Resistive Instabilities in Rapidly Rotating Fluids:
Linear Theory of the Tearing Mode

Weijia Kuang
Paul H. Roberts

March 1990
CAM Report 90-07
RESISTIVE INSTABILITIES IN RAPIDLY ROTATING FLUIDS:
LINEAR THEORY OF THE TEARING MODE

WEIJIA KUANG
Department of Mathematics, University of California, Los Angeles, California, CA 90024

PAUL H. ROBERTS
Center for Earth and Planetary Interiors, Institute of Geophysics and Planetary Physics,
University of California, Los Angeles, California, CA 90024

Received:

The linear stability of the sheet pinch is examined for a rapidly rotating fluid. The sheet
pinch is a horizontal layer of uniform, incompressible, inviscid fluid of density $\rho$, electrical
conductivity $\sigma$, magnetic permeability $\mu$, confined between two perfectly conducting planes
$z = 0, d$, where $z$ is height. The prevailing magnetic field, $B_0(z)$, is horizontal; at each
height it is unidirectional, but that direction turns continuously (in one sense only) as $z$
increases; the system is in magnetostatic equilibrium. The layer rotates about the vertical
with an angular velocity, $\Omega$, that is large: $\Omega \gg V_A/d$, where $V_A = B_0/\sqrt{(\mu \rho)}$ is the Alfvén
velocity. The dimensionless number measuring $\sigma$ is the Elsasser number $\Lambda = \sigma B_0^2/2 \Omega \rho$.

The principal example studied here is force-free ($B_0 = \text{constant}$); the direction of $B_0$
rotates uniformly with height ($\nabla \times B_0 = -qB_0/d$, where $q$ is a constant), turning through
$q$ radians between $z = 0$ and $z = d$.

The growth rate, $s$, of small perturbations of horizontal wave vector $k$, is determined
in a number of cases. It is found that instabilities do not occur if $B_0$ makes less than
one quarter turn between the boundaries. For $q > \frac{1}{2} \pi$, stability is lost when $\Lambda$
exceeds a critical value, $\Lambda_c$, the instability being direct (i.e. the largest $\Re s$ is real, and this $s$
is zero for $\Lambda = \Lambda_c$); as $q$ increases, $\Lambda_c$ decreases and $s$ increases for any $\Lambda$. As $\Lambda$ increases
beyond $\Lambda_c$, $s$ attains a maximum and then decreases monotonically to zero as $\Lambda \to \infty$.

The asymptotic form of the eigenmodes in the limit $\Lambda \to \infty$ is analysed in detail for
general $B_0$, especially their structure within the critical layers [of thickness $O(\Lambda^{-1/4}d)$]
surrounding the critical levels, at which $k$ is orthogonal to $B_0$. The equilibrium is found
to be more unstable when $J_0 \times B_0/J_0B_0$ is antiparallel to $\Omega$ than when it is parallel,
$J_0 = \nabla \times B_0/\mu$ being the (horizontal) electric current density generating $B_0$. It is
shown that, provided this critical level is not asymptotically within one of the boundary
layers [thickness $O(d \Lambda^{-1/2})$] at a wall, $s = O(\Lambda^{1/4} \tau^{-1}_\eta)$, i.e. the instability develops
more rapidly than the rate $\tau^{-1}_\eta (= 1/\mu \sigma d^2)$ at which $B_0$ evolves through ohmic diffusion.
Numerical evidence is presented, however, which indicates that, for the mode of maximum
instability, the critical level moves into a boundary layer as $\Lambda \to \infty$.

KEY WORDS: resistive instability, tearing mode, sheet pinch, rotating magnetohydrodynamics.

---

1 Contribution no.1 of the Center for Earth and Planetary Interiors, UCLA.
2 Also at Department of Mathematics, University of California, Los Angeles, California, CA 90024.
owes its existence to the finiteness of $\sigma$; it would be absent in a perfectly conducting fluid. It is theoretically perhaps the most fascinating and most important of the resistive modes, the so-called "tearing mode" (Furth, Killeen and Rosenbluth, 1963). When $\sigma = \infty$, the topology of the magnetic field lines cannot be altered by fluid motions; they are "frozen" to the conductor (Alfvén's theorem). When $\sigma$ is finite, the field lines can diffuse relative to the fluid and can reconnect in topologically new ways. If these result in highly bent field lines, there is a possibility that, by straightening up and carrying the (almost) frozen fluid with them as they do so, they can transform the magnetic energy of the equilibrium state into kinetic energy, i.e. create an instability that could not arise without the change in topology.

An instability of this type exists in the sheet pinch, the configuration studied in this paper. A uniform, electrically conducting, incompressible fluid occupies the "horizontal" layer $0 \leq z \leq d$, bounded by impermeable walls that are perfect electrical conductors. A horizontal field, $B_0$, is present given by

$$B_0 = B_{0x}(z)\hat{x} + B_{0y}(z)\hat{y},$$  \hspace{1cm} (1.1)

where $\hat{x}$ and $\hat{y}$ are the unit vectors in the coordinate directions. It is supposed that $\phi'(z)$ is nowhere zero in $[0, d]$, where

$$\phi(z) = \tan^{-1}(B_{0y}/B_{0x})$$  \hspace{1cm} (1.2)

is the angle between $B_0$ and the $x$-axis. The field (1.1) then turns continuously about the "vertical" in one sense as $z$ increases. The example of (1.1) on which we shall principally concentrate is the constant strength field,

$$B_0 = B_0[\hat{x} \cos qz/d + \hat{y} \sin qz/d],$$  \hspace{1cm} (1.3)

for which $B_0$ and $q$ are nonzero constants, and

$$\phi = qz/d.$$

Associated with $B_0$ is a horizontal electric current, $J_0$, given by

$$\mu J_0 = \nabla \times B_0 = -DB_{0y} \hat{x} + DB_{0x} \hat{y},$$  \hspace{1cm} (1.4)

where $D = d/dz$. The Lorentz force, $J_0 \times B_0$, is balanced by the gradient of the pressure

$$P_0(z) = \text{constant} - B_0^2/2\mu.$$  \hspace{1cm} (1.5)

Hydrostatic equilibrium prevails; the velocity $V$ of the fluid is everywhere zero. The field (1.3) is in fact "force-free"; since

$$J_0 = (qB_0/\mu d)[\hat{x} \cos qz/d + \hat{y} \sin qz/d] = qB_0/\mu d,$$  \hspace{1cm} (1.6)

we have $J_0 \times B_0 = 0$. 

3
The differences between the MHD of non-rotating and rapidly rotating fluids are so substantial as to invite us to re-examine the tearing mode when Coriolis forces are large and the layer is rotating about the vertical: $\Omega = \Omega \hat{z}$. Experts in MHD will correctly surmise that one of the more significant differences is that of timescales. In the “classical theory” (as we shall call the study of tearing in a non-rotating fluid) the two timescales of interest are the hydromagnetic (or Alfvén) timescale, $\tau_A$, and the electromagnetic decay time, $\tau_\eta$, where

$$\tau_A = \frac{H}{V_A}, \quad \tau_\eta = \frac{H^2}{\eta},$$

(1.7, 1.8)

where $H$ is the horizontal scale of the instability (say the wavelength of the normal mode under study). The instability is studied in the conceptually simplest limit, $\Lambda \to \infty$, where $\Lambda$ is the “Lundquist number,” i.e. the ratio of timescales (1.8) and (1.7):

$$\Lambda = \frac{\tau_\eta}{\tau_A} = \frac{HV_A}{\eta}. \quad (1.9)$$

In a rapidly rotating fluid ($\Omega \gg V_A/H$) the role of the Alfvén wave is divided between the slow\(^3\) and fast waves, with which are associated the timescales

$$\tau_s = \frac{2\Omega H^2}{V_A^2}, \quad \tau_f = \frac{1}{2\Omega}, \quad (1.10, 1.11)$$

the former of which replaces $\tau_A$ as the important non-dissipative timescale. The limit of greatest interest is $\Lambda \to \infty$, where $\Lambda$ is now the “Elsasser number”, i.e. the ratio of the two timescales (1.8) and (1.10):

$$\Lambda = \frac{\tau_\eta}{\tau_s} = \frac{V_A^2}{2\Omega \eta} = \frac{\sigma B_0^2}{2\Omega \rho},$$

which is scale independent. Coriolis forces dominate inertial forces, and take over their role in the resistive modes of classical theory. The Elsasser number replaces the Lundquist number as the key non-dimensional parameter measuring electrical conductivity.

Unless $\Lambda$ is large it is hard to visualize resistive modes in the terms we chose above, for when $\Lambda = O(1)$ the picture of field lines frozen to the conductor is seriously marred. Moreover, when $\Lambda = O(1)$, the growth rate of the instability (if any) can only be $O(\tau_\eta^{-1})$, the very same rate at which the field $B_0$ is itself diffusively evolving. The development of $B_0$ in time and its resistive instability would then, in general, be inextricably mixed, a serious theoretical handicap. The only exception to this statement would be if $B_0$ had already completed its diffusive evolution to an electrodynamically steady state consistent with the boundary conditions applied. But, if that configuration were of minimum magnetic energy, no resistive instability could ensue; this is also true for such a $B_0$ even when $\Lambda$ is large (Barston, 1969).

It is now apparent that theoretically the clearest case arises when $\Lambda \gg 1$, and when $\tau_{\text{instab}}$ the timescale of the instability is short compared with $\tau_\eta$, for it is only when $\tau_{\text{instab}} \ll \tau_\eta$.

