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Abstract. The aim of this paper is to present a spectral analysis of several block preconditioners : line SSOR (LSSOR), INV and MINV, for model elliptic problems in two dimensions. We use the Fourier analysis developed in [4] for point preconditioners corresponding to constant coefficient problems with periodic boundary conditions. This technique allows us to derive analytical expressions for the eigenvalues of the preconditioned system, from which various useful information can be obtained, such as the asymptotic behavior of the condition number K, the distribution of the eigenvalues and the estimation of optimal parameters. In particular, we prove that $K = O(h^{-2})$ for INV and $K = O(h^{-1})$ for MINV and LSSOR with appropriate choices of the parameters in the methods, where h is the mesh spacing. The Fourier analysis also identifies a "MINV condition", which characterizes the condition under which modifications of INV (of which MINV is just one possibility) give $K = O(h^{-1})$. We then provide extensive numerical results which show that the predictions from the Fourier analysis agree extremely well with the results for the corresponding problem with Dirichlet boundary conditions.

1. Introduction

In this paper we are primarily interested in considering the model problem

$$-\Delta u = f \quad \text{in } \Omega =]0,1[\text{ x }]0,1[$$

with Dirichlet boundary conditions

$$u|_{\partial\Omega} = 0$$

Using the standard five point finite difference stencil with a stepsize $h = \frac{1}{n+1}$, this gives a symmetric linear system of order n^2,

$$A_D \, x \; = \; b,$$

with

$$A_D = \begin{pmatrix} T_D & -I & & & \\ -I & T_D & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & T_D & -I \\ & & & -I & T_D \end{pmatrix} \quad \text{and} \quad T_D = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ & & & -1 & 4 \end{pmatrix}.$$

We are solving this problem either with a descent method or with the conjugate gradient method (CG). It is well known [2,7] that the rate of convergence of both methods depends on the condition number of the matrix $\kappa(A_D) = \lambda_{max}/\lambda_{min}$, where λ_{max} (resp. λ_{min}) is the largest (resp. smallest) eigenvalue of A_D. Moreover, the CG convergence rate is dependent on the eigenvalue distribution. This is why these methods are usually applied, not directly to the linear system to be solved, but to the symmetric form of the following preconditioned system

$$M_D^{-1} A_D x = M_D^{-1} b.$$

The matrix M_D is called the preconditioner and is in general chosen heuristically to get a condition number $\kappa(M_D^{-1} A_D)$ smaller than $\kappa(A_D)$ and a clustering of the eigenvalues around 1.

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While there are some theoretical results for bounding the condition number of the preconditioned system as a function of n, very few theoretical results are known for the eigenvalue distributions, even for problems as simple as the model problem. Therefore, it is interesting to try to gain some insights in this area; this could help in understanding the convergence properties and determining the optimal values of parameters which appear in some methods.

Our main goal is to analyze some block preconditioners which have been proposed during the last several years [1,5,6]. We derive the eigenvalue distributions using the Fourier analysis technique developed in [4]. In particular, we prove that $K = O(h^{-2})$ for INV and $K = O(h^{-1})$ for MINV and LSSOR with appropriate choices of the parameters in the methods. We note that these bounds on the condition numbers and the eigenvalue distributions were not previously known. The Fourier analysis also leads us to a "MINV condition", which characterizes the condition under which modifications of INV (of which MINV is just one possibility) give $K = O(h^{-1})$. Finally, we provide extensive numerical results which show that the predictions from the Fourier analysis agree extremely well with the results for the corresponding problem with Dirichlet boundary conditions.

The outline of the paper is as follows. Section 2 introduces the Fourier analysis technique. As a first example, in Section 3 we consider the LSSOR preconditioner, giving an "optimal" value for the ω parameter. Section 4 and 5 derive results for two very efficient preconditioners INV and MINV which have been heavily used recently (for Dirichlet boundary conditions). Finally, in Section 6, we present numerical results which show the Fourier results agree extremely well with the corresponding results for the problem with Dirichlet boundary conditions. Technical proofs are given in Appendices A–D.

2. Fourier Analysis Framework

Fourier analysis has been used quite extensively in the past to analyze the performance of discrete numerical methods. The main idea is to compute the effect of a discrete difference operator on the Fourier modes (i.e. to compute its *symbol*.) If the difference operator is periodic and has constant coefficients, then its eigenfunctions are often precisely these Fourier modes and its symbol can be easily computed and analyzed algebraically. Examples are the classical von Neumann stability analysis for difference schemes for time dependent PDEs [8] and the smoothing rate analysis in multigrid methods [3].

Rather surprisingly, the use of Fourier analysis for the study of iterative methods for *elliptic* problems has not been wide-spread, especially for the analysis of preconditioners. Part of this reason is that many preconditioners (such as the incomplete factorization methods [2]) for non-periodic problems do not have constant coefficients. However, as is well-known the coefficient often approach constant values in the interior of the domain away from the boundaries. These asymptotic values can often be computed easily by considering the corresponding periodic problem. It is thus natural to expect the Fourier analysis to produce meaningful results for these preconditioners as well. In [4], this approach is applied to many of the *point* incomplete factorization preconditioners (such as ILU, MILU, RILU, SSOR and ADDKR), as well as to the classical stationary iterative methods (such as Jacobi, Gauss Seidel, SOR and SSOR). In all of these cases, the Fourier analysis reproduces the classical convergence results. Moreover, it reveals much more details (such as the eigenvalue distribution) about these methods than was known before. In this paper, we apply the Fourier analysis to study several block preconditioners. For some of these, even classical convergence theory is lacking and therefore the Fourier analysis allow us for the first time to understand their performance, especially in terms of the distribution of eigenvalues of the preconditioned system and the determination of optimal parameters.

We now illustrate the Fourier technique by applying it to several basic difference operators which we shall employ later. Consider the unit square in 2D and a uniform grid with n interior grid points in both the x and y directions. The Fourier modes on this grid are grid functions $u^{(s,t)}$ whose (j,k)-th component is given by:

$$(u^{(s,t)})_{j,k} = e^{ij\theta_s} e^{ik\phi_t}$$

where $\theta_s = \frac{2\pi s}{n+1}$ and $\phi_t = \frac{2\pi t}{n+1}$.

Consider as an example the difference operator $T_1(\alpha, \beta)$ defined by:

$$(T_1(\alpha,\beta)v)_{j,k} = -\beta v_{j,k-1} + \alpha v_{j,k} - \beta v_{j,k+1}.$$

The Fourier transform $F(T_1)$ (i.e. its symbol) can be computed by applying T_1 to the Fourier mode $u^{(s,t)}$ given earlier and we get:

$$T_1 u^{(s,t)} = \beta e^{ij\theta_s}(e^{i(k-1)\phi_t} + e^{i(k+1)\phi_t}) + \alpha e^{ij\theta_s} e^{ik\phi_t},$$

which after some simplifications yields:

$$T_1 u^{(s,t)} = ((\alpha - 2\beta) + 4\beta \sin^2(\phi_t/2)) u^{(s,t)}.$$

Thus,

$$F(T_1(\alpha,\beta)) = (\alpha - 2\beta) + 4\beta \sin^2(\phi_t/2).$$

Similarly, the Fourier transform of the following operators:

$$(T_2(\alpha,\beta,\gamma)v)_{j,k} = -\beta v_{j,k-1} + \alpha v_{j,k} - \beta v_{j,k+1} - \gamma v_{j+1,k},$$

$$(T_3(\alpha,\beta,\gamma)v)_{j,k} = -\beta v_{j,k-1} + \alpha v_{j,k} - \beta v_{j,k+1} - \gamma v_{j-1,k},$$

are given by:

$$F(T_2(\alpha,\beta,\gamma)) = (\alpha - 2\beta) + 4\beta \sin^2(\phi_t/2) - \gamma e^{i\theta_s}.$$

$$F(T_3(\alpha,\beta,\gamma)) = (\alpha - 2\beta) + 4\beta \sin^2(\phi_t/2) - \gamma e^{-i\theta_s}.$$

Finally, the Fourier transform of the standard 5-point discrete Laplacian A_P is given by:

$$F(A_P) = 4(\sin^2(\theta_s/2) + \sin^2(\phi_t/2)).$$

We also note the Fourier transform of a product (composition) of operators is the product of the transforms of the individual factors and the transform of the inverse of an operator is the inverse of the transform.

A good way to view the Fourier technique is to think of the difference operators as constant coefficient operators with periodic boundary conditions, which are diagonalizable by the Fourier transform. In matrix form, these difference operators are represented by order $(n+1)^2$ block circulant matrices. For example,

$$A_P = \begin{pmatrix} T & -I & & & -I \\ -I & T & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & T & -I \\ -I & & & -I & T \end{pmatrix}, \quad \text{with } T = \begin{pmatrix} 4 & -1 & & & -1 \\ -1 & 4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 4 & -1 \\ -1 & & & -1 & 4 \end{pmatrix}.$$

For further reference, we will denote

$$L_P = \begin{pmatrix} 0 & & & -I \\ -I & 0 & & \\ & \ddots & \ddots & \\ & & -I & 0 \end{pmatrix},$$

and by D_P the block diagonal of A_P. The matrix A_P is singular, as the solution of the problem with periodic boundary conditions is only defined up to a constant. Following the approach in [4], we will ignore the zero eigenvalue for our purposes.

3. LSSOR

The first block preconditioner we consider is the LSSOR method [1]. The preconditioning matrix M for this method is given by

$$M = \frac{1}{\omega(2-\omega)} \, (\Delta + \omega L_P) \, \Delta^{-1} \, (\Delta + \omega L_P^T),$$

with

$$\Delta = \begin{pmatrix} T & & & \\ & \ddots & & \\ & & \ddots & \\ & & & T \end{pmatrix},$$

ω being a parameter between 1 and 2.

In terms of the difference operators defined in Section 2,

$$M = \frac{1}{\omega(2-\omega)} T_3(4,1,\omega) T_1^{-1}(4,1) T_2(4,1,\omega).$$

Using the techniques of Section 2, we can compute the eigenvalues μ_{st} of $M^{-1} A_P$ as:

$$\begin{aligned}\mu_{st} &= \frac{\omega(2-\omega)F(T_1(4,1))F(A_P)}{F(T_3(4,1,\omega))F(T_2(4,1,\omega))} \\ &= \frac{4\omega(2-\omega)(\sin^2(\theta_s/2)+\sin^2(\phi_t/2))(4\sin^2(\theta_s/2)+2)}{(4\sin^2(\theta_s/2)+2-\omega e^{i\phi_t})(4\sin^2(\theta_s/2)+2-\omega e^{-i\phi_t})}\end{aligned}$$

which "simplifies" to

$$\mu_{st} = \frac{4\omega(2-\omega)(\sin^2(\theta_s/2)+\sin^2(\phi_t/2))(4\sin^2(\theta_s/2)+2)}{(4\sin^2(\theta_s/2)+2)^2+\omega^2-2\omega\cos(\phi_t)(4\sin^2(\theta_s/2)+2)},$$

where

$$\theta_s = 2\pi sh, s=1,\ldots,n; \quad \phi_t = 2\pi th, t=1,\ldots,n; \quad h=\frac{1}{n+1}; \quad \omega\in[1,2).$$

Using these values, it can be seen, for instance, that for $\omega = 1$, $\kappa(M^{-1}A_P) \geq O(h^{-2})$, but we have:

Theorem 3.1 There exists an "optimal" value $\omega^* = 2 - 2\sqrt{2}\pi h + O(h^2)$ of ω for which

$$\kappa(M^{-1}A_P) \leq O(h^{-1}).$$

Proof : This result is proven in Appendix A. ◇

In Figures 3.1, 3.2 we plot the eigenvalues μ_{st} as a function of θ_s and ϕ_t for $\omega = 1$ and $\omega = 1.74$ respectively, both for $n = 40$. Figure 3.3 gives the condition number $\kappa(M^{-1} A_P)$ as a function of ω from which it can be seen that the optimal value of ω is close to 1.74. From Figure 3.4, it is verified that κ grows linearly with n for the optimal ω and like n^2 for other values of ω as predicted by the theory.

4. INV

The motivation behind this preconditioner comes, for Dirichlet boundary conditions, from the complete block Choleski decomposition [5].

Here, with periodic boundary conditions, A_P can be decomposed as

$$A_P = (\Delta + L_P)\, \Delta^{-1}\, (\Delta + L_P^T),$$

with

$$\Delta = \begin{pmatrix} \Sigma & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \Sigma \end{pmatrix},$$

and the diagonal blocks Σ are defined by the following equation:

$$\Sigma = T - \Sigma^{-1}.$$

The matrix Σ is in general dense. The main idea of the INV preconditioner is to approximate the inverse that appear in the formula above by sparse matrices to get an incomplete decomposition. In the special case we consider, we will use tridiagonal circulant matrices. To derive this approximation, we need a few preliminary results.

Theorem 4.1 Let $\gamma > 2$ and

$$S = \begin{pmatrix} \gamma & -1 & & & -1 \\ -1 & \gamma & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & \gamma & -1 \\ -1 & & & -1 & \gamma \end{pmatrix},$$

be a symmetric tridiagonal circulant matrix. Then

a)

$$S = L_S \, D_S^{-1} \, L_S^T,$$

where

$$D_S = \begin{pmatrix} d & & & \\ & d & & \\ & & \ddots & \\ & & & d \end{pmatrix}, \quad L_S = \begin{pmatrix} d & & & -1 \\ -1 & d & & \\ & \ddots & \ddots & \\ & & -1 & d \end{pmatrix},$$

with $d = \frac{\gamma + \sqrt{\gamma^2 - 4}}{2}$.

b) S^{-1} is a symmetric circulant matrix.

c) The elements of the main diagonal of S^{-1} are $\chi = \frac{d}{d^2-1}\left[\frac{d^n+1}{d^n-1}\right]$.

The second diagonal of S^{-1} is given by $\xi = \frac{1}{2}(\gamma\chi - 1) = \frac{1}{d^2-1}\left[1 + \gamma\frac{d}{d^n-1}\right]$.

Proof : see Appendix B. $\diamondsuit$

From these results we can get the desired approximation.

Definition 4.2

$$Ttrid(S^{-1}) = \begin{pmatrix} \chi & \xi & & & \xi \\ \xi & \chi & \xi & & \\ & \ddots & \ddots & \ddots & \\ & & \xi & \chi & \xi \\ \xi & & & \xi & \chi \end{pmatrix},$$

with the values of χ and ξ given in Theorem 4.1. $\diamondsuit$

This definition can be extended by scaling to a more general tridiagonal circulant matrix. Let

$$S(\alpha, \beta) = \begin{pmatrix} \alpha & -\beta & & & -\beta \\ -\beta & \alpha & -\beta & & \\ & \ddots & \ddots & \ddots & \\ & & -\beta & \alpha & -\beta \\ -\beta & & & -\beta & \alpha \end{pmatrix}.$$

with $\alpha > 2\beta$.

Definition 4.3 Define the operator $Ttrid$, which approximates the inverse of a circulant tridiagonal matrix $S(\alpha, \beta)$ by a circulant tridiagonal matrix $S(\chi, \xi)$, by:

$$Ttrid(S^{-1}(\alpha, \beta)) \equiv S(\chi, \xi),$$

where

$$\chi = \frac{d}{\beta(d^2-1)}\left[\frac{d^n+1}{d^n-1}\right],$$

$$\xi = \frac{1}{\beta(d^2-1)}\left[1 + \frac{\alpha}{\beta}\frac{d}{d^n-1}\right],$$

and where

$$d = \frac{\alpha + \sqrt{\alpha^2 - 4\beta^2}}{2\beta}.$$

$\diamond$

Now we can define the INV preconditioner as :

$$M = (\Delta + L_P)\ \Delta^{-1}\ (\Delta + L_P^T),$$

with

$$\Delta = \begin{pmatrix} \Lambda & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \Lambda \end{pmatrix},$$

and Λ is a tridiagonal circulant matrix satisfying the following equation :

$$\Lambda = T - Ttrid(\Lambda^{-1}).$$

With the definition of $Ttrid$, this equation can be solved for Λ.

