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Yann Brenier

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

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Yann Brenier*

INRIA, BP 105, 78153 Le Chesnay, Cedex, France

Abstract

The first order linear equation $\frac{\partial u_\varepsilon}{\partial t} + a_\varepsilon(t,x) \cdot \nabla u_\varepsilon = 0$ is discussed when a_ε is an oscillatory function, in particular when $a_\varepsilon = a(\frac{x}{\varepsilon})$, where a is 1-periodic in all space variables. The concept of generalized flow is used to describe the behavior of u^ε when ε approaches 0.

1. Introduction.

We are concerned with first order linear equations of the type:

$$\frac{\partial u_\varepsilon}{\partial t} + a_\varepsilon(t,x) \cdot \nabla u_\varepsilon = 0, \quad x \in K,\ t > 0;\ u_\varepsilon(0,x) = u_0(x), \tag{1.1}$$

where K is $\mathbf{R}^d$ or $\mathbf{R}^d/\mathbf{Z}^d$, u_0 is a given compactly supported continuous function on K and a_ε is a smooth globally Lipschitz continuous divergence free vector field on K.

It is well known that (1.1) is easily solved once the flow X_ε associated with a_ε and defined by

$$X_\varepsilon(0,x) = x,\ \dot{X}_\varepsilon(t,x) = a_\varepsilon(t, X_\varepsilon(t,x)),\ t \in \mathbf{R},\ x \in K, \tag{1.2}$$

is known. (Here $\cdot$ denotes the time partial derivative.) Indeed we get from (1.1-2):

$$u_\varepsilon(t, X_\varepsilon(t,x)) = u_0(x),\ \forall t \in \mathbf{R},\ \forall x \in K. \tag{1.3}$$

We are interested in knowing the behavior of u_ε when (a_ε) is a sequence of oscillating functions, with more and more oscillations when ε approaches 0. To

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describe the limit of $u_\varepsilon(t,\cdot)$, for each fixed $t>0$, it is convenient to introduce (in the spirit of L.C. Young's generalized functions) nonlinear integrals of the form:

$$\int_K \varphi(x, u_\varepsilon(t,x))dx, \tag{1.4}$$

where φ is an arbitrary compactly supported continuous function defined on $K\times\mathbf{R}$. Because a_ε is divergence free, X_ε is measure preserving ($\det D_x X_\varepsilon(t,x)\equiv 1$) and (1.4) can be rewritten as:

$$\int_K \varphi(X_\varepsilon(t,x),\ u_\varepsilon(t,X_\varepsilon(t,x)))dx = \int_K \varphi(X_\varepsilon(t,x),\ u_0(x))dx, \tag{1.5}$$

by using (1.3) and the change of variable $x\to X_\varepsilon(t,x)$ in (1.4). Thus, we are now mainly interested in knowing the behavior of nonlinear integrals of the form

$$\int_K \psi(X_\varepsilon(t,x),x)dx, \quad \text{for } t>0, \tag{1.6}$$

where ψ is an arbitrary compactly supported continuous function defined on $K\times K$.

An elementary example.

Let us assume $K=\mathbf{R}^2$ and :

$$a_\varepsilon(t,x_1,x_2) = \begin{pmatrix}\alpha(\frac{x_2}{\varepsilon})\\ 0\end{pmatrix} \tag{1.7}$$

where α is a given smooth 1-periodic function. Then, one immediately gets:

$$X_\varepsilon(t,x_1,x_2) = \begin{pmatrix}x_1+t\alpha(\frac{x_2}{\varepsilon})\\ x_2\end{pmatrix},$$

and, therefore,

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^2}\psi(X_\varepsilon(t,x),x)dx = \int_{\mathbf{R}^2}\int_0^1 \psi(x_1+t\alpha(y),x_2,x)dydx, \tag{1.8}$$

which shows that a new integration variable y has to be introduced to describe the limit when ε goes to 0. In the general case (1.1), we will say that the flow $(t,x)\in\mathbf{R}\times K\to X_\varepsilon(t,x)\in K$ converges to a "generalized flow":

$$(t,x,y)\in\mathbf{R}\times K\times\Omega\to X(t,x,y)\in K,$$

if we find a probability space (Ω, dy) and a mapping X such that:

$$\lim_{\varepsilon \to 0} \int_K \psi(X_\varepsilon(t,x),x)dx = \int_K \int_\Omega \psi(X(t,x,y),x)dydx \tag{1.9}$$

for all continuous compactly supported function ψ on $K \times K$.