\(^3\)Because $\tau_s \ll \tau_\eta$, we earlier described $\tau_s$ as a “fast” timescale; since $\tau_f \ll \tau_s$, we here call $\tau_s$ “slow”.\n
4
\( \tau_\eta \) that one can ignore the diffusive evolution of \( B_0 \) over the time during which it is destroyed by the instability. One may then, as we did in (1.1) or (1.2), select any \( B_0 \) that is magnetostatic in the sense of ideal MHD (\( \sigma = \infty \)). One is left wondering however whether such a rapidly growing instability exists. It is clear that, to develop so quickly, the instability must operate in a region [which in the case of the field (1.1) is a horizontal “critical layer”] having a dimension, \( \delta \), small compared with \( H \). The velocity, \( V_{\text{drift}} \), with which lines of force can drift relative to the conductor and reconnect is of order \( V_{\text{drift}} \approx \eta/\delta \) (e.g. see Roberts, 1967, p. 55). One can therefore expect the growth rate of the instability to be
\[
\tau_{\text{instab}} = \frac{d}{V_{\text{drift}}} = \frac{d\delta}{\eta}.
\] (1.12)
The process is nevertheless slow compared with that of ideal wave propagation. Confining attention to the highly rotating case, \( V_{\text{drift}} \ll V_s (= d/\tau_s) \), so that
\[
\tau_s \ll \tau_{\text{instab}} \ll \tau_\eta.
\] (1.13)

To determine \( \delta \), and to learn more about the structure of the critical layer of reconnection, consider a “normal mode” of instability, in which the perturbation field,
\[
b = B - B_0,
\] (1.14)
is proportional to
\[
\exp[i(k_x x + k_y y) + st],
\] (1.15)
where
\[
k_x = k \cos \theta, \quad k_y = k \sin \theta,
\] (1.16)
are the horizontal wavenumbers of the mode and \( s \) is its growth rate. The phase fronts make an angle of \( \theta - \phi \) with \( B_0 \) so that, at a “critical level”, at which \( \theta = \phi \pm \frac{1}{2} \pi \), the slow wave propagates in the direction perpendicular to that in which the instability moves. Within a critical layer of width, \( \delta \), surrounding the critical level, the effective strength of \( B_0 \) is therefore effectively reduced by a factor of order \( \delta/H \), i.e. to \( B_0 \delta/H \). The fluid motions \( v \) (required in the critical layer, to balance by electromagnetic induction the rate, \( \eta b/\delta^2 \), at which field diffuses and reconnects) is therefore correspondingly larger. Thus \( \eta b/\delta^2 \) is not of order \( v B_0/d \) but is \( O(\nu B_0 \delta/H d) \), which implies that \( v = O(\eta H db/\delta^3) \). This is true also for the classical tearing mode. The dynamical balances in the non-rotating and highly rotating layers are, however, very different. In the former, Lorentz force is balanced by inertial forces; in the latter, it is balanced by Coriolis force. Because of the effective reduction in \( B_0 \) in the critical layer, the Lorentz force is not of order \( B_0 j \approx O(\eta b/\delta^3) \), but is of order \( B_0 \delta/\mu H d \). When this is set against a Coriolis force that is \( O(2\Omega \rho v) \), we obtain
\[
b = O(2\Omega \mu H d v / B_0 \delta).
\] Combining this with our earlier estimate, \( v = O(\eta H db/\delta^3) \), we see that
\[
\delta = O(H^{1/2} d^{1/2} \Lambda^{1/4}), \quad \tau_{\text{instab}} = O \left[ (d/H)^{3/2} \tau_s^{1/4} \tau_\eta^{3/4} \right].
\] (1.17, 1.18)

The latter follows from (1.12); obviously (1.13) is satisfied.
2. Basic Equations and Boundary Conditions

We study the motion of a uniform, invisid, incompressible fluid in the so-called "magnetostrophic approximation", in which inertial forces are neglected:

\[ 2\Omega \times \mathbf{V} = -\nabla P + \frac{1}{\rho} \mathbf{J} \times \mathbf{B}, \quad (2.1) \]

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \quad (2.2) \]

\[ \nabla \cdot \mathbf{V} = 0, \quad \nabla \cdot \mathbf{B} = 0. \quad (2.3, 2.4) \]

Here \( P \) is the reduced pressure (pressure divided by the constant density \( \rho \) and including centrifugal forces), and \( \mathbf{J} = \mu^{-1} \nabla \times \mathbf{B} \) is the electric current density.

The field (1.1) satisfies the hydrostatic form of (2.1):

\[ \mathbf{V}_0 \equiv 0, \quad 0 = -\nabla P_0 + \frac{1}{\rho} \mathbf{J}_0 \times \mathbf{B}_0, \quad (2.5) \]

but not (2.2). As explained in Section 1 we suppose that \( \Lambda \gg 1 \) so that the diffusive evolution of \( \mathbf{B}_0 \) is too slow to be a factor. We study the linear stability of field (1.1) by writing

\[ \mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad \mathbf{J} = \mathbf{J}_0 + \mathbf{j}, \quad \mathbf{V} = \mathbf{V}_0 + \mathbf{v}, \quad P = P_0 + p, \quad (2.6) \]

substituting into (2.1)–(2.4), simplifying with the help of (2.5), and linearizing in the usual way. We obtain

\[ \dot{\mathbf{v}} = -\nabla p + \mathbf{B}_0 \cdot \nabla \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{B}_0, \quad (2.7) \]

\[ \frac{\partial \mathbf{b}}{\partial t} = \frac{1}{\Lambda} \nabla^2 \mathbf{b} + \mathbf{B}_0 \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{B}_0, \quad (2.8) \]

\[ \nabla \cdot \mathbf{v} = 0, \quad \nabla \cdot \mathbf{b} = 0. \quad (2.9) \]

We have here absorbed the perturbed magnetic pressure \( \mathbf{B}_0 \cdot \mathbf{b}/\mu \) into \( p \), and have made the equation dimensionless by the transformation

\[ x \to dx, \quad t \to \tau_s t, \quad \mathbf{v} \to (d/\tau_s) \mathbf{v}, \quad (2.10) \]

\[ \mathbf{B} \to \tilde{\mathbf{B}} \mathbf{B}, \quad \mathbf{b} \to \tilde{\mathbf{B}} \mathbf{b}, \quad p \to (2\Omega d^2/\tau_s) p, \]

where \( \tilde{\mathbf{B}} \) is a typical scale for \( \mathbf{B}_0 \) and from now onward

\[ \tau_s = \frac{2\Omega \mu \rho d^2}{B^2}, \quad \tau_\eta = \frac{d^2}{\eta}, \quad \Lambda = \frac{\tau_\eta}{\tau_s}. \quad (2.11) \]

After eliminating \( p \), we find that (2.7)–(2.9) require that

\[ D \mathbf{v}_z = \dot{\mathbf{z}} \cdot \tilde{\mathbf{D}} \mathbf{B}_0 \times \nabla \mathbf{b}_z - (\mathbf{B}_0 \cdot \nabla) j_z, \quad (2.12) \]
\[
D\omega_z = (B_0 \cdot \nabla)\nabla^2 b_z - (D^2 B_0) \cdot \nabla b_z, \\
\left( \frac{\partial}{\partial t} - \frac{1}{\Lambda} \nabla^2 \right) b_z = (B_0 \cdot \nabla) v_z, \\
\left( \frac{\partial}{\partial t} - \frac{1}{\Lambda} \nabla^2 \right) j_z = (B_0 \cdot \nabla) \omega_z + \hat{z} \cdot DB_0 \times \nabla v_z,
\]
where \( \omega = \nabla \times \mathbf{v} \) and \( \mathbf{j} = \nabla \times \mathbf{b} \) are the dimensionless vorticity and perturbed current density; \( D = d/dz \).

We seek normal mode solution of (2.12)–(2.15) of the form (1.15) so that
\[
Dv_z = -i(Fj_z + \bar{F}b_z), \\
D\omega_z = i[F(D^2 - k^2)b_z - (D^2 F)b_z], \\
[\Lambda^{-1}(D^2 - k^2) - s]b_z = -iFv_z, \\
[\Lambda^{-1}(D^2 - k^2) - s]j_z = -i(F\omega_z - \bar{F}v_z),
\]
where
\[
F = k_x B_{0x} + k_y B_{0y}, \quad \bar{F} = D(k_x B_{0y} - k_y B_{0x}).
\]

We must solve the sixth order system (2.16)–(2.19) subject to three conditions at each of \( z = 0 \) and \( z = 1 \). We shall suppose that these boundaries are impermeable, perfectly conducting walls, so that
\[
v_z = b_z = Dj_z = 0, \quad z = 0, 1.
\]

3. THE LIMIT \( \Lambda \to \infty \). CRITICAL LAYER ANALYSIS
If we formally set \( \Lambda = \infty \) in (2.16)–(2.20), that sixth order system system collapses to the second order problem of finding \( s_0^2 \) such that
\[
s_0^2 D \left[ \frac{1}{F^2} D \left( \frac{b_z^{(0)}}{F} \right) \right] + F(D^2 - k^2)b_z^{(0)} - (D^2 F)b_z^{(0)} = 0, \\
\]

\[
\text{together with} \\
b_z^{(0)}(0) = b_z^{(0)}(1) = 0,
\]
possesses a nontrivial solution. Here \( s_0 \) is the growth rate to leading order in \( \Lambda \).

By multiplying (3.1) by \( b_z^{(0)*}/F \), integrating by parts and applying (3.2), it is easily seen that, if \( F \) has no zero in \([0, 1] \), i.e. if there is no critical level, then
\[
\int_0^1 \left[ \left( F^2 + \frac{s_0^2}{F^2} \right) \left| D \left( \frac{b_z^{(0)}}{F} \right) \right|^2 + k^2 |b_z^{(0)}|^2 \right] dz = 0.
\]

It follows that \( s_0^2 \) is real and negative, i.e. there is no instability. Supposing \( s_0 \neq 0 \), it is conceivable that this oscillation, which is purely sinusoidal for \( \Lambda = \infty \), might be growing
slowly for \( \Lambda \gg 1 \), i.e. \( s \) might have a positive real part of order \( \Lambda^{-1} \) for \( \Lambda \to \infty \). We have been unable to establish this; nor have we been able to show that the principle of the exchange of stabilities holds for the system (2.16)–(2.20), although our numerical work (see Section 5) in no case located a growing overstable mode. From now on we shall assume that \( s_0 = 0 \), so that

\[
F(D^2 - k^2)^{b_x^{(0)}} - (D^2 F)b_x^{(0)} = 0.
\]

(3.4)

More precisely, we suppose that \( s < O(1) \), so that it does not appear in the leading order equation (3.4). We shall regard (3.4) as governing the "outer solution" away from the boundaries at \( z = 0 \) and \( z = 1 \), and away also from critical levels.

There may be more than one critical level, but for simplicity we shall here suppose that there is only one, and that it is situated at \( z = z_c \), where \( 0 < z_c < 1 \). We then have

\[
F(z_c) = 0,
\]

(3.5)

and, in a sufficiently small neighborhood of \( z = z_c \),

\[
F(z) = F'(z - z_c),
\]

(3.6)

where \( F' \equiv F'(z_c) \neq 0 \). In this neighborhood, the solution of (3.4) we require may be written as a linear combination of the two solutions

\[
b_x^{(0)} = 1, \quad b_x^{(0)} = z - z_c,
\]

(3.7)

i.e. \( b_x^{(0)} \) is proportional to \( 1 + A(z - z_c) \). As for the classical tearing mode, \( A \) changes discontinuously at \( z = z_c \), i.e.

\[
b_x^{(0)} = \begin{cases} 1 + A_+(z - z_c), & z \to z_c^+, \\ 1 + A_-(z - z_c), & z \to z_c^- \end{cases}.
\]

(3.8)

It is to this discontinuity that tearing modes owe their instability.