Theorem 4.2 The INV preconditioner is given by

$$M = (\Delta + L_P)\ \Delta^{-1}\ (\Delta + L_P^T) \equiv T_3(\alpha,\beta,1)T_1^{-1}(\alpha,\beta)T_2(\alpha,\beta,1),$$

where $\Lambda = S(\alpha,\beta)$, with

$$\alpha = 4 - \frac{d}{\beta(d^2-1)}\left[\frac{d^n+1}{d^n-1}\right],$$

$$\beta = 1 + \frac{1}{\beta(d^2-1)}\left[1 + \frac{\alpha}{\beta}\frac{d}{d^n-1}\right],$$

and where

$$d = \frac{\alpha + \sqrt{\alpha^2 - 4\beta^2}}{2\beta}.$$

$\diamond$

Using the Fourier technique of Section 2, we can compute the eigenvalues μ_{st} of $M^{-1}A_P$ as a function of α and β.

Theorem 4.3 The eigenvalues of INV are given by:

$$\mu_{st} = \frac{4\beta(\sin^2(\theta_s/2) + \sin^2(\phi_t/2))(4\sin^2(\theta_s/2) + \zeta)}{(\beta[4\sin^2(\theta_s/2) + \zeta] - e^{i\phi_t})(\beta[4\sin^2(\theta_s/2) + \zeta] - e^{-i\phi_t})}.$$

with $\zeta = \alpha/\beta - 2$, which simplifies to:

$$\mu_{st} = \frac{4\eta(\sin^2(\theta_s/2) + \sin^2(\phi_t/2))}{\eta^2 + 1 - 2\eta\cos(\phi_t)},$$

where $\eta = \beta(4\sin^2(\theta_s/2) + \zeta)$. $\diamond$

It can be shown (c.f. Appendix C) that $4 > \alpha > 2\beta > 2$. This means that as $n \to \infty$ (i.e. $h \to 0$), $\alpha/\beta = O(1)$.

From this, we can deduce :

Theorem 4.4 For the INV preconditioner,

$$\kappa(M^{-1}A_P) \geq O(h^{-2}).$$

Proof: Consider first μ_{st} for $\theta_s = \phi_t = \pi$, then $\eta = \beta(4+\zeta) = O(1)$;

$$\mu_{st} = \frac{\eta}{\eta^2 + 2\eta + 1} = O(1).$$

Hence, $\lambda_{max} \geq O(1)$. Now, consider $\theta_s = \phi_t = 2\pi h$, then $\eta = O(1)$, $\cos(\phi_t) = O(1)$ and $\mu_{st} = O(h^2)$. Therefore $\lambda_{min} \leq O(h^2)$. This implies $\kappa(M^{-1}A_P) \geq O(h^{-2})$. ◇

This result was first observed in numerical results for Dirichlet boundary conditions in [5].

As we are mainly interested in Dirichlet boundary conditions, it is useful to consider the asymptotic case, when $n \rightarrow \infty$, as this reflects what happens inside the domain, far enough from the boundaries. Then, the values for α and β we obtain are the same as for Dirichlet boundary conditions. The limiting values of α and β are solution of the equations

$$\alpha = 4 - \frac{d}{\beta(d^2-1)},$$

$$\beta = 1 + \frac{1}{\beta(d^2-1)}.$$

Numerically, it can be computed that $\alpha = 3.6539$ and $\beta = 1.1138$.

In Figure 4.1, we plot the eigenvalues μ_{st} as a function of θ_s and ϕ_t for $n = 40$. Figure 4.2 gives the condition number $\kappa(M^{-1}\ A_P)$ as a function of n from which it can be seen that κ grows like n^2 as predicted by the theory.

5. MINV

The MINV preconditioner has been introduced in [5] for Dirichlet boundary conditions. Here, we derive it for periodic boundary conditions.

It is expressed in the same way as for INV; the only difference being that the values of diagonal elements of M are modified such that the sum of the elements of each row is equal to the sum of the elements of the corresponding row of A_P (plus possibly an $O(h^2)$ term). Hence, to be able to define the MINV preconditioner, we must first compute the row sums of $S^{-1}(\alpha,\beta)$ where $S(\alpha,\beta)$ has been defined in the previous section.

Lemma 5.1 Let $r(S^{-1})$ be the vector of row sums of S^{-1} and $e = (1,1,\ldots,1)^T$. Then

$$r(S^{-1}(\alpha,\beta)) = \frac{1}{\alpha - 2\beta}\ e.$$

Proof : It is obvious that $r(S^{-1}) = S^{-1}\ e$; but

$$Se = (\alpha - 2\beta)\ e,$$

and hence

$$S^{-1}e = \frac{1}{\alpha - 2\beta}\ e.$$

◇

Now we can define the preconditioner as for INV; except now we must compute Λ such that: (i) Δ is an approximation of the corresponding matrix in the complete decomposition and (ii) our row sum criterion is satisfied.

First, we compute the remainder R_P, defined by:

$$M = A_P + R_P.$$

By direct computation,

$$M = \Delta + L_P + L_P^T + L_P\Delta^{-1}L_P^T = \Delta - D_P + L_P\Delta^{-1}L_P^T + A_P.$$

It is easy to see that $L_P\Delta^{-1}L_P^T$ is a block diagonal matrix with all the blocks equal to Λ^{-1}; hence R_P is also block diagonal with blocks R whose value is

$$R = \Lambda - T + \Lambda^{-1}.$$

Let $\Lambda = S(\alpha, \beta)$ and $Mtrid(\Lambda^{-1})$ be the circulant tridiagonal approximation we are going to use for Λ^{-1}. Analogous to INV, Λ is defined by

$$\Lambda = T - Mtrid(\Lambda^{-1}),$$

and therefore we have

$$R = \Lambda^{-1} - Mtrid(\Lambda^{-1}).$$

The row sum condition we require is

$$r(R) = ch^2,$$

where c is a constant independent of h. This implies that:

$$r(Mtrid(\Lambda^{-1})) = r(\Lambda^{-1}) - ch^2 = \frac{1}{\alpha - 2\beta} - ch^2.$$

Therefore, using the approximation $Ttrid$ of Λ^{-1} in Definition 4.3 and modifying its diagonal to satisfy the row sum condition above, we arrive at the following approximation of the inverse of a circulant tridiagonal matrix.

Definition 5.2 Define the operator $Mtrid$, which approximates the inverse of the circulant tridiagonal matrix $S(\alpha, \beta)$ by a circulant tridiagonal matrix $S(\chi, \xi)$, by:

$$Mtrid(S^{-1}(\alpha, \beta)) \equiv S(\chi, \xi)$$

with

$$\chi = \frac{1}{\alpha - 2\beta} - \frac{2}{\beta(d^2-1)}\left[1 + \frac{\alpha}{\beta}\frac{d}{d^n - 1}\right] - ch^2,$$

$$\xi = \frac{1}{\beta(d^2-1)}\left[1 + \frac{\alpha}{\beta}\frac{d}{d^n - 1}\right],$$

and

$$d = \frac{\alpha + \sqrt{\alpha^2 - 4\beta^2}}{2\beta}.$$

◇

The condition $\Lambda = T - Mtrid(\Lambda^{-1})$ now gives a set of nonlinear equations for α and β.

Theorem 5.3 The MINV preconditioner is given by

$$M = (\Delta + L_P)\Delta^{-1}(\Delta + L_P^T) \equiv T_3(\alpha, \beta, 1)T_1^{-1}(\alpha, \beta)T_2(\alpha, \beta, 1)$$

where $\Lambda = S(\alpha, \beta)$ and α and β are the solutions of the following equations:

$$\alpha = 4 + \frac{2}{\beta(d^2-1)}\left[1 + \frac{\alpha}{\beta}\frac{d}{d^n - 1}\right] - \frac{1}{\alpha - 2\beta} + ch^2,$$

$$\beta = 1 + \frac{1}{\beta(d^2-1)}\left[1 + \frac{\alpha}{\beta}\frac{d}{d^n - 1}\right],$$

where

$$d = \frac{\alpha + \sqrt{\alpha^2 - 4\beta^2}}{2\beta}.$$

$\diamondsuit$

As before the limiting values of α and β when $n \to \infty$ can be found numerically; for example, $\alpha = 3.3431$ and $\beta = 1.1715$ when $c = 0$. However, even without knowledge of the actual values of α and β, we can deduce the following:

Theorem 5.4 For any n, the α and β computed by the MINV preconditioner satisfy the "MINV condition":

$$(\alpha - 2\beta - 1)^2 = (\alpha - 2\beta)ch^2.$$

Proof : From Theorem 5.3, we have

$$\alpha - 2\beta = 2 - \frac{1}{\alpha - 2\beta} + ch^2.$$

Define $\nu = \alpha - 2\beta$, then

$$\nu^2 = (2 + ch^2)\nu - 1$$

and therefore

$$(\nu - 1)^2 = \nu^2 - 2\nu + 1 = (2 + ch^2)\nu - 2\nu = \nu ch^2.$$

$\diamondsuit$

It should be emphasized that this "MINV condition" arises independently of the approximation we choose for Λ^{-1} , as long as the row sum criterion is satisfied.

The expression for the eigenvalues μ_{st} of $M^{-1}A_P$ is exactly the same as for INV, the only difference being the actual values of α and β. So we have:

$$\mu_{st} = \frac{4\eta(\sin^2(\theta_s/2) + \sin^2(\phi_t/2))}{\eta^2 + 1 - 2\eta\cos(\phi_t)},$$

$$\eta = \beta(4\sin^2(\theta_s/2) + \zeta), \quad \zeta = \frac{\alpha}{\beta} - 2.$$

The main result of this section is:

Theorem 5.5 Let the MINV preconditioner be as defined in Theorem 5.3, with α, β being any values that satisfy the "MINV condition"

$$(\alpha - 2\beta - 1)^2 = (\alpha - 2\beta)ch^2,$$

then

$$\kappa(M^{-1}A_P) \leq O(h^{-1}).$$

Proof : The proof is given in Appendix D. $\diamondsuit$

We note that the bound on the condition number holds for any value of $c \geq 0$, which is different than the analogous situation with MILU whose behaviour depends on whether $c = 0$ or $c > 0$. The important condition is the "MINV" condition, not the value of c, nor the values of α and β.

In Figures 5.1, 5.2 we plot the eigenvalues μ_{st} as a function of θ_s and ϕ_t for $c = 0$ and $c = 30$ (close to the optimal value) respectively, both for $n = 41$. Figure 4.3 gives the condition number $\kappa(M^{-1}\ A_P)$ as a function of c from which it can be seen that the optimal value of c is close to 30. From Figure 5.4, it is verified that κ grows linearly with n for $c = 0$ as predicted by the theory.

In closing this section, we would like to explain why, with the same expression for the eigenvalues, the result for MINV is so much different from the one for INV. The $O(h^{-2})$ behaviour for INV comes from the numerator of μ_{st}, as the denominator is $O(1)$. In MINV, because of the "MINV condition", there is a cancellation in the denominator, which gives the $O(h^{-1})$ condition number. It should be stressed that the proof of Theorem 5.5 is independent of the actual values of α and β and only uses the "MINV condition"; this means we can use other approximations for α and β and still retains the $O(h^{-1})$ condition number, although the eigenvalue distribution is dependent on α and β.

6. Comparison with the Dirichlet Problem

In this section, we compare the Fourier results with the actual numerical results for the *Dirichlet* problem. To do this, we follow [4] and use the Fourier eigenvalues with $h_p = h_d/2$ (or equivalently $n_p = 2n_d + 1$) to predict the corresponding eigenvalues of the Dirichlet problem with grid size h_d. While this correspondence is exact for only a few simple cases, it has proven to be a very good heuristic and excellent agreements were obtained for the problems treated in [4].

We consider the Dirichlet problem with the right hand side of the discrete problem chosen such that the exact solution of the discrete problem is given by:

$$u = xy(1-x)(1-y)e^{xy}$$

on a uniformly spaced n_d by n_d grid. We solve the discrete problem by the conjugate gradient method preconditioned by the Dirichlet versions of the three block preconditioners using zero as initial guess. The largest and smallest eigenvalues of the preconditioned system are estimated by information collected during the conjugate gradient iteration using the well-known correspondence between CG and the Lanczos process [7]. The CG iterations are stopped when these estimates have converged.

We first present results for the LSSOR preconditioner. In Fig. 6.1 and 6.2, we compare the condition number $\kappa(M^{-1}A)$ and the maximum and minimum eigenvalues of the preconditioned system for the Dirichlet problem and the corresponding values predicted by the Fourier analysis, for $\omega = 1$ and for a range of values of n. In Fig. 6.3 and 6.4, similar comparisons are presented for the optimal value of ω, given by the formula $\omega_{opt}^{(d)} = 2 - \sqrt{2}\pi h_d$, which corresponds to the formula given in Theorem 3.1, but with h there replaced by $\frac{h_d}{2}$ for the Dirichlet problem. In Fig. 6.5 and 6.6, similar results are presented with ω varying and $n_d = 20$ (corresponding to $n_p = 41$.) We note in all cases the agreements are quite good. In general, the quality of agreement is not sensitive to n, but more sensitive when ω is near its optimal value, where disageement in the maximum eigenvalue becomes noticeable.

We next present results for the INV preconditioner. In Figs. 6.7 and 6.8, we compare the condition number $\kappa(M^{-1}A)$ and the maximum and minimum eigenvalues of the preconditioned system for the Dirichlet problem and the corresponding values predicted by the Fourier analysis, for a range of values of n. The agreement is excellent.

Finally, we present similar comparisons for the MINV preconditioner. In Fig. 6.9 and 6.10, we compare the condition number $\kappa(M^{-1}A)$ and the maximum and minimum eigenvalues of the preconditioned system for the Dirichlet problem and the corresponding values predicted by the Fourier analysis, for $c = 0$ and for a range of values of n. In Fig. 6.11 and 6.12, similar comparisons are presented for the optimal value of $c_d = 7$. In Fig. 6.13 and 6.14, similar results are presented with c varying and $n_d = 20$ (corresponding to $n_p = 41$.) We note in all cases the agreements are quite good, but less well for the case $c = 0$, where disagreement in the maximum eigenvalue and the condition number becomes very noticeable. Even with this disagreement, the optimal value of c as predicted by the Fourier analysis is rather close to the optimal value for the Dirichlet problem. This situation is similar to that for the MILU preconditioner [4]. Again, the important difference is that for MINV both the Fourier analysis and the actual performance indicates that $\kappa(M^{-1}A) = O(h^{-1})$ for $c = 0$.

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Appendix A: Proof of Theorem 3.1

We derive upper and lower bounds for the eigenvalues of the LSSOR preconditioned matrix. This allows us to compute an "optimal" value of ω which minimizes an upper bound of the condition number.

