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The eigenvalue spectrum of domain decomposed preconditioners

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Abstract

In the past several years, domain decomposition has been a very popular topic, motivated by the ease of parallelisation. In a large class of domain decomposition methods, the continuity of the solution over the internal subdomain boundaries is obtained through an iterative process. When solving the Schur complement equations for the unknowns on the internal boundaries, an elliptic problem has to be solved on each subdomain in each iteration step. Therefore, doubts have been raised as to the efficiency of these methods. An alternative solution is using preconditioners for the original problem that can be inverted in a "domain decomposed" way.

In this paper, we describe and study a class of preconditioners that are combinations of preconditioners for the subdomain and the interface problems. We derive some properties of the eigenvalue spectrum of the preconditioned system, relating it to the eigenspectra of the subdomain problems.

We show some numerical examples to illustrate these properties.

Keywords : parallel algorithms, domain decomposition, partial differential equations, preconditioned conjugate gradients

1 Introduction

In the majority of domain decomposition methods, the matching of the solution on the subdomains to an overall solution is done by an iterative process. A large

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class is based on the preconditioned conjugate gradient method for solving the reduced equations on the interfaces between the subdomains. The efficiency of these methods is determined by the preconditioner used. This approach involves a solve on each subdomain in each iteration step and the cost could be expensive if the number of iterations is not kept at a minimum. Therefore, doubts have been raised on the efficiency of these methods as compared to a parallelisation of traditional preconditioned conjugate gradient iterations on the whole domain.

An approach that avoids the exact solves on the subdomains, consists of constructing preconditioners for the original problem but that can be inverted in a "domain decomposed" way. Specifically, given a method for constructing a preconditioner on the whole domain (such as the ILU- and $MILU$-type methods), a domain decomposed preconditioner can be constructed by applying the same method in the subdomains and using an appropriately chosen preconditioner for the interface unknowns. Experimentally, it has been shown that this class of preconditioners for the original domain can yield a faster convergence rate as compared to traditional preconditioners, [8, 19, 22].

In this paper, we proof some properties of the eigenvalue distribution of the preconditioned system. More specifically, we show that the eigenspectrum is a perturbation of the union of the spectra of the preconditioned systems on the subdomains and on the interfaces. The construction of the preconditioner and the proofs of the eigenspectrum are purely algebraically.

In section 2, we describe how to construct this kind of preconditioners and we proof some basic lemmas. In section 3, general properties that are valid for any decomposition are proven. These are elaborated more in section 4 for the special case of separable equations on a rectangle split into two strips. Based on this, we rederive and refine some results that were already mentioned in the literature, [4]. We illustrate these results with numerical examples in section 5.

2 Construction of domain decomposed preconditioners

We consider the solution of the linear, second order elliptic partial differential equation :

$$\begin{aligned} Lu &= f \qquad (x,y) \in \Omega \\ u &= u_b \qquad (x,y) \in \partial\Omega \end{aligned} \tag{1}$$

where Ω is a two-dimensional domain. After defining a grid on the domain Ω, the equation (1) is discretised using a finite element or a finite difference method. This gives rise to a linear system :

$$Au = f. \tag{2}$$

In (2), u and f refer to the discrete solution and the discrete right hand side. From now on, we always refer to this algebraical system. When L is a positive definite and self-adjoint operator, A is a symmetric, positive definite matrix. Given a symmetric and positive definite preconditioner M for A, (2) can be solved with the preconditioned conjugate gradient method, [10]. The convergence properties of the preconditioned conjugate gradient method are determined by the eigenvalue spectrum of the preconditioned matrix $M^{-1}A$. Let λ_m and λ_M be the smallest and the largest eigenvalue of $M^{-1}A$, and let $\kappa = \lambda_M/\lambda_m$ be the spectral condition number. The following properties are well known, [10].

- $$||u^k - u||_A \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k ||u^0 - u||_A. \tag{3}$$

- If $M^{-1}A$ has n distinct eigenvalues, the preconditioned conjugate gradient iteration converges in at most n iterations in the absence of rounding errors.

Furthermore, (3) only gives an upper bound for the A-norm of the error. The actual rate of convergence is faster as the components of the error are selectively eliminated during the iteration process. This is positively influenced by a clustering of the eigenvalues, [1].

Besides having good spectral properties yielding a fast convergence, preconditioners should be cheap to invert. In a parallel environment, "cheap" also includes "in parallel". The preconditioners studied here are based on the domain decomposition principle. They therefore have a natural parallelism. Furthermore, domain decomposition is characterised by a large locality giving better local approximations.

