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Abstract A layer of stagnant water and ice containing sources of heat is thermally insulated at one end, and is maintained at a subzero temperature at the other. The evolution of an arbitrary initial state to the final equilibrium state, in which one side of the layer is water (or in some circumstances ice) and the other side is (necessarily) ice, is studied in the limit of large Stefan number. It is shown that, depending on the degree of heating and the initial state of the system, the system may pass through a transient phase in which an internal region is filled with mush, i.e. is a mixture of both ice and water. The manner in which this slushy region is born, evolves and eventually dies is studied both numerically and analytically.

1. Introduction.

It is well known that melting/freezing processes in alloys and aqueous solutions will sometimes produce an interface between the solid and liquid phases which is so thin that it is usually modelled as a sharp transition surface. Seemingly equally possible is a state in which the two phases may be separated by a finite region of mixed phase, often termed a mush or mushy zone. Which of these two situations actually arises in practice depends upon material parameter ranges and delicate thermal balances. For pure substances such as an ice-water system the sharp interface is much more prevalent, so much so that there is perhaps the feeling that it is the only possibility. We shall show, however, that sharp interfaces are not inevitable in this case either.

To illustrate we consider in this paper the melting evolution of an ice-water system contained in a one-dimensional finite region which is being warmed by volumetric heating. For definiteness, we assume that one boundary of the system is maintained at constant temperature while the other is thermally isolated. This is, of course, a Stefan problem and the case when the associated Stefan number is large was previously considered by Ockendon (1975). His initial state was one in which the system was originally all ice. Here we shall include more general initial data.

The analysis of this paper confirms that, for a certain range of positions of the initial phase interface, a sharp surface will persist throughout the whole evolution of the two-phase system: on one side of the interface the ice temperature is sub-zero (on the Centigrade scale) while on the other, the liquid has a positive temperature distribution. More interesting is the range of positions of the initial interface for which this does not happen for all time. For these cases it is possible to find a mathematical solution in which, after a certain onset-time, the temperature of the ice rises above the melting temperature without any release of latent heat (see Ockendon (1975), Lacey and Shillor (1983)). It is sometimes said that the ice has taken up a quasi-equilibrium state being superheated.

Statistical mechanics offers a compelling explanation for the existence of quasi-equilibrium states such as supercooled water, and provides methods (in terms of the "escape over a barrier" from the quasi-equilibrium state to the nearby state of complete thermodynamic equilibrium) by which the lifetime of a quasi-equilibrium state can be estimated. Supercooled water occurs because the surface energy required to nucleate a grain of ice increases with the radius, R, of the grain. Only if R exceeds some critical radius, R_c, will the grain increase in size and solidification occur; otherwise the grain will dissolve back into the fluid to be replaced by other grains that attempt to escape back over the barrier in turn. The top of the barrier itself is defined by the energy required to produce a single grain of radius R_c, and this is so large that escape over the barrier is a statistical rarity, thereby explaining the longevity of the supercooled state.

There is a complete theoretical symmetry between nucleation of a small ice grain in supercooled water and the nucleation of a small water cavity in superheated ice, both being quasi-equilibrium states that owe their existences to interfacial energy between phases. The practical difficulties in creating superheated ice appear to be the greater (see, for instance, Landau and Lifshitz, 1969, §150), although there have been reports that such states have been observed even existing long enough to be experimentally studied (Chalmers, 1964; Woodruff, 1980). In the context of the present one-dimensional problem, neither supercooled water nor superheated ice are possible states, since both rely on nonzero surface energies between phases, and these are conspicuously absent when those interfaces are planar!

Thus, instead of the appearance of superheated ice we show that a more physically acceptable solution has a transitory region of mixed phase. An earlier treatment of a mushy zone has been presented by Atthey (1974). Here we are particularly concerned with the manner in which the mush is born and the fashion of its eventual demise. There can be no question that it persists for all time: the continual warming by volumetric heating would systematically deplete the mass fraction of the solid phase in the mush in contradiction to the assumption that the solution was steady. Thus the system ultimately moves to a configuration with a sharp interface in which the heat being supplied by volumetric sources is in balance with the heat flux out of the non-insulated end. Governing the evolution to this final state there are two intrinsic time scales, one associated with latent heat and the other with thermal diffusion processes. The assumption of large latent heat (and hence large Stefan number) means that its associated time scale is so slow that the diffusion time is much shorter.

For the cases when a mush does occur its life cycle is intimately bound up with these characteristic rates. There are two main scenarios. For definiteness let us assume that ice lies to the left (smaller x). The mush is always formed by an ice/mush interface appearing and moving left on the fast time scale towards a critical position that is a function of the rate of volumetric heating. As this edge moves across it effectively lays down a distribution of solid and liquid phase in the mush behind it. The morphology of the mush continually changes: the volumetric heating works to diminish the proportion of solid present. Although the leading edge initially moves quickly, it actually approaches the critical position with increasing lethargy. Ultimately the mush vanishes by the trailing (mush/liquid) edge overhauling the leading edge in an exponentially small neighbourhood of the critical position. The essential difference between the scenarios lies in the way in which the trailing edge follows across. When the initial configuration is an ice-water system, the trailing edge immediately follows across but only on the *slow* time scale. Since the leading edge approaches the critical position moving slower and slower, there will come a time when it is moving as slow as the trailing edge. Thereafter the mush/liquid interface is actually the faster moving and so the width of the mush must decrease and eventually vanish. The other type of mush evoluton occurs when the initial phase configuration has only ice present. In this case the Stefan condition that applies at phase interfaces at first dictates that the trailing edge does not move at all. It has to remain motionless until all the solid at the right hand end of the mush has melted. Since the solid in the mush is eroded at a constant rate and the extreme right hand end is born first, it must be obliterated first due to the heating. Once this happens the trailing edge becomes free to move but, in contrast to the case above, it does so on the *fast* time scale. This again follows from the Stefan condition and reflects the fact that, in the period of time required to melt all the solid at the stationary trailing edge, the solid fraction becomes very small elsewhere (of the order of the reciprocal of the Stefan number) and an O(1) velocity for the mush/liquid interface is needed in order to satisfy the Stefan condition.

The mathematical definition of the problem discussed above is stated in Section 2. In particular we show why a mush is born and in the following two sections discuss aspects of the mush evolution for the first of the cases above. Section 5 is concerned with the evolution of a system that is initially all ice and so is pertinent to the second scenario.

2. Ockendon's Problem.

The domain $0 \le x \le 1$ (dimensionless units) is filled with H_2O either in its liquid phase (water) or its solid phase (ice) and the pressure is everywhere maintained constant. Uniformly distributed heat sources would cause the temperature, $u(x,t)$, to rise uniformly everywhere at the rate $r(>0)$, were it not for heat conduction which removes heat from the plane $x = 0$ which is maintained at temperature $u = -1$, the zero of u being the melting temperature; the plane $x = 1$ is a thermal insulator. The thermal diffusivity is everywhere 1 and the Stefan number is denoted by $\lambda = L/c_pT_m$ where L is the latent heat, c_p the specific heat and T_m a reference temperature on the absolute scale. The difference in the densities of ice and water is ignored so that questions of volume preservation do not arise. Given the initial u and the initial distribution of the phases, it is required to determine these at all subsequent times.

In mathematical terms, this one dimensional Stefan problem may, in the first instance, be stated as follows. Solve

$$u_t = u_{xx} + r, \tag{2.1}$$

subject to

$$u(0,t) = -1, \qquad u_x(1,t) = 0, \tag{2.2,3}$$

and at any zero, $x = s(t)$, of u,

$$\lambda \dot{s} = [u_x]^{\text{SOLID}}_{\text{LIQUID}}. \tag{2.4}$$

Here a superposed dot denotes time differentiation, $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$ and $[u_x]$ denotes the discontinuity in u_x in the limit $x \to s$ from the two sides. Implicit in this first approach is the assumption that u vanishes only at isolated x, and not over intervals of x, in [0,1]. This is assumed also to be true of the initial distribution,

$$u_0(x) \equiv u(x,0), \tag{2.5}$$

which is otherwise arbitrary.

Depending on the choice of u_0, there may be many distinct regions of ice ($u < 0$) and water ($u > 0$) in [0,1], but in the final steady state defined by (2.1)–(2.3), there is at most one region of each. We have

$$u_\infty(x) \equiv u(x,\infty) = -1 + rx - \tfrac{1}{2}rx^2. \tag{2.6}$$

Thus, if $r < 2$ the material is ultimately solid everywhere. We shall confine attention to the case $r \geq 2$, and will write (2.6) as

$$u_\infty = -\tfrac{1}{2}r\,(x - s_\infty)\,(x - \bar{s}_\infty)\,, \tag{2.7}$$

where $s_\infty \leq 1$, $\bar{s}_\infty \geq 1$ and

$$\left.\begin{matrix} s_\infty \\ \bar{s}_\infty \end{matrix}\right\} = 1 \mp \left(1 - \frac{2}{r}\right)^{\frac{1}{2}}\,, \tag{2.8}$$

We see from (2.7) that ice ultimately fills $x < s_\infty$ and water fills $x > s_\infty$, and that $s_\infty = s(\infty)$.