In the elementary case (1.7), we get:

$$\Omega = [0,1] \text{ and } X(t,x,y) = x + t\bar{a}(y) \tag{1.10}$$

where $\bar{a}(y) = (\alpha(y), 0)^T$.

The goal of this short paper is to give some examples when the generalized flows can be more or less explicitly found. Let us quote, among the most recent papers devoted to these topics, the works of Amirat, Hamdache and Ziani [AHZ] and the paper of Gérard [Gé] where the concept of microlocal defect measure, which is the same as Tartar's H measure [Ta], is used.

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2. The case of purely oscillatory velocity fields.

In this section, we investigate the case:

$$K = \mathbf{R}^d, \quad a_\varepsilon(t,x) = a(\frac{x}{\varepsilon}), \tag{2.1}$$

where a is a smooth divergence free velocity given, 1-periodic in all space variables, which also means that a can be defined as a smooth vector field given on the torus $\mathbf{T}^d = \mathbf{R}^d/\mathbf{Z}^d$.

In that case, we can prove

Proposition 1. *There is a bounded measurable mapping $\bar{a} : \mathbf{T}^d \to \mathbf{R}^d$ such that the flow X_ε tends to the "generalized" flow*

$$(t,x,y) \in \mathbf{R} \times \mathbf{R}^d \times \mathbf{T}^d \to X(t,x,y) = x + t\bar{a}(y) \in \mathbf{R}^d. \tag{2.2}$$

Moreover $\int_{\mathbf{T}^d} \bar{a}(y)dy = \int_{\mathbf{T}^d} a(y)dy$.

Proof. The proof is a simple application of Birkhoff's pointwise ergodic theorem that asserts:

Let be a smooth divergence free vector field on the torus $K = \mathbf{T}^d$ and $(t, y) \to \xi(t, y)$ be the associated flow on K. Then, for each $f \in L^1(K)$, there is $\bar{f} \in L^1(K)$ s.t.

$$\int_K \bar{f}(y)dy = \int_K f(y)dy \tag{2.3a}$$

$$\text{for a.e. } y \in K \lim_{T\to\infty} \frac{1}{T} \int_0^T f(\xi(t, y))dt = \bar{f}(y). \tag{2.3b}$$

Here ξ is the flow associated with a, on the torus. By using an elementary scaling X_ε, which is the flow associated with a_ε, on the whole space $\mathbf{R}^d$, can be easily recovered from ξ. Indeed, let us fix $x \in \mathbf{R}^d$. x can be written as

$$x = \varepsilon k + \varepsilon y, \text{ where } k \in \mathbf{Z}^d,\ y \in \mathbf{T}^d. \tag{2.4}$$

It is not true (as shown on figure 1) that,

$$X_\varepsilon(t, x) = \varepsilon k + \varepsilon \xi(\frac{t}{\varepsilon}, y)(\text{because } X_\varepsilon \text{ is defined on } \mathbf{R}^d \text{ and } \xi \text{ on } \mathbf{T}^d)$$

However, there is a simple correspondance between the velocities:

$$\dot{X}_\varepsilon(t, x) = \dot{\xi}(\frac{t}{\varepsilon}, y). \tag{2.5}$$

Thus, by integrating (2.5):

$$X_\varepsilon(t, x) = x + \int_0^t \dot{\xi}(\frac{s}{\varepsilon}, y)ds = x + \varepsilon \int_0^{t/\varepsilon} \dot{\xi}(s, y)ds \tag{2.6}$$

where $x = \varepsilon k + \varepsilon y$.