They are constructed as follows. The domain is split into a number of nonoverlapping subdomains with the internal boundaries of the subdomains coinciding with grid lines. This divides the nodes of the finite element mesh or the grid points of the finite difference grid in points that are internal to the subdomains and in separator nodes. Quantities related to these sets of points are denoted by a subscript d, respectively s. In the latter set, we distinguish between nodes on the internal boundaries (called *edges* or *interfaces*, denoted by the subscript e) and nodes on the intersection of edges. We call the last class of nodes *corners*, denoted by subscripts c. We consider the case where the unknowns for the nodes in different subdomains are not directly coupled. In a finite element setting, this means that the supports of the finite element basis functions for a particular node are restricted to the elements of which this node is a vertex. In finite difference methods, this implies that the discretisation stencil extends at most one grid line at each side of the point under consideration. We order first the subdomain unknowns, secondly the edge unknowns and finally

the corners. With this ordering, the discretisation matrix A has the form

$$\begin{pmatrix} A_{dd} & A_{ds} \\ A_{ds}^T & A_{ss} \end{pmatrix}.$$

The submatrix A_{dd} is a block diagonal matrix with each block corresponding to the discretisation of an elliptic problem on the corresponding subdomain. As only the nodes adjacent to the internal boundaries are directly coupled to the separator nodes, the off-diagonal blocks are very sparse. The structure of the matrix A_{ss} depends on the division in subdomains. The matrix A can be factorised as

$$A = \begin{pmatrix} I & 0 \\ A_{ds}^T A_{dd}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{dd} & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I & A_{dd}^{-1} A_{ds} \\ 0 & I \end{pmatrix} \tag{4}$$

with

$$S = A_{ss} - A_{ds}^T A_{dd}^{-1} A_{ds}.$$

S is the so-called Schur complement of A_{dd} in A. Given a preconditioning method for the subdomain problems and for the Schur complement system, we construct a preconditioner M for A by replacing the diagonal blocks in (4) by the corresponding preconditioners :

$$M = \begin{pmatrix} I & 0 \\ A_{ds}^T B^{-1} & I \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & M_s \end{pmatrix} \begin{pmatrix} I & B^{-1} A_{ds} \\ 0 & I \end{pmatrix}. \tag{5}$$

B is a block diagonal matrix where the blocks are preconditioners on the subdomains. They can come from any class of symmetric positive definite preconditioners that have been derived for a single domain, such as incomplete factorisations, constant coefficient approximations, or separable approximations. Possibly, they are defined in an algorithmical way, as *e.g.,* a number of multigrid cycles. M_s is a preconditioner for the interface system as expressed by S. Several possibilities have been proposed in the two subdomain case, [11, 16, 3, 7, 9, 2, 22]. By neglecting the coupling between the interfaces, these preconditioners can be combined into a block diagonal preconditioner M_s for a decomposition in strips, *i.e.,* a decomposition with no corners [12, 13, 22]. In box wise decompositions, *i.e.,* with corners, few is known about good preconditioners for the Schur complement S. As the derivation of the properties of the eigenspectrum does not depend on the actual form of the preconditioners B and M_s, we do not elaborate here further on possible choices for M_s in the box case. In section 5, we present a possible way of constructing M_s. For a more extensive discussion, we refer to [17].

Preconditioners of this kind were first proposed by Bramble, Pasciak and Schatz, [6, 5]. They are constructed by replacing the bilinear form of the weak formulation of the differential equation by a spectrally equivalent form that can

be inverted in a domain decomposed way. Algebraically, it results in a preconditioner of the form (5) where also the off-diagonal blocks are replaced. As subdomain preconditioners, constant coefficient operators or separable approximations are used. The preconditioner for the interface system is a combination of scaled trace preconditioners, [11], and the system resulting from discretising the differential equation on the coarse grid defined by the corner points. Numerical experiments with these preconditioners are reported in [5] and in the survey paper by Keyes and Gropp, [19]. This method is also studied in [24]. Preconditioners of the form (5), with the Neumann-Dirichlet preconditioner, [3], and with the trace preconditioner, [11], for two subdomains are studied in [4]. In [22, 21], Meurant proposes preconditioners of the form (5) derived from block preconditioners. The interface preconditioner here follows naturally from the recursive construction of sparse approximations to the Schur complements as they arrive in the block Cholesky decomposition of the discretisation matrices. Experiments are reported for the multiple strip case. In [8], the efficiency of a combination of incomplete factorisation preconditioners on the subdomains with the boundary probe preconditioner on the interface is discussed for two subdomains. Efficiency results of an implementation on a distributed memory computer for a general decomposition of the domain are reported in [18].

We first give some basic lemmas that will be used in the sequel. Let A be a general block (2×2) matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix},$$

and let $S = A_{22} - A_{12}^T A_{11}^{-1} A_{12}$.