To determine the discontinuity \( A_+ - A_- \) in \( Db_x \) at \( z = z_c \), or more precisely to find

\[
\Delta \equiv \left[ \frac{Db_x^{(0)}}{b_x^{(0)}} \right]_{z_c^+} - \left[ \frac{Db_x^{(0)}}{b_x^{(0)}} \right]_{z_c^-},
\]

(3.9)

it is necessary to analyse the structure of the critical layer. This will occupy the remainder of this Section.

It follows from (2.16)–(2.19) by elimination that

\[
D \left[ \frac{1}{F} \left\{ \frac{1}{\Lambda} (D^2 - k^2) - s \right\} \frac{1}{F} D \left[ \frac{1}{F} \left\{ \frac{1}{\Lambda} (D^2 - k^2) - s \right\} b_x \right] \right] \\
+ \frac{1}{\Lambda} D \left[ \frac{1}{F} \left\{ 2D \left( \frac{\bar{F}}{F} \right) Db_x + D^2 \left( \frac{\bar{F}}{F} \right) b_x \right\} \right] + F(D^2 - k^2)b_x - (D^2 F)b_x = 0.
\]

(3.10)
In the critical layer we may replace $F$ by (3.6) and use the stretched coordinate $\zeta$ in place of $z$, where

$$z = z_c + \zeta \delta. \quad (3.11)$$

Understanding $D$ now to be $d/d\zeta$, we find that, in the leading order,

$$D \left[ \frac{1}{\zeta^2} D \left( \frac{1}{\zeta^2} D \right)^2 \zeta^2 \right] + \left( \frac{1}{\zeta^2} \frac{2\alpha}{\zeta^2} \right) D \left( \frac{b_z}{\zeta} \right)$$

$$= p D \left( \frac{1}{\zeta^2} D^2 \right) \left( \frac{b_z}{\zeta} \right), \quad (3.12)$$

where

$$p = s\Lambda \delta^2 \quad (3.13)$$

is the scaled growth rate; $p \ll 1$, by the arguments of Section 1. The boundary layer thickness is

$$\delta = \frac{1}{|F_c'|^{1/2} \Lambda^{1/4}}, \quad (3.14)$$

and

$$\alpha = -\frac{\tilde{F}_c}{F_c'} = \left( \frac{\hat{\mathsf{J}} \cdot J_0 \times B_0}{J_0 \cdot B_0} \right) \zeta_c. \quad (3.15)$$

In our model (1.3), $\alpha \equiv 0$, and in what follows we shall suppose not only that $p \ll 1$ but also that $\alpha \ll 1$. We shall therefore replace (3.12) by

$$D \left[ \frac{1}{\zeta^2} D \left( \frac{1}{\zeta^2} D \right)^2 \zeta^2 \right] + \zeta D \left( \frac{b_z}{\zeta} \right)$$

$$= 2p D \left[ \frac{1}{\zeta^2} D^3 \left( \frac{b_z}{\zeta} \right) \right] - 2\alpha D \left[ \frac{1}{\zeta^2} D \left( \frac{b_z}{\zeta} \right) \right], \quad (3.16)$$

and regard the right-hand side as a perturbation.

The homogeneous equation,

$$D \left( \frac{1}{\zeta^2} D \right) \left( \frac{1}{\zeta^2} D \right) + \zeta \left( \frac{1}{\zeta} D^2 w \right) = 0, \quad (3.17)$$

obtained by setting the right-hand side of (3.16) zero, admits the solutions

$$w_1 = 1, \quad w_2 = \zeta. \quad (3.18)$$

In fact $w_2$ is also an exact solution of (3.16) and (3.12). It corresponds to the solution $b_z^{(0)}$ given in (3.7), and demonstrates that this solution passes through the critical layer unaffected, i.e. it is only the difference, $\Delta$, between $A_+$ and $A_-$ that is significant in (3.8). We may therefore take $A_+ = -A_- (= A, \text{say})$ without loss of generality, so that in the critical layer

$$b_z(-\zeta) \equiv b_z(\zeta), \quad (3.19)$$

9
and

\[ b_z \sim A|\zeta|, \quad \zeta \to \pm \infty, \quad (3.20) \]

where \( A = \frac{1}{2} \Delta \delta; \) see (3.8), (3.9) and (3.11). The objective is to determine \( A(p, \alpha). \)

Since we are interested only in even \( b_z, \) we need consider only two of the remaining four solutions of (3.17), namely

\[
w_3 = \sum_{n=0}^{\infty} \frac{(-1)^n(\zeta/2^{3/2})^{8n}}{n! \Gamma(n + \frac{1}{4}) \Gamma(n + \frac{3}{8}) \Gamma(n + \frac{5}{8}) (n - \frac{1}{8})} \quad (3.21)
\]

\[
w_4 = \sum_{n=0}^{\infty} \frac{(-1)^n(\zeta/2^{3/2})^{8n+6}}{n! \Gamma(n + \frac{3}{4}) \Gamma(n + \frac{9}{8}) \Gamma(n + \frac{11}{8}) (n + \frac{5}{8})}. \quad (3.22)
\]

It may be noted that these are entire functions, and that the series converge for all complex \( \zeta. \) Also, if \( k \) is an integer,

\[
w_3(\zeta e^{k\pi i/4}) \equiv w_3(\zeta), \quad w_4(\zeta e^{k\pi i/4}) \equiv e^{3k\pi i/4} w_4(\zeta). \quad (3.23)
\]

They therefore satisfy (3.19); \( k = 4. \) We shall therefore confine attention to the sector

\[
|\arg \zeta| < \frac{1}{8} \pi. \quad (3.24)
\]

We may write \( w_3 \) and \( w_4 \) as contour integrals:

\[
w_3 = \frac{1}{2\pi i} \int_{C_3} \frac{(\zeta/2^{3/2})^{8s} \csc s \pi}{\Gamma(s+1) \Gamma(s+\frac{1}{4}) \Gamma(s+\frac{3}{8}) \Gamma(s+\frac{5}{8}) (s-\frac{1}{8})} ds \quad (3.25)
\]

\[
w_4 = \frac{1}{2\pi i} \int_{C_4} \frac{(\zeta/2^{3/2})^{8s+6} \csc s \pi}{\Gamma(s+1) \Gamma(s+\frac{7}{4}) \Gamma(s+\frac{9}{8}) \Gamma(s+\frac{11}{8}) (s+\frac{5}{8})} ds \quad (3.26)
\]

Here \( C_4 \) passes from \( s = \infty + 0i \) to \( s = \infty - 0i, \) to the left of \( s = 0 \) but to the right of \( s = -\frac{5}{8}; \) \( C_3 \) is similar but is distorted to avoid surrounding \( s = \frac{1}{8}. \) See Figures 1. Both contours surround the poles of the integrands at \( s = n, \) where \( n \) is a non-negative integer; because of the \( \Gamma(s+1) \) factors, the negative integers are not poles of the integrands. Both integrands have saddle points near

\[
s = \frac{1}{8} \zeta^2 e^{\pm i\pi/4}, \quad (3.27)
\]

and, by standard applications of the method of steepest descent, it is found that, to leading order as \( \zeta \to \infty \) in sector (3.24),

\[
w_3 \sim \frac{1}{(2\pi)^{3/2}} \left( \frac{8}{\zeta^2} \right)^{3/4} e^{\zeta^2/2\sqrt{2}} \cos \left( \frac{\zeta^2}{2\sqrt{2}} - \frac{3\pi}{16} \right), \quad (3.28)
\]

\[
w_4 \sim \frac{1}{(2\pi)^{3/2}} \left( \frac{8}{\zeta^2} \right)^{3/4} e^{\zeta^2/2\sqrt{2}} \cos \left( \frac{\zeta^2}{2\sqrt{2}} - \frac{15\pi}{16} \right). \quad (3.29)
\]
We now return to the inhomogeneous equation (3.16), and write its solution as
\[ b_\zeta = 1 + pb_\zeta - \alpha b_{\zeta \alpha}, \tag{3.30} \]
where, to leading order in \( p \) and \( \alpha \),
\[ D \left( \frac{1}{\zeta^2} D \right) \left( \frac{1}{\zeta^2} D \right) + \zeta^2 \right] \frac{1}{\zeta} D^2 b_\zeta = \frac{72}{\zeta^7}, \tag{3.31} \]
\[ D \left( \frac{1}{\zeta^2} D \right) \left( \frac{1}{\zeta^2} D \right) + \zeta^2 \right] \frac{1}{\zeta} D^2 b_{\zeta \alpha} = \frac{8}{\zeta^5}. \tag{3.32} \]
The right-hand sides are obtained by setting \( b_\zeta = 1 \), the dominant part of (3.30), into the right-hand side of (3.16). Particular solutions satisfying (3.19) are
\[ b^P_{\zeta \zeta} = c_\rho \sum_{n=0}^\infty \frac{(-1)^n (\zeta / 2^{3/2})^{8n+2}}{\Gamma \left( n + \frac{3}{2} \right) \Gamma \left( n + \frac{5}{8} \right) \Gamma \left( n + \frac{7}{8} \right) \Gamma \left( n + \frac{9}{4} \right)} \tag{3.33} \]
\[ b^P_{\zeta \alpha} = c_\alpha \sum_{n=0}^\infty \frac{(-1)^n (\zeta / 2^{3/2})^{8n+4}}{\Gamma \left( n + \frac{3}{4} \right) \Gamma \left( n + \frac{7}{8} \right) \Gamma \left( n + \frac{9}{8} \right) \Gamma \left( n + \frac{3}{2} \right) \Gamma \left( n + \frac{3}{8} \right)}. \tag{3.34} \]
where
\[ c_\rho = \frac{\sqrt{\pi}}{8} \Gamma \left( \frac{1}{4} \right) \Gamma \left( \frac{5}{8} \right) \Gamma \left( \frac{7}{8} \right), \tag{3.35} \]
\[ c_\alpha = \frac{\pi^{3/2}}{8 \sin(\pi/8)} \Gamma \left( \frac{3}{4} \right). \tag{3.36} \]