To avoid the complications of following the evolution and demise of many regions of ice and water, we shall concentrate on simple initial states that have at most one of each. Typical of such an initial state is

$$u_i(x) = -1 + 2\alpha x - \alpha x^2. \tag{2.9}$$

If $\alpha < 1$, ice is initially present everywhere. If $\alpha \geq 1$, we may write (2.9) as

$$u_i = -\alpha(x - s_i)(x - \bar{s}_i), \tag{2.10}$$

where $s_i \leq 1$, $\bar{s}_i \geq 1$ and

$$\left.\begin{matrix} s_i \\ \bar{s}_i \end{matrix}\right\} = 1 \mp \left(1 - \frac{1}{\alpha}\right)^{\frac{1}{2}}. \tag{2.11}$$

Then ice initially fills $0 < x < s_i$ and water fills $s_i < x < 1$, where $s_i = s(0)$. We should emphasize that the picture that emerges from (2.9) is robust, *i.e.* other simple choices of u_i, that have one or no zeros, evolve in a way similar to (2.9). Ockendon (1975) focussed on an initial state for which $u \leq 0$ and $s_i = 1$; this can be simulated by taking $\alpha = 1$ in (2.9).

One other 'critical' value of x will be significant in what follows, namely $x = s_c$, where

$$s_c = \sqrt{2/r}. \tag{2.12}$$

In the cases $r \geq 2$ of greatest interest, s_c falls within [0,1]. It is clear that $s_c > s_\infty$.

Suppose that, for all t during the evolution of u, from (2.9) to (2.6), there exists a single interface $x = s(t)$ between ice in $[0, s)$ and water in $(s, 1]$. Denote u in these intervals by u_1 and u_2 respectively and write (2.4) as

$$\lambda \dot{s}(t) = u_{1x}(s,t) - u_{2x}(s,t). \tag{2.13}$$

Ockendon (1975) noted that Stefan problems of the present type are easily solved in the limit $\lambda \to \infty$. There are then two timescales, the "fast" $O(1)$ timescale of thermal diffusion and the "slow" $O(\lambda)$ timescale of latent heat processes. According to (2.13) the interface moves on the slow timescale, and during this time heat conduction processes are effectively instantaneous, *i.e.* on the slow timescale, (2.1) gives, to leading order

$$0 = u_{xx} + r, \qquad r = 2/s_c^2. \tag{2.14}$$

Let us attempt to solve Ockendon's problem in the limit $\lambda \to \infty$. By (2.2), (2.3) and (2.14) we have, to leading order,

$$u_1 = -(x-s)(xs - s_c^2)/ss_c^2, \qquad u_2 = (x-s)(2-x-s)/s_c^2 \qquad (2.15,\ 16)$$

It may be noted that, even taking $s = s_i$, (2.15) and (2.16) do not coincide with (2.9). This is because there is a fast initial phase, $O(1)$ in duration, during which (2.9) evolves to (2.15) and (2.16) by heat conduction; this initial phase cannot be monitored by the slow timescale approximation (2.14) of (2.1).

Using (2.15) and (2.16), we see that the interface condition (2.13) is

$$\lambda \dot{s} = (s - s_\infty)(s - \bar{s}_\infty)/ss_c^2. \tag{2.17}$$

Thus, if $s_i < s_\infty$, then s increases until ultimately $s = s_\infty$; if $s_i > s_\infty$, then $\dot{s} < 0$ and s decreases monotonically to s_∞. Equation (2.17) may be integrated to give

$$t = \frac{\lambda s_c^2}{(\bar{s}_\infty - s_\infty)} \left[\bar{s}_\infty \ln\left(\frac{\bar{s}_\infty - s}{\bar{s}_\infty - s_i}\right) - s_\infty \ln\left(\frac{s - s_\infty}{s_i - s_\infty}\right)\right], \tag{2.18}$$

so that

$$s \sim s_\infty + (s_i - s_\infty)\exp\left[-\frac{(\bar{s}_\infty - s_\infty)}{\lambda s_c^2} t\right], \quad \text{as } t \to \infty. \tag{2.19}$$

This analysis provides a physically uncontentious solution if $s_i \le s_c$. When however $s_i > s_c$, the second zero, $x = \tilde{s}$ where $\tilde{s} = s_c^2/s$, of (2.15) falls within the interval $0 \le x < s$ in which (2.15) is supposed to hold. It remains in that interval until s has decreased to s_c; thereafter it exceeds s_c, *i.e.* it is then irrelevant. Thus, during the time in which $s > s_c$, the temperature in $\tilde{s}(t) < x < s(t)$ exceeds zero but, since the latent heat condition (2.4) is not obeyed at $x = \tilde{s}$ (u_x is in fact continuous there), it does *not* reflect a change of phase. It is for this reason that the material in such a region has been called "superheated ice" (Ockendon (1975); Lacey and Shillor (1983); Lacey and Tayler (1983)). For the physical reasons adumbrated in the introduction we prefer to investigate the appearance of a mushy region between the pure phases.

3. The mush: overview of a life cycle and details of the birth.

In this and the following section we develop a new solution in which a mushy zone makes a temporary appearance. The initial position of the (sharp) phase interface, $x = s_i$, occurs at an interior point of $[0,1]$ and this corresponds to $\alpha > 1$ in (2.9). The case $\alpha \leq 1$ is considered in detail in Section 5. The morphology of the mixed phase region is characterized by the mass (or volume) fraction, $\phi(x,t)$, of ice at point x at time t. On the microscope level, ϕ can only be 1 (within an ice grain) or 0 (in the water surrounding the grains). In the description of the mush used here, we assume that an average over many grains surrounding x at time t has been taken to obtain a smoothly varying $\phi(x,t)$ that can take any value in $[0,1]$, *i.e.* we describe the mush using the approach of mixture theory.

Before becoming enmeshed in the details of the phase evolution we first give an outline of the life cycle for the case $\alpha > 1$. Four distinct stages can be identified in our solution.

(1) **The gestation and birth of the mush** (*described in this section*).

During $0 < t < t_1 = O(1)$, the initial state (2.9) evolves until at $t = t_1$ (say)

$$u_{1x}(s, t_1) = 0, \tag{3.1}$$

for the first time. Because $t_1 = O(1)$ and $\lambda \gg 1$, we have $s(t_1) = s_i$, to leading order. To the same accuracy, we may replace s in (3.1) by s_i.

(2) **Growth of the mush** (*Section 4*).

During $t_1 < t < t_2 = O(1)$, $s_1(t)$ moves towards s_c and the mush spreads quickly from s_i, and fills $s_1(t) < x < s_2(t)$, where s_1 and s_2 are to be determined. The solution now consists of three distinct regions:

(i) **ice.** In $0 < x < s_1(t)$ we have

$$u = u_1(x,t) < 0, \quad \phi \equiv 1;$$

(ii) **mush.** In $s_1(t) < x < s_2(t)$ we have

$$u = u_3(x,t) \equiv 0, \quad 0 < \phi \equiv \phi_3(x,t) < 1;$$

(iii) **water.** In $s_2(t) < x < 1$ we have

$$u = u_2(x,t) > 0, \quad \phi \equiv 0.$$

We may not assume, *a priori*, that ϕ is continuous at s_1 and s_2, and we define

$$\phi_1 = \lim_{x \to s_1+} \phi_3, \qquad \phi_2 = \lim_{x \to s_2-} \phi_3,$$

to be the values of ϕ at the edges of the mush. Since domain (ii) is everywhere in phase equilibrium, $u_3 \equiv 0$, so that (2.13) is replaced by the demands

$$\lambda(1 - \phi_1)\dot{s}_1 = u_{1x}(s_1, t), \tag{3.2}$$

$$\lambda \phi_2 \dot{s}_2 = -u_{2x}(s_2, t). \tag{3.3}$$

Because $t_2 = O(1)$ and $\lambda \gg 1$, the latter shows that $s_2(t_2) = s_i$ to leading order. The rapid motion of s_1 occurs, despite the enormous size of λ, because the system satisfies (3.2) by [*cf.* (3.1)]

$$\phi_1 = 1, \qquad u_{1x}(s_1, t) = 0, \tag{3.4,5.}$$

(3) **Death of the mush** (*Section 4*).
During $t_2 < t < t_3 = O(\lambda)$, the mush contracts slowly in the following manner. While s_1 continues to approach s_c it does so ever more slowly. The trailing edge, s_2, has meanwhile been moving towards s_c on the slow time scale. Eventually there comes a time, t_2, when s_1 in approaching s_c is travelling as slowly as, and later even more slowly than, s_2. In a very definite sense, stage 2 is a part of stage 3, rather than being anterior to it.