The LSSOR eigenvalues are :

$$\mu_{st} = \frac{4\omega(2-\omega)(\sin^2(\theta_s/2)+\sin^2(\phi_t/2))(4\sin^2(\theta_s/2)+2)}{(4\sin^2(\theta_s/2)+2)^2+\omega^2-2\omega\cos(\phi_t)(4\sin^2(\theta_s/2)+2)},$$

with

$$\theta_s = 2\pi sh, s=1,\ldots,n; \quad \phi_t = 2\pi th, t=1,\ldots,n; \quad h=\frac{1}{n+1}; \quad \omega\in[1,2[.$$

Let $x=\sin^2(\theta_s/2)$, $y=\sin^2(\phi_t/2)$. Clearly x and y take values in $[\sin^2(\pi h), \sin^2(\frac{\pi n}{2(n+1)})]$, but for convenience, we will sometimes consider $x,y\in[0,1]$, excluding the zero eigenvalue.

We have

$$\mu_{st} = \mu(x,y) = \frac{4\omega(2-\omega)(x+y)(4x+2)}{(4x+2)^2+\omega^2-2\omega(4x+2)(1-2y)}.$$

Our aim is to find lower and upper bounds for $\mu(x,y)$.

Lemma A.1 For a fixed value of x, $\mu(x,y)$ is either increasing or decreasing as a function of y, so the minimum and maximum of μ are located on the boundaries with constant y in the x,y plane.

Proof : Let $D=(4x+2)^2+\omega^2-2\omega(4x+2)(1-2y)$. A little algebra shows that, for fixed x,

$$\frac{\partial\mu}{\partial y} = \frac{4\omega(2-\omega)(4x+2)}{D^2}[(4x+2)^2+\omega^2-2\omega(4x+2)(1+2x)].$$

So the sign of $\frac{\partial\mu}{\partial y}$ is the same as that of $(4x+2-\omega)^2-4\omega x$ which is independent of y. Therefore, to find an upper bound for μ, it is enough to look at values for $y=0$ and $y=1$. ◇

Lemma A.2

$$\mu(x,1)<1,$$
$$\mu(x,0)\leq 1.$$

Proof : At $y=1$, we have:

$$\mu(x,1) = \frac{4\omega(2-\omega)(1+x)(4x+2)}{(4x+2+\omega)^2}.$$

As $\omega\in[1,2[$, $\omega(2-\omega)$ is a decreasing function of ω and $4x+2+\omega$ is an increasing function of ω; and so $\mu(x,1)$ is a decreasing function of ω with x fixed. Therefore,

$$\mu(x,1)\leq\frac{4(1+x)(4x+2)}{(4x+3)^2}<1.$$

Now, at $y=0$, we have:

$$\mu(x,0)=\frac{4\omega(2-\omega)x(4x+2)}{(4x+2-\omega)^2}$$

and

$$\frac{\partial\mu}{\partial x}\Big|_{y=0} = \frac{8\omega(2-\omega)}{(4x+2-\omega)^3}[4(1-\omega)x+(2-\omega)].$$

But, $4x+2-\omega>0$, so the sign of $\frac{\partial\mu}{\partial x}|_{y=0}$ is the same as that of $4(1-\omega)x+(2-\omega)$.

If $\omega = 1$, $\frac{\partial \mu}{\partial x} |_{y=0} > 0$ and so $\mu(x,0)$ is increasing and

$$\mu(x,0) \leq \frac{24\omega(2-\omega)}{(6-\omega)^2} = \frac{24}{25} < 1.$$

If $\omega \neq 1$, there is a zero of $\frac{\partial \mu}{\partial x} |_{y=0}$ for $x = \bar{x} = \frac{2-\omega}{4(\omega-1)}$. Since at $x = 0$, $\frac{\partial \mu}{\partial x} |_{y=0} > 0$, the zero of the derivative corresponds to a maximum as long as $\bar{x} \in [0,1]$ i.e. for $\omega \geq \frac{6}{5}$.

For $\omega \in]1, \frac{6}{5}]$, the same argument as before applies; the maximum is given for $x = 1$ and $\mu \leq 1$.

For the remaining case, the maximum occurs for $x = \frac{2-\omega}{4(\omega-1)}$ and the minimum occurs at the boundaries.

Since $4(1-\omega)\bar{x} + 2 - \omega = 0$, we have $4\bar{x} + 2 = \omega(4\bar{x}+1)$. Therefore, since $\omega > \frac{6}{5}$, we have:

$$\begin{aligned}\mu(\bar{x},0) &= \frac{4\omega^2(2-\omega)\bar{x}(4\bar{x}+1)}{(4\omega\bar{x})^2} = \frac{(2-\omega)}{4}(4\bar{x}+1) \\ &= \frac{(2-\omega)}{4(\omega-1)} \leq \frac{5}{4}(2-\omega) \leq 1.\end{aligned}$$

It should be noted that there are values of x and ω for which μ is equal to 1 or at least $\mu = O(1)$. ♢

Now, we are interested in finding lower bounds of μ for $x, y, \in [\sin^2(\pi h), \sin^2(\frac{\pi}{2}\frac{n}{n+1})]$.

Lemma A.3 $\mu(x,1)$ is an increasing function of x.

Proof : A little algebra shows that:

$$\frac{\partial \mu(x,1)}{\partial x} = \frac{4\omega(2-\omega)}{4x+2+\omega}[8(\omega-1)x + 6\omega - 4].$$

As $\omega \geq 1$, $\mu(x,1)$ is an increasing function. ♢

To summarize at this point, we already know that:
(i) for $1 \leq \omega \leq \frac{6}{5}$, $\mu(x,0)$ is an increasing function
(ii) for $\frac{6}{5} < \omega < 2$, $\mu(x,0)$ is increasing and then decreasing.

As the partial derivatives are continuous functions of y, the same is true (for small enough h) for $y = \sin^2(\pi h)$ and $y = \sin^2(\frac{\pi}{2}\frac{n}{n+1}) = 1 - O(h^2)$.

In all cases, we only have three points in the x, y plane to look for the minimum of μ :

$$(\sin^2(\pi h), \sin^2(\pi h)), \left(\sin^2(\frac{\pi}{2}\frac{n}{n+1}), \sin^2(\pi h)\right) \text{ and } \left(\sin^2(\pi h), \sin^2(\frac{\pi}{2}\frac{n}{n+1})\right).$$

Lemma A.4 Let $x = \theta^2 = \sin^2(\pi h)$ and $\omega_0^2 = (\sqrt{5\theta^2+2} - \theta)^2 = 2 - \sqrt{2}\pi h + O(h^2)$.

If $\omega < \omega_0^2$, $\mu(\theta^2, y)$ is an increasing function of y, the minimum of $\mu(\theta^2, y)$ occurs for $y = \sin^2(\pi h)$.

If $\omega \geq \omega_0^2$, $\mu(\theta^2, y)$ is a decreasing function of y, the minimum of $\mu(\theta^2, y)$ occurs for $y = \sin^2(\frac{\pi}{2}\frac{n}{n+1})$ and a lower bound is obtained for $y = 1$.

Proof : As we have seen before, the sign of the derivative is the sign of

$$(4\theta^2 + 2 - \omega)^2 - 4\omega\theta^2 = (4\theta^2 + 2 - \omega - 2\theta\sqrt{\omega})(4\theta^2 + 2 - \omega + 2\theta\sqrt{\omega}).$$

So, it is also the sign of $4\theta^2 + 2 - \omega - 2\theta\sqrt{\omega}$. Let $\bar{\omega} = \sqrt{\omega}$, we look for the sign of $4\theta^2 + 2 - \bar{\omega}^2 - 2\theta\bar{\omega}$. Therefore, if $\omega < \omega_0^2$, the derivative is positive and it is negative elsewhere. ♢

Let us now distinguish between $x = \sin^2(\pi h)$ and $x = \sin^2(\frac{\pi}{2}\frac{n}{n+1})$.

Lemma A.5 For sufficiently small h,

$$\mu(\sin^2(\pi h), \sin^2(\pi h)) \leq \mu(\sin^2\left(\frac{\pi}{2}\frac{n}{n+1}\right), \sin^2(\pi h)).$$

Proof : A Taylor expansion of $\sin(\pi h)$ and $\sin(\frac{\pi}{2}\frac{n}{n+1})$ shows that the left hand side tends to 0 as $h \to 0$; as the right hand side is equal to

$$4\omega(2-\omega)\frac{6-4\pi^2h^2+O(h^4)}{(6-4\pi^2h^2)\omega^2-2\omega(6-4\pi^2h^2+O(h^4))(1-2\pi^2h^2+O(h^4))} = \frac{24\omega(2-\omega)}{(6-\omega)^2}+O(h^2).$$

◇

Hence, only two points remains as candidates for the minimum: $x = \sin^2(\pi h)$ and either $y = \sin^2(\pi h)$ or $y = \sin^2(\frac{\pi}{2}\frac{n}{n+1})$.

As the maximum value is bounded by 1, our final goal is to find a value of ω which maximizes the lower bound.

Lemma A.6 The lower bound of μ_{min} is maximized at $\omega = \omega^* \equiv \omega_0^2$.

Proof : We know that
(i) for $\omega < \omega_0^2$,

$$\mu_{min} = 4\omega(2-\omega)\frac{2\sin^2(\pi h)(4\sin^2(\pi h)+2)}{4\sin^2(\pi h)/2+4\omega\sin^2(\pi h)},$$

(ii) for $\omega \geq \omega_0^2$,

$$\begin{aligned}\mu_{min} &\geq 4\omega(2-\omega)\frac{(1+\sin^2(\pi h))(4\sin^2(\pi h)+2)}{(4\sin^2(\pi h)+2+\omega)^2} \\ &= 4\omega(2-\omega)\frac{2+O(h^2)}{(2+\omega)^2+O(h^2)}\end{aligned}.$$

For the latter, the lower bound for μ_{min} is a decreasing function of ω; so, its maximal value is given for $\omega = \omega_0^2$. For $\omega \leq \omega_0^2$, the lower bound for μ_{min} is also a decreasing function whose maximum value (for $\omega = 1$) tends to 0 as $h \to 0$; so, the value that maximizes the lower bound is $\omega^* = \omega_0^2$,

$$\omega^* = 2 - \sqrt{2}\pi h + O(h^2).$$

Then the lower bound is equal to

$$4(2-\sqrt{2}\pi h)2\sqrt{2}\pi h\frac{2+O(h^2)}{(4-2\sqrt{2}\pi h)^2+O(h^2)} = \frac{32\sqrt{2}\pi h+O(h^2)}{16-8\sqrt{2}\pi h+O(h^2)} = O(h).$$

◇

Appendix B: Proof of Theorem 4.1

a) Since

$$S = L_S\ D_S^{-1}\ L_S^T = \begin{pmatrix} d+\frac{1}{d} & -1 & & & -1 \\ -1 & d+\frac{1}{d} & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & d+\frac{1}{d} & -1 \\ -1 & & & -1 & d+\frac{1}{d} \end{pmatrix},$$

d satisfies the equation:

$$d + \frac{1}{d} = \gamma,$$

or

$$d^2 - \gamma d + 1 = 0,$$

the positive root of which is:

$$d = \frac{\gamma + \sqrt{\gamma^2 - 4}}{2}.$$

b) It is well known that the inverse of a symmetric circulant matrix is also symmetric circulant.

c) Now we must determine some elements of S^{-1}. The easiest way is to first compute L_S^{-1}, where

$$L_S = \begin{pmatrix} d & & & -1 \\ -1 & d & & \\ & \ddots & \ddots & \\ & & -1 & d \end{pmatrix}.$$

Let us solve $L_S x = (1, 0, \ldots, 0)^T$ to compute the first column of L_S^{-1}.

It is easily seen that $x_n = x_1/d^{n-1}$; as $dx_1 - x_n = 1$, we have

$$x_1 = \frac{d^{n-1}}{d^n - 1} \quad \text{and} \quad x_j = \frac{d^{n-j}}{d^n - 1}\ j = 2, \ldots, n.$$

More generally, for the i-th column of L_S^{-1}, we solve

$$L_S x = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T,$$

where the 1 is in the ith position.

As before, it can be seen that

$$x_{i-1} = \frac{1}{d^{i-1}} x_n \quad \text{and} \quad x_n = \frac{1}{d^{n-i}} x_i,$$

so $x_{i-1} = x_i/d^{n-1}$; as $dx_i - x_{i-1} = 1$, we have $x_{i-1} = 1/(d^n - 1)$.

Therefore

$$x_i = \frac{d^{n-1}}{d^n - 1}, \quad x_{i+j} = \frac{d^{n-j-1}}{d^n - 1}, \quad x_{i-j} = \frac{d^j}{d^n - 1}.$$

Hence, we explicitly know the inverse of L_S,

$$L_S^{-1} = \frac{1}{d^n - 1} \begin{pmatrix} d^{n-1} & 1 & d & \ldots & d^{n-2} \\ d^{n-2} & d^{n-1} & 1 & \ldots & d^{n-3} \\ d^{n-3} & d^{n-2} & d^{n-1} & \ldots & d^{n-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & d & d^2 & \ldots & d^{n-1} \end{pmatrix}.$$

From this, we can find the elements of $L_S^{-1} D_S^{-1} L_S^{-T}$ we need. For instance the $(1,1)$ entry is

$$\frac{1}{(d^n - 1)^2}(d^{2n-1} + d^{2n-3} + \ldots + d^3 + d) = \frac{1}{(d^n - 1)^2} \frac{d(d^{2n-1} - 1)}{d^2 - 1} = \frac{d}{d^2 - 1} \frac{d^n + 1}{d^n - 1}.$$

The $(1,2)$ term is

$$\frac{1}{(d^n-1)^2}\left(d^n+\frac{d^2}{d^2-1}(d^{2n-2}-1)\right).$$

This can also be expressed as

$$\xi=\frac{1}{2}(\gamma\chi-1)=\frac{1}{d^2-1}\left[1+\frac{\gamma d}{d^n-1}\right].$$

$\diamondsuit$

Appendix C

Let

$$\Lambda = \begin{pmatrix} \alpha & -\beta & & & -\beta \\ -\beta & \alpha & -\beta & & \\ & \ddots & \ddots & \ddots & \\ & & -\beta & \alpha & -\beta \\ -\beta & & & -\beta & \alpha \end{pmatrix},$$

satisfies the following equation:

$$\Lambda = T - Ttrid(\Lambda^{-1}),$$

where $Ttrid(\Lambda^{-1})$ is defined in Definition 4.3. We have the following result :

Theorem The values of α and β defined by INV satisfy:

$$4 > \alpha > 2\beta > 2.$$

Proof : We want to prove that Λ is strictly diagonally dominant. Let us define the following sequence

$$\Lambda_1 = T,$$

$$\Lambda_i = T - Ttrid(\Lambda_{i-1}^{-1}), \quad i = 2, \ldots$$

Since T is strictly diagonally dominant (sdd), so is Λ_1. Using the same technique as in [5], it can be proven by induction that Λ_i is also sdd. We are therefore exactly in the hypothesis as for Theorem 3 of [5] as for constructing $Ttrid$, we drop some elements of the true inverse. So, all Λ_i's are sdd and by continuity the limit is also. Moreover, it is obvious that $\alpha < 4$ and $\beta > 1$ as $d > 1$. ◊

Appendix D: Proof of Theorem 5.5

We recall that the eigenvalues are

$$\mu_{st} = \frac{4\eta(\sin^2(\theta_s/2) + \sin^2(\phi_t/2))}{\eta^2 + 1 - 2\eta\cos(\phi_t)},$$
$$\eta = \beta(4\sin^2(\theta_s/2) + \zeta), \quad \zeta = \frac{\alpha}{\beta} - 2$$

and that α and β satisfy the "MINV condition" : $(\alpha - 2\beta - 1)^2 = (\alpha - 2\beta)ch^2$.

