments to confirm this. As we will show in the next section, with these preconditioners the condition number of the preconditioned system on the interface seems to grow only slowly as the grid size h goes to zero. Furthermore, all except a few eigenvalues remain bounded as h tends to zero. Hence if a PCG type of iterative method is used, only a bounded number of iterations are needed to satisfy a given tolerance of the error. We'd like to add that since the completion of this paper, some theoretical results have been obtained regarding a similar preconditioner in a finite element setting [15], confirming the effectiveness of preconditioners obtained using our approach.

4. Numerical Results. We now present the results of some numerical experiments carried out to assess the performance of the interface preconditioners that we have proposed in this paper. As a preliminary test, the preconditioners are used on the interface operator corresponding to Problems 1 and 3. The parameter t in the discretization (2) was set to zero. The PCG method is used to solve the interface system. The iterations are stopped when the initial residual is reduced to a factor of 10^{-7}. The subdomain solver is chosen to be an SSOR preconditioned PCG method with the same stopping criteria.

Figs. 2 - 5 show our numerical results for the case when both sub-domains have the same size. Fig. 2 gives the iteration numbers for M and M_∞ on Problems 1 and 3. Both preconditioners perform quite well and there is practically no difference between the two. Note that the preconditioner M, which should be exact for Problem 3, took more than one iteration because of the inexact subdomain solve. Fig. 3 gives similar results for the preconditioners M_d and M_c. It can be seen that a more accurate preconditioner for second order problems (M_c versus M_d) can lead to a correspondingly more effective preconditioner for fourth order problems. Figure 4 and 5 show the eigenvalue distribution for the preconditioned systems using M and M_∞. One can see that only two distinct eigenvalues grow unbounded as h tends to zero. A numerical data fit yields an approximate order of $\log(h^{-1})$ for the growth rate. This clustering of eigenvalues can be exploited by the PCG iteration.

Next we show some results for Problem 1 where the sizes of the two sub-domains are different. Fig. 6 shows a similar situation as in Fig. 2. We have the domain Ω discretized by a 32×16 grid and vary the aspect ratio of the two sub-domains by moving the interface. As can be expected, M performs slightly better than M_∞ because the aspect ratio is incorporated in the construction of M. Fig. 7 shows the results for an L-shaped domain, with a fixed number of 31 grid points on one side of the square . The picture gives the PCG iteration numbers as the width of the small sub-square varies. The result suggests that the performance of the preconditioner M is quite insensitive to the aspect ratios.

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Fig. 2 Comparison of preconditioners for Problem 1 and 3