(4) **Post–mush evolution** (*Section 4*).
During the final phase $t > t_3$, the solution at last evolves to the state (2.6) on the $O(\lambda)$ timescale and s becomes s_∞. The solution essentially coincides with that obtained in Section 2 for $s_i \leq s_c$.

In the remainder of this section we develop the solution for stage (1), that is, we solve (2.1) – (2.3) subject to (2.5) and (2.9). Since $\lambda >> 1$, we replace (2.13) by

$$s = s_i,$$

so that for this period of the evolution, to leading order, the interface remains at the initial position. We are therefore effectively seeking the temperature distribution on a fixed interval and this can be obtained using standard Laplace transforms. Since $s_i < 1$ an appropriate representation is

$$u_1(x,t) = -\frac{(x-s_i)(x-s_c^2/s_i)}{s_c^2} + \Lambda s_i^2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\left(\frac{(2n+1)\pi x}{s_i}\right) \exp\left[-\frac{(2n+1)^2\pi^2 t}{s_i^2}\right], \quad 0 < x < s_i, \tag{3.7}$$

$$u_2(x,t) = \frac{(x-s_i)(2-s_i-x)}{s_c^2} + 4\Lambda(1-s_i)^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cos\left(\frac{(2n+1)\pi(1-x)}{2(1-s_i)}\right) \exp\left[-\frac{(2n+1)^2\pi^2 t}{4(1-s_i)^2}\right], \quad s_i < x < 1. \tag{3.8}$$

where $\Lambda = 8(\alpha s_i^2 - 1)/\pi^3 s_c^2$. This solution holds until (3.1) is obeyed for the first time at $t = t_1$. This onset time for the appearance of the mush can be easily found from (3.7) to be the solution of

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \exp\left[-\frac{(2n+1)^2\pi^2 t_1}{s_i^2}\right] = \frac{\pi^2(s_i^2 - s_c^2)}{8 s_i^2(1-\alpha s_c^2)}. \tag{3.9}$$

In Table I we list onset times obtained by numerical solution of (3.9) for various ranges of r and α and in Figure 1 we show stages of a temperature distribution evolution up to the onset time t_1.

	$\alpha = 1 + p$				
r	p=1/64	p=1/16	p=1/4	p=1/2	p=1
5	5.6566e-03	2.5264e-02	NO MUSH	NO MUSH	NO MUSH
10	7.8499e-04	3.3640e-03	1.7533e-02	NO MUSH	NO MUSH
15	2.9632e-04	1.2585e-03	6.2832e-03	1.6994e-02	NO MUSH
20	1.5429e-04	6.5291e-04	3.2057e-03	8.1630e-03	NO MUSH
50	2.1480e-05	9.0927e-05	4.3511e-04	1.0666e-03	2.9699e-03
100	4.9584e-06	2.1636e-05	1.0324e-04	2.5040e-04	6.8176e-04
150	2.0172e-06	9.3877e-06	4.5076e-05	1.0902e-04	2.9477e-04

Table I. Non-dimensional onset times for the appearance of a mushy zone for various values of the volumetric heating, r, and the parameter α, that characterises the initial state of the temperature field [see (2.9)].

4. Growth and death of the mush.

As indicated in Section 3, at the onset time t_1, the leading edge of the mush $x = s_1(t)$ moves to the left towards $x = s_c$. The position of this solid–mush interface and the temperature distriburion in the solid phase are determined by solving on $(x,t) \in [0, s_1(t)] \times [t_1, t)$

$$u_{1t} = u_{1xx} + r, \qquad r = 2/s_c^2, \tag{4.1}$$

subject to

$$u_1(0,t) = -1, \quad u_1(s_1,t) = 0, \quad u_{1x}(s_1,t) = 0. \tag{4.2,3,4}$$

The initial data, $u_1(x,t_1)$, is given by (3.7) and, according to (3.2) and (4.4),

$$\phi_1 = 1. \tag{4.5}$$

These equations have been solved for $r = 15$, $\alpha = 1.25$ (which is the standard case of this section). Standard numerical techniques were used. In Figure 2 we illustrate the manner in which the interface moves across and show the development of the temperature field.

The rate at which s_1 approaches s_c decreases with $s_1 - s_c$. If s_1 were not eventually overtaken by the trailing edge, $x = s_2(t)$ at time t_3 (say) with the concomitant loss of the mush, the evolution of s_1 to s_c of the solution would take an infinite time to complete. To investigate this point, we develop the solution to (4.1)–(4.4) for small $s_1 - s_c$ and small $\dot{s}_1$. The solution we seek is asymptotic to the steady solution that exists when $s_1 = s_c$, namely

$$u_1(x) = -\left(1 - x/s_c\right)^2. \tag{4.6}$$

For small $s_1 - s_c$, we seek solutions of the form

$$u_1(x,t) = -\left(1 - x/s_c\right)^2 - v(x)e^{-k^2 t}, \tag{4.7}$$

where k and v are to be determined and v is of order $s_1 - s_c$. Substituting (4.7) into (4.1) and (4.2), we obtain ($' = d/dx$)

$$v'' + k^2 v = 0, \tag{4.8}$$

$$v(0) = 0. \tag{4.9}$$

We expand conditions (4.3) and (4.4) about $s_1 = s_c$, to obtain

$$0 = u_1(s_1, t) = u_1(s_c, t) + (s_1 - s_c)u_{1x}(s_c, t),$$
$$0 = u_{1x}(s_1, t) = u_{1x}(s_c, t) + (s_1 - s_c)u_{1xx}(s_c, t),$$

with $O(s_1 - s_c)^2$ errors. After substitution from (4.7) we find, to the same accuracy

$$v(s_c) = 0, \tag{4.10}$$

$$s_1 - s_c = -\tfrac{1}{2}s_c^2 v'(s_c)e^{-k^2 t}. \tag{4.11}$$

By (4.8)–(4.10) we have

$$v(x) = A\sin kx, \qquad k = \pi/s_c, \tag{4.12, 13}$$

for some constant A, so that by (4.11)

$$s_1 - s_c = \tfrac{1}{2}\pi s_c A e^{-(\pi/s_c)^2 t}. \tag{4.14}$$

In short, we now have

$$u_1(x, t) = -\left(1 - \frac{x}{s_c}\right)^2 - \frac{2}{\pi s_c}\sin\left(\frac{\pi x}{s_c}\right)(s_1 - s_c), \tag{4.15}$$

$$\dot{s}_1 = -(\pi/s_c)^2 (s_1 - s_c), \tag{4.16}$$

both expressions being in error at order $(s_1 - s_c)^2$. For the illustrative case $r = 15$, $\alpha = 1.25$, if we determine for $s_1 - s_c = 0.01$ the constant A by matching the asymptotic solution, s_{1a}, of (4.14) with the numerical solution, s_1, then the error satisfies $\|(s_{1a} - s_1)/s_1\| \simeq 10^{-4}$. Matching near s_c gives concomitantly improved errors.

In the mush itself, that is for $s_1(t) < x < s_2(t)$, there is no heat conduction, as $u_3 \equiv 0$, so the heat sources act only to melt the ice. Thus

$$\phi_{3t} = -r/\lambda = -2/\lambda s_c^2, \tag{4.17}$$

which gives

$$\phi_3(x, t) = \phi_0(x) - 2t/\lambda s_c^2. \tag{4.18}$$

Here $\phi_0(x)$ must be such that $\phi_3(s_1(t), t) = 1$ for all relevant t, i.e. such that (4.5) is obeyed. We therefore introduce the function, $T(s_1)$ say, that is the inverse of $s_1(t)$, so that $s_1\big(T(x)\big) \equiv x$. (We note that s_1 is a monotonically decreasing function of t for $t > t_1$ so that the inverse exists.) The statement that $\phi_3\big(s_1(t), t\big) = 1$ for all relevant t can then be restated as $\phi_3\big(x, T(x)\big) = 1$ for all relevant x. Then, by (4.18),

$$\phi_0(x) = 1 + 2T(x)/\lambda s_c^2. \tag{4.19}$$

Since by (4.18) and (4.19) $T(x)$ is monotonically decreasing for $s_c \le x \le s_i$, ϕ_0 and ϕ_3 are also monotonically decreasing functions of x, and

$$\phi_3(x,t) = 1 - 2[t - T(x)]/\lambda s_c^2. \tag{4.20}$$

In Figure 3, for $r = 15$, $\alpha = 1.25$, we graph ϕ_3 versus x for various times.