After forming integral representations similar to (3.25) and (3.26) we find from a steepest descent analysis of the saddle points near (3.27) that, as \( \zeta \to \infty \) in sector (3.24),
\[ b^P_{\zeta \zeta} \sim \frac{c_\rho}{(2\pi)^{3/2}} \left( \frac{8}{\zeta^2} \right)^{3/4} e^{\zeta^2 / 2 \sqrt{2}} \cos \left( \frac{\zeta^2}{2 \sqrt{2}} - \frac{7\pi}{16} \right), \tag{3.37} \]
\[ b^P_{\zeta \alpha} \sim \frac{c_\alpha}{(2\pi)^{3/2}} \left( \frac{8}{\zeta^2} \right)^{3/4} e^{\zeta^2 / 2 \sqrt{2}} \cos \left( \frac{\zeta^2}{2 \sqrt{2}} - \frac{11\pi}{16} \right). \tag{3.38} \]

Evidently the particular integrals (3.33) and (3.34) do not match to (3.20), but by adding appropriate multiples of \( w_3 \) and \( w_4 \) we can eliminate the exponentially increasing terms (3.37) and (3.38). We therefore replace \( b^P_{\zeta \zeta} \) and \( b^P_{\zeta \alpha} \) by
\[ b_{\zeta \zeta} = b^P_{\zeta \zeta} - 2^{1/2} c_\rho w_3 - c_\rho w_4, \tag{3.39} \]
\[ b_{\zeta \alpha} = b^P_{\zeta \alpha} - c_\alpha w_3 - 2^{1/2} c_\alpha w_4. \tag{3.40} \]
Returning to the series representation (3.21), (3.22), (3.33) and (3.34) we find the more convenient forms
\[ b_{\zeta \zeta} = -\frac{c_\rho}{\sqrt{2}} (W + W^*), \tag{3.41} \]
\[ b_{\zeta \alpha} = -\frac{c_\alpha}{\sqrt{2}} [e^{3i\pi/4}W + e^{-3i\pi/4}W^*], \tag{3.42} \]
where
\[ W = \sum_{m=0}^{\infty} \frac{(-1)^m (\zeta^2 e^{-\pi i/4}/8)^m}{\Gamma \left( \frac{m}{4} + \frac{1}{4} \right) \Gamma \left( \frac{m}{4} + \frac{3}{8} \right) \Gamma \left( \frac{m}{4} + \frac{5}{8} \right) \Gamma \left( \frac{m}{4} + 1 \right) \left( \frac{m}{4} - \frac{1}{8} \right)} . \]  
(3.43)

It may be shown (see Appendix) that
\[ W \sim \frac{8(2\pi)^{1/2} e^{\pi i/8}}{\Gamma \left( \frac{1}{8} \right) \Gamma \left( \frac{3}{8} \right) \Gamma \left( \frac{5}{8} \right)} \zeta , \quad \zeta \to \infty . \]  
(3.44)

It follows from (3.20), (3.30), (3.41) and (3.42) that
\[ A = A_p p + A'_\alpha \alpha , \]  
(3.45)

where
\[ A_p = 2\pi^2 \frac{\Gamma \left( \frac{1}{8} \right) \Gamma \left( \frac{3}{8} \right)}{\Gamma \left( \frac{3}{4} \right) \Gamma \left( \frac{1}{8} \right) \Gamma \left( \frac{5}{8} \right)} \frac{1}{2} \approx 1.5032 \ldots , \]  
(3.46)
\[ A'_\alpha = \frac{2\pi^2}{\Gamma \left( \frac{1}{8} \right) \Gamma \left( \frac{3}{8} \right)} \approx 1.1053 \ldots \]  
(3.47)

These results, which are valid only for \(|p| \ll 1\) and \(|\alpha| \ll 1\), were verified numerically by using a specially constructed program for critical layer integration (see Appendix B). It was found that, for larger values of \(\alpha\), (3.45) is replaced by
\[ A = A_p(\alpha)p + A_{\alpha}(\alpha) , \]  
(3.48)

where \(A_p(\alpha)\) and \(A_{\alpha}(\alpha)\) took the values shown in Table 1 for \(-10 \leq \alpha \leq 10\). The numerical value of \(A'_\alpha(0)\) was 1.1056, which should be compared with (3.47).

Unlike classical tearing, for which the growth rate for given \(k\) is determined when \(\Lambda \gg 1\) not by \(B_{0x}\) and \(B_{0y}\) individually but only in the combination, \(F\), the rate of tearing in a highly rotating fluid is affected by the magnetic helicity at each critical level, as measured by \(\alpha\). Let us compare tearing for \(k_x = k_y\) in the field (1.3) for \(q = \frac{3}{2} \pi\), i.e.
\[ B_{0}^{(1)} = \hat{x} \cos \frac{3}{2} \pi z + \hat{y} \sin \frac{3}{2} \pi z , \]  
(3.49)

with tearing in the field
\[ B_{0}^{(2)} = \hat{x} \left( \cos \frac{3}{2} \pi z + \frac{3\alpha}{2\sqrt{2}} \cos \pi z \right) + \hat{y} \left( \sin \frac{3}{2} \pi z - \frac{3\alpha}{2\sqrt{2}} \cos \pi z \right) . \]  
(3.50)

Since \(F\) is the same for each, marginal stability is decided by the same value of \(A = \frac{1}{2} \Delta \delta\) at the single critical point \(z_c = \frac{1}{2}\). By (3.15) we have \(\alpha^{(1)} = 0\), \(\alpha^{(2)} = \alpha\). For \(\alpha > 0\) we see from (3.45) or (3.48) that \(p\) is larger in case (3.49) than in case (3.50), as therefore is the growth rate, \(s\), of the instability.
4. The Limit $\Lambda = \infty$. Examples

The principal form for $B_0$ studied in this paper is (1.3) for which [see also (1.16)]

$$F = k \cos(\theta - qz), \quad \bar{F} = -qF.$$  \hspace{1cm} (4.1, 4.2)

Clearly $F$ is unchanged under the transformation

$$z \to 1 - z, \quad \theta \to q - \theta,$$  \hspace{1cm} (4.3S)

while its sign is reversed if

$$z \to 1 - z, \quad \theta \to \pi + q - \theta.$$  \hspace{1cm} (4.3A)

In either case (3.10) is unaltered. At first sight it might seem that, once one has obtained a solution for one choice of $k$, we could obtain a second by one of the transformations (4.3S) or (4.3A), and that in this new solution

$$b_z(z) \to b_z(1 - z).$$  \hspace{1cm} (4.4)

This hope is, however, frustrated by the boundary conditions on $b_z$ implied by (2.21). This may also be seen directly from (2.16)–(2.19): the symmetries

$$b_z \to b_z, \quad v_z \to v_z, \quad \omega_z \to -\omega_z, \quad j_z \to -j_z,$$  \hspace{1cm} (4.5S)

and

$$b_z \to -b_z, \quad v_z \to -v_z, \quad \omega_z \to \omega_z, \quad j_z \to j_z,$$  \hspace{1cm} (4.5A)

expected from (4.3S) and (4.3A) are destroyed by the $\bar{F}$ terms in (2.16) and (2.19). There are however exceptional choices of $k$, namely

$$\theta = \frac{1}{2} q + m\pi, \quad \bar{\theta} = \frac{1}{2} q + \left(m + \frac{1}{2}\right)\pi,$$  \hspace{1cm} (4.6S)

for which $F$ is symmetric or antisymmetric when $m$ is an integer. In these cases solutions satisfying symmetries (4.5S) or (4.5A) exist. Moreover, even for general $q$, the effectiveness of the $\bar{F}$ terms in (2.16) and (2.19) diminishes continually as $\Lambda$ increases and, to leading order in the limit $\Lambda \to \infty$, (4.3S) are (4.3A) do imply new solutions obtained respectively from (4.5S) and (4.5A).

According to (4.1), if $n\pi < q < (n + 1)\pi$ where $n(\geq 0)$ is an integer, there may be either $n$ or $n + 1$ critical levels in $(0, 1)$, depending on the choice of $\theta$. The existence of a critical level does not guarantee that an instability exists in the limit $\Lambda \to \infty$, as the case $n = 0$ described below will show. In general the greater the number of critical levels, the more magnetic energy will become available to an instability, and the greater the growth rate, $s$.

From now onwards, suppose that

$$0 < q \leq 2\pi.$$  \hspace{1cm} (4.7)
There is only one critical level (at \( z = z_c \), say) if either
\[(i)\] \( 0 < q \leq \pi \), or
\[(ii)\] \( \pi < q < 2\pi \) and \( 1 - \pi/q < z_c < \pi/q \).
There are two critical levels (at \( z = z_{c1}, z_{c2} \), say) if
\[(iii)\] \( \pi < q \leq 2\pi \), and \( 0 < z_{c1} < 1 - \pi/q \); then \( z_{c2} = z_{c1} + \pi/q \), so that \( \pi/q < z_{c2} < 1 \).
In all these cases
\[\alpha = 0, \quad |F_0'| = kq. \tag{4.8, 4.9}\]

We shall now find, in the limit \( \Lambda \to \infty \), the unstable modes in cases (i)-(iii). Away from critical points and boundary layers, the solution to leading order obeys (3.4), which reduces for field (1.3) to
\[D^2 b^{(0)}_z = (k^2 - q^2) b^{(0)}_z. \tag{4.10}\]

First consider cases (i) and (ii).

(A) Suppose that \( k > q \) and let \( \beta = (k^2 - q^2)^{1/2} \). Then, by (3.2),
\[b^{(0)}_z = \begin{cases} \sinh \beta z / \sinh \beta z_c, & \text{if } z \leq z_c; \\ \sin \beta (1 - z) / \sin \beta (1 - z_c), & \text{if } z \geq z_c, \end{cases} \tag{4.11}\]
so that
\[\Delta = \left[ \frac{D b^{(0)}_z}{b^{(0)}_z} \right]_c = -\beta [\coth \beta z_c + \coth \beta (1 - z_c)]; \tag{4.12}\]
see (3.9). But according to the critical layer analysis of Section 3, we must have
\[\Delta > 0, \quad \text{for instability.} \tag{4.13}\]
It follows that disturbances of short wavelength are always stable.

(B) Suppose next that \( k < q \) and let \( \gamma = (q^2 - k^2)^{1/2} \). In place of (4.11) and (4.12) we have
\[b^{(0)}_z = \begin{cases} \sin \gamma z / \sin \gamma z_c, & \text{if } z \leq z_c; \\ \sin \gamma (1 - z) / \sin \gamma (1 - z_c), & \text{if } z \geq z_c, \end{cases} \tag{4.14}\]
\[\Delta = \frac{\gamma \sin \gamma}{\sin \gamma z_c \sin \gamma (1 - z_c)}. \tag{4.15}\]
Since \( \gamma < q \), we have
\[0 < \gamma z_c < q z_c < \pi, \quad 0 < \gamma (1 - z_c) < q (1 - z_c) < \pi. \tag{4.16}\]
The denominator of (4.15) is therefore necessarily positive. Thus instability can arise only if \( \pi < \gamma < 2\pi \). Since \( \gamma < q \), this means that an instability cannot occur in case (i), but may arise in case (ii).