Lemma 1

$$|A| = |A_{11}| \times |S|.$$

It follows directly that the product of the eigenvalues of A is equal to the product of the eigenvalues of A_{11} and S.

Let $(\pi(A), \nu(A), \zeta(A))$ be the inertia of A, *i.e.,* the triplet with the number of positive, negative and zero eigenvalues of A.

Lemma 2 *The inertia of A is equal to the sum of the inertia of A_{11} and S,* i.e.,

$$\begin{aligned} \pi(A) &= \pi(A_{11}) + \pi(S) \\ \nu(A) &= \nu(A_{11}) + \nu(S) \\ \zeta(A) &= \zeta(A_{11}) + \zeta(S). \end{aligned}$$

This is a direct application of Sylvester's theorem, [23]. From this, we derive that if B and M, are symmetric positive definite matrices, M is also symmetric positive definite.

3 Eigenstructure of the preconditioned operator

The convergence behaviour of preconditioned conjugate gradient iteration is determined by the eigenvalue spectrum of the preconditioned operator

$$M^{-1}A = \begin{pmatrix} B^{-1}A_{dd} - B^{-1}A_{ds}M_s^{-1}A_{ds}^T\left(I - B^{-1}A_{dd}\right) & B^{-1}A_{ds}\left(I - M_s^{-1}\tilde{S}\right) \\ M_s^{-1}A_{ds}^T\left(I - B^{-1}A_{dd}\right) & M_s^{-1}\tilde{S} \end{pmatrix}$$

where

$$\tilde{S} = A_{ss} - A_{ds}^T B^{-1} A_{ds}.$$

Applying a similarity transformation $VM^{-1}AV^{-1}$ with

$$V = \begin{pmatrix} B^{1/2} & B^{-1/2}A_{ds} \\ 0 & M_s^{1/2} \end{pmatrix},$$

we arrive at the symmetric matrix

$$\begin{pmatrix} B^{-1/2}A_{dd}B^{-1/2} & \left(I - B^{-1/2}A_{dd}B^{-1/2}\right)B^{-1/2}A_{ds}M_s^{-1/2} \\ M_s^{-1/2}A_{ds}^T B^{-1/2}\left(I - B^{-1/2}A_{dd}B^{-1/2}\right) & M_s^{-1/2}\hat{S}M_s^{-1/2} \end{pmatrix}, \tag{6}$$

where

$$\begin{aligned} \hat{S} &= \tilde{S} - A_{ds}^T B^{-1/2}\left(I - B^{-1/2}A_{dd}B^{-1/2}\right)B^{-1/2}A_{ds} \\ &= S + A_{ds}^T A_{dd}^{-1/2}\left(I - A_{dd}^{1/2}B^{-1}A_{dd}^{1/2}\right)^2 A_{dd}^{-1/2}A_{ds} \\ &= S + A_{ds}^T B^{-1/2}\left(I - B^{-1/2}A_{dd}B^{-1/2}\right)B^{1/2}A^{-1}B^{1/2} \times \\ &\qquad \left(I - B^{-1/2}A_{dd}B^{-1/2}\right)B^{-1/2}A_{ds}. \end{aligned} \tag{7}$$

We use the following shorthands for the blocks of (6) :

$$\begin{aligned} \Delta &= B^{-1/2}A_{dd}B^{-1/2} \\ \Delta_S &= M_s^{-1/2}SM_s^{-1/2} \\ \Delta_{\tilde{S}} &= M_s^{-1/2}\tilde{S}M_s^{-1/2} \\ \widetilde{B} &= B^{-1/2}A_{ds}M_s^{-1/2}. \end{aligned}$$

We rewrite (6) in this notation for later use :

$$\begin{pmatrix} \Delta & (I - \Delta)\widetilde{B} \\ \widetilde{B}^T(I - \Delta) & \hat{\Delta} \end{pmatrix}, \tag{8}$$

with

$$\begin{aligned}\hat{\Delta} &= \Delta_{\tilde{S}} - \tilde{B}^T (I - \Delta) \tilde{B} \\ &= \Delta_S + \tilde{B}^T (I - \Delta) \Delta^{-1} (I - \Delta) \tilde{B}.\end{aligned}$$

Let A_{dd} be of order N and let S be of order n, so A is of order $\bar{N} = N + n$. Note that $N = \mathrm{O}(n^2)$. We introduce the following notations for the eigenvalues of the various blocks :