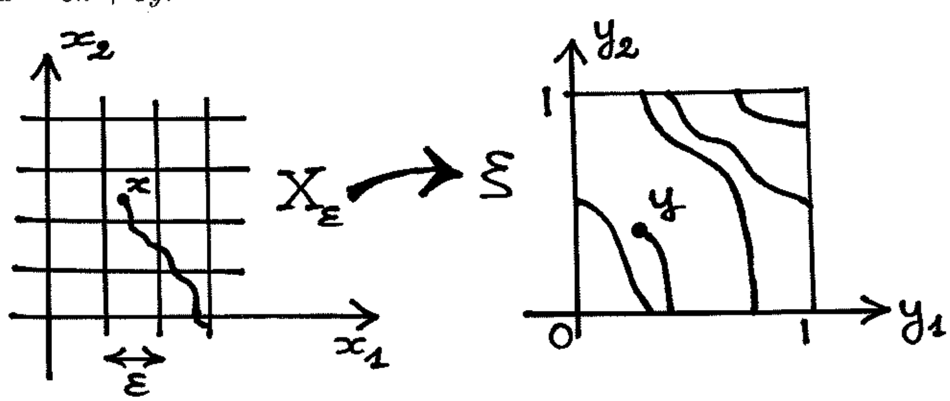

Figure 1. Correspondance between X_ε and ξ

Let us now consider $I_\varepsilon = \int_{\mathbf{R}^d} \psi(X_\varepsilon(t,x),x)dx$ for a compactly supported continuous function ψ defined on $\mathbf{R}^d \times \mathbf{R}^d$. We have (by splitting $\mathbf{R}^d$ in cells of length ε):

$$I_\varepsilon = \sum_{k\in\mathbf{Z}^d} \varepsilon^d \int_{[0,1]^d} \psi(X_\varepsilon(t,\varepsilon k+\varepsilon y),\ \varepsilon k+\varepsilon y)dy$$

$$= \sum_{k\in\mathbf{Z}^d} \varepsilon^d \int_{[0,1]^d} \psi(\varepsilon k+\varepsilon y+\varepsilon \int_0^{\frac{t}{\varepsilon}} \dot{\xi}(s,y)ds,\ \varepsilon k+\varepsilon y)dy,$$

(by (2.6))

that can be (artificially!) rewritten as

$$I_\varepsilon = \sum_k \varepsilon^d \int\int_{[0,1]^d} \psi(\varepsilon k+\varepsilon y+\varepsilon \int_0^{\frac{t}{\varepsilon}} \dot{\xi}(s,y)ds, \varepsilon k+\varepsilon y)dydz.$$

Since ψ is uniformly continuous, changing εy into εz in I_ε does not modify much I_ε:

$$I_\varepsilon = \sum_k \varepsilon^d \int\int_{[0,1]^d} \psi(\varepsilon k+\varepsilon z+\varepsilon \int_0^{\frac{t}{\varepsilon}} \dot{\xi}(s,y)ds,\ \varepsilon k+\varepsilon z)dydz$$
$$+ r_\varepsilon$$

where $r_\varepsilon \to 0$ when ε goes to 0.

Then, we can go backward to recover an integral over the whole space $\mathbf{R}^d$:

$$I_\varepsilon = \int_{\mathbf{R}^d} \int_{[0,1]^d} \psi(x+\varepsilon \int_0^{\frac{t}{\varepsilon}} \dot{\xi}(s,y)ds, x)dxdy + r_\varepsilon.$$

Since $\dot{\xi}(s,y) = a(\xi(s,y))$, this can be rewritten as:

$$I_\varepsilon = \int_{\mathbf{R}^d} \int_{[0,1]^d} \psi(s+\varepsilon \int_0^{\frac{t}{\varepsilon}} a(\xi(s,y))ds, x)dydx + r_\varepsilon.$$

According to Birkhoff's theorem, there is a bounded measurable function $\bar{a}$ associated with a, such that $\bar{a}(y) = \lim_{T\to\infty} \int_0^T \frac{1}{T} a(\xi(s,y))ds$, for a.e. y in $\mathbf{T}^d$. Thus, it follows from Lebesgue's theorem that:

$$I_\varepsilon \to \int_{\mathbf{R}^d} \int_{[0,1]^d} \psi(x+t\bar{a}(y), x)dydx.$$

Moreover, Birkhoff's theorem asserts that $\int_{\mathbf{T}^d} \bar{a}(y)dy = \int_{\mathbf{T}^d} a(x)dx$, which achieves the proof of Proposition 1.

Remark 1.