To find the growth rate of the instability in case (ii), it is merely a matter of equating \( \Delta \) is (4.15) to the expression \( 2\Lambda/\delta \) implied by (3.20), \( \Lambda \) being given by (3.45), (3.46) and (4.8). We obtain, using also (3.14) and (4.9),
\[p \sim \frac{\gamma |\sin \gamma|}{3.0065(kq)^{1/2} \sin \gamma z_c \sin \gamma (1 - z_c)} \Lambda^{-1/4}, \quad \Lambda \to \infty, \tag{4.17}\]
or by (3.13)

\[ s \sim \frac{\gamma |\sin \gamma| (kq)^{1/2}}{3.0065 \sin \gamma z_c \sin \gamma (1 - z_c)} \Lambda^{-3/4}, \quad \Lambda \to \infty. \tag{4.18} \]

As an example, suppose that

\[ q = \frac{3}{2} \pi, \quad k_x = k_y = 1. \tag{4.19} \]

Then \( z_c = \frac{1}{2} \), and we obtain from (4.17) and (4.18)

\[ p \sim \frac{0.93057}{\Lambda^{1/4}}, \quad s \sim \frac{6.2016}{\Lambda^{3/4}}, \quad \Lambda \to \infty. \tag{4.20, 4.21} \]

For \( \Lambda = 10^6 \) these give

\[ p \doteq 0.029427, \quad s \doteq 1.9611 \times 10^{-4}, \tag{4.22, 4.23} \]

the latter of which should be compared with the numerically determined value of \( s = 1.6549 \times 10^{-4} \).

Despite the small value (4.22) of \( p \), the error (4.23) is a surprisingly large 20%. There are three reasons for this. First, our expansion is in powers of \( \delta \), i.e. in powers of \( \Lambda^{-1/4} \), see (3.14). It is \( \Lambda^{-1/4} \), and not \( \Lambda^{-1} \), that measures the approach to the asymptotic limit \( \Lambda \to \infty \). For \( \Lambda = 10^6 \), \( \Lambda^{-1/4} \) is only about 0.032. Second, (3.45) gives only the first term in the expansion of the critical layer solution of (3.12) is powers of \( p \). This particular source of error could be remedied by solving (3.12) numerically for arbitrary \( p \). This would, however, be pointless because of the third, and most serious, source of error. Equation (3.12) is derived from the full equation (3.10) by retaining only the first nonzero term in the expansion (3.6) of \( F \). An error of relative order \( \delta \) in \( s \) enters instantly. A numerical study of the present case (4.19) has shown that, in the next order, (4.21) is replaced by

\[ s = \frac{6.2016}{\Lambda^{3/4}} - \frac{30.58}{\Lambda}, \tag{4.24} \]

and this gives \( s = 1.6553 \times 10^{-4} \) for \( \Lambda = 10^6 \), which is in much better agreement with the numerically determined value \( s = 1.6549 \times 10^{-4} \). The basis for (4.25) is shown in Table 2.

The development of a critical layer structure around \( z = z_c = \frac{1}{2} \) is clearly seen in Figures 2–5 in which \( b_z, j_z, -iv_z, -i\omega_z \) are shown for \( \Lambda = 10^3, 10^4, 10^5 \) and 10^6. Figures 5, for \( \omega_z \), are particularly striking. Also to be discerned in several of these figures are the boundary layers, of thickness \( O(\Lambda^{-1/2}) \), adjacent to \( z = 0 \) and \( z = 1 \), but which affect \( s \) and the instability only passively, i.e. in no essential way.

Figures 6–9 show further comparisons between the solutions for \( \Lambda = 10^4 \) of the full equations (2.16)–(2.19), and the asymptotic theory. In panels (a) and (b) in each of these figures, the numerical solutions are compared with the mainstream solutions, which for \( b_z^{(0)} \) is given by (4.14) and which for the remaining functions are given by the \( \Lambda = \infty \) forms of (2.16)–(2.19), namely

\[ j_z^{(0)} = -q z^{(0)} + \frac{s}{F} D \left( \frac{b_z^{(0)}}{F} \right), \quad -iv_z^{(0)} = -s \left( \frac{b_z^{(0)}}{F} \right), \quad -i\omega_z^{(0)} = -\frac{s^2}{F^2} D \left( \frac{b_z^{(0)}}{F} \right). \tag{4.25} \]
Although the similarities between panels (b) and (c) are striking in each case, a detailed comparison is impossible for two reasons. First, it should not be forgotten that, according to the analysis of Section 3, the solutions of the critical layer equations are non-unique; an arbitrary multiple of \( w_2 = \zeta \) can be added to \( b_2 \). The addition of \( \bar{A} \delta \zeta [\text{where } \bar{A} = \frac{1}{2}(A_+ + A_-); \text{see (3.8))] \) leaves \( \omega_z \) and \( j_z \) unaltered to leading order in the critical layer, but adds \(-i\bar{A} p F_c^2 \delta^2 \) to \( v_z \), i.e. it displaces the zero of \(-iv_z\) at \( \zeta = 0 \) by \(-s\bar{A}/F_c^2\); see (4.25). Since \( \bar{A} \) is almost zero in the present, almost symmetric case, this displacement is not apparent in panel 8(b). Second, in the critical layer scaling (3.11) — see also (B9) later — \( b_z = O(1) \) but \( j_z = O(\delta^{-1}) \) to leading order. In the next approximation, correction terms are added that are \( O(\delta) \) and \( O(1) \), respectively. The fact that \( j_z \) is unaffected to leading order by the addition of \( \bar{A} \delta \zeta \) to \( b_z \) means that \( j_z \) is in error by \( O(1) \), rather than by \( O(\delta^{-1}) \). But this error cannot be derived from the analysis of Section 3, although its existence is clearly seen when panels 7(a) and (b) are compared with panel 7(c).

We now turn to the case (iii) of two critical points. In place of (4.15) we now have, for some \( \lambda_1/\lambda_2 \),

\[
b_z^{(0)} = \begin{cases} \\
\lambda_1 \sin \frac{\gamma z}{\sin \gamma z_{c_1}}, & \text{for } 0 \leq z \leq z_{c_1}; \\
\lambda_2 \sin \frac{\gamma (1 - z)}{\sin \gamma (1 - z_{c_2})}, & \text{for } z_{c_2} \leq z \leq 1; \\
[\lambda_1 \sin \frac{\gamma (z_{c_2} - z)}{\sin \gamma (z_{c_2} - z_{c_1})} + \lambda_2 \sin \frac{\gamma (z - z_{c_1})}{\sin \gamma (z_{c_2} - z_{c_1})}], & \text{for } z_{c_1} < z < z_{c_2}.
\end{cases}
\]  

(4.26)

From each critical point we get one expression for \( \Delta \), namely

\[
\frac{D b_z^{(0)}}{b_z^{(0)}} = \gamma \left[ \frac{\lambda_2}{\lambda_1} \cosec \gamma (z_{c_2} - z_{c_1}) - \cot \gamma (z_{c_2} - z_{c_1}) - \cot \gamma z_{c_1} \right].
\]  

(4.27)\text{1}

\[
\frac{D b_z^{(0)}}{b_z^{(0)}} = \gamma \left[ \frac{\lambda_1}{\lambda_2} \cosec \gamma (z_{c_2} - z_{c_1}) - \cot \gamma (z_{c_2} - z_{c_1}) - \cot \gamma (1 - z_{c_2}) \right].
\]  

(4.27)\text{2}

Since \( \delta \) is the same for each boundary layer by (3.14) and (4.10), these two expression must both equal \( 2A/\delta \), and therefore must be equal to each other, so that

\[
\frac{\lambda_1}{\lambda_2} - \frac{\lambda_2}{\lambda_1} = \frac{\sin (\gamma \pi / q) \sin \gamma (z_{c_1} + z_{c_2} - 1)}{\sin \gamma z_{c_1} \sin \gamma (1 - z_{c_2})},
\]  

(4.28)

\[
\Delta = \frac{\gamma}{2 \sin (\gamma \pi / q) \sin \gamma z_{c_1} \sin \gamma (1 - z_{c_2})}
\]

\[
\cdot \left\{ \left[ (\sin (\gamma \pi / q) \sin \gamma (z_{c_1} + z_{c_2} - 1))^2 + (2 \sin \gamma z_{c_1} \sin \gamma (1 - z_{c_2}))^2 \right]^{1/2} + \cos \gamma - \cos (\gamma \pi / q) \cos \gamma (z_{c_1} - z_{c_2} - 1) \right\}.
\]

(4.29)

Equating this to \( 2A/\delta = 3.0065p/\delta = 3.0065sA\delta \) with \( \delta = (qk)^{-1/2}A^{-1/4} \), we obtain \( p \) and \( \delta \) as before.

\text{4}See, however, the displacement of \( v_z \) away from zero at the \( z = \frac{1}{\delta} \) critical point in Figures 13(c) and 13(d) below.
For example, if
\[ q = \frac{3}{2} \pi, \quad k_x = -k_y = 1, \quad (4.30) \]
then \( z_{c1} = \frac{1}{6} \) and \( z_{c2} = \frac{5}{6} \). By (4.28) and (4.29), \( \lambda_1 = \lambda_2 \) and
\[ \Delta = \frac{-2\gamma \sin \frac{1}{3} \gamma \cos \frac{1}{2} \gamma}{\sin \frac{2}{3} \gamma \sin \frac{1}{6} \gamma} \cong 57.143, \quad (4.31) \]
since \( \gamma \approx 4.4952 \). So we obtain
\[ p \sim \frac{7.3624}{\Lambda^{1/4}}, \quad s \sim \frac{49.065}{\Lambda^{3/4}}, \quad \Lambda \rightarrow \infty. \quad (4.32, 4.33) \]
For \( \Lambda = 10^6 \) these give
\[ p = 0.23282, \quad s = 1.5516 \times 10^{-3}, \quad (4.34, 4.35) \]
the latter of which should be compared with the numerically determined value of \( s = 1.1013 \times 10^{-3} \). The agreement is not as good as in the case of one critical layer; the error is about 40%. This may be attributed to the reasons given below (4.23) combined with the fact that \( p \) given by (4.34) is larger than the \( p \) for the earlier case (4.22).

It is noteworthy how much greater the growth rate (4.35) is than (4.23). This is because two critical layers are twice as effective as one in reconnecting field lines of \( B_0 \) and converting their magnetic energy into perturbation energy.

5. Numerical Results
We carried out extensive numerical integrations of the basic equations (2.16)-(2.20), both for finite values of \( \Lambda \) and, in a suitably scaled form, for \( \Lambda = \infty \), i.e. for the critical layer structure. We shall postpone until Appendix B the description of the numerical techniques we employed and the tests of accuracy that we applied. Here we describe only our principal findings.

For every choice of \( k, \Lambda \) and \( \alpha \), our programs located both the real and complex eigenvalues \( s \). Although complex \( s \) were found, we discovered that in all cases the most unstable mode (i.e. the one for which \( \text{Re}(s) \) is greatest) is direct. In our discussion below, we focus on this real eigenvalue alone.

We always adopted (1.2) for \( B_0 \). We investigated three cases:
\[ q = \frac{1}{2} \pi, \quad q = \frac{3}{2} \pi, \quad q = 2\pi. \quad (5.1) \]
In the first of these we were unsuccessful in locating any \( k \) or \( \Lambda \) for which \( s > 0 \). We conjecture therefore that unstable modes do not exist when \( q = \frac{1}{2} \pi \). This conclusion is consistent with the results of the asymptotic analysis developed in Sections 3 and 4.

We subjected the case \( q = \frac{3}{2} \pi \) to special scrutiny. We searched for the maximum growth rate, \( s_c(\Lambda) \), over the range \( 0 < \Lambda < 4000 \), i.e. we sought, for a number of values of \( \Lambda \), the value of \( k (= k_c(\Lambda), \text{say}) \) for which \( s(k, \Lambda) \) is a maximum, and for which therefore
\[ (\nabla ks)_c = 0. \quad (5.2) \]
There were two such local maxima, which we call ‘mode 1’ and ‘mode 2’. As \( \Lambda \) is increased from zero, ‘mode 1’ is the first to become unstable, \( s_c \) becoming positive at \( \Lambda = \Lambda_c \approx 22.17 \). As \( \Lambda \) is increased further, mode 2 becomes the more unstable; see Figure 10 where \( s_c(\Lambda) \) is plotted for both modes.

Of particular interest is the fact that, as \( \Lambda \) increases, \( k_c \) evolves in such a way that one of the two critical levels approaches a boundary, that critical layer eventually merging with the boundary layer. At that stage, the asymptotic analysis of Section 3 becomes invalid, and it is not even certain that \( s_c \) is proportional to \( \Lambda^{-3/4} \), as \( \Lambda \to \infty \), though it clearly cannot tend to zero more rapidly than that. In the case of mode 1, it is the critical layer at \( z = z_{c1} \) that moves into the boundary layer at \( z = 1 \); in the case of mode 2, it is the critical layer at \( z = z_{c2} \) that moves to \( z = 0 \); see Table 3.

The evolution of the eigenfunction for \( k = k_c \) is shown as \( \Lambda \) increases in Figures 11–14 for mode 1. The development of the critical layer at \( z = z_{c1} \) near \( z = \frac{1}{3} \) is apparent, while at the same time the approach of the critical layer at \( z = z_{c2} \) towards \( z = 1 \) is obvious.

When \( q = 2\pi \) there are always two critical layers in the interval. Because of the greater shearing of \( B_0 \), the growth rate of the instability is always larger than when \( q = \frac{3}{2} \pi \). For instance \( \Lambda_c \) is 16.78, instead of 22.17 for \( q = \frac{3}{2} \pi \). The maximum attainable growth rate is 0.3435, instead of 0.1140 for \( q = \frac{3}{2} \pi \). When \( q = 2\pi \), mode 1 is always more unstable than mode 2.

6. Conclusions
In this paper we have studied a classical phenomenon in plasma physics: resistive instabilities. We have had in mind, however, geophysical and astrophysical systems which, though in principle prone to such instabilities, are rotating rapidly, and in which therefore Coriolis forces greatly exceed inertial forces. Such systems are generally geometrically complicated, but we have felt that, in a first analysis, we should study the canonical example of tearing: the magnetic instability of the sheet pinch.

We have found that the tearing mode is strongly suppressed by rotation; the growth rate of the instability is only \( O(\tau_\eta^{-3/4} \tau_\tau^{-1/4}) \), where \( \tau_\eta = 2\Omega \mu \rho L^2 / B^2 \) is the relevant dynamical timescale and \( \tau_\eta = \mu \sigma L^2 \) is the diffusive timescale for a field of strength \( B \) and characteristic scale \( L \). This may be compared with the \( O(\tau_\eta^{-3/5} \tau_A^{-2/5}) \) growth rate of classical theory, where \( \tau_A = L / V_A \) is the appropriate dynamical timescale when inertial forces dominate Coriolis forces; \( V_A \) is the Alfvén velocity. The ratio of these two growth rates is \( \Lambda^{-3/10} (V_A / \Omega L)^{2/5} \), where \( \Omega L \gg V_A \) when rotation is large. Instability can then only arise if the conductivity of the fluid is sufficiently high, i.e. if the Elsasser number, \( \Lambda = \sigma B^2 / 2 \Omega \rho \) is \( O(1) \). It follows that when tearing occurs in a rapidly rotating fluid, the instability grows much more slowly than it would in a non-rotating fluid.

We have found that, when \( \Lambda \to \infty \), the structure of the tearing mode consists of “exterior” regions in which the solution satisfies, to a first approximation, the equations governing an ideal (perfectly conducting) fluid, and “interior” regions of two types: boundary layers of thickness \( O(\Lambda^{-1/2}) \) at the walls, and critical layers of thickness \( O(\Lambda^{-1/4}) \) surrounding critical levels (if any), at which the perturbation is of interchange type (\( k \cdot B_0 = 0 \)). We have studied in detail the case in which the critical layers and boundary layers do not overlap. In this case, the boundary layers are completely passive (i.e.
do not influence the growth rate to leading order); the instability draws its energy from reconnection within the critical layers. We have presented numerical evidence, however, that energy is extracted with slightly greater efficiency when a boundary lies asymptotically within a critical layer, for in this case the growth rate is slightly larger, so that the mode of maximum instability occurs when a critical level is at a wall.

We have also shown that the sheet pinch is more unstable when $\mathbf{J}_0 \times \mathbf{B}_0 / \mathbf{J}_0 \cdot \mathbf{B}_0$ is antiparallel to $\mathbf{Ω}$ in the equilibrium state than when it is parallel.

Acknowledgements

We are grateful to our colleague, Nick Grossman, for pointing out to us how Mellin transforms could be elegantly used to remove gamma functions from (3.43), so making it possible to obtain (3.44), and also to our colleague Heinz Kreiss for his advice on the integrations described in Appendix B. We thank Dr. Steve Moler for supplying the subroutines we used in applying the QZ algorithm to our numerical work. One of us (WK) is grateful to the National Science Foundation for support under grant EAR-8846267; the other of us (PHR) thanks NSF for the same reason, and also thanks the Office of Naval Research for partial support under a URI grant N00014-86-K-0641.

References


APPENDIX A: BEHAVIOR OF $b_2$ AT THE EDGE OF THE CRITICAL LAYER.

We have seen in Section 3 that the solution at the edge of the critical layer is determined by the behavior as $\zeta \to +\infty$ of

$$W = \sum_{m=0}^{\infty} \frac{(-1)^m (\zeta^2 e^{-i\pi/4}/8)^m}{\Gamma\left(\frac{m}{4} + \frac{1}{4}\right) \Gamma\left(\frac{m}{4} + \frac{1}{2}\right) \Gamma\left(\frac{m}{4} + \frac{5}{8}\right) \Gamma\left(\frac{m}{4} + 1\right) \left(\frac{m}{4} - \frac{1}{8}\right)^m}.$$  \hfill (A1)

The leading term in its asymptotic expansion is $K\zeta$, where $K$ is to be found. To this end we write

$$K = \left(\frac{dW}{d\zeta}\right)_{\zeta=\infty} = \int_0^{\infty} \frac{d^2W}{d\zeta^2} d\zeta = -2\sqrt{2} e^{-i\pi/4} \int_0^{\infty} P_0 \frac{dz}{z^{7/8}} = -2\sqrt{2} e^{-i\pi/4} \mathcal{M}(P_0; \frac{1}{8}),$$  \hfill (A2)

where $\mathcal{M}(P_0; s)$ is the Mellin transform of

$$P_0 = \sum_{m=0}^{\infty} \frac{(-1)^m e^{-i\pi m/4} z^{m/4}}{\Gamma\left(\frac{m}{4} + \frac{1}{4}\right) \Gamma\left(\frac{m}{4} + \frac{1}{2}\right) \Gamma\left(\frac{m}{4} + \frac{5}{8}\right) \Gamma\left(\frac{m}{4} + \frac{7}{8}\right)}. \hfill (A3)$$

The evaluation of $\mathcal{M}(P_0; s)$ rests on the result that, if $\mathcal{L}(P, t)$ is the Laplace transform of a function $P(z)$, then

$$\mathcal{M}\left[\mathcal{L}(z^{\nu-1}P(z), t); s\right] = \Gamma(s) \mathcal{M}[P(z); \nu - s];$$  \hfill (A4)

e.g. see the result (o') on p3 of Oberhettinger (1974). Evidently

$$\mathcal{L}\left(z^{\nu-1}P_0(z), t\right) = \sum_{m=0}^{\infty} \frac{(-1)^m e^{-i\pi m/4}}{\Gamma\left(\frac{m}{4} + \frac{1}{4}\right) \Gamma\left(\frac{m}{4} + \frac{1}{2}\right) \Gamma\left(\frac{m}{4} + \frac{5}{8}\right) \Gamma\left(\frac{m}{4} + \frac{7}{8}\right) \frac{\Gamma\left(\frac{m}{4} + \nu\right)}{t^{m/4} + \nu}},$$  \hfill (A5)

a series that converges for all $|t| > 0$. Taking the Mellin transform of (A5), we obtain

$$\mathcal{M}\left[\mathcal{L}(z^{\nu-1}P_0(z), t); s\right] = \mathcal{M}[P_1(z); \nu - s],$$  \hfill (A6)

where

$$P_1 = \sum_{m=0}^{\infty} \frac{(-1)^m \Gamma\left(\frac{m}{4} + \nu\right) e^{-i\pi m/4} z^{m/4}}{\Gamma\left(\frac{m}{4} + \frac{1}{4}\right) \Gamma\left(\frac{m}{4} + \frac{1}{2}\right) \Gamma\left(\frac{m}{4} + \frac{5}{8}\right) \Gamma\left(\frac{m}{4} + \frac{7}{8}\right) \Gamma\left(\frac{m}{4} + \nu\right)}. \hfill (A7)$$

Thus, by (A4)

$$\mathcal{M}(P_0; s) = \frac{\mathcal{M}(P_1; s)}{\Gamma(\nu - s)}. \hfill (A8)$$

By choosing $\nu = \frac{1}{4}$, the first of the gamma functions that appears in $P_0$ is eliminated from the denominator of (A7).