Finally, in this section we consider the trailing edge of the mush, that is the mush/liquid interface at $x = s_2(t)$. Throughout it moves slowly much in the manner of the persistent, sharp interface of the case $s_i < s_c$ discussed in Section 2. The explanation of this follows from (3.3): both $u_{2x}(s_2, t)$ and ϕ_2 are non-zero so that $\dot{s}_2 = O(\lambda^{-1})$. Compared with this slow evolution, diffusion processes are effectively instantaneous and it follows that (*cf.* (2.16))

$$u_2(x,t) = (x - s_2)(2 - x - s_2)/s_c^2, \tag{4.21}$$

By (3.3) we have

$$\lambda\phi_2\dot{s}_2 = -2(1 - s_2)/s_c^2, \tag{4.22}$$

and from (4.20) we have

$$\lambda\left\{1 - 2[t - T(s_2)]/\lambda s_c^2\right\}\dot{s}_2 = -2(1 - s_2)/s_c^2. \tag{4.23}$$

The numerical solution of (4.23) for the illustrative case used in this paper is represented in Figure 4 and shows that $t_3 = 0.534$.

The mush having disappeared at $t = t_3$, the subsequent evolution from $s = s_c$ is given by the marginal case $s_i = s_c$ of the $s_i \le s_c$ theory developed in Section 2. The relevant solution to (2.17) is now, instead of (2.18),

$$t = t_3 + \frac{\lambda s_c^2}{2(\bar{s}_\infty - s_\infty)}\left[\bar{s}_\infty \ln\left(\frac{\bar{s}_\infty - s}{\bar{s}_\infty - s_c}\right) - s_\infty \ln\left(\frac{s - s_\infty}{s_c - s_\infty}\right)\right]. \tag{4.24}$$

5. The case $\alpha \leq 1$.

Suppose that water is absent initially, that is u_0 is everywhere negative, but nevertheless still satisfies the endpoint conditions (2.2) and (2.3). A typical example is provided by (2.9) with $\alpha \leq 1$. If $\alpha < 1$, the first stage of the evolution is again on the fast time scale with u increasing until, at time $t = t_1$, u first becomes zero at $x = 1$. Thereafter a mush rapidly develops as in the previous section but the third and fourth stages dealing with the diminution and demise of the mush have marked differences.

The central observation is that, from (4.22), since s_2 is initially 1, $\dot{s}_2 = 0$ until ϕ_2, the mass fraction of solid at $x = 1$, vanishes. Hence the trailing edge must languish motionless until the volumetric heating has dissolved the ice at $x = 1$. According to (4.20), since $T(1) = t_1$, ϕ_2 will vanish at $t = t_2$, where

$$t_2 = t_1 + \frac{\lambda s_c^2}{2}, \tag{5.1}$$

and

$$\phi_3(x, t_2) = 2\left[T(x) - t_1\right]/\lambda s_c^2. \tag{5.2}$$

Since $T(x)$ is monotonically decreasing and $T(1) = t_1$, it follows from (5.2) that $\phi_3 \geq 0$ at $t = t_2$, within the mush and equality only occurring at $x = 1$. Clearly, from (5.2), $\phi_3 = O(1/\lambda s_c^2)$ at $t = t_2$ and the left-hand side of (3.3) is, therefore, at most of order $2\dot{s}_2/s_c^2$. This means that s_2 will evolve on the fast time scale in contrast to the situation of the previous section, where s_2 always moved on the slow scale. Because the evolution is rapid, we cannot adopt the approximation (2.14) but must retain the u_{2t} term in (2.1). Thus, for $t > t_2$ we must solve

$$u_{2\hat{t}} = u_{2xx} + r, \quad r = 2/s_c^2, \tag{5.3}$$

$$u_2(s_2, \hat{t}) = 0, \quad u_{2x}(1, \hat{t}) = 0, \tag{5.4,5}$$

$$2\left[T(s_2) - t_1 - \hat{t}\right]\dot{s}_2/s_c^2 = -u_{2x}(s_2, \hat{t}), \tag{5.6}$$

where $t = t_2 + \hat{t}$. We observe that both sides of equation (5.6) vanish when $\hat{t} = 0$. Consequently, some care will be needed to determine the manner in which the solution starts. It is clear that if s_2 moves off at $\hat{t} = 0$ with a speed that is less than or equal to the speed of s_1 initially, then s_2 must exactly mimic the journey of s_1 since ϕ_2 will always then vanish at the trailing edge. This is not the case and to illustrate we concentrate on the case $\alpha = 1$ so that stage 1 does not occur and the onset time, t_1, is zero. To investigate the solution for $1 - s_2 \ll 1$, $\hat{t} \ll 1$, we assume the first few terms of a double power series solution with

$$u_2 = 2a_1 t - a_2(1-x)^2, \tag{5.7}$$

where a_1, a_2 are constants to be determined. Condition (5.5) is identically satisfied and (5.3,4,6) give

$$2a_1 = r - 2a_2, \qquad 2a_1\hat{t} = a_2(1-s_2)^2, \qquad \left[T(s_2) - \hat{t}\right]\dot{s}_2 = -s_c^2 a_2(1-s_2). \tag{5.8}$$

To make further progress, we need information regarding $T(x)$ which is related to the motion of the leading edge. In the appendix to this paper A M Soward has shown that when $\alpha = 1$ it is possible to construct a similarity solution for the leading edge by using a boundary layer analysis. hIS result shows that, for $s_2 \to 1$,

$$T(s_2) \sim (1-s_2)^2/4\Gamma^2, \tag{5.9}$$

where Γ is the solution of the transcendental equation

$$s_c^2 \Gamma \operatorname{erfc} \Gamma + (1 - s_c^2)\left[\Gamma \operatorname{erfc} \Gamma - e^{-\Gamma^2}/\sqrt{\pi}\right] = 0. \tag{5.10}$$

From (5.8) we find that

$$a_2 = \left[(1 + 20\Gamma^2 + 4\Gamma^4)^{1/2} - (1 + 2\Gamma^2)\right]/8\Gamma^2 s_c^2. \tag{5.11}$$

Equation $(5.8)_2$ shows that

$$s_2(t) \sim 1 - (2a_1 t/a_2)^{1/2}. \tag{5.12}$$

By comparing with the result (5.9), which implies that

$$s_1(t) \sim 1 - 2\Gamma\sqrt{t}, \tag{5.13}$$

we see that, since $a_1/2a_2\Gamma^2 = 2s_c^2 a_2 + 1 > 1$, the trailing edge starts out faster than the leading edge did. The asymptotic results (5.7), $(5.8)_1$, (5.11), (5.12) can be used as an initial solution state to numerically solve the system (5.3)–(5.6). In Figure 5 we illustrate the evolution of the edges of the mush for $r = 15$, $\alpha = 1$.

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Figure captions.

Figure 1. The evolution of the temperature distribution $u(x,t)$ for $r = 15$, $\alpha = 1.25$, from an initial state to just prior to the onset of a mush at $t = t_1$ ($= 6.62832\text{e-}03$). Since $\lambda \gg 1$, the position of the ice/water interface at $x = s_i$ ($= 0.5528$) does not move to leading order and at onset t_1, $u_{1x}(s_i, t_1) = 0$.

Figure 2. (a) The evolution of the leading edge $x = s_1(t)$ of the mushy zone for the case $r = 15$, $\alpha = 1.25$, showing that the edge approaches the critical position $x = s_c$ ($= 0.3652$) increasingly slowly; see (4.14).
(b) The temperature distribution $u_1(x,t)$ within the solid for $r = 15$, $\alpha = 1.25$ showing how the solid/mush interface moves towards the critical position $x = s_c$ ($= 0.3652$) for times $\tau = t - t_1$ subsequent to the onset time t_1.

Figure 3. The distribution of the mass fraction of the solid, $\phi_3(x,t)$, in the two-phase mush at various times $\tau = t - t_1$ (> 0), where t_1 is the onset time.

Figure 4. The evolution of the position of the edges of the mushy region for the case $r = 15$, $\alpha = 1.25$. The region quickly forms at time t_1 by the leading edge $x = s_1(x,t)$ moving across towards $x = s_c$ and the trailing edge moves across on the slow time scale with the mush disappearing at $t = t_3$. Thereafter the system has a sharp phase boundary that moves towards the equilibrium position $x = s_\infty$ ($= 0.06905$).

Figure 5. The evolution of the position of the edges of the mushy region for the case $r = 15$, $\alpha = 1$. For this case the onset time $t_1 = 0$ and the region forms by the leading edge $x = s_1(x,t)$ moving across towards $x = s_c$. In order to satisfy the Stefan condition (4.22), the trailing edge must remain motionless at $x = 1$ until the solid fraction there has been melted by volumetric heating. Thereafter the trailing edge $x = s_2(t)$ moves on the fast time scale.