This result can be considerably improved in the 2-dimensional case, as shown by Liu [Li] (personnal communication). If $A_1 = \int_{\mathbf{T}^d} a_1(y)dy$ and $A_2 = \int_{\mathbf{T}^d} a_2(y)dy$ are not rationally linearly dependent then the limit flow is a classical flow defined by $X(t,x) = x + tA$. Otherwise, there is a scalar function $\beta : [0,1] \to \mathbf{R}$ such that $\int_0^1 \beta(\theta)d\theta = 1$ and the limit flow is a generalized flow defined by:

$$X(t,x,\theta) = x + t\beta(\theta)A.$$

Remark 2.

By definition, the flow ξ associated with a on the torus is ergodic if and only if, for each $f \in L^1(K)$, $\bar{f}$ is a constant and $\bar{f}(x) = cst = \int_K f(y)dy$.

It follows that, if the flow is ergodic, then the flow X_ε converges to a classical flow, namely:

$$X(t,x) = x + tA, \text{ where } A = \int_K a(y)dy. \tag{2.7}$$

3. Examples of complicated behavior.

Let us consider the 2-dimensional case. Then a_ε must be a curl. Let us assume that

$$a_\varepsilon(t,x) = \text{curl } \psi_\varepsilon(x), \text{ where} \tag{3.1}$$

$$\psi_\varepsilon(x) = \psi_0(x) + \varepsilon\psi_1(x, \frac{x}{\varepsilon}) \tag{3.2}$$

and ψ_0 is, say, a uniformly strictly convex function, namely $\psi_0(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

The dynamics of the ODE $\dot{x} = a_\varepsilon$ is governed by the level contours of the "stream function" ψ_ε:

$$\psi_\varepsilon(X_\varepsilon(t,x)) = \psi_\varepsilon(x), \quad \forall t \in \mathbf{R}, \ x \in \mathbf{R}^2 \tag{3.3}$$

If ψ_1 is assumed to be uniformly bounded, the level contours of ψ_ε will stay very close to the level contours of ψ_0 (with a distance of $O(\varepsilon)$). Thus, if it is assumed that X_ε converges to some generalized flow $(t, x, y) \in \mathbf{R} \times \mathbf{R}^2 \times \Omega \rightarrow X(t, x, y) \in \mathbf{R}^2$, we will get when ε approaches 0:

$$\psi_0(X(t, x, y)) = \psi_0(x). \tag{3.4}$$

But, beside this information, it is quite clear that essentially any kind of behavior can be expected, in the limit, for the trajectories that live near a given level contour of ψ_0, as shown on figure 2: some of then can stay trapped in a small level island, when ε goes to 0, that converges to a loop along the level contour of ψ_0.

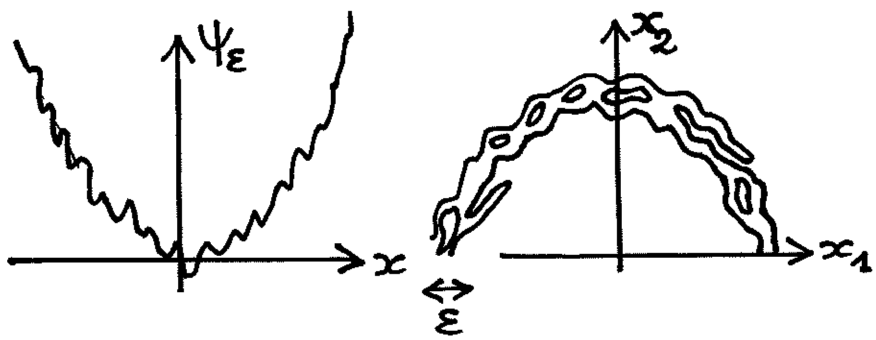

Figure 2. Behavior of trajectories

4. Generalized flows and higher order ODEs.

If we get back to the results of section 2, we see that the generalized flow, obtained in the limit,

$$X(t, x, y) = x + t\bar{a}(y), \tag{4.1}$$

trivially satisfies the 2nd order ODE:

$$\ddot{X} = 0. \tag{4.2}$$

Thus, one may wonder whether this behaviour has some more generality. An example, that was considered in an earlier work [Bre] on the Euler equations of perfect incompressible fluids, shows a similar behavior. Here, we have:

$$d = 2,\ a_\varepsilon = \text{curl}\ \psi_\varepsilon(x) \text{ where } \psi_\varepsilon(x) = \varepsilon \sin x_1 \sin \frac{x_2}{\varepsilon}. \tag{4.3}$$

a_ε solves the (stationary) Euler equations:

$$(a_\varepsilon \cdot \nabla) a_\varepsilon = -\nabla p_\varepsilon\ \nabla \cdot a_\varepsilon = 0, \tag{4.4}$$

where the pressure field is given by:

$$p_\varepsilon(x) = \frac{1}{4} \cos 2x_1 + \frac{\varepsilon^2}{4} \cos 2\frac{x_2}{\varepsilon}. \tag{4.5}$$