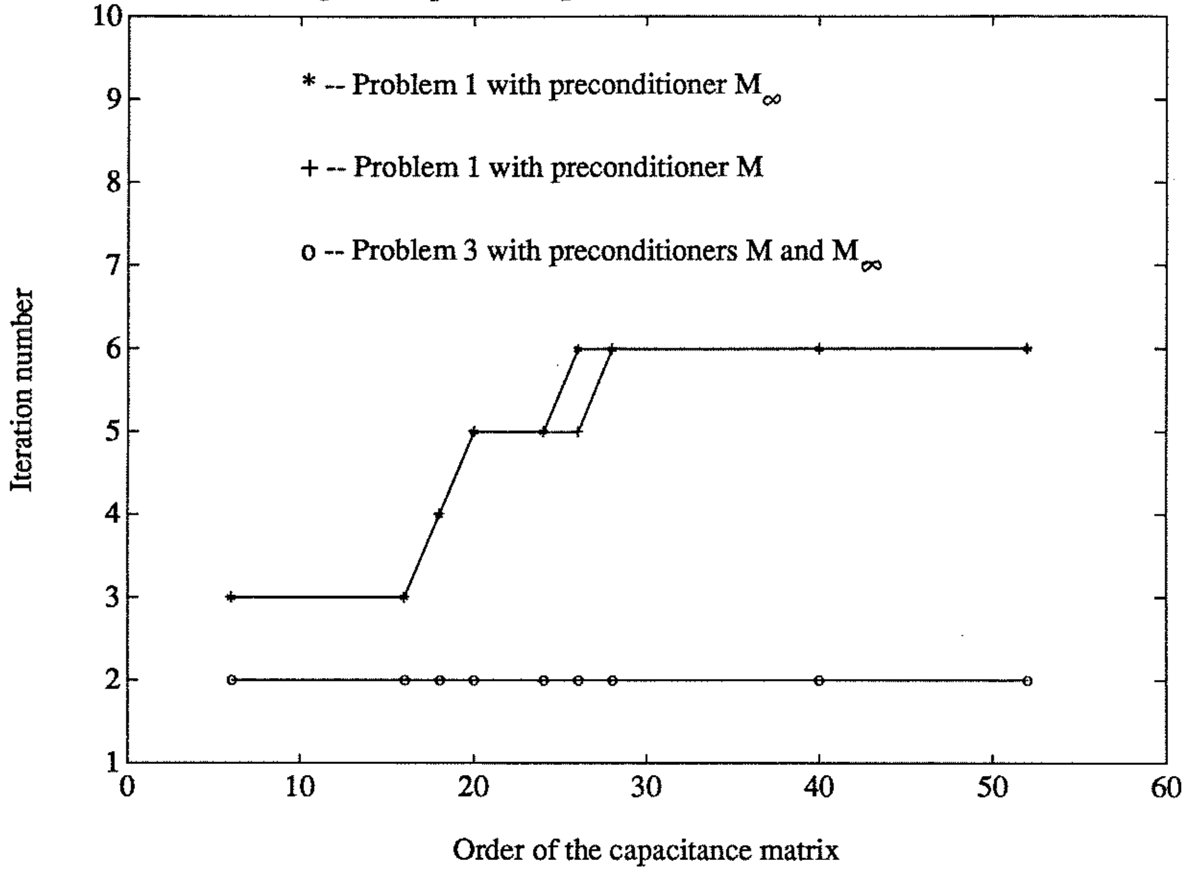

Fig.3 Comparison of M and M_d

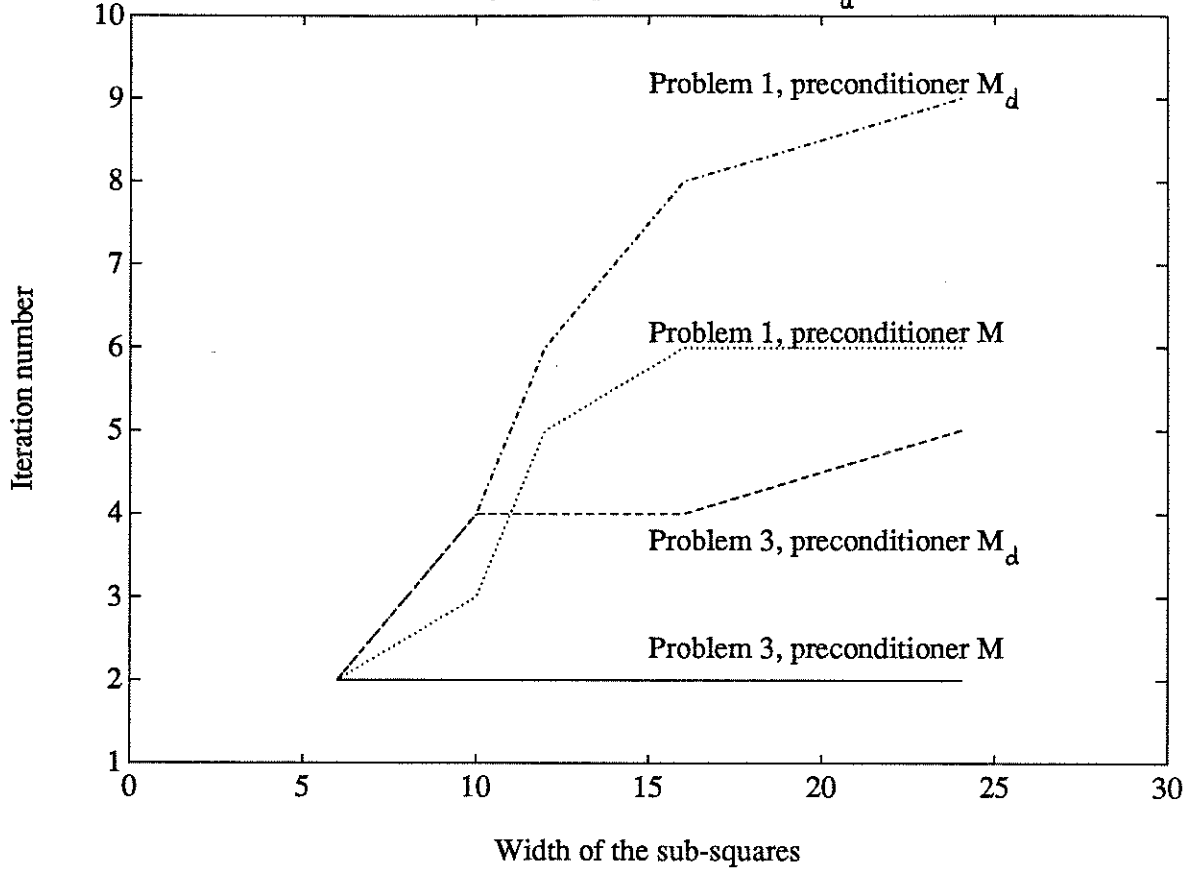

Fig. 4 Eigenvalue distribution for the preconditioned system $M^{-1}C$

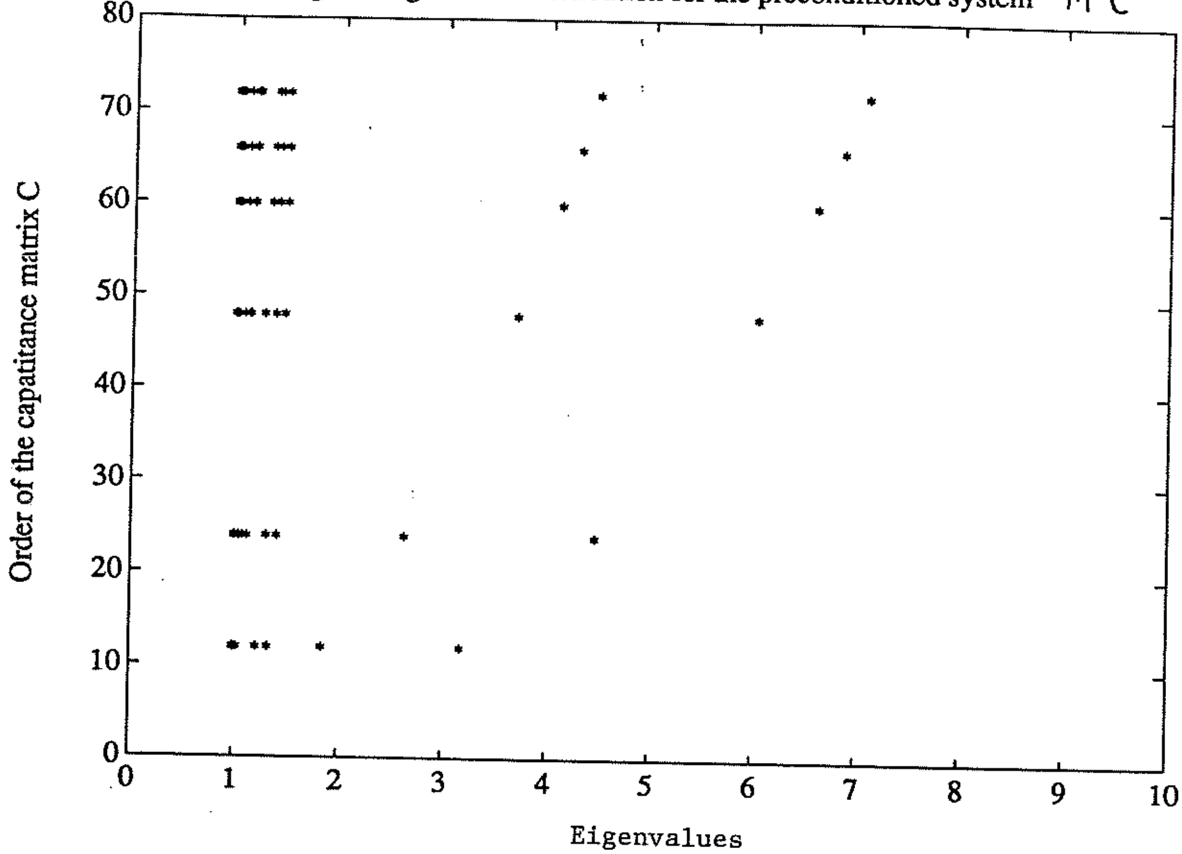

Fig. 5 Eigenvalue distribution for the preconditioned system $M_{\infty}^{-1}C$

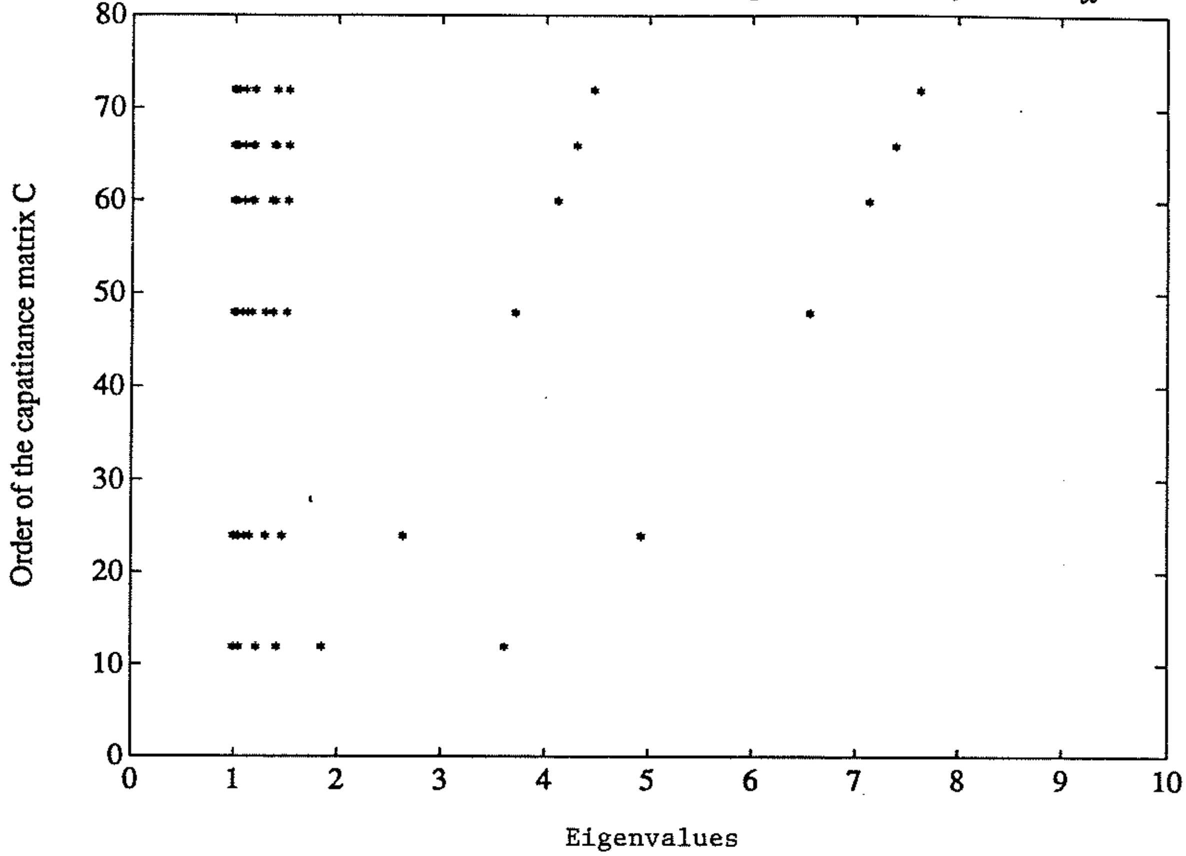

Fig. 6 Problem 1 over two rectanguler sub-domains

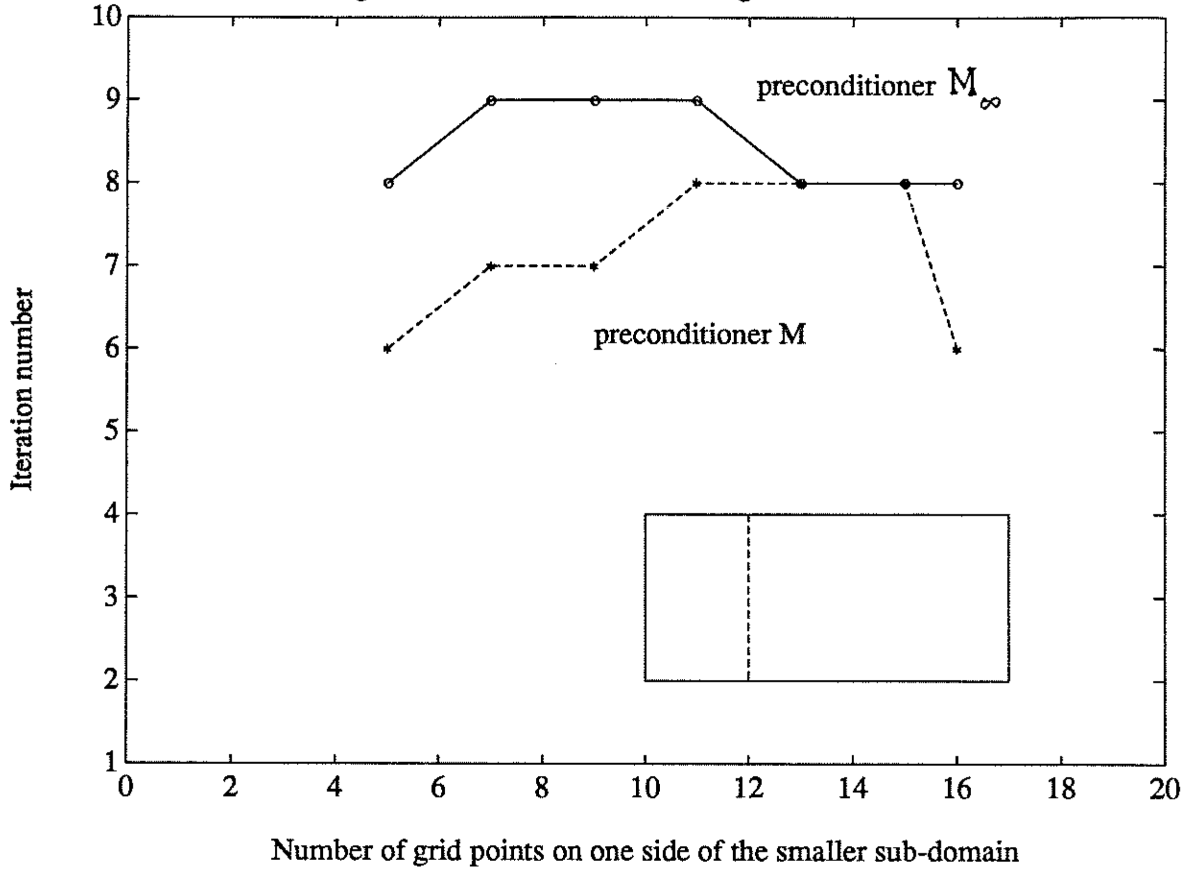

Fig. 7 PCG with preconditioner M for L-shaped domain