Appendix. Early time solution for $\alpha = 1$.

This appendix aims to re-inforce the analysis of the case $\alpha = 1$ described in §5, but only minor modifications are required to make the analysis applicable also to the more general case $\alpha > 1$ considered in §§2–4.

According to (2.9) for $\alpha = 1$, the initial state,

$$u_i(x) = -(1 - x)^2, \tag{A1}$$

is one in which ice is present everywhere, but in which melting is about to be initiated at $x = 1$.

By writing

$$u_1(x, t) = u_i(x) + v(x, t), \tag{A2}$$

we introduce the departure, $v(x, t)$, of the temperature from (A2) at later times; evidently

$$v(x, 0) = 0. \tag{A3}$$

By (4.1) and the boundary conditions (4.2 – 4.4), v must be the solution of

$$v_t = v_{xx} + 2Q, \qquad (0 < x < s_1(t)) \tag{A4}$$

that satisfies, for $t > 0$,

$$v(0, t) = 0, \tag{A5}$$

$$v(s_1, t) = (1 - s_1)^2, \tag{A6}$$

$$v_x(s_1, t) = -2(1 - s_1), \tag{A7}$$

where

$$Q = \frac{1}{s_c^2} - 1, \qquad r = \frac{2}{s_c^2}. \tag{A8}$$

For sufficiently small times, the solution in the interior of the region $0 < x < s_1(t)$ away from the end points is

$$v = V(t) = 2Qt. \tag{A9}$$

This does not satisfy the boundary conditions at $x = 0$ and $x = s_1(t)$, and in their immediate neighbourhoods boundary layers form, whose widths are of order $t^{1/2}$ and in which solution (A9) adjusts to conditions (A5) – (A7). So at $x = 0$, for example, the solution takes the similarity form

$$v = V(t)[1 - \psi(\xi)], \tag{A10}$$

where $\xi = x/2t^{1/2}$. By (A4), $\psi(\xi)$ satisfies

$$\psi'' + 2\xi\psi' - 4\psi = 0, \tag{A11}$$

where the prime denotes differentiation with respect to ξ. Condition (A4) and matching to (A9) require

$$\psi(0) = 1, \tag{A12}$$

$$\psi(\xi) \to 0, \quad \text{as} \quad \xi \to \infty. \tag{A13}$$

The solution to (A11) – (A13) is

$$\psi(\xi) = (1 + 2\xi^2)\operatorname{erfc}\xi - \frac{2}{\pi^{1/2}}\xi e^{-\xi^2}, \tag{A14}$$

where erfc denotes the complementary error function:

$$\operatorname{erfc}\xi = \frac{2}{\pi^{1/2}}\int_\xi^\infty e^{-x^2}\,dx.$$

Of greater interest is the nature of the boundary layer at the moving end point, $x = s_1(t)$. To understand the structure of this layer, it is helpful to consider the comparison problem defined on the *fixed* interval $0 < x < 1$, and to solve (A4) subject to the boundary condition (A5) with an *evolving* value for $v(1, t)$. That value is determined by the demand that the resulting solution obeys the two conditions (A6) and (A7) at the interior point $x = s_1(t)$.

When

$$v(1, t) = V(t)(1 - \mu), \tag{A15}$$

where μ is a constant as yet unknown, a similarity solution of (A4) can again be constructed which obeys (A15) and matches to (A9):

$$v = V(t)[1 - \mu\psi(\xi)], \tag{A16}$$

where ψ is again given by (A14) but now $\xi = (1 - x)/2t^{1/2}$. Fortuitously, (A16) can also satisfy both (A6) and (A7) at an interior point

$$s_1(t) = 1 - 2\Gamma t^{1/2}, \tag{A17}$$

where Γ is another constant to be determined. Specifically, (A6) and (A7) are met when respectively

$$1 - \mu\psi(\Gamma) = \frac{2}{Q}\Gamma^2, \tag{A18}$$

$$-\mu\psi'(\Gamma) = \frac{4}{Q}\Gamma. \tag{A19}$$

By (A14) we therefore have

$$Q = \frac{\Gamma\operatorname{erfc}\Gamma}{\pi^{-1/2}e^{-\Gamma^2} - \Gamma\operatorname{erfc}\Gamma}. \tag{A20}$$

$$\mu = \frac{1}{\operatorname{erfc}\Gamma}, \tag{A21}$$

The value of Γ is obtained by solving (A20), and then μ is derived from (A21).

We will examine the nature of the solution for the two extremes: (i) $Q \ll 1$ and (ii) $Q \gg 1$. In both cases the results will be valid only during the time in which the boundary layers at $x = 0$ and $x = s_1$ are non-interacting, which is the case when t is sufficiently small:

$$t \ll 1. \tag{A22}$$

(i). ***The Case $Q \ll 1$.***

When Q is small, it follows that

$$0 < 1 - s_c \ll 1, \tag{A23}$$

so that the initial state $u_i(x)$ is very close to the state (4.2) to which the temperature in the ice tends during the growth of the mush, namely $u_1 = (1 - x/s_c)^2$. The solutions of (A20) and (A21) are

$$\Gamma = \frac{1}{\pi^{1/2}}Q + O(Q^2), \tag{A24}$$

$$\mu = 1 + \frac{2}{\pi}Q + O(Q^2). \tag{A25}$$

Since μ is close to unity, the boundary layer at $x = s_1(t)$ is approximately the same as that at the stationary boundary $x = 0$. Furthermore since Γ is small the displacement,

$$1 - s_1 \sim 2Q\left(\frac{t}{\pi}\right)^{1/2}, \tag{A26}$$

of the mush-ice interface away from $x = 1$ is ever small compared with the boundary layer thickness $t^{1/2}$.

(ii). ***The Case $Q \gg 1$.***

When Q is large, it follows that

$$0 < s_c \ll 1, \tag{A27}$$

so that the ice-mush interface eventually moves to a location far from its initial position at $x = 1$. The asymptotic solutions of (A20) and (A21) are

$$\Gamma \sim \left(\tfrac{1}{2}Q\right)^{1/2} + O\left(Q^{-1/2}\right), \tag{A28}$$

$$\mu \sim \left[\left(\tfrac{1}{2}\pi Q\right)^{1/2} + O\left(Q^{-1/2}\right)\right] e^{Q/2}. \tag{A29}$$

In contrast to the case of small Q, the displacement,

$$1 - s_1 \sim (2Qt)^{1/2}, \tag{A30}$$

of the mush/ice interface is large compared with the boundary layer thickness. Only the large ξ asymptotic behaviour is relevant to the interval $0 < x < s_1(t)$, and that gives

$$v \sim V(t)\left[1 - \frac{2}{Q}e^{-\zeta}\right], \tag{A31}$$

where

$$\zeta = \left(\frac{Q}{2t}\right)^{1/2}(s_1 - x). \tag{A32}$$

Since the interface $x = s_1(t)$ moves very rapidly, the boundary layer in front of it is very thin, of order $(t/Q)^{1/2}$ in width. Indeed, the location of the moving interface can be found by ignoring the boundary layer entirely, and simply applying the boundary condition (A6) directly to the mainstream, so obtaining

$$(1 - s_1)^2 = V(t). \tag{A33}$$

This gives the result (A30) directly. The boundary layer is passive, and is only required so that condition (A7) on $v_x(s_1, t)$ can be met.

Clearly the solution (A30) for $s_1(t)$ is valid only for the very short times for which Qt is small. At later times we can make further progress by ignoring the boundary layer (A31) generated by the moving interface. Thus we apply condition (A6) directly to the mainstream solution, as modified by the boundary layer at the stationary wall, $x = 0$. We therefore substitute (A10) into (A6), set

$$s_1 = 2\eta t^{1/2}, \qquad \tau = 2Qt, \tag{A34, A35}$$

and solve for τ in terms of η, so obtaining

$$\frac{1}{\tau^{1/2}} = \left(\frac{2}{Q}\right)^{1/2}\eta + [1 - \psi(\eta)]^{1/2}. \tag{A36}$$

When η is large ($\psi(\eta) \ll 1$) and the moving interface $x = s_1$ has not penetrated the boundary layer significantly, we recover solution (A33) in the form

$$\frac{1}{\tau^{1/2}} - 1 = \left(\frac{2}{Q}\right)^{1/2}\eta, \tag{A37}$$

valid provided that

$$Q^{1/2}(1 - \tau) \gg 1. \tag{A38}$$

An interesting feature of the result (A36) is that the advance of the rapidly moving interface is arrested sharply in the tail of the boundary layer (A10), close to $s_1 = 2\eta_e t_e^{1/2}$ at time $t_e = \tau_e/2Q$, where η_e is the solution of

$$\eta_e^2 e^{\eta_e^2} = Q^{1/2}, \qquad (\eta_e \gg 1), \tag{A39}$$

and τ_e is the corresponding solution of (A36). To see how this is achieved, we note that

$$\psi(\eta) \sim \pi^{-1/2}\eta^{-3}e^{-\eta^2}, \qquad \text{as} \quad \eta \to \infty,$$

and set

$$\eta = \eta_e + \tfrac{1}{2}\rho. \tag{A40}$$

The lowest order approximation of (A36) then becomes

$$(2Q)^{1/2}\eta_e\left(\frac{1}{\tau^{1/2}} - \frac{1}{\tau_e^{1/2}}\right) = \eta_e\rho + \frac{1}{(2\pi)^{1/2}}\left(1 - e^{-\eta_e\rho}\right), \tag{A41}$$

where, correct to leading order,

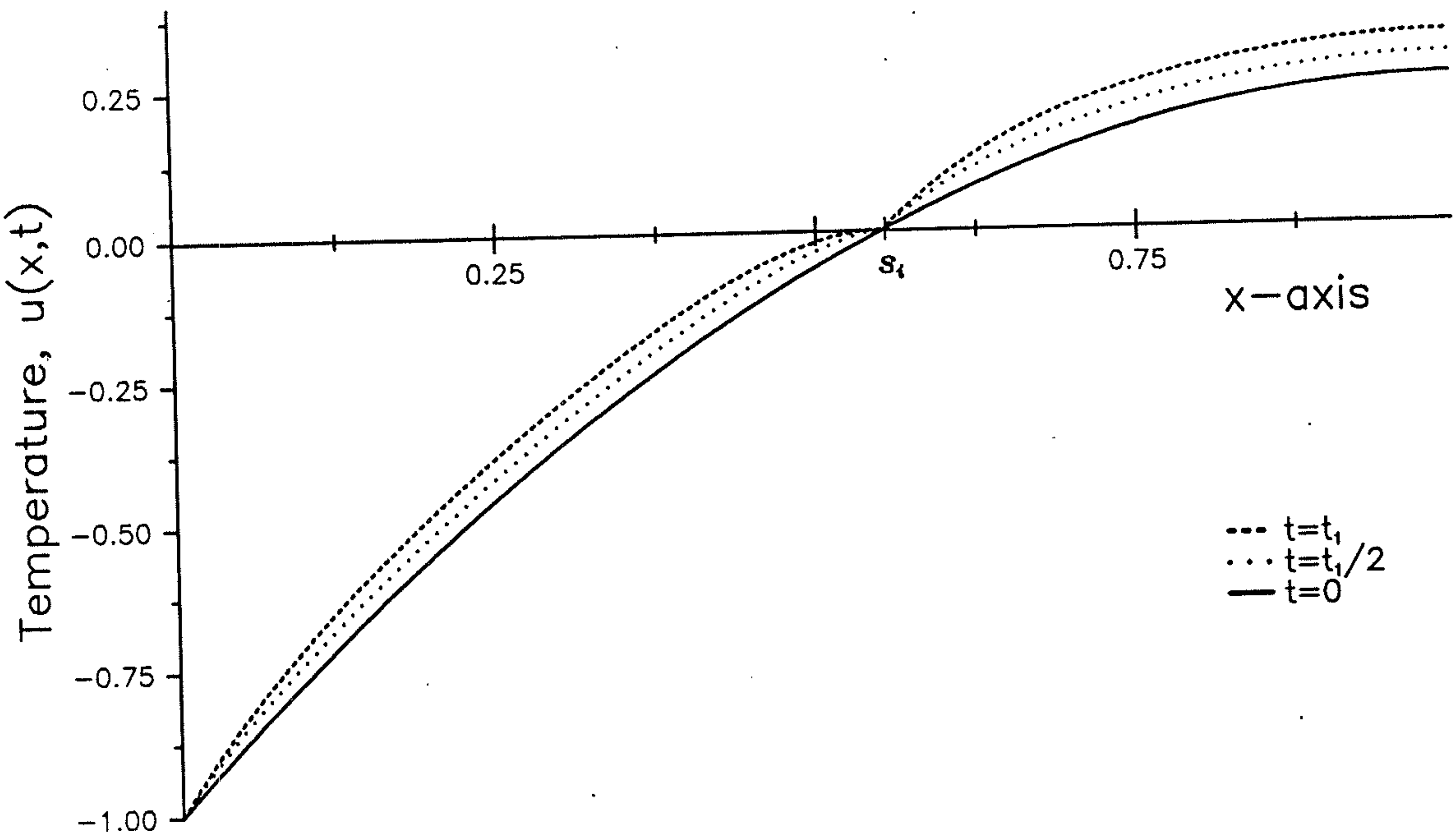
$$\frac{1}{\tau_e^{1/2}} - 1 = \left(\frac{2}{Q}\right)^{1/2}\eta_e, \tag{A42}$$

as in (A37). The asymptotic solutions of (A41) for positive and negative ρ are given by

$$2(\eta - \eta_e) \sim \begin{cases} \left(\dfrac{Q}{2\tau_e}\right)^{1/2}\left(1 - \dfrac{\tau}{\tau_e}\right), & \dfrac{1}{Q^{1/2}} \ll \tau_e - \tau \ll 1, \\ -\dfrac{2}{\eta_e}\ln\left[\left(\dfrac{\pi Q}{\tau_e}\right)^{1/2}\left(\dfrac{\tau}{\tau_e} - 1\right)\eta_e\right], & \dfrac{1}{Q^{1/2}} \ll \tau - \tau_e \ll 1. \end{cases} \tag{A43}$$

This result highlights the nature of the transition that occurs at the time $t = t_e$.

FIGURE I



Temperature, u(x,t)
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0.25
s_t
0.75
x−axis
t=t1
t=t1/2
t=0