By equation (4.4), the flow X_ε associated with a_ε satisfies the 2nd order ODE:

$$\ddot{X}_\varepsilon(t, x) = -\nabla p_\varepsilon(X_\varepsilon(t, x)). \tag{4.6}$$

Let us introduce, for each fixed ε, the auxiliary Hamiltonian system in $\mathbf{R}^4$:

$$\dot{x} = v,\ \dot{v} = -\nabla p_\varepsilon(x),\ x, v \in \mathbf{R}^2 \tag{4.7}$$

and the corresponding flowmap:

$$(t, x, v) \in \mathbf{R} \times \mathbf{R}^2 \times \mathbf{R}^2 \to (\xi_\varepsilon(t, x, v),\ \eta_\varepsilon(t, x, v)) \in \mathbf{R}^2 \times \mathbf{R}^2 \tag{4.8}$$

such that $\xi_\varepsilon(0, x, v) = x,\ \eta_\varepsilon(0, x, v) = v$.

Then, it follows from (4.6) that:

$$X_\varepsilon(t, x) = \xi_\varepsilon(t, x, \dot{X}_\varepsilon(0, x)),$$

that is

$$X_\varepsilon(t, x) = \xi_\varepsilon(t, x, a_\varepsilon(x)). \tag{4.9}$$

In this formula, ξ_ε and a_ε behave very differently a_ε is an highly oscillatory function:

$$a_\varepsilon(x) = \begin{pmatrix} -\sin x_1 \cos \frac{x_2}{\varepsilon} \\ \varepsilon \cos x_1 \sin \frac{x_2}{\varepsilon} \end{pmatrix} \tag{4.10}$$

and its limit (in the sense of Young's measures) is the generalized function:

$$a(x,y) = \begin{pmatrix} -\sin x_1 \cos 2\pi y \\ 0 \end{pmatrix}, \ x \in \mathbf{R}^2, \ y \in [0,1]. \tag{4.11}$$

More precisely, for any compactly supported continuous function on $\mathbf{R}^2$:

$$\int_{\mathbf{R}^2} \phi(x, a_\varepsilon(x))dx \to \int_{\mathbf{R}^2} \int_0^1 \phi(x, a(x,y))dydx.$$

In a very different way, ξ_ε behaves nicely when ε approaches 0. Indeed, (p_ε) is bounded in C^2, uniformly in ε, and ∇p_ε converges strongly, in C^0, to ∇p where:

$$p(x) = \frac{1}{4}\cos 2x_1. \tag{4.12}$$

According to classical results on ODEs, this shows that $(\xi_\varepsilon, \eta_\varepsilon)$ converges, in C^0, to the flow map (ξ, η) associated with the limit Hamiltonian system

$$\dot{x} = v, \ \dot{v} = -\nabla p(x). \tag{4.13}$$

Thus, for any compactly supported continuous function ϕ defined on $\mathbf{R}^2 \times \mathbf{R}^2$, we get, for each fixed $t > 0$,

$$\int_{\mathbf{R}^2} \psi(X_\varepsilon(t,x), x)dx = \int_{\mathbf{R}^2} \psi(\xi_\varepsilon(t,x,a_\varepsilon(x)), x)dx$$
$$\to \int_{\mathbf{R}^2} \int_0^1 \psi(\xi(t,x,a(x,y)), x)dydx,$$

which means that X_ε converges to the generalized flow

$$(t,x,y) \in \mathbf{R} \times \mathbf{R}^2 \times [0,1] \to \xi(t,x,a(x,y)) \in \mathbf{R}^2. \tag{4.14}$$

This flow $X(t,x,y) = \xi(t,x,a(x,y))$ solves, by definition, the 2nd order ODE:

$$\ddot{X} = -\nabla p(X), \tag{4.15}$$

as expected.