Let $x = \sin^2(\theta_s/2)$, $y = \sin^2(\phi_t/2)$. Clearly, $C_0 h^2 \leq x \leq 1$, and $C_0 h^2 \leq y \leq 1$. (In this proof, C_i denotes a generic constant independent of h.) Now,

$$\cos(\phi_t) = 1 - 2\sin^2(\phi_t/2) = 1 - 2y \quad \text{and} \quad \eta = \beta(4x + \zeta) = 4\beta x + C_1;$$

so,

$$\mu(x,y) = \frac{4\beta(x+y)(4x+\zeta)}{\eta^2 + 1 - 2\eta(1-2y)} = \frac{4(x+y)(4\beta x + C_1)}{(\eta-1)^2 + 4\eta y}.$$

But, because of the "MINV condition"

$$\eta - 1 = 4\beta x + \alpha - 2\beta - 1 = 4\beta x + C_2 h.$$

Therefore

$$\mu(x,y) = \frac{4(x+y)(4\beta x + C_1)}{(4\beta x + C_2 h)^2 + 4y(4\beta x + C_1)}.$$

Our aim is to find upper and lower bounds for $\mu(x,y)$ in the range $C_0 h^2 \leq x, y \leq 1$.

For the lower bound, we have:

$$\mu \geq \frac{4C_1(x+y)}{(4\beta x + C_2 h)^2 + 4y(4\beta + C_1)}.$$

Suppose that $4\beta x \geq C_2 h$, then since $x^2 < x$,. we have

$$\mu \geq \frac{4C_1(x+y)}{(8\beta x)^2 + 4y(4\beta + C_1)} \geq \frac{4C_1(x+y)}{C_3 x + C_4 y} \geq \frac{4C_1}{\max(C_3, C_4)} = C.$$

Suppose now that $4\beta x < C_2 h$, then

$$\mu \geq \frac{4C_1(x+y)}{C_5 h^2 + C_6 y} \geq \frac{4C_1 y}{C_5 h^2 + C_6 y}.$$

The function on the right hand side is an increasing function of y. Since $y = C_0 h^2$, this function is bounded from below by a constant, we have

$$\mu \geq C.$$

That is to say, the eigenvalues are bounded away from zero when $h \to 0$.

Next we find an upper bound for μ. We first have:

$$\mu \leq \frac{C_7(x+y)}{C_8 x^2 + C_9 y} \leq C_{10}\frac{x+y}{x^2+y}.$$

Let $g(x,y) = \frac{x+y}{x^2+y}$,

$$\frac{\partial g}{\partial y} = \frac{x^2 + y - x - y}{(x^2+y)^2} = \frac{x^2 - x}{(x^2+y)^2} \leq 0 \quad \text{as} \quad x \leq 1.$$

Hence, in the range $C_0 h^2 \leq x, y \leq 1$, we have

$$g(x,y) \leq \frac{x + C_0 h^2}{x^2 + C_0 h^2} \leq \frac{2x}{x^2 + C_0 h^2}.$$

Let $m(x) = \frac{2x}{x^2 + C_0 h^2}$, m has a maximum for $x = C_0 h$; hence

$$g(x,y) \leq \frac{2C_0 h}{C_0^2 h^2 + C_0 h^2} = \frac{2}{C_0^2 + C_0}\frac{1}{h}.$$

Altogether these results implies that

$$\kappa(M^{-1}A_P) \leq O(h^{-1}).$$

◇

Fig. 3.1 LSSOR: Fourier Eigenvalues, n=41, w=1

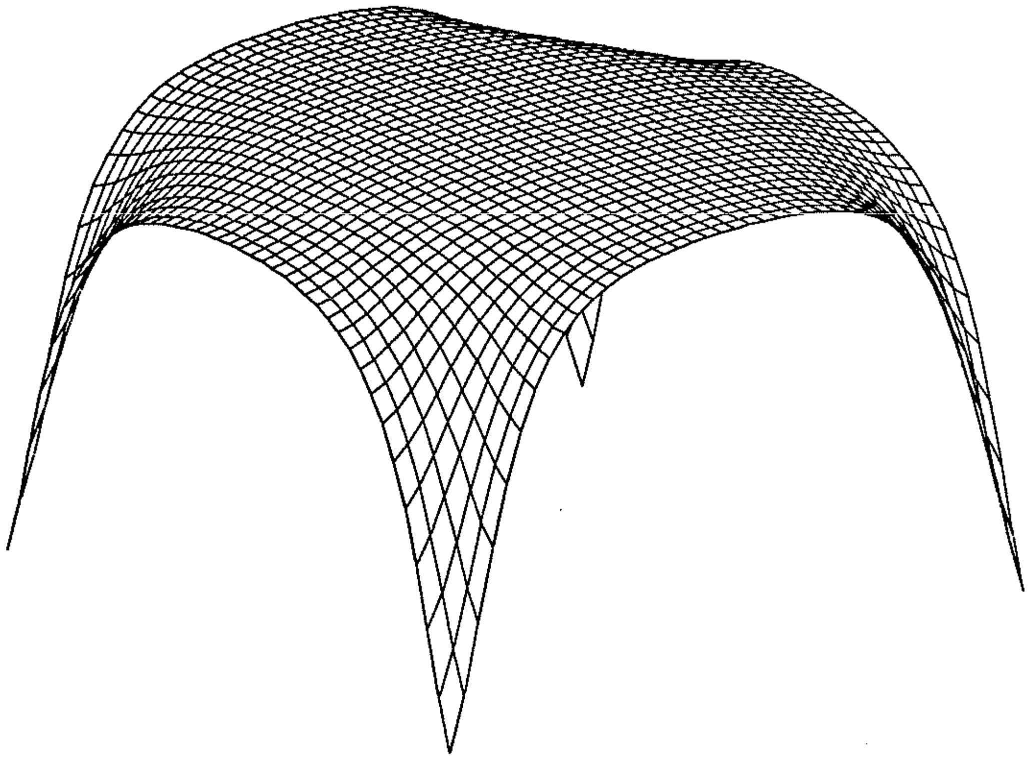


Fig. 3.2 LSSOR: Fourier Eigenvalues, n=41, w=1.74

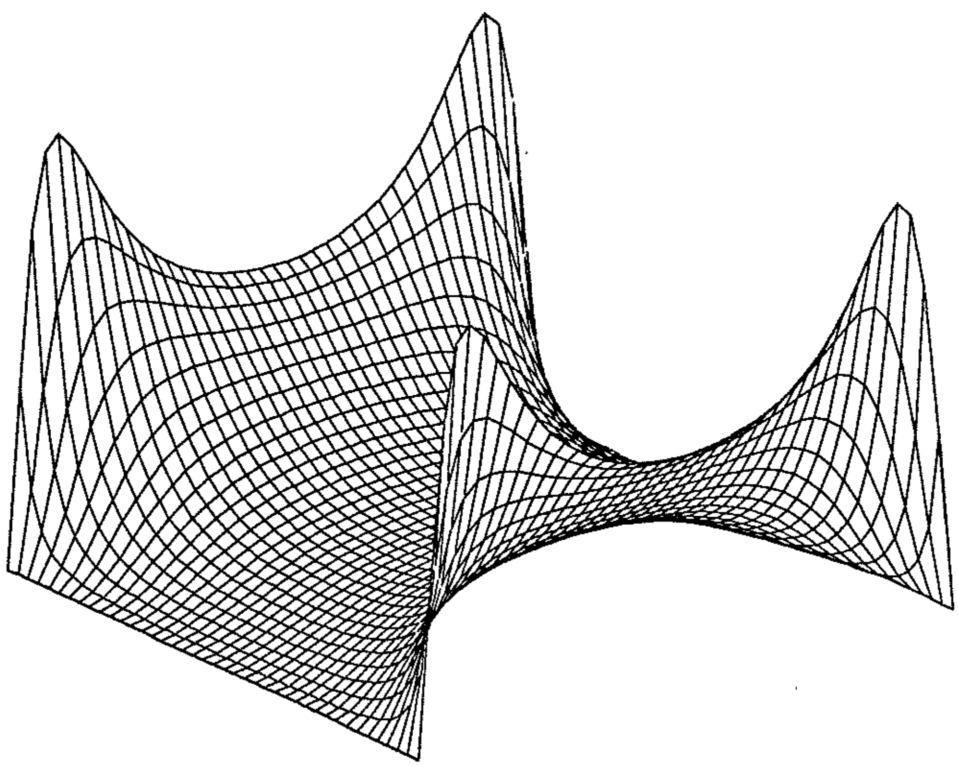


Fig. 3.3 LSSOR: Cond. No. vs w, n=41

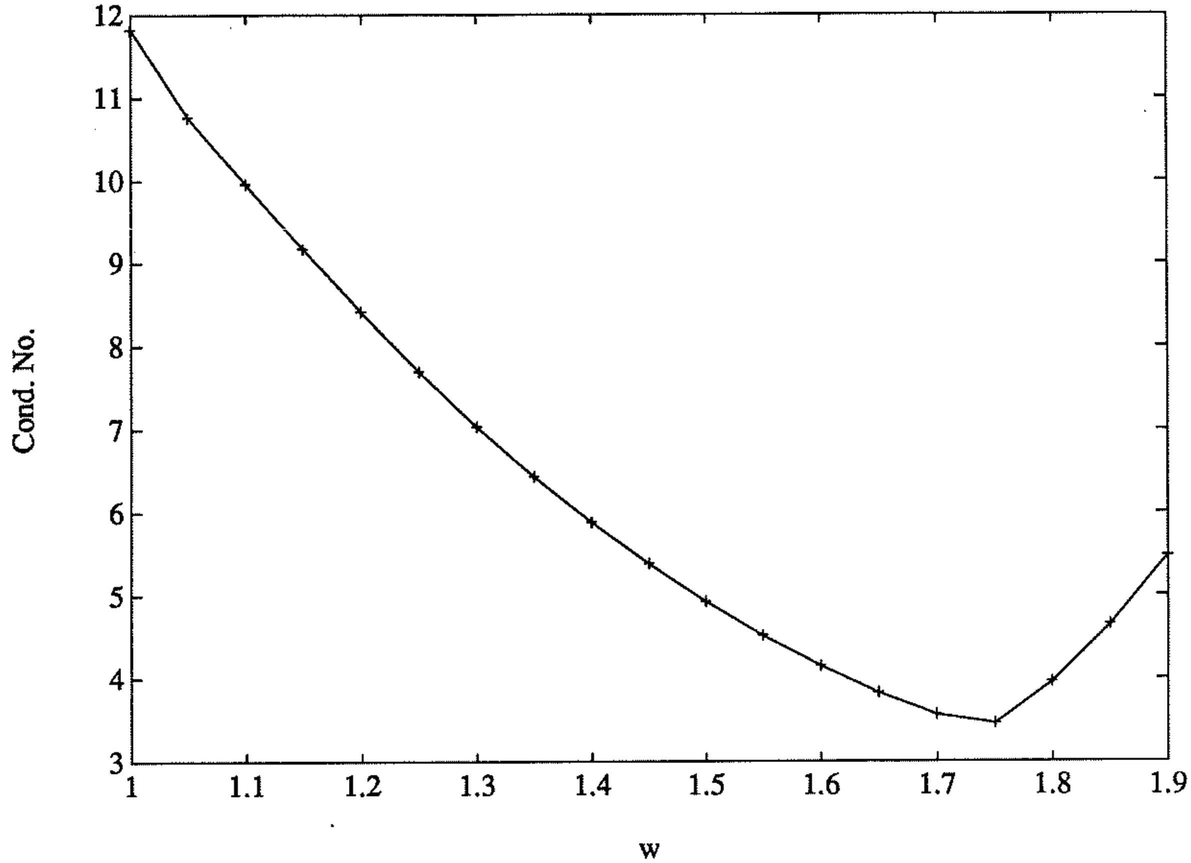


Fig. 3.4 LSSOR: Cond. No. vs n. + : w=1, * : w=~~1.74~~ w_{opt}

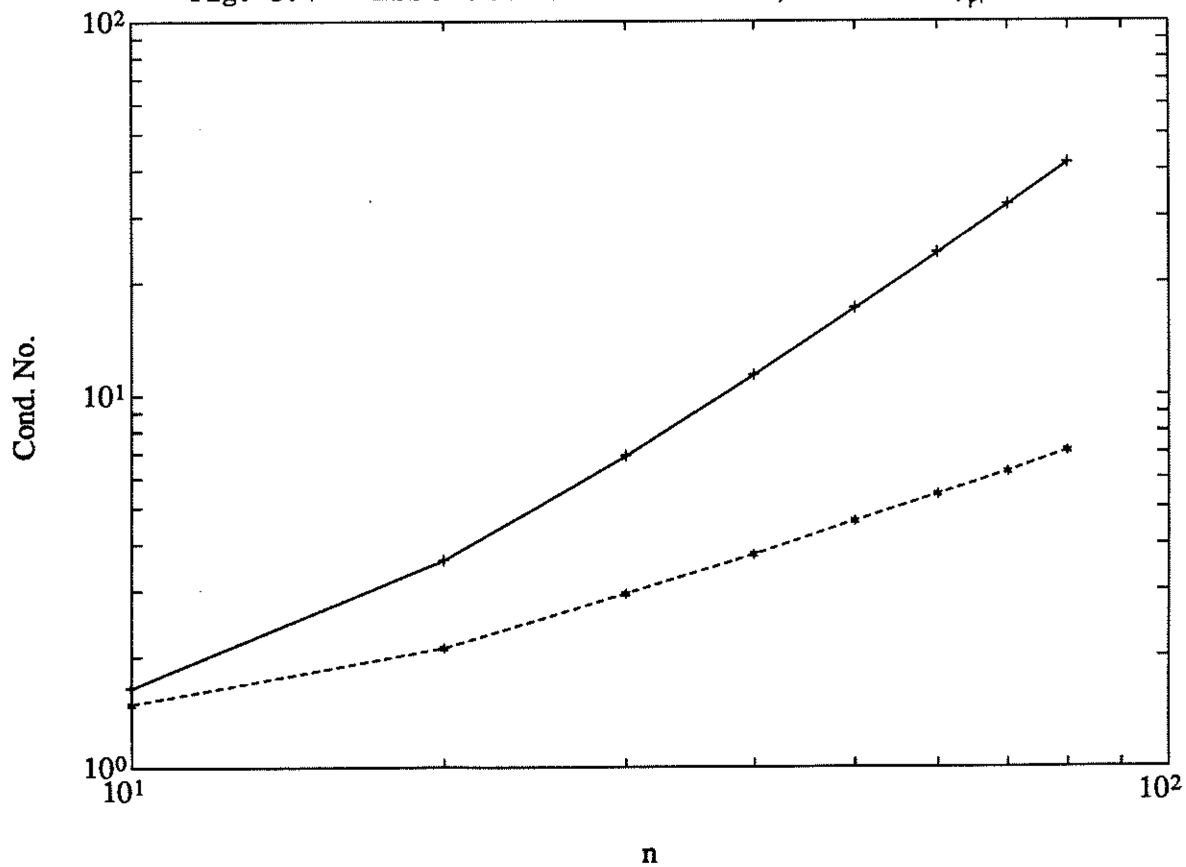


Fig. 4.1 INV: Fourier Eigenvalues, n=41

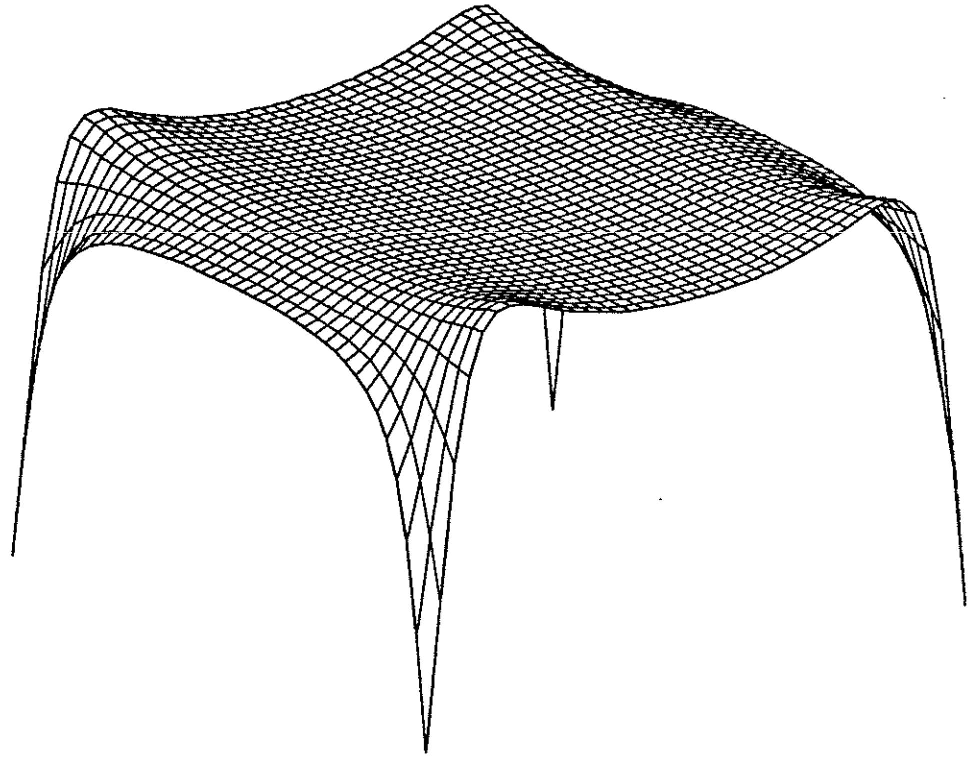


Fig. 4.2 INV: Cond. No. vs n.