FIGURE 2a

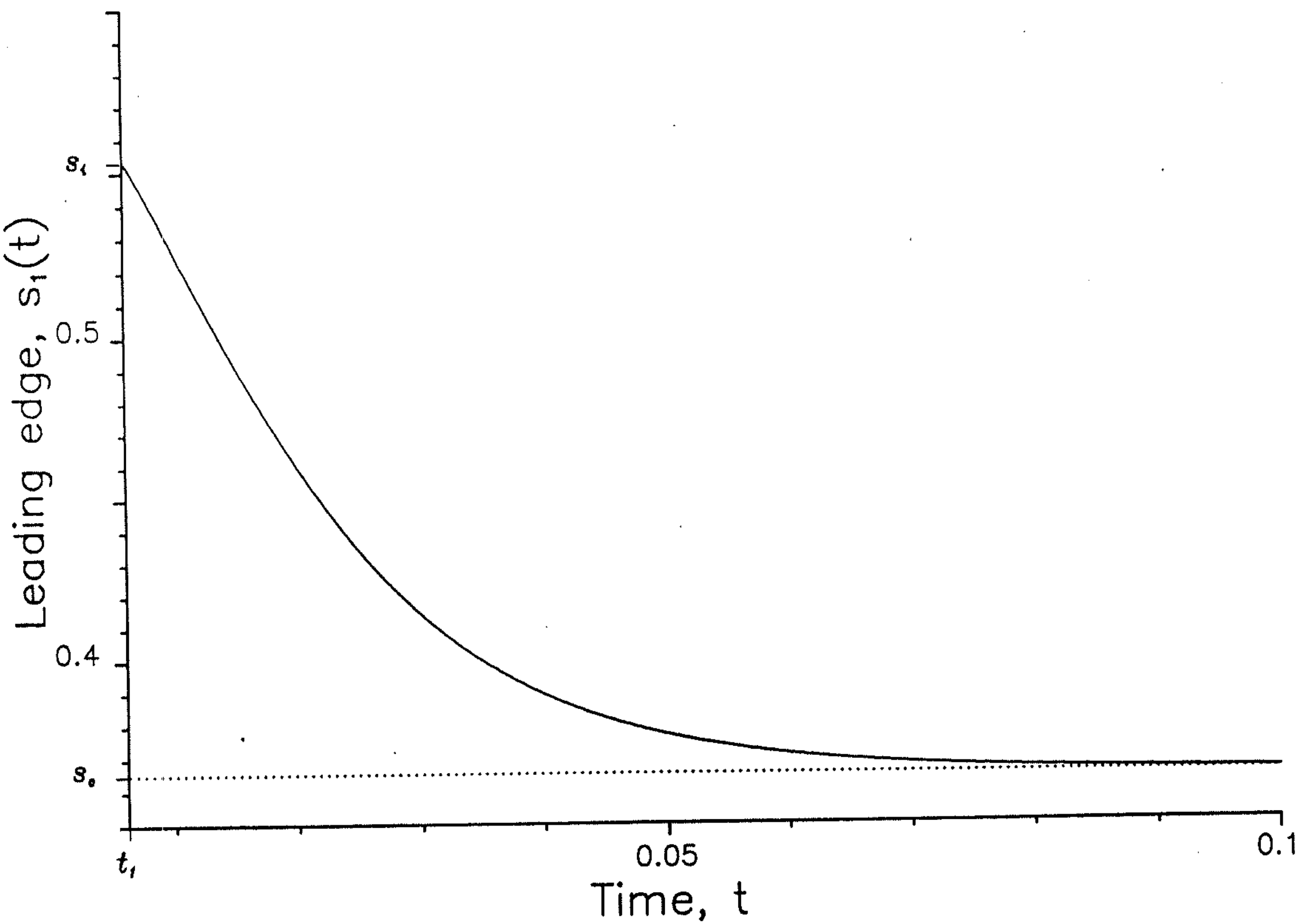
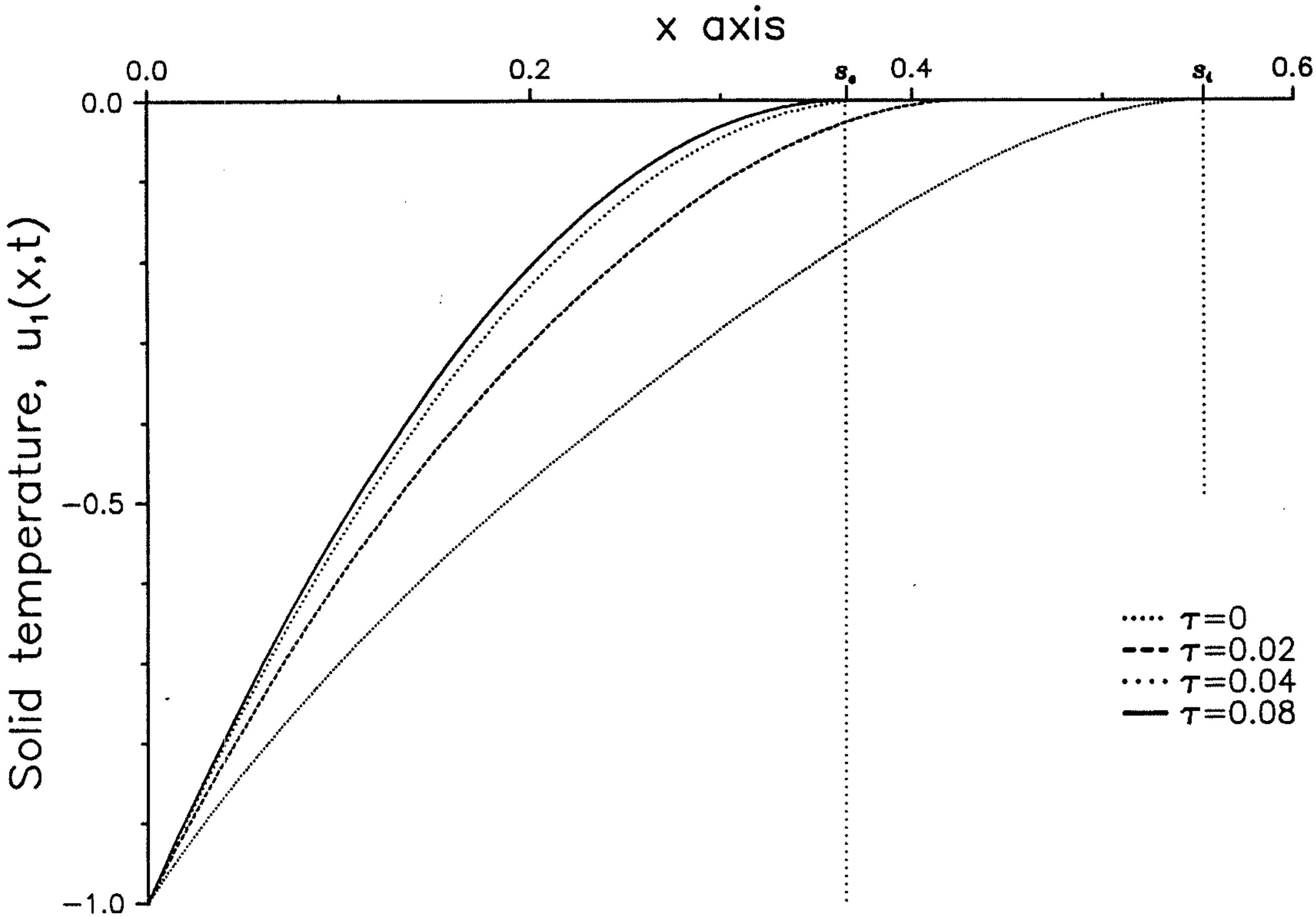


FIGURE 2b



x axis
0.0
0.2
s_0
0.4
s_t
0.6
Solid temperature, $u_1(x,t)$
0.0
-0.5
-1.0
$\tau=0$
$\tau=0.02$
$\tau=0.04$
$\tau=0.08$

FIGURE 3

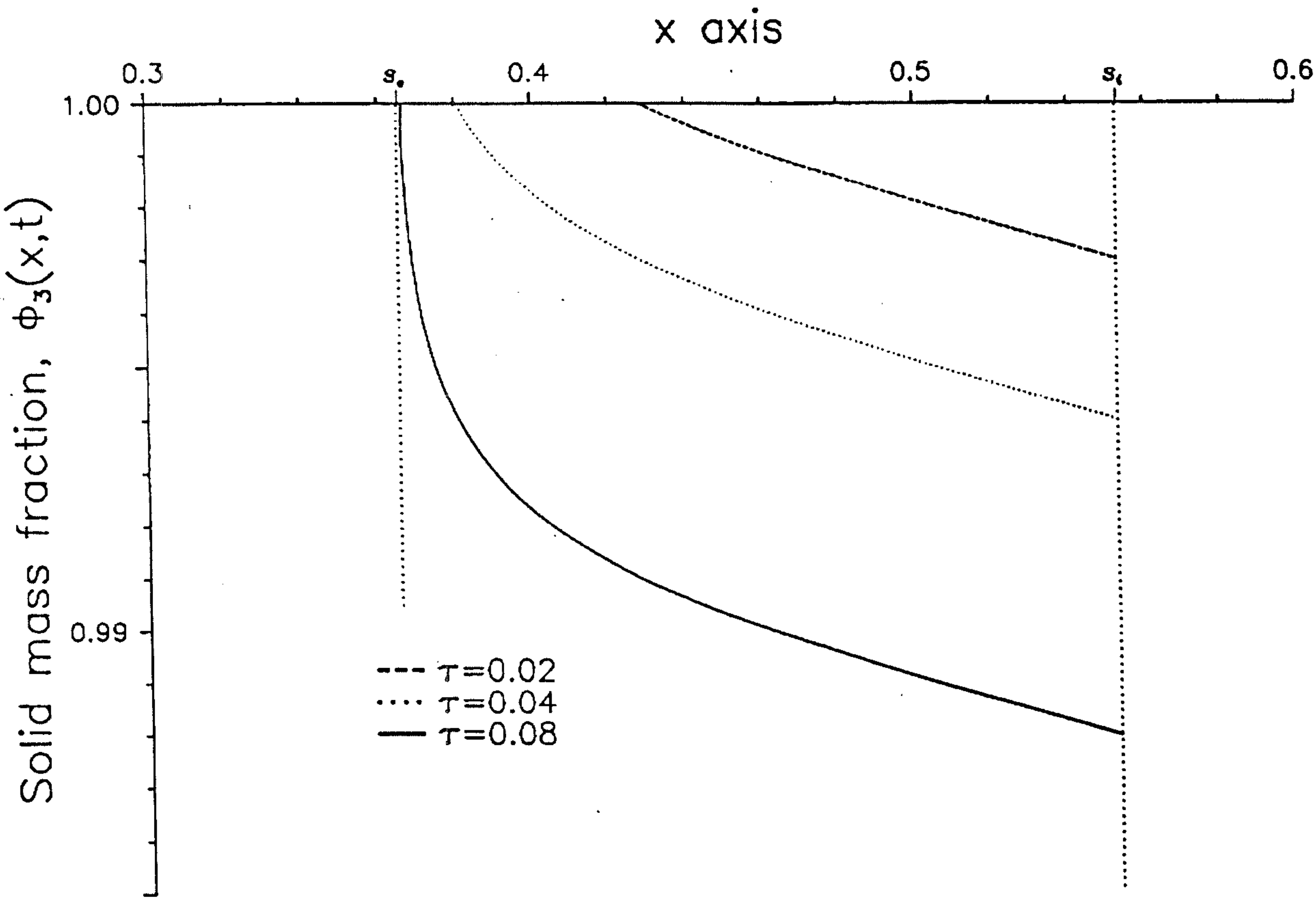
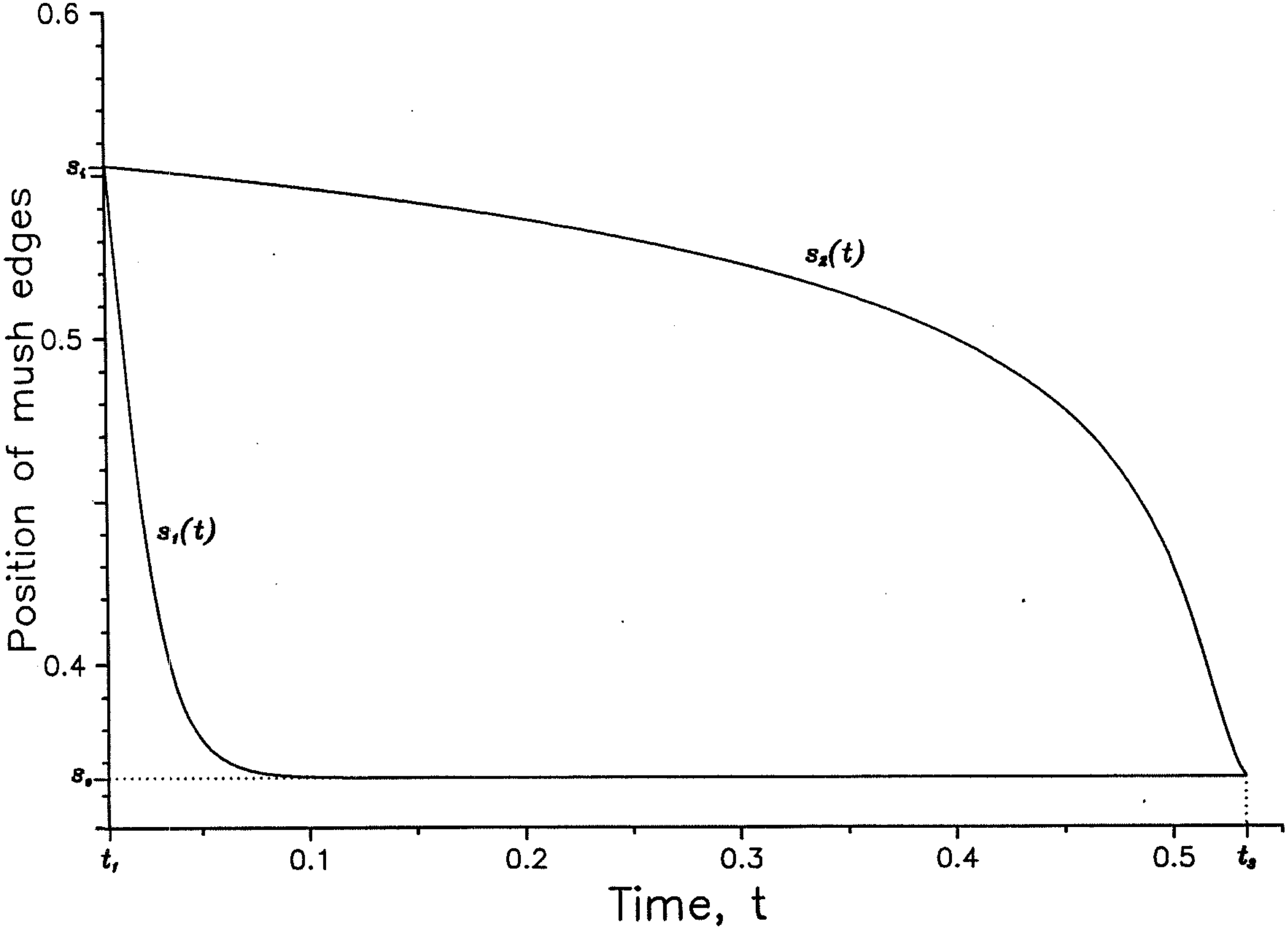
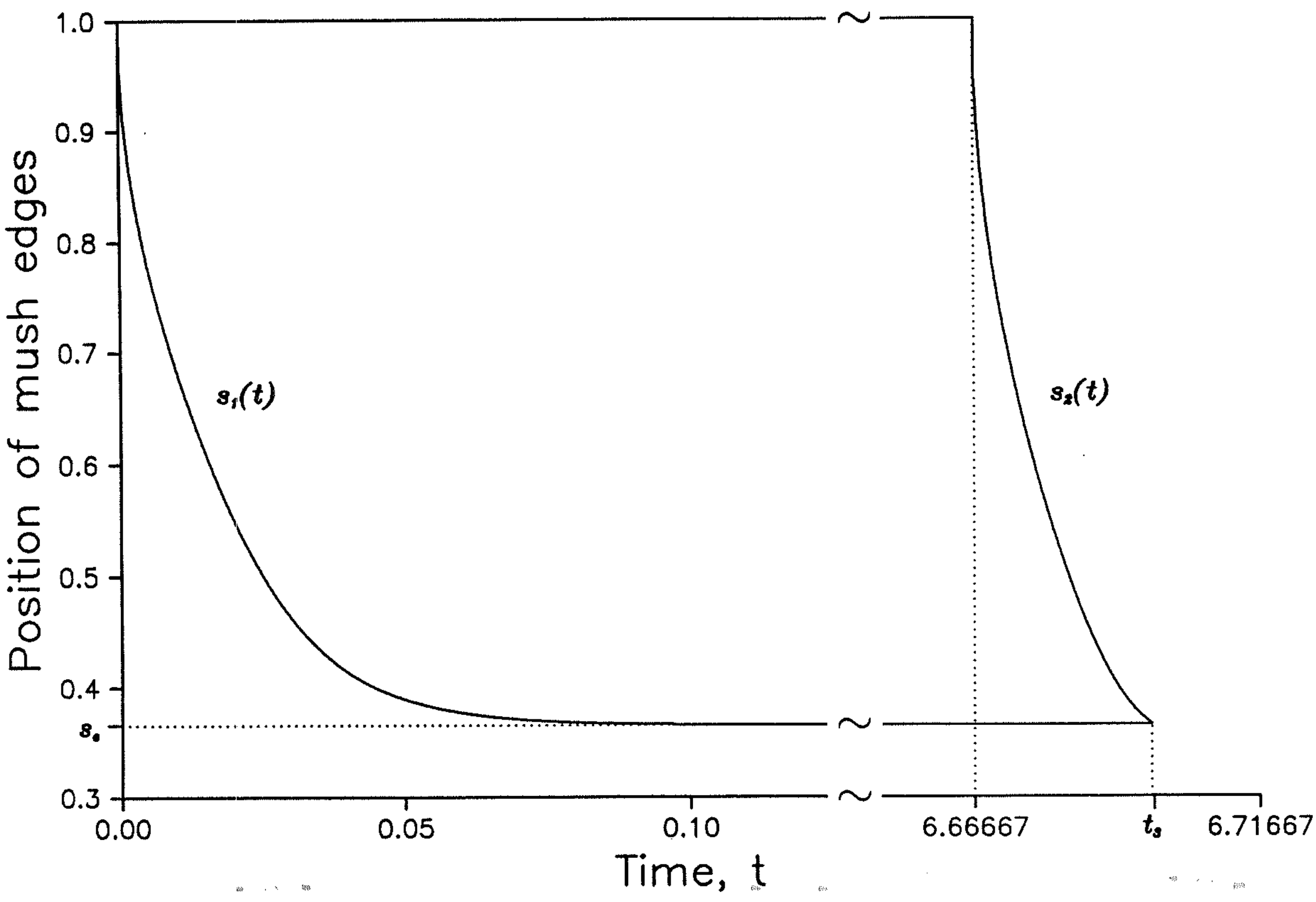

x axis
0.3
0.4
0.5
0.6
s
s
1.00
0.99
Solid mass fraction, φ3(x,t)
τ=0.02
τ=0.04
τ=0.08

FIGURE 4



0.6
s_i
0.5
0.4
s_e
Position of mush edges
$s_2(t)$
$s_1(t)$
t_1
0.1
0.2
0.3
0.4
0.5
t_s
Time, t

FIGURE 5


Position of mush edges
1.0
0.9
0.8
0.7
0.6
0.5
0.4
s_e
0.3
$s_1(t)$
$s_2(t)$
0.00
0.05
0.10
6.66667
t_s
6.71667
Time, t