Symbol		Matrix
$\tilde{\lambda}^l$,	$l = 1, \ldots, \bar{N}$	A
$\tilde{\alpha}^l$,	$l = 1, \ldots, \bar{N}$	$M^{-1}A$
α^j ,	$j = 1, \ldots, N$	$B^{-1}A_{dd}$, $B^{-1/2}A_{dd}B^{-1/2}$, Δ
λ_S^k ,	$k = 1, \ldots, n$	S
μ_M^k ,	$k = 1, \ldots, n$	M_s
σ^k ,	$k = 1, \ldots, n$	$M_s^{-1}S$, $M_s^{-1/2}SM_s^{-1/2}$, Δ_S
$\lambda_{\tilde{S}}^k$,	$k = 1, \ldots, n$	$\tilde{S}$
$\tilde{\sigma}^k$,	$k = 1, \ldots, n$	$M_s^{-1}\tilde{S}$, $M_s^{-1/2}\tilde{S}M_s^{-1/2}$, $\Delta_{\tilde{S}}$

We sometimes change the superscript to denote another way of numbering the eigenvalues. The symbols themselves always refer to the same type of block. We assume that all the eigenvalues of a block are numbered in ascending order :

$$\lambda_1 \leq \lambda_2 \leq \cdots,$$

where λ is any of the symbols just defined.

Applying Cauchy's interlace theorem, [23, pages 186–187], to the block matrix (6) gives

Theorem 3 *There are at most n eigenvalues of $M^{-1}A$ smaller than α^1, and analogously, there are at most n eigenvalues of $M^{-1}A$ larger than α^N. Furthermore, in each interval $[\alpha^j, \alpha^{j+n}]$ lies at least one eigenvalue $\tilde{\alpha}^l$.*

Note that the eigenspectrum of $B^{-1/2}A_{dd}B^{-1/2}$ is the union of the spectra of the preconditioned operators on the subdomains. It follows from theorem 3 that the spectra on the subdomains have to be clustered around the same value. If these spectra are shifted relatively to each other, the condition number of $M^{-1}A$ can

be very large even though the condition numbers for the subdomain problems are small.

In the following two theorems, we give an indication for the amplitude by which the extreme eigenvalues of the subdomain spectra are shifted. Applying the trace theorem for matrices to (6) gives

Theorem 4

$$\begin{aligned}\sum_{l=1}^{\bar{N}} \tilde{\alpha}^l &= \sum_{j=1}^{N} \alpha^j + \sum_{k=1}^{n} \tilde{\sigma}^k \\ &\quad + \sum_{k=1}^{n} (e_k^n)^T M_s^{-1/2} A_{ds}^T B^{-1/2} \left(I - B^{-1/2} A_{dd} B^{-1/2}\right) \times \\ &\qquad B^{-1/2} A_{ds} M_s^{-1/2} e_k^n \\ &= \sum_{j=1}^{N} \alpha^j + \sum_{k=1}^{n} \sigma^k \\ &\quad + \sum_{k=1}^{n} (e_k^n)^T M_s^{-1/2} A_{ds}^T A_{dd}^{-1/2} \left(I - A_{dd}^{1/2} B^{-1} A_{dd}^{1/2}\right)^2 \times \\ &\qquad A_{dd}^{-1/2} A_{ds} M_s^{-1/2} e_k^n \end{aligned} \tag{9}$$

with e_k^n the k^{th} unit vector of length n.

The perturbation (9) is equal to the sum of the eigenvalues of

$$M_s^{-1/2} A_{ds}^T A_{dd}^{-1/2} \left(I - A_{dd}^{1/2} B^{-1} A_{dd}^{1/2}\right)^2 A_{dd}^{-1/2} A_{ds} M_s^{-1/2}. \tag{10}$$

Let

$$P_1 = (I - A_{dd}^{1/2} B^{-1} A_{dd}^{1/2}) A_{dd}^{-1/2} A_{ds} M_s^{-1/2}.$$

The eigenvalues of the matrix (10) are the squares of the singular values of P_1. Let ρ_1 be the largest singular value. The sum of the eigenvalues of $M^{-1}A$ exceeds the sum of the eigenvalues of Δ and Δ_S by at least $(\rho_1)^2$. Let

$$\begin{aligned} P_2 &= \left(I - A_{dd}^{1/2} B^{-1} A_{dd}^{1/2}\right) \\ P_3 &= A_{dd}^{-1/2} A_{ds} M_s^{-1/2}, \end{aligned}$$

and let ρ_2 be the spectral radius of P_2, and ρ_3 the largest singular value of P_3. An upper bound for ρ_1 is

$$\rho_1 \leq \rho_2 \rho_3. \tag{11}$$

The singular values of P_3 are equal to the square roots of the eigenvalues of $P_3^T P_3$, or equivalently of $A_{ds}^T A_{dd}^{-1} A_{ds} M_s^{-1}$. Let ρ_4 be the largest eigenvalue of $A_{ds}^T A_{dd}^{-1} A_{ds}$. This alleviates the upper bound (11) :

$$(\rho_1)^2 \leq (\rho_2)^2 (\rho_3)^2 \leq (\rho_2)^2 \rho_4 (\mu_M^1)^{-1}. \tag{12}$$