We found a similar example in the 3-dimensional case, with the same behavior. We have:

$$a_\varepsilon(x) = e^{-\frac{1}{2}(x_1^2+x_2^2)} \begin{pmatrix} \cos \frac{x_3}{\varepsilon} \\ \sin \frac{x_3}{\varepsilon} \\ \varepsilon(-x_2 \cos \frac{x_3}{\varepsilon} + x_1 \sin \frac{x_3}{\varepsilon}) \end{pmatrix}, \tag{4.16}$$

that solves the 3-dimensional Euler equations:

$$(a_\varepsilon \cdot \nabla)a_\varepsilon = -\nabla p_\varepsilon, \ \nabla \cdot a_\varepsilon = 0, \tag{4.17}$$

together with the pressure field

$$p_\varepsilon(x) = p(x) = -\frac{1}{2}\exp(-(x_1^2 + x_2^2)), \tag{4.18}$$

which actually is ε-independent.

Thus, the flow associated with X_ε, already satisfies $\ddot{X}_\varepsilon = -\nabla p(X_\varepsilon)$ and the limit generalized flow is of the form:

$$X(t, x, \theta) = \xi(t, x, a(x, \theta)), \ t \in \mathbf{R}, \ x \in \mathbf{R}^3, \ \theta \in [0, 1] \tag{4.19}$$

where

$$a(x, \theta) = e^{-\frac{1}{2}(x_1^2+x_2^2)} \begin{pmatrix} \cos 2\pi\theta \\ \sin 2\pi\theta \\ O \end{pmatrix} \tag{4.20}$$

and (ξ, η) is the flow associated with the Hamiltonian system:

$$\dot{x} = v, \ \dot{v} = -\nabla p(x). \tag{4.21}$$

5. More on the concept of generalized flows.

There is a very general and vague answer to the problem we are interested in. Let us assume that $K = \mathbf{R}^d/\mathbf{Z}^d$ and introduce Ω as the compact product space $K^{\mathbf{R}}$, that is the space of all paths $t \to z(t) \in K$. Then, to each classical flow X_ε on K, we can associate a probability measure g_ε on Ω, defined as follows. Take any finite sequence $t_1 < \cdots < t_n$ in $\mathbf{R}$ and any continuous function ϕ on K^n. Then define the path functional $\Phi(z) = \phi(z(t_1), \ldots, z(t_n))$, for $z \in \Omega$. We set:

$$\langle g_\varepsilon, \Phi \rangle = \int_\Omega \Phi(z) g_\varepsilon(dz) = \int_K \phi(X_\varepsilon(t_1, x), \ldots, X_\varepsilon(t_n, x))dx.$$

Following [Bre] or [Bre2], one can show that (5.1) defines a unique probability measure g_ε on Ω (namely, a unique $g_\varepsilon \in C(\Omega)'$, nonnegative such that $\langle g_\varepsilon, 1 \rangle = 1$).

(Notice that g_ε is not necessarily a regular Borel measure, since Ω is not separable nor metric.)

Now we can use the fact that the unit ball is weakly compact in $C(\Omega)'$ to deduce that there is at least a cluster point g for the sequence (g_ε). If (a_ε) is uniformly bounded in $L^\infty(\mathbf{R}\times K)$, then g has the following properties: g - a.e. z in Ω belongs to $H^1_{\text{loc}}(\mathbf{R}; K) \subset C^0(\mathbf{R}, K)$

$$\int_\Omega \int_{t_0}^{t_1} ||\dot{z}(t)||^2 dt g(dz) < +\infty, \ \forall t_0 < t_1.$$

Because a_ε is supposed to be divergence free, X_ε is measure preserving on K and it easily follows that g is a generalized incompressible flow (see [Bre2]) in the sense that:

$$\int_\Omega \phi(z(t_0))g(dz) = \int_K \phi(x)dx \tag{5.4}$$

holds for any time t_0 and any $\phi \in C(K)$.

Moreover, we have, for any compactly supported continuous function ϕ defined on $K \times K$, a subsequence (ε_k) such that:

$$\int_K \phi(X_{\varepsilon_k}(t,x), x)dx \to \int_\Omega \phi(z(t), z(0))g(dz).$$

For more information on the concept of generalized flow, we refer to [Bre2].

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