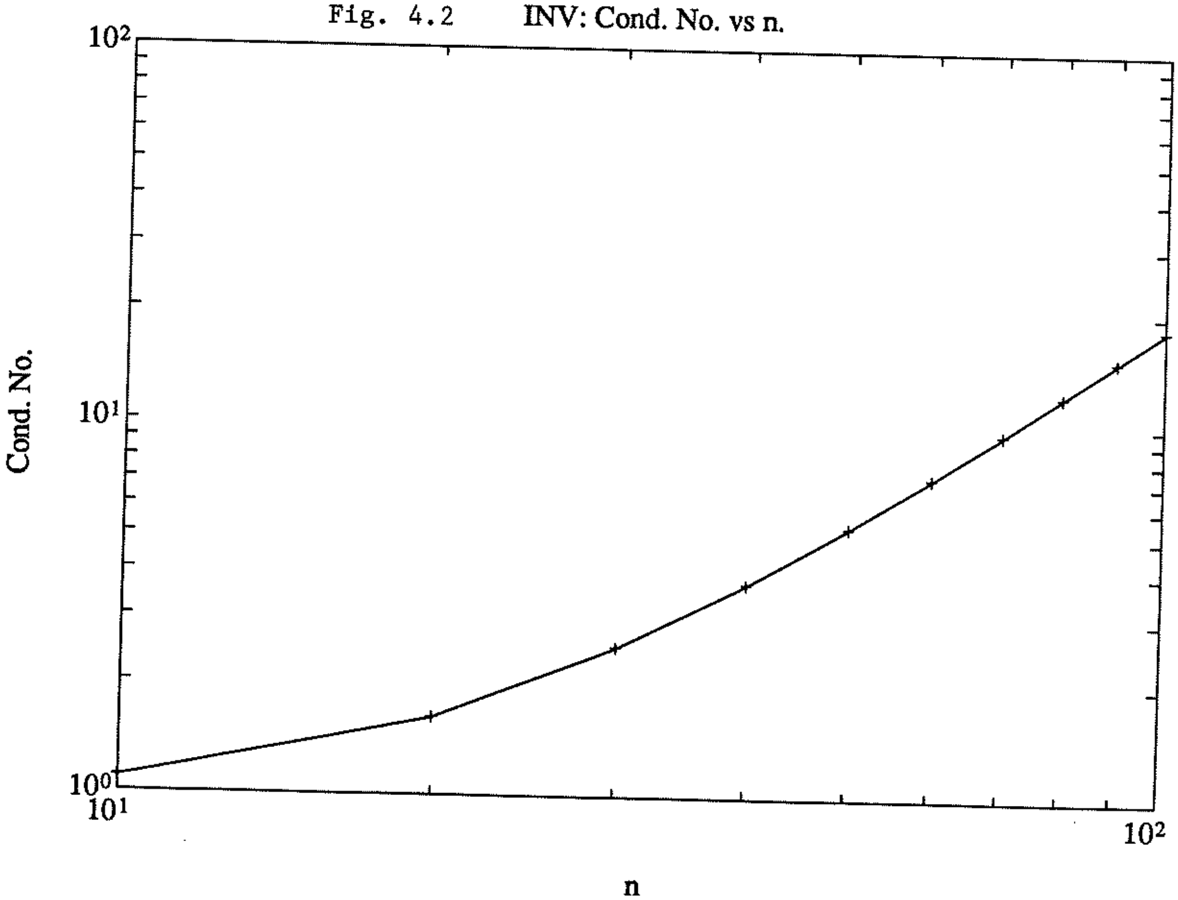


Fig. 5.1 MINV: Fourier Eigenvalues, n=41, c=0

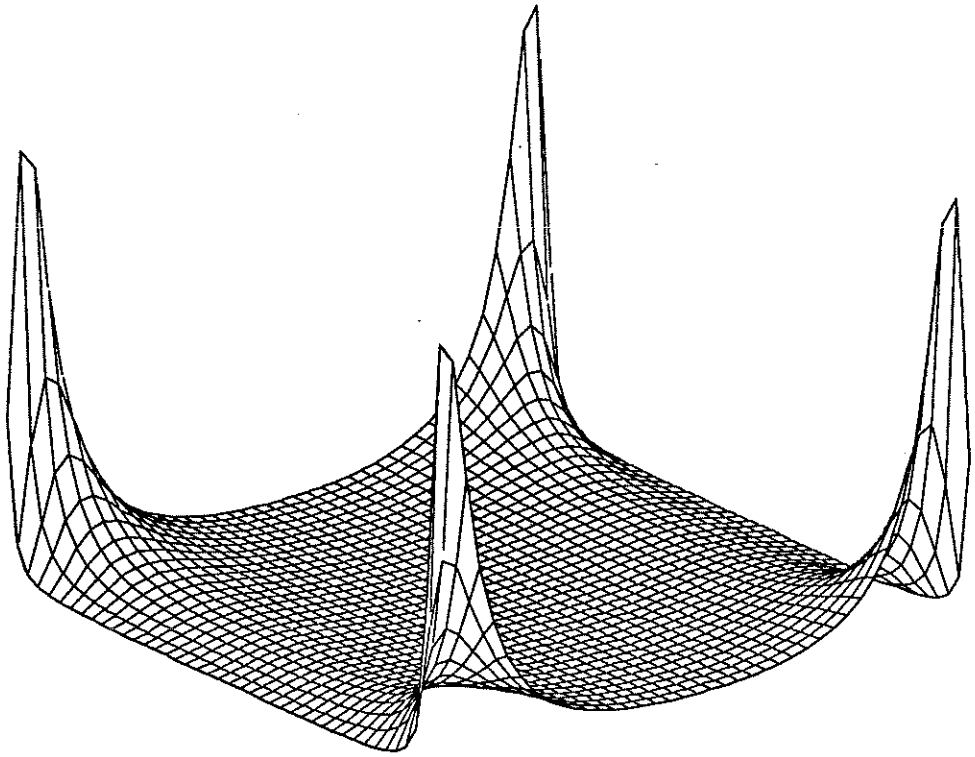


Fig. 5.2 MINV: Fourier Eigenvalues, n=41, c=30

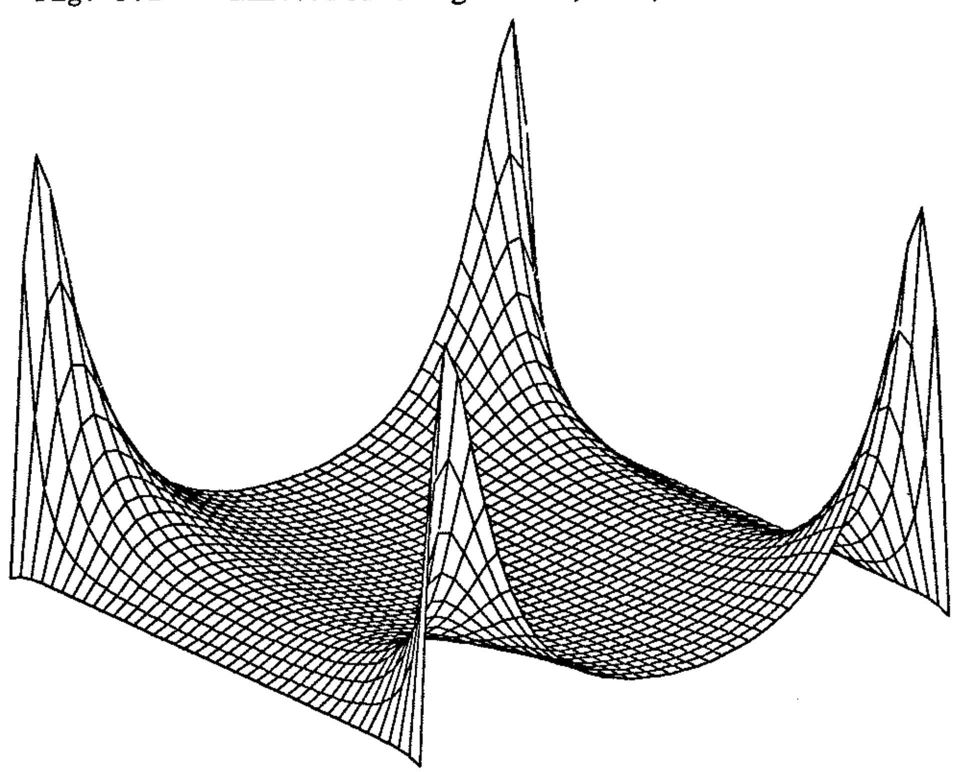


Fig. 5.3 MINV: Cond. No. vs c, n=41

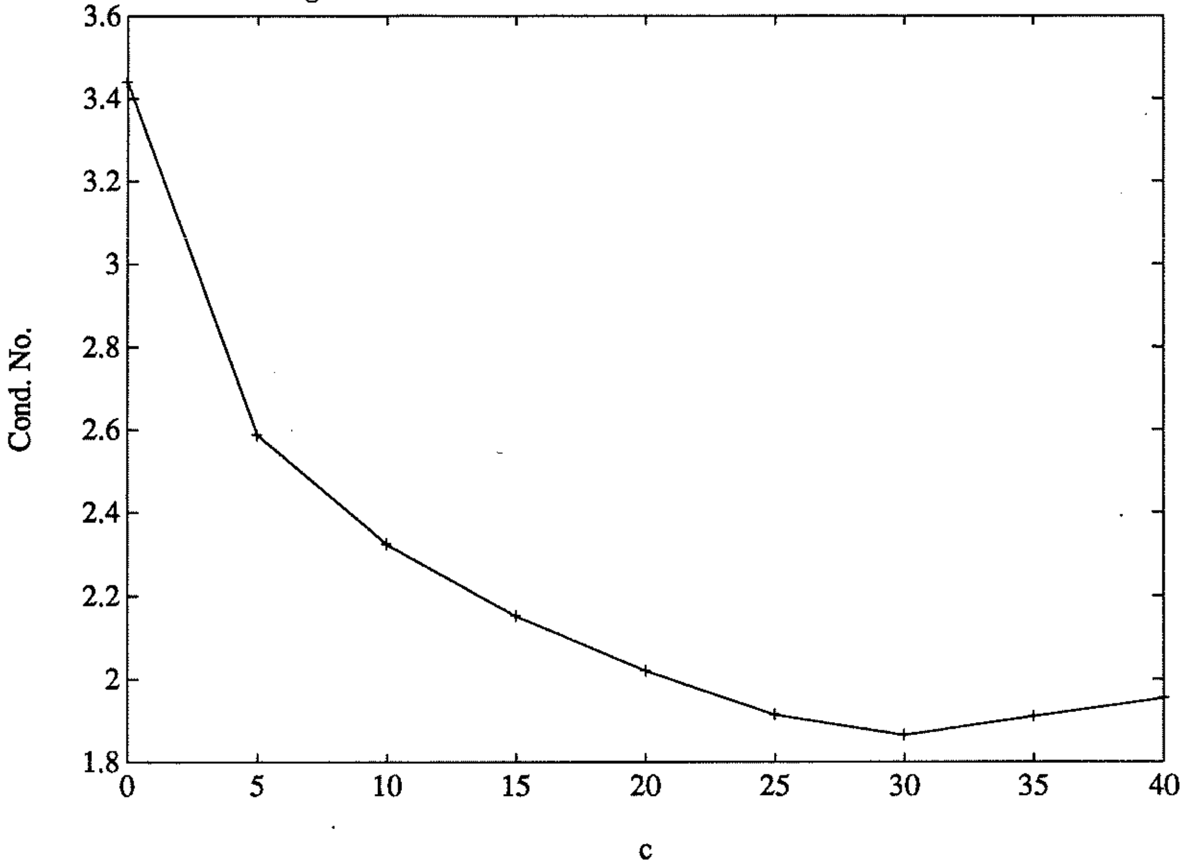


Fig. 5.4 MINV: Cond. No. vs n., c=0

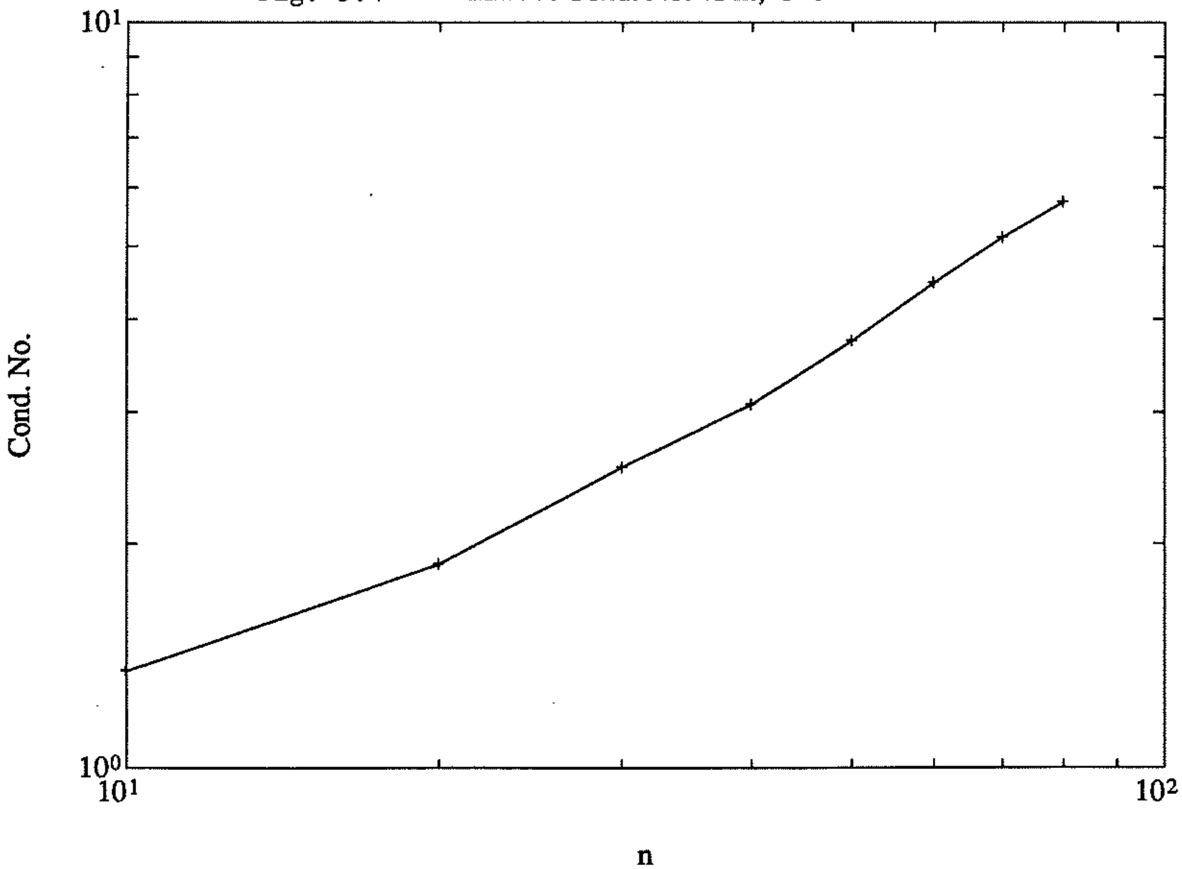


Fig. 6.1 LSSOR: Cond No vs n, Dir(+) vs Fourier(o), w=1

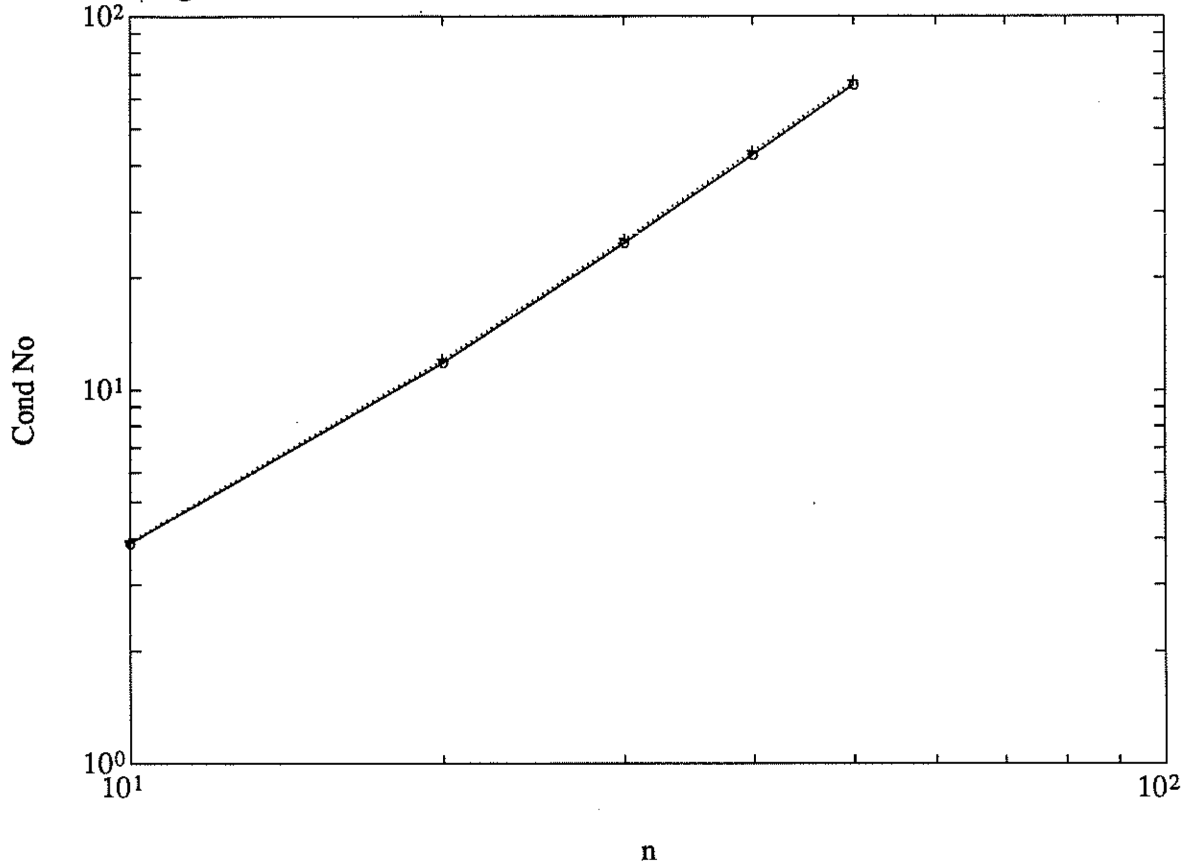


Fig. 6.2 LSSOR: Min Max EV , Dir(+) vs Fourier(o), w=1

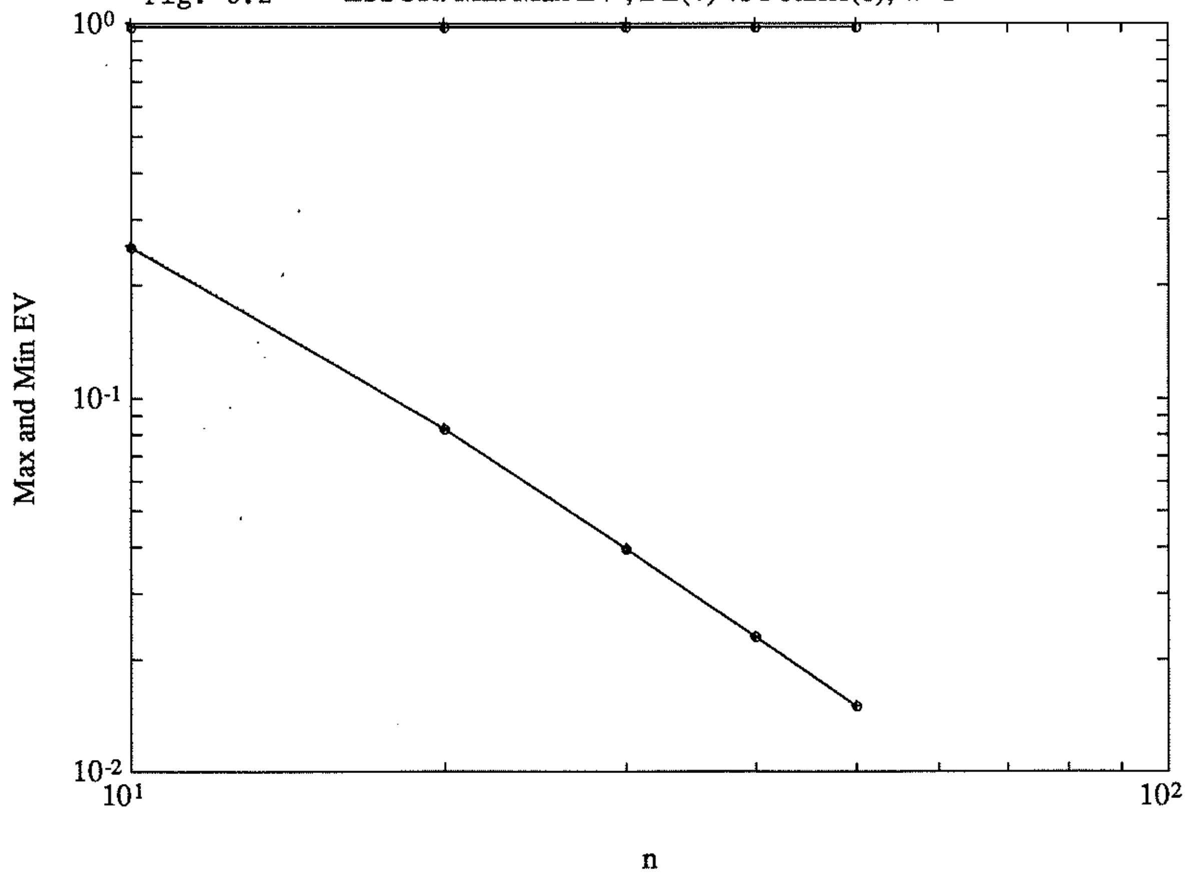


Fig. 6.3 LSSOR: Cond No vs n, Dir(+) vs Fourier(o), w(opt)

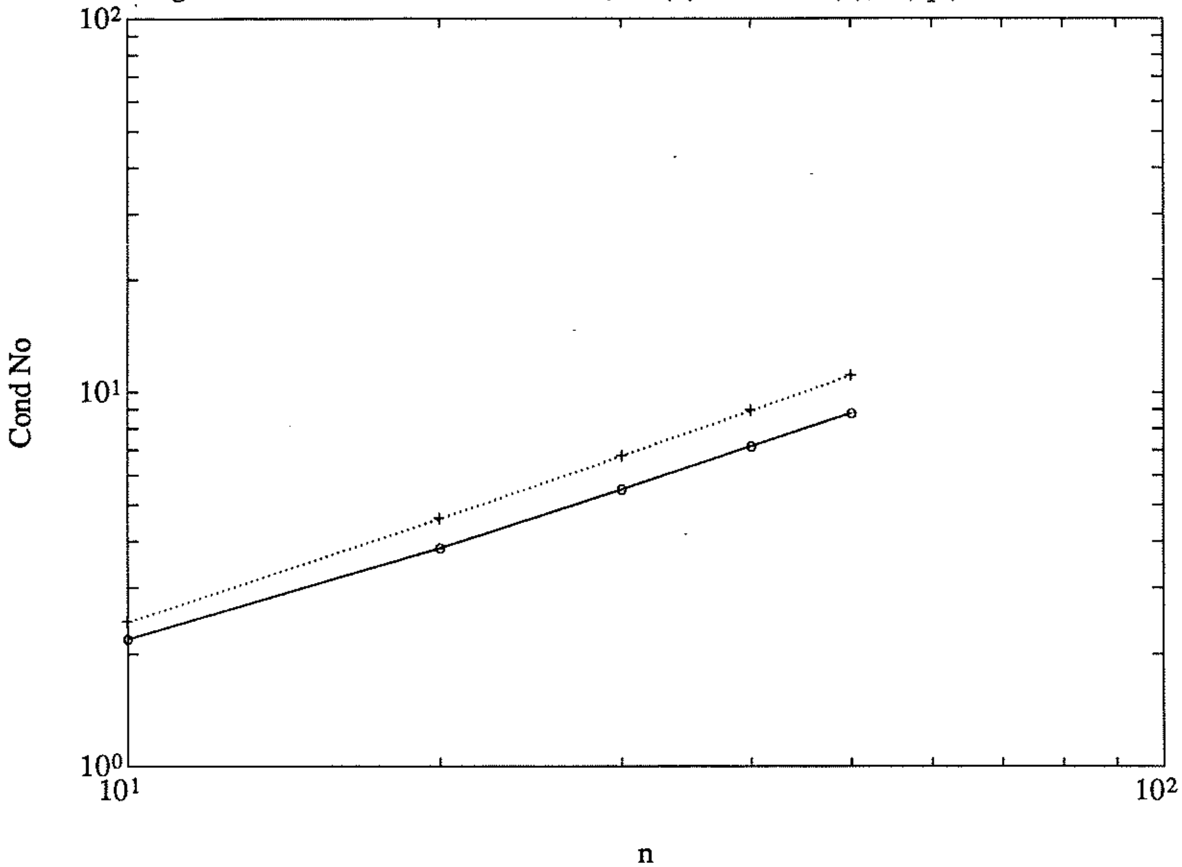
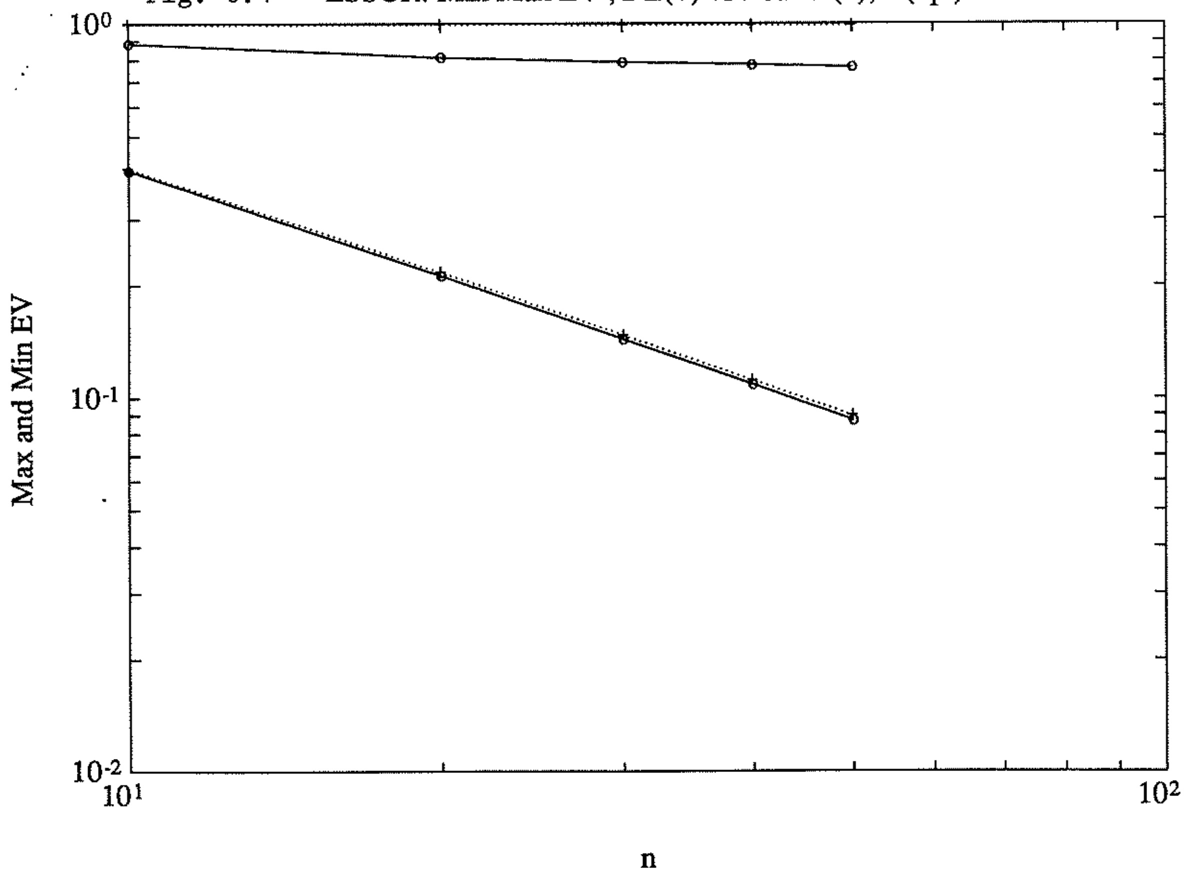


Fig. 6.4 LSSOR: Min Max EV , Dir(+) vs Fourier(o), w(opt)



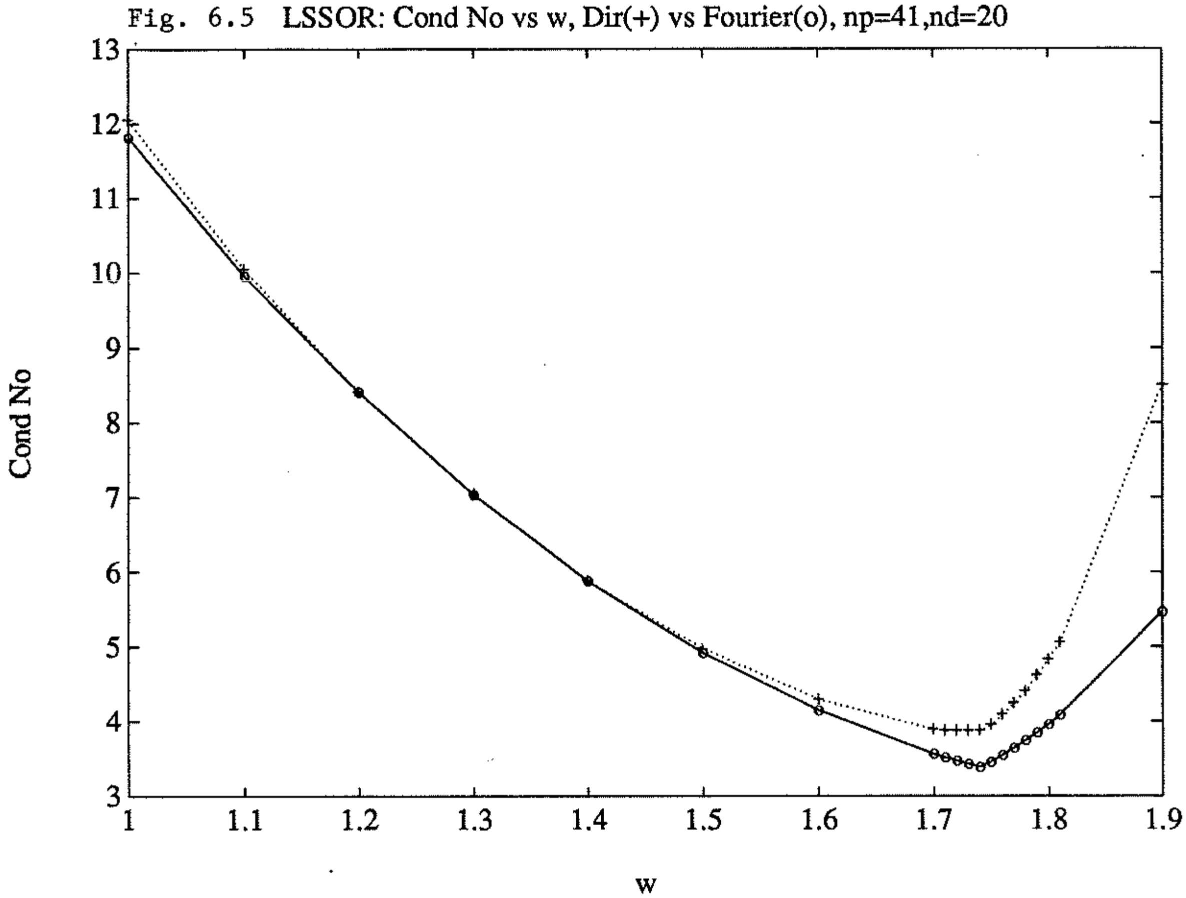


Fig. 6.5 LSSOR: Cond No vs w, Dir(+) vs Fourier(o), np=41,nd=20
Cond No
w
3
4
5
6
7
8
9
10
11
12
13
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9

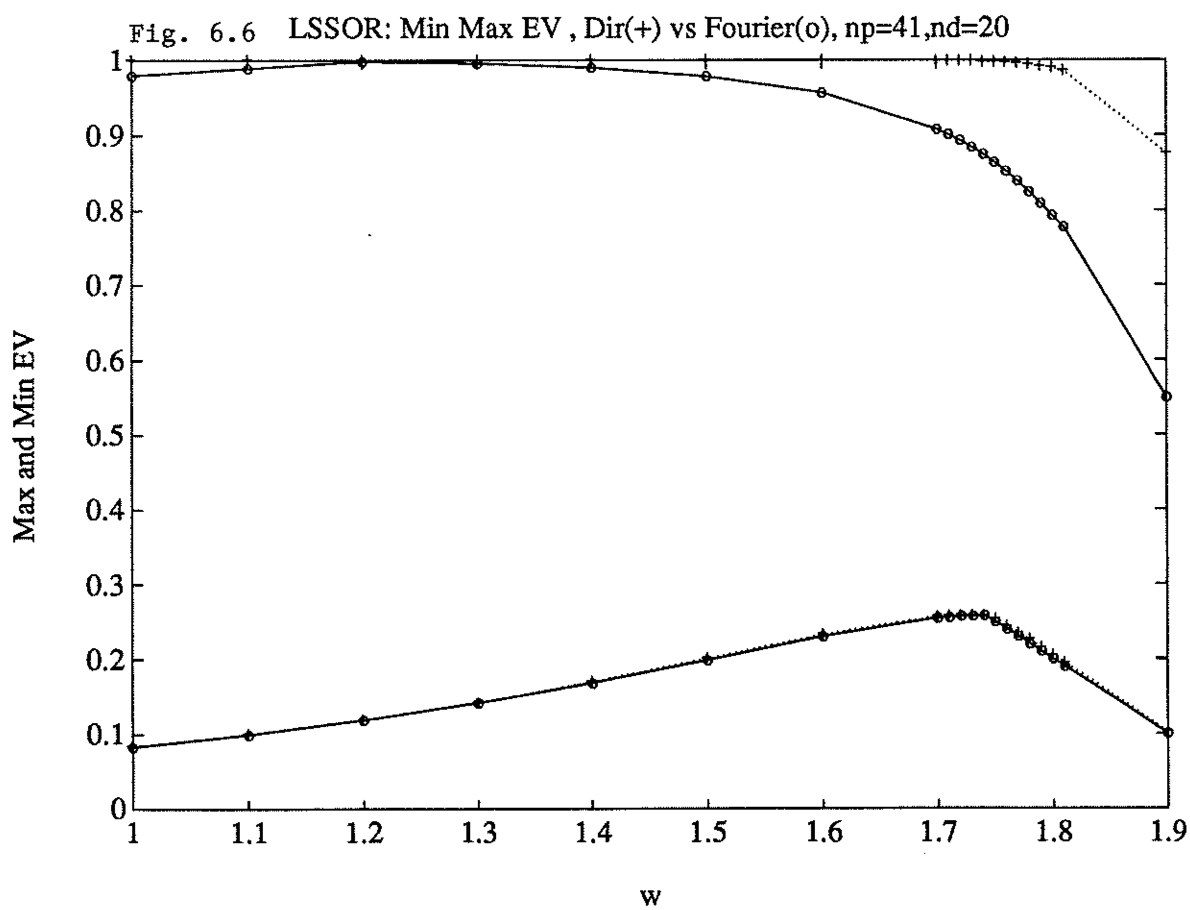


Fig. 6.6 LSSOR: Min Max EV , Dir(+) vs Fourier(o), np=41,nd=20
Max and Min EV
w
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9

Fig. 6.7 INV: Cond No vs n, Dir(+) vs Fourier(o)

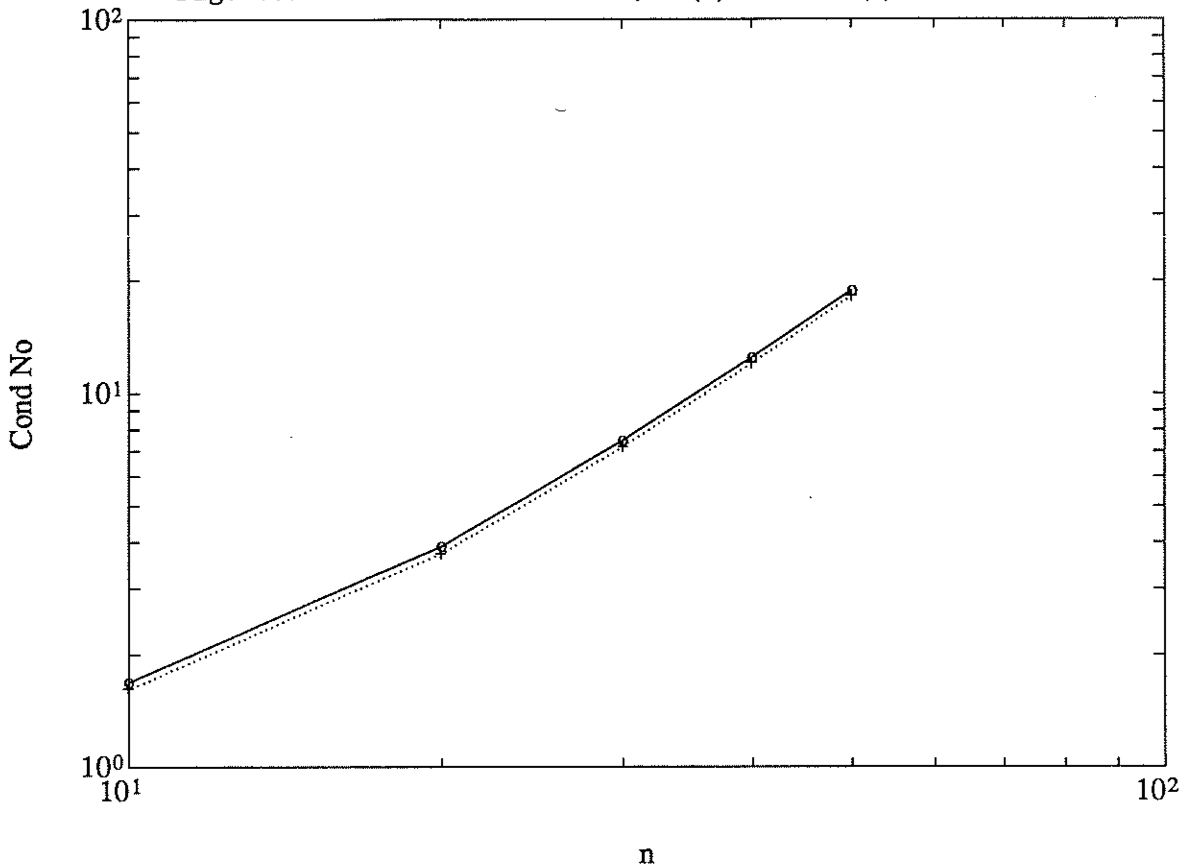


Fig. 6.8 INV: Min Max EV , Dir(+) vs Fourier(o)

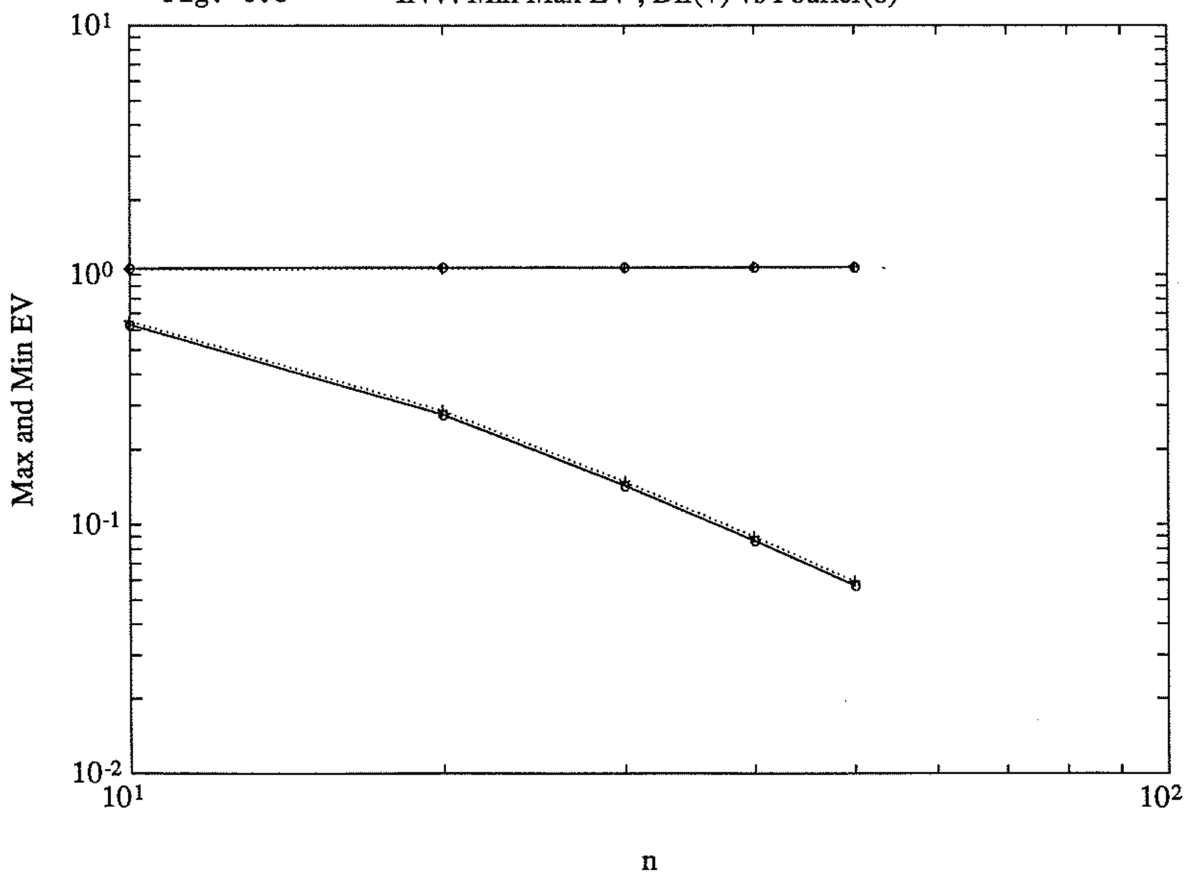


Fig. 6.9 MINV: Cond No vs n, Dir(+) vs Fourier(o), c=0

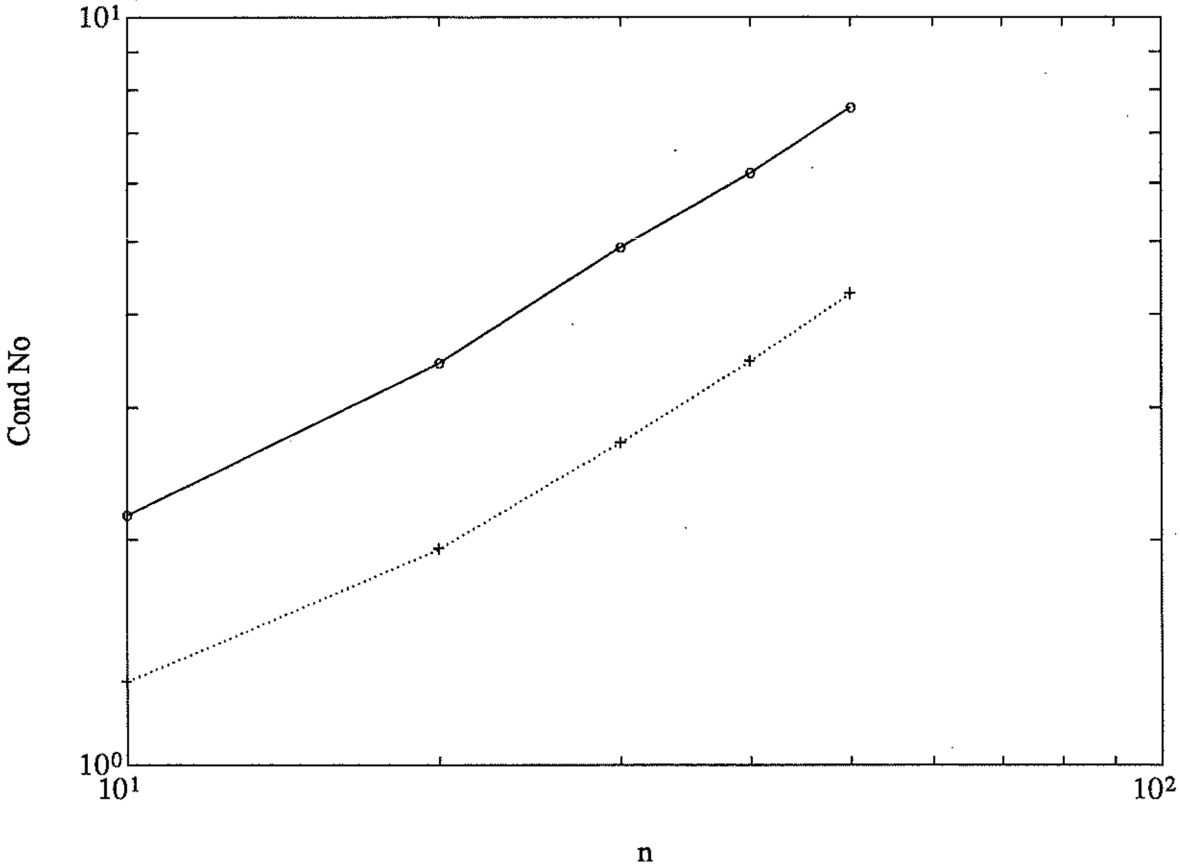


Fig. 6.10 MINV: Min Max EV , Dir(+) vs Fourier(o), c=0

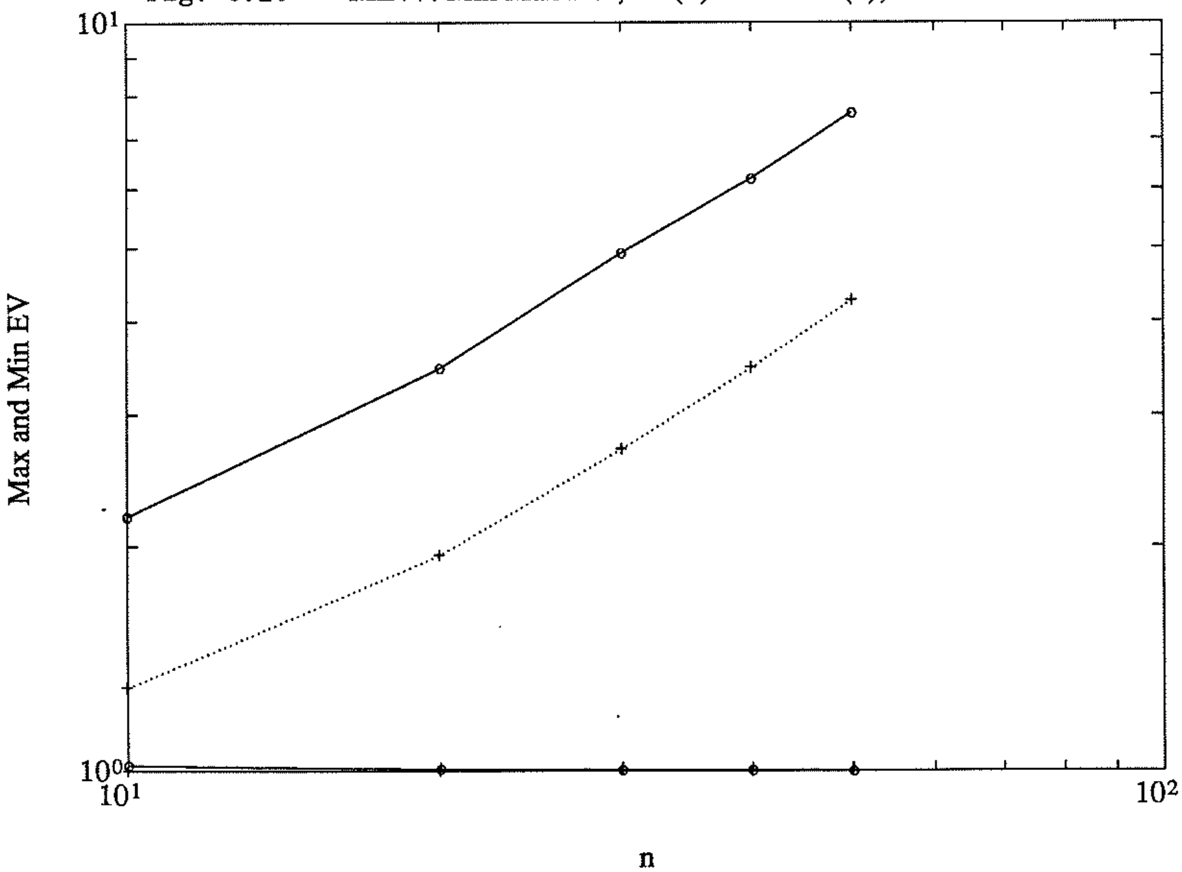


Fig. 6.11 MINV: Cond No vs n, Dir(+) vs Fourier(o), c=7

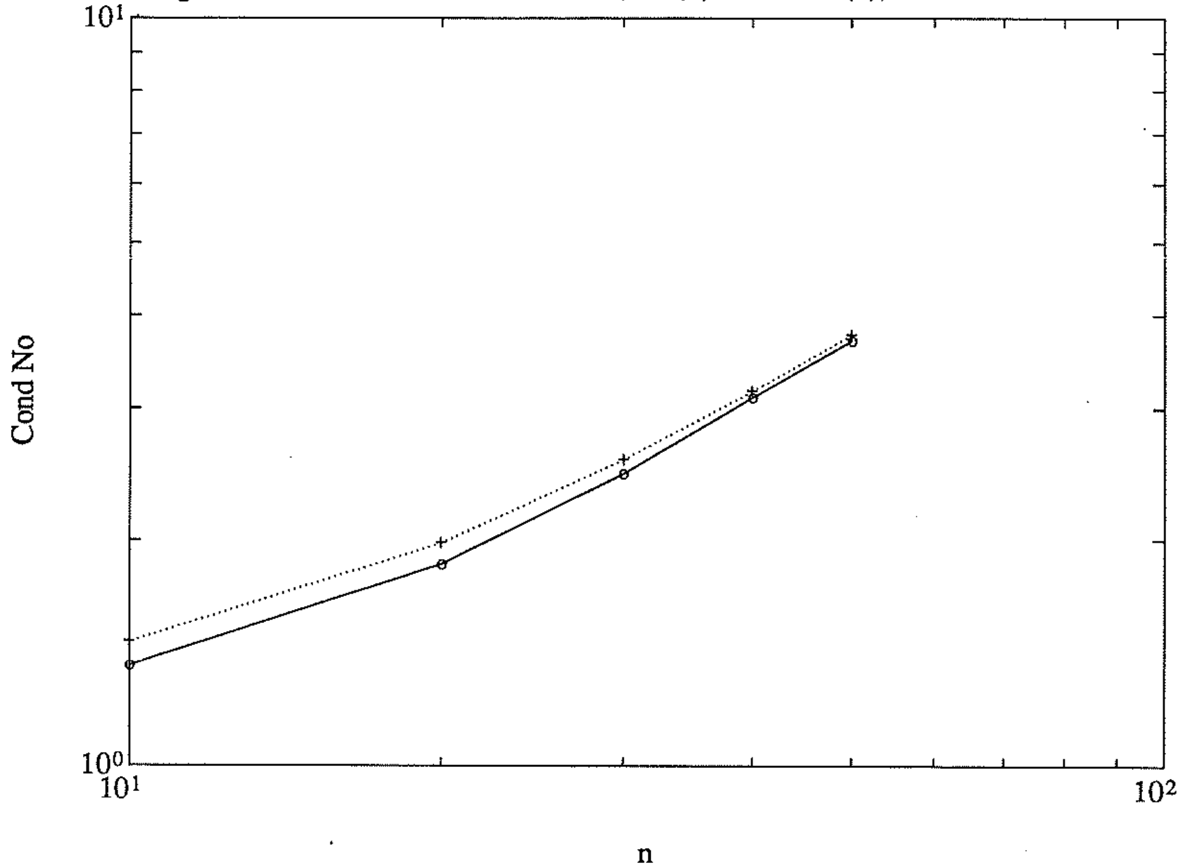


Fig. 6.12 MINV: Min Max EV , Dir(+) vs Fourier(o), c=7

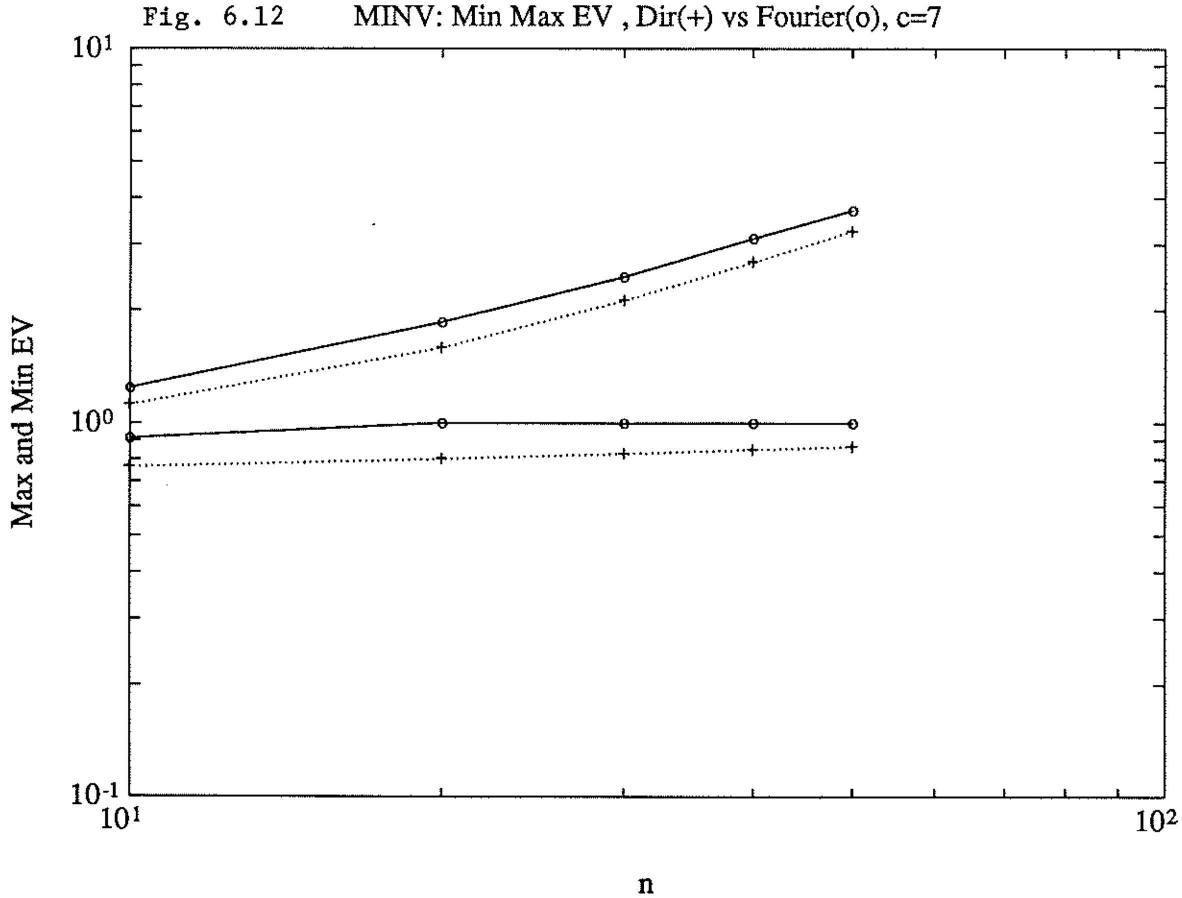


Fig. 6.13 MINV: Cond No vs c, Dir(+) vs Fourier(o), np=41,nd=20

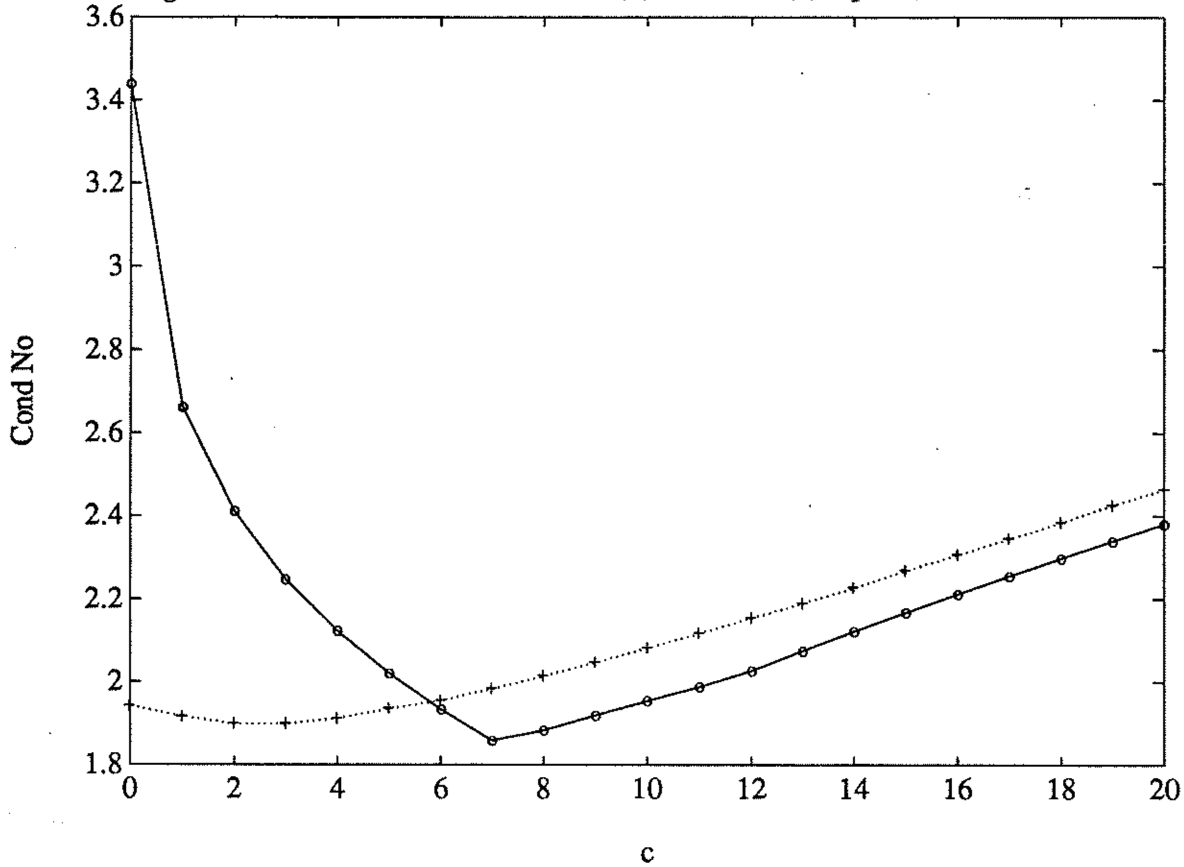


Fig. 6.14 MINV: Min Max EV , Dir(+) vs Fourier(o), np=41,nd=20