The upper bound (12) becomes sharp when :

- the eigenvector of P_2, belonging to the spectral radius, is the same as the left singular vector of $A_{dd}^{-1/2} A_{d\bullet}$ belonging to the largest singular value, and
- the right singular vector of $A_{dd}^{-1/2} A_{d\bullet}$, belonging to the largest singular value (or equivalently the eigenvector of $A_{d\bullet}^T A_{dd}^{-1} A_{d\bullet}$ with the largest eigenvalue), is the same as the eigenvector of $M_\bullet$, corresponding to the smallest eigenvalue.

We conclude that the trace of $M^{-1}A$ differs from the sum of the traces of $B^{-1/2} A_{dd} B^{-1/2}$ and $M_\bullet^{-1/2} S M_\bullet^{-1/2}$, by at least the amount $(\rho_1)^2$. This term is bounded from above by the spectral radius of $(I - A_{dd}^{1/2} B^{-1} A_{dd}^{1/2})$, by the largest eigenvalue of $A_{d\bullet}^T A_{dd}^{-1} A_{d\bullet}$ and by the smallest eigenvalue of $M_\bullet$. The upper bound is attained under rather special conditions. To keep the distance small over which the extreme eigenvalues are shifted, the subdomain spectra should be clustered around one. This is a requirement that does not hold when we use B_{ii} in a preconditioned conjugate gradient iteration on the subdomains, for which only the condition number $\kappa(B_{ii}^{-1} A_{ii})$ and the relative spacing of the eigenvalues are important.

Using (7) it is easily shown that the Schur complement of $B^{-1/2} A_{dd} B^{-1/2}$ in (6) is equal to S. From lemma 1 follows

Theorem 5

$$\prod_{l=1}^{\tilde{N}} \tilde{\alpha}^l = \prod_{j=1}^{N} \alpha^j \prod_{k=1}^{n} \sigma^k.$$

When more information on the spectrum of Δ is available, we can further refine these general results.

3.1 B spectrally equivalent to A_{dd}

Suppose we combine spectrally equivalent preconditioners on the subdomains with a spectrally equivalent preconditioner for S. It is a natural question to ask if the resulting domain decomposed preconditioner is spectrally equivalent to A. Spectral equivalence implies that the eigenvalues of the preconditioned system are of the order O(1), independent of the grid size h, so :

$$\begin{aligned} \alpha^j &= \mathrm{O}(1) \\ \sigma^k &= \mathrm{O}(1). \end{aligned}$$

The trace of $M^{-1}A$ differs from the sum of the traces of Δ and of Δ_S, by at least $(\rho_1)^2$. Suppose that this perturbation is $\mathrm{O}(h^{-\beta})$, with $\beta > 0$. This perturbation

results in an eigenvalue of $M^{-1}A$ of order $O(h^{-\beta})$. It follows from theorem 5 that $\Pi_{l=1}^{\tilde{N}} \tilde{\alpha}^l = O(1)$. So, there must also be an eigenvalue $M^{-1}A$ of the order $O(h^{\beta})$. We conclude that in this situation

$$\kappa(M^{-1}A) \geq O(h^{-2\beta}).$$

The combination of spectrally equivalent preconditioners for A_{dd} and S can result in preconditioned operator whose spectral condition number grows with $h^{-2\beta}$.

To have a spectral equivalent preconditioner M, the perturbation term for the trace of $M^{-1}A$ should be independent of the grid size. A necessary condition is that $(\rho_1)^2$ is independent of h. From the upper bound (12) we derive the sufficient condition

$$(\rho_2)^2 \rho_4 (\mu_M^1)^{-1} \leq O(1).$$

All the eigenvalues of $A_{ds}^T A_{dd}^{-1} A_{ds}$ are $O(1)$. For the Laplace operator and a decomposition in two strips, this is derived in [3]. For more general splittings, consider the Raleigh quotient

$$\frac{x^T A_{ds}^T A_{dd}^{-1} A_{ds} x}{x^T x}. \tag{13}$$

The nominator of (13) is equal to

$$\sum_{i=1}^{n_d} [x]_i^T \left(A_{ds}^T\right)_i \left((A_{dd})_i\right)^{-1} (A_{ds})_i [x]_i, \tag{14}$$

where $(A_{dd})_i$ is the diagonal block corresponding to subdomain i, $[x]_i$ is the subvector of x with the components for points on the internal boundaries of Ω_i, and n_d is the number of subdomains. The terms in (14) correspond to solving homogeneous problems with homogeneous Dirichlet conditions on the external edges and x as Dirichlet boundary conditions on the internal boundaries, and subsequently making the inner product of the solution in the points adjacent to the internal boundaries, with the vector of the Dirichlet boundary conditions. As the grid is refined, these points move closer and closer to the internal boundaries or equivalently

$$[x]_i^T \left(A_{ds}^T\right)_i \left((A_{dd})_i\right)^{-1} (A_{ds})_i [x]_i = O(([x]_i)^T [x]_i),$$