Continue this process three times more, eliminating all gamma functions from the denominator of series (A3), so obtaining

$$\mathcal{M}(P_0; s) = \frac{\mathcal{M}(P_4; s)}{\Gamma\left(\frac{1}{4} - s\right) \Gamma\left(\frac{1}{2} - s\right) \Gamma\left(\frac{5}{8} - s\right) \Gamma\left(\frac{7}{8} - s\right)}, \hfill (A9)$$
where
\[ P_4 = \sum_{m=0}^{\infty} (-1)^m e^{-i\pi m/4} z^{m/4} = \frac{1}{1 + e^{-i\pi/4} z^{1/4}}. \tag{A10} \]

Clearly, from the definition of the Mellin transform,
\[ \mathcal{M}(P_4; s) = \int_0^\infty \frac{t^{s-1} dt}{1 + e^{-i\pi/4} t^{1/4}} = 4\pi e^{i\pi s} \csc 4\pi s. \tag{A11} \]

By (A9) and (A11), we now have
\[ \mathcal{M}(P_0; s) = \frac{4\pi e^{i\pi s} \csc 4\pi s}{\Gamma(\frac{1}{4} - s) \Gamma(\frac{1}{2} - s) \Gamma(\frac{5}{8} - s) \Gamma(\frac{7}{8} - s)}. \tag{A12} \]

Returning to (A2), we finally obtain
\[ K = -\frac{8(2\pi)^{1/2} e^{-i\pi/8}}{\Gamma(\frac{1}{8}) \Gamma(\frac{3}{8}) \Gamma(\frac{3}{4})}. \tag{A13} \]

To make use of this result, we appeal to (3.20), (3.30), (3.41) and (3.42), and obtain
\[ A = A_p \rho + A'_a \alpha, \tag{A14} \]

where
\[ A_p = -\frac{1}{8} \left(\frac{\pi}{2}\right)^{1/2} \Gamma(\frac{1}{4}) \Gamma(\frac{3}{8}) \Gamma(\frac{7}{8}) (K + K^*), \tag{A15} \]
\[ A'_a = \frac{1}{8} \left(\frac{\pi}{2}\right)^{1/2} \Gamma(\frac{1}{8}) \Gamma(\frac{3}{4}) \Gamma(\frac{5}{8}) \left[e^{-i\pi/4} K + e^{i\pi/4} K^*\right]. \tag{A16} \]

On substituting from (A13), we obtain (3.46) and (3.47).

It may also be noted that if, as in our representations of series (3.21) and (3.22) by integrals (3.25) and (3.26), we express series (A1) as a contour integral, using a contour of the type shown in Figures 1 and 2, the residue at the pole \(s = -\frac{1}{2}\) outside the contour is, when multiplied by \(-2\pi i\), equal to \(K\zeta\), the minus sign arising because a distorted contour would pass round \(s = -\frac{1}{2}\) in a clockwise sense.
APPENDIX B: NUMERICAL METHODS

Finite \( \Lambda \)

The eigenvalue problem for \( s \) posed by (2.16)-(2.19) and boundary conditions (2.21) was solved by finite difference methods. On writing

\[
(f_1, f_2, f_3, f_4) = (v_2, \omega_2, i b_2, ij_z),
\]

these become

\[
\begin{align*}
Df_1 + \bar{F}f_3 + & = 0, \\
-\Lambda F^2 f_1 + Df_2 + (D^2 F - \Lambda s F)f_3 & = 0, \\
-Ff_1 + [\Lambda^{-1}(D^2 - k^2) - s]f_3 & = 0, \\
\bar{F}f_1 - Ff_2 + [\Lambda^{-1}(D^2 - k^2) - s]f_4 & = 0,
\end{align*}
\]

and

\[
f_1 = f_3 = Df_4 = 0, \quad \text{at } z = 0, 1;
\]

here \( D = d/dz \).

The range \([0, 1]\) of integration was divided into \( N \) equal intervals of length \( h = 1/N \), and the derivatives in (B2)-(B6) were replaced by centered differences; the derivatives of \( f_4 \) in (B6) were represented by right-hand and left-hand differences in the obvious way. This gave \( 4N + 2 \) difference equations in \( 4N + 4 \) unknowns, namely the values of the four functions \( f_n \) at the \( N + 1 \) grid points. The system was closed by two further relations, obtained from

\[
Df_2 = 0, \quad \text{at } z = 0, 1,
\]

which are obvious consequences of (B3) and (B6). The values of \( f_n \) at the end points \( z = 0, 1 \) could then be eliminated and an algebraic eigenvalue problem, of the form

\[
AX = sBX,
\]

could be posed, where \( A \) and \( B \) are banded \( 4(N-1) \times 4(N-1) \) matrices and \( X \) is a \( 1 \times 4(N-1) \) column vector consisting of the values of \( f_n \) at all grid values except those at \( z = 0, 1 \). As mentioned in Section 5, we did not presuppose that \( s \) is real, and performed all calculations in complex arithmetic.

The matrix \( B \) is singular, and (B8) therefore poses a so-called generalized eigenvalue problem, which can be solved by the QZ algorithm (Moler and Stewart, 1973). The advantage of this method is that it finds all nonsingular eigenvalues of (B6). It does not, however, take advantage of the banded structures of \( A \) and \( B \). Moreover, we are really only interested in the eigenvalue, \( s_1 \), that has the largest real part. We therefore nearly always employed the inverse iteration method (e.g. Golub and van Loan, 1989). Provided both \( s_1 \) and the initial guess, \( \mu \), for \( s_1 \) are real, and provided \( \mu \) is sufficiently large, inverse iteration always converges to \( s_1 \). Confirmation that the value of \( s_1 \) obtained was the one having the largest real part could always be verified by an application of the QZ algorithm.
The accuracy of the finite difference scheme leading to (B8) is $O(h^2)$. Convergence tests were carried out with the results shown in Figures 15 and 16 and Table 4.

Infinite $\Lambda$

The critical layer equation (3.12), attacked by analytic methods in Section 3, was also studied numerically. We did this by rescaling (B2)-(B5) by

$$(f_1, f_2, f_3, f_4) \rightarrow (F'_c \delta f_1, F'_c f_2, f_3, f_4/\delta),$$  \hspace{1cm} (B9)

and then taking the limit $\Lambda \rightarrow \infty$, so obtaining

$$D f_1 - \alpha f_3 + \zeta f_4 = 0, \hspace{1cm} (B10)$$
$$-\zeta^2 f_1 + D f_2 - p \zeta f_3 = 0, \hspace{1cm} (B11)$$
$$-\zeta f_1 + (D^2 - p) f_3 = 0, \hspace{1cm} (B12)$$
$$-\alpha f_1 - \zeta f_2 + (D^2 - p) f_4 = 0. \hspace{1cm} (B13)$$

Here $D = d/d\zeta$, where $\zeta$ is $z$ scaled as in (3.11), the critical level itself being $\zeta = 0$; $p$ is the scaled growth rate (3.13), and $\alpha$, the magnetic helicity at the critical level, is defined by (3.15). Solutions to (B10)-(B13) must satisfy boundary conditions that follow from (3.18)-(3.19):

$$f_1(0) = 0, \hspace{1cm} f_2(0) = 1, \hspace{1cm} f_4(0) = 0, \hspace{1cm} Df_3(0) = 0, \hspace{1cm} (B14)$$
$$f_2(X) = 0, \hspace{1cm} Df_4(X) = 0. \hspace{1cm} (B15)$$

Here $X$ is a numerical substitute for $\infty$; it is clear from the analysis of Section 3 that this will introduce errors of order $\zeta^{-3/2} \exp(-\zeta^2/2\sqrt{2})$. These are negligible if, as we assumed, $X = 5$. (In some cases we chose $X = 10$, and obtained the same results as in Table 1, to the accuracy shown.) The prime objective is that of evaluating $f_3(\infty)$, or effectively

$$A = f_3(X).$$  \hspace{1cm} (B16)

Our finite difference scheme for (B10)-(B15) followed that described by Kreiss (1972), a paper in which the stability and consistency of the method is analyzed in detail.
TABLE 1
Values of $A_p(\alpha)$ and $A_\alpha(\alpha)$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>-10</th>
<th>-5</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_p(\alpha)$</td>
<td>1.8506</td>
<td>0.4067</td>
<td>0.7311</td>
<td>1.1193</td>
<td>1.5032</td>
</tr>
<tr>
<td>$A_\alpha(\alpha)$</td>
<td>1.8118</td>
<td>5.2505</td>
<td>3.4337</td>
<td>1.9595</td>
<td>0.0</td>
</tr>
</tbody>
</table>

| $\alpha$ | 1 | 2 | 5 | 10 |
| $A_p(\alpha)$ | 1.8784 | 2.2411 | 3.2245 | 4.4179 |
| $A_\alpha(\alpha)$ | 2.47101 | 5.4768 | 17.893 | 50.666 |

TABLE 2
Numerical basis of (4.25)

<table>
<thead>
<tr>
<th>$\Lambda \times 10^{-6}$</th>
<th>$s \times 10^4$</th>
<th>$6.2016\Lambda^{1/4} - s\Lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2.6885</td>
<td>30.485</td>
</tr>
<tr>
<td>1.0</td>
<td>1.6550</td>
<td>30.613</td>
</tr>
<tr>
<td>1.5</td>
<td>1.2424</td>
<td>30.675</td>
</tr>
<tr>
<td>2.0</td>
<td>0.8634</td>
<td>30.713</td>
</tr>
<tr>
<td>3.0</td>
<td>0.7578</td>
<td>30.759</td>
</tr>
<tr>
<td>4.0</td>
<td>0.6164</td>
<td>30.785</td>
</tr>
<tr>
<td>5.0</td>
<td>0.5249</td>
<td>30.801</td>
</tr>
<tr>
<td>6.0</td>
<td>0.4602</td>
<td>30.811</td>
</tr>
<tr>
<td>7.0</td>
<td>0.4117</td>
<td>30.817</td>
</tr>
</tbody>
</table>
TABLE 3
The most unstable modes for $q = \frac{3}{2}\pi$

Mode 1

<table>
<thead>
<tr>
<th>$\Lambda$</th>
<th>$k_{xc}$</th>
<th>$k_{yc}$</th>
<th>$\sigma_c \times 10^4$</th>
<th>$\zeta_{c1}$</th>
<th>$\zeta_{c2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>22.17</td>
<td>1.70987</td>
<td>-0.39001</td>
<td>0.10262</td>
<td>0.2857</td>
<td>0.9524</td>
</tr>
<tr>
<td>75.60</td>
<td>1.63151</td>
<td>-0.32520</td>
<td>0.11405</td>
<td>0.2916</td>
<td>0.9582</td>
</tr>
<tr>
<td>1000</td>
<td>1.45943</td>
<td>-0.21715</td>
<td>0.05059</td>
<td>0.3019</td>
<td>0.9686</td>
</tr>
<tr>
<td>2000</td>
<td>1.39553</td>
<td>-0.19984</td>
<td>0.03617</td>
<td>0.3032</td>
<td>0.9695</td>
</tr>
<tr>
<td>3000</td>
<td>1.35768</td>
<td>-0.19070</td>
<td>0.02942</td>
<td>0.3037</td>
<td>0.9704</td>
</tr>
<tr>
<td>4000</td>
<td>1.33079</td>
<td>-0.18578</td>
<td>0.02532</td>
<td>0.3039</td>
<td>0.9706</td>
</tr>
</tbody>
</table>

Mode 2

<table>
<thead>
<tr>
<th>$\Lambda$</th>
<th>$k_{xc}$</th>
<th>$k_{yc}$</th>
<th>$\sigma_c \times 10^4$</th>
<th>$\zeta_{c1}$</th>
<th>$\zeta_{c2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>41.5</td>
<td>1.51995</td>
<td>-1.50941</td>
<td>0.00557</td>
<td>0.1674</td>
<td>0.8341</td>
</tr>
<tr>
<td>125.2</td>
<td>1.23400</td>
<td>-1.41142</td>
<td>0.15455</td>
<td>0.1528</td>
<td>0.8191</td>
</tr>
<tr>
<td>1000</td>
<td>0.80384</td>
<td>-1.28158</td>
<td>0.07118</td>
<td>0.1189</td>
<td>0.7855</td>
</tr>
<tr>
<td>2000</td>
<td>0.67358</td>
<td>-1.25873</td>
<td>0.05001</td>
<td>0.1043</td>
<td>0.7709</td>
</tr>
<tr>
<td>3000</td>
<td>0.60495</td>
<td>-1.24294</td>
<td>0.04032</td>
<td>0.0961</td>
<td>0.7628</td>
</tr>
<tr>
<td>4000</td>
<td>0.56011</td>
<td>-1.22847</td>
<td>0.03452</td>
<td>0.0908</td>
<td>0.7574</td>
</tr>
</tbody>
</table>

TABLE 4
Numerical Convergence for some $q = \frac{3}{2}\pi$ cases

Fixed $k. \ k_x = k_y = 1, \ \Lambda = 10^6$

$$
\begin{array}{ccc}
  h & 0.004 & 0.002 & 0.001 \\
  s & 1.6573 \times 10^{-4} & 1.6554 \times 10^{-4} & 1.6549 \times 10^{-4} \\
\end{array}
$$

Most Unstable Mode, $\Lambda = 2000$

$$
\begin{array}{ccc}
  h & 0.004 & 0.002 & 0.001 \\
  k_{xc} & 1.39567 & 1.39550 & 1.39546 \\
  k_{yc} & -0.199827 & -0.199846 & -0.199852 \\
  \sigma_c & 3.6190 \times 10^{-2} & 3.6169 \times 10^{-2} & 3.6163 \times 10^{-2} \\
\end{array}
$$
LEGENDS FOR FIGURES

Figure 1. The contours for the integral representation of solutions of the homogeneous equation (3.17). (a) The contour $C_3$ for $w_3$ as given by (3.25), (b) the contour $C_4$ for $w_4$ as given by (3.26).

Figure 2. The function $b_z$ for the case $k_x = k_y = 1$ for (a) $\Lambda = 10^3$, (b) $\Lambda = 10^4$, (c) $\Lambda = 10^5$ and (d) $\Lambda = 10^6$ showing the development of the asymptotic structure.

Figure 3. The function $j_z$ for the cases shown in Figure 2.

Figure 4. The function $v_z$ (multiplied by $-i$) for the cases shown in Figure 2.

Figure 5. The function $\omega_z$ (multiplied by $-i$) for the cases shown in Figure 2.

Figure 6. The function $b_z$ for the case $k_x = k_y = 1$ and $\Lambda = 10^4$, (a) as computed by integration of the full equations (2.16)–(2.19) (these points are marked with a +) and compared with the mainstream from (4.20); (b) the same, but showing the critical region on an enlarged scale; (c) the result of integration of the critical layer equations, $\zeta = 0$ corresponding to the critical point itself.

Figure 7. The function $j_z$ for the same three cases as shown in Figure 6. Here the mainstream form of $j_z$ is obtained from (4.26).

Figure 8. The function $v_z$ (multiplied by $-i$) for the cases shown in Figure 7.

Figure 9. The function $\omega_z$ (multiplied by $-i$) for the cases shown in Figure 7.

Figure 10. The growth rate, $s_c(\Lambda)$, for the most unstable mode, i.e. the local maxima of $s(k, \Lambda)$ as a function of $\Lambda$, obtained by solving (5.2). Two such maxima exist in the case $q = \frac{3}{2} \pi$ shown: mode 1 ($M_1$) and mode 2 ($M_2$).

Figure 11. The evolution of $b_z$ as $\Lambda$ increases, shown for four values of $\Lambda$ for mode 1 in the case $q = \frac{3}{2} \pi$: (a) $\Lambda = \Lambda_c = 22.17$, close to the marginal state for the tearing mode, $k_{xc} = 1.7100$, $k_{yc} = 0.3900$; (b) $\Lambda = 75.6$, close to the maximum attainable growth rate for the tearing mode, $k_{xc} = 1.6315$, $k_{yc} = -0.3252$, $\sigma_c = 0.1140$; (c) $\Lambda = 2,000$, $k_{xc} = 1.3955$, $k_{yc} = -0.1998$, $\sigma_c = 0.03617$; (d) $\Lambda = 4,000$, $k_{xc} = 1.3308$, $k_{yc} = -0.1858$, $\sigma_c = 0.02532$.

Figure 12. The evolution of $j_z$ as $\Lambda$ increases, for the cases shown in Figure 11.

Figure 13. The evolution of $v_z$ as $\Lambda$ increases, for the cases shown in Figure 11.

Figure 14. The evolution of $\omega_z$ as $\Lambda$ increases, for the cases shown in Figure 11.

Figure 15. Convergence of $b_z$ as the step size, $h$, is reduced; $\Lambda = 10^6$, $k_x = k_y = 1$, $q = \frac{3}{2} \pi$. Results for $h = 0.005, 0.0025$ and $0.00125$ are marked by $+$, 0 and $-$, respectively. In (a) the full range $0 \leq z \leq 1$ is shown; in (b) only the vicinity of the critical point, $z_c = \frac{1}{2}$, is shown.

Figure 16. Convergence of $j_z$ as the step size, $h$, is reduced; the same cases as in Figure 15.
Figure 1. The contours for the integral representation of solutions of the homogeneous equation (3.17): (a) The contour $C_3$ for $w_3$ as given by (3.25), (b) the contour $C_4$ for $w_4$ as given by (3.26).
Figure 2. The function $b_z$ for the case $k_x = k_y = 1$ for (a) $\Lambda = 10^3$, (b) $\Lambda = 10^4$, (c) $\Lambda = 10^5$ and (d) $\Lambda = 10^6$ showing the development of the asymptotic structure.
Figure 3. The function $j_z$ for the cases shown in Figure 2.
Figure 4. The function $u_z$ (multiplied by $-i$) for the cases shown in Figure 2.
Figure 5. The function $\omega_z$ (multiplied by $-i$) for the cases shown in Figure 2.
Figure 6. The function $b_z$ for the case $k_x = k_y = 1$ and $\Lambda = 10^4$, (a) as computed by integration of the full equations (2.16)–(2.19) (these points are marked with a +) and compared with the mainstream form (4.20); (b) the same, but showing the critical region on an enlarged scale; (c) the result of integration of the critical layer equations, $\zeta = 0$ corresponding to the critical point itself.
FIGURE 7. The function $j_x$ for the same three cases as shown in Figure 6. Here the mainstream form of $j_x$ is obtained from the $b_x$ of Figure 6 by using (4.26).
Figure 8. The function $u_x$ (multiplied by $-i$) for the cases shown in Figure 7; see (4.26).
FIGURE 9. The function $\omega_z$ (multiplied by $-i$) for the cases shown in Figure 7; see (4.26).
Figure 10. The growth rate, \( s_c(\Lambda) \), for the most unstable mode, i.e. the local maxima of \( s(k, \Lambda) \) as functions of \( \Lambda \), obtained by solving (5.2). Two such maxima exist in the case \( q = \frac{3}{2} \pi \) shown: mode I (\( M_1 \)) and mode II (\( M_2 \)).
Figure 11. The evolution of $b_z$ as $\Lambda$ increases, shown for four values of $\Lambda$ for mode I in the case $q = \frac{3}{2} \pi$: (a) $\Lambda = \Lambda_c = 22.17$, close to the marginal state for the tearing mode, $k_{zc} = 1.7100$, $k_{yz} = 0.3900$; (b) $\Lambda = 75.6$, close to the maximum attainable growth rate for the tearing mode, $k_{zc} = 1.6315$, $k_{yz} = -0.3252$, $\sigma_c = 0.1140$; (c) $\Lambda = 2,000$, $k_{zc} = 1.3955$, $k_{yz} = -0.1998$, $\sigma_c = 0.03617$; (d) $\Lambda = 4,000$, $k_{zc} = 1.3308$, $k_{yz} = -0.1858$, $\sigma_c = 0.02532$. 
Figure 12. The evolution of $j_z$ as $\Lambda$ increases, for the cases shown in Figure 11.
Figure 13. The evolution of $v_z$ as $\Lambda$ increases, for the cases shown in Figure 11.
Figure 14. The evolution of $\omega_z$ as $\Lambda$ increases, for the cases shown in Figure 11.
Figure 15. Convergence of $b_z$ as the step size, $h$, is reduced; $\Lambda = 10^6$, $k_x = k_y = 1$, $q = \frac{3}{2} \pi$. Results for $h = 0.005, 0.0025$ and $0.00125$ are marked by $+$, $O$ and $-$, respectively. In (a) the full range $0 \leq z \leq 1$ is shown; in (b) only the vicinity of the critical point, $z_c = \frac{1}{2}$, is shown.

Figure 16. Convergence of $j_z$ as the step size, $h$, is reduced; see caption to Figure 15.