So, the Raleigh quotient (13) and thus also ρ_4, are $O(1)$. We conclude that a sufficient condition for $(\rho_1)^2$ to be of order $O(1)$ is

$$(\rho_2)^2 \leq O(\mu_M^1).$$

The question on the behaviour of the combination of spectrally equivalent preconditioners, was first raised and answered by Börgers in [4], for a splitting

in two subdomains. In section 4, we come back to this problem and we extend the results from [4].

We next consider the situation where all the eigenvalues of Δ lie at the same side of one. This situation arises when using incomplete factorisation preconditioners on the subdomains. In what follows, we use the simple lemma that the triplet with the numbers of eigenvalues of a matrix A that are larger, smaller, and equal to a constant λ is equal to the inertia of $(A - \lambda I)$. Lemma 2 applied to the matrix

$$(M^{-1}A - \lambda I) = \begin{pmatrix} \Delta - \lambda I & (I - \Delta)\widetilde{B} \\ \widetilde{B}^T(I - \Delta) & \hat{\Delta} - \lambda I \end{pmatrix},$$

gives that the inertia of $(M^{-1}A - \lambda I)$ is equal to the sum of the inertia of $(\Delta - \lambda I)$ and of $\check{\Delta}(\lambda)$, with

$$\begin{aligned} \check{\Delta}(\lambda) &= (\Delta_{\tilde{S}} - \lambda I) - (1 - \lambda)\widetilde{B}^T(\Delta - \lambda I)^{-1}(I - \Delta)\widetilde{B} \\ &= (\Delta_S - \lambda I) - \lambda\widetilde{B}^T(I - \Delta)\Delta^{-1}(\Delta - \lambda I)^{-1}(I - \Delta)\widetilde{B}, \end{aligned}$$

where we assume that λ is not an eigenvalue of Δ. For λ equal to 1, we find that

$$\begin{aligned} \pi(M^{-1}A - I) &= \pi(\Delta - I) + \pi(\Delta_{\tilde{S}} - I) \\ \nu(M^{-1}A - I) &= \nu(\Delta - I) + \nu(\Delta_{\tilde{S}} - I) \\ \zeta(M^{-1}A - I) &= \zeta(\Delta - I) + \zeta(\Delta_{\tilde{S}} - I). \end{aligned}$$

3.2 Eigenspectrum of Δ greater than 1, $\alpha^j > 1$

When all the eigenvalues of Δ are larger than one, $\nu(\Delta - I)$ is equal to zero and $\nu(M^{-1}A - I) = \nu(\Delta_{\tilde{S}} - I)$. If $\tilde{\sigma}^1 > 1$, then $\nu(\Delta_{\tilde{S}} - I) = 0$, and all the eigenvalues of $M^{-1}A$ are larger than one. If $\tilde{\sigma}^1 < 1$, consider $\check{\Delta}(\tilde{\sigma}^1)$. The smallest eigenvalue of $\check{\Delta}(\tilde{\sigma}^1)$ is equal to

$$\min \frac{x^T\check{\Delta}x}{x^Tx}.$$

The eigenvalues of the matrix $\left(-(1 - \tilde{\sigma}^1)(\Delta - \tilde{\sigma}^1 I)^{-1}(I - \Delta)\right)$ are equal to

$$\left(-(1 - \tilde{\sigma}^1)(\alpha^j - \tilde{\sigma}^1)^{-1}(1 - \alpha^j)\right),$$

with $j = 1, \ldots, N$. This expression is always positive, so

$$x^T\check{\Delta}x > x^T(\Delta_{\tilde{S}} - \tilde{\sigma}I)x > 0.$$

So, $\nu(\check{\Delta}) = 0$, and from this we conclude that all the eigenvalues of $M^{-1}A$ are larger than $\tilde{\sigma}^1$. In summary :

Theorem 6 *When the spectrum of Δ lies to the right of one, all the eigenvalues of $M^{-1}A$ are larger than the minimum of $(1, \tilde{\sigma}^1)$.*

3.3 Eigenspectrum of Δ smaller than 1, $\alpha^j < 1$

In an analogous fashion as above, we derive:

Theorem 7 *When the spectrum of Δ lies to the left of one, all the eigenvalues of $M^{-1}A$ are smaller than the maximum of $(1, \tilde{\sigma}^n)$.*

4 Separable equations on a rectangle

We illustrate the theoretical results of the previous section, which are valid for any decomposition and any choice for B or M_s, with a splitting of a rectangle in two strips. We use the academic example where the subdomain preconditioners B_{ii} have the same eigenvectors as the discretisation matrices on the subdomains. In practice, this restriction will hardly ever be satisfied. However, this choice allows more quantitative bounds to be derived for the eigenvalues of $M^{-1}A$. This helps to understand the interplay between the eigendecomposition of the subdomain blocks and the eigendecomposition of the Schur complement block. We also rederive and refine the conclusions from [4], and give a possible explanation for the numerical results reported in that paper.

We consider the special case of a separable operator over a rectangle divided in two horizontal strips. The differential equation is discretised over an equidistant grid with n internal grid lines in the x direction and $m_1 + m_2 + 1$ internal grid lines in the y direction. Grid line $m_1 + 1$ forms the interface between the two subdomains. The matrix A has the form

$$A = \begin{pmatrix} A_{11} & & A_{13} \\ & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{pmatrix}$$

The indices 1 and 2 refer to the subdomain unknowns and the index 3 to the interface unknowns. For a separable operator

$$L = L_x + L_y$$

where

$$\begin{aligned} L_x &= \frac{\partial}{\partial x}\left(f_{10}(x)\frac{\partial}{\partial x}\right) + f_{00x}(x) \\ L_y &= \frac{\partial}{\partial y}\left(f_{01}(y)\frac{\partial}{\partial y}\right) + f_{00y}(y), \end{aligned}$$

the discretisation matrices can be expressed as the tensor product of the discretisation matrices for the one-dimensional problems with the operators L_x and L_y. Ordering the unknowns in the subdomains by lines and the lines from

bottom to top in the lower subdomain and from top to bottom in the upper subdomain, these matrices are :

$$\begin{aligned} A_{ii} &= A_{y_i} \otimes I_n + I_{m_i} \otimes A_x \\ A_{i3} &= \beta_i e_{m_i}^{m_i} \otimes I_n \\ A_{33} &= \beta I_n + A_x, \end{aligned}$$

where β_i and β are constants, determined by the functions $f_{01}(y)$ and $f_{00y}(y)$, and $e_{m_i}^{m_i}$, I_n are the m_i^{th} unit vector of length m_i and the identity matrix of order n respectively. The eigendecomposition of the subdomain blocks is expressed in a similar tensor product form

$$A_{ii} = (V_{y_i} \otimes V_x)(\Delta_{y_i} \otimes I_n + I_{m_i} \otimes \Delta_x)(V_{y_i}^T \otimes V_x^T)$$

with V_{y_i} and V_x the eigenvectors of the one-dimensional problems in the $y-$ and $x-$direction. The Schur complement S can be factorised as, [3],

$$\begin{aligned} S &= A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23} \\ &= V_x \Bigg(\beta I_n + \Delta_x \\ &\quad - \sum_{i=1}^{2} (\beta_i)^2 \sum_{j=1}^{m_i} V_{y_i}(m_i, j) V_{y_i}^T(j, m_i) (\Delta_{A_i}^j)^{-1} \Bigg) V_x^T \\ &= V_x \left(\beta I_n + \Delta_x - \sum_{i=1}^{2} \Delta_d^i \right) V_x^T \\ &= V_x \mathrm{diag}\left(\lambda_S^k\right)_{k=1,\ldots,n} V_x^T, \end{aligned}$$

with

$$\begin{aligned} \Delta_d^i &= \mathrm{diag}\left(d_i^k\right)_{k=1,\ldots,n} \\ d_i^k &= \beta_i^2 \sum_j^{m_i} \frac{V_{y_i}^2(m_i, j)}{\lambda_i^{j,k}}. \end{aligned}$$

It is well known, [3, 7], that

$$\begin{aligned} d_i^k &= \mathrm{O}(1) \\ \lambda_S^1 &= \mathrm{O}(h_x) \\ \lambda_S^n &= \mathrm{O}(1). \end{aligned}$$

The matrix (6) expands to

$$\begin{pmatrix} B_{11}^{-1/2} A_{11} B_{11}^{-1/2} & 0 & \tilde{P}_1 B_{11}^{-1/2} A_{13} M_s^{-1/2} \\ 0 & B_{22}^{-1/2} A_{22} B_{22}^{-1/2} & \tilde{P}_2 B_{22}^{-1/2} A_{23} M_s^{-1/2} \\ M_s^{-1/2} A_{31} B_{11}^{-1/2} \tilde{P}_1 & M_s^{-1/2} A_{32} B_{22}^{-1/2} \tilde{P}_2 & M_s^{-1/2} \hat{S} M_s^{-1/2} \end{pmatrix} \qquad (15)$$

where

$$\tilde{P}_i = \left(I - B_{ii}^{-1/2} A_{ii} B_{ii}^{-1/2}\right).$$

Let the preconditioners B_{ii} have the same (tensor product) eigenvectors as A_{ii}. This gives the following decomposition :

$$B_{ii}^{-1/2} A_{ii} B_{ii}^{-1/2} = (V_{y_i} \otimes V_x) \text{diag}\left(\Delta_i^j\right)_{j=1,\ldots,m_i} (V_{y_i}^T \otimes V_x^T)$$

with

$$\Delta_i^j = \text{diag}\left(\alpha_i^{j,k}\right)_{k=1,\ldots,n}.$$

Let V_x be also the eigenvectors of M_s :

$$\begin{aligned} M_s &= V_x \Delta_{M_s} V_x^T \\ \Delta_{M_s} &= \text{diag}\left(\mu_M^k\right)_{k=1,\ldots,n}. \end{aligned}$$

The matrix (15) is similar to

$$\begin{pmatrix} \tilde{\Delta}_{11} & & \tilde{\Delta}_{13} \\ & \tilde{\Delta}_{22} & \tilde{\Delta}_{23} \\ \tilde{\Delta}_{13}^T & \tilde{\Delta}_{23}^T & \tilde{\Delta}_{33} \end{pmatrix} \tag{16}$$

where all the elements in this block decomposition have themselves diagonal matrices as blocks. After a similarity transformation by a permutation of the rows and columns, (16) becomes :

$$\text{diag}\left(\bar{\Delta}^k\right)_{k=1,\ldots,n}$$

with

$$\bar{\Delta}^k = \begin{pmatrix} \bar{\Delta}_{11}^k & & \bar{\Delta}_{13}^k \\ & \bar{\Delta}_{22}^k & \bar{\Delta}_{23}^k \\ (\bar{\Delta}_{13}^k)^T & (\bar{\Delta}_{23}^k)^T & \bar{\Delta}_{33}^k \end{pmatrix} \tag{17}$$

and

$$\begin{aligned} \bar{\Delta}_{ii}^k &= \text{diag}\left(\alpha_i^{j,k}\right)_{j=1,\ldots,m_i} \\ \bar{\Delta}_{33}^k &= \tilde{\sigma}^k - (\mu_M^k)^{-1} \sum_{i=1}^{2} (\beta_i)^2 \sum_{j=1}^{m_i} V_{y_i}^2(m_i, j)(1 - \alpha_i^{j,k})(\mu_i^{j,k})^{-1} \\ \bar{\Delta}_{i3}^k &= \beta_i (\mu_M^k)^{-1/2} \text{col}\left(V_{y_i}^T(j, m_i)(1 - \alpha_i^{j,k})(\mu_i^{j,k})^{-1/2}\right)_j, \end{aligned}$$

where $\text{col}\left(a^j\right)_j$ denotes a column vector of length m_i, with a^j as elements.

From the special structure of $\bar{\Delta}^k$ we can derive some more properties on the position of the eigenvalues. Let α_j be a generic notation for the elements of the

blocks $\bar{\Delta}_{11}^k$ and $\bar{\Delta}_{22}^k$. The matrix (17) is a symmetric rank-two perturbation of a diagonal matrix of the general form :

$$C_m = \begin{pmatrix} \operatorname{diag}(\alpha_j)_{j=1,\ldots,m} & \operatorname{col}(c_j(1-\alpha_j))_{j=1,\ldots,m} \\ \operatorname{row}(c_j(1-\alpha_j))_{j=1,\ldots,m} & \alpha_{m+1} - \sum_{j=1}^m c_j^2(1-\alpha_j) \end{pmatrix}.$$

The following properties of the eigenvalues of such matrices are proven in [25, pages 103–104], or can be derived straightforwardly.

Property 1 *If $c_j = 0$ or $\alpha_j = 1$, for $j = 1, \ldots, m$, then is α_j an eigenvalue of C_m. If $\alpha_{m+1} = 1$, then it also is an eigenvalue.*

From now on we consider the case where $c_i \neq 0$ and $\alpha_i \neq 1$.

Property 2 *If α_j appears l times in the row $\alpha_1, \ldots, \alpha_m$ then α_j is a $(l-1)$ fold eigenvalue of C_m.*

Let $\tilde{\alpha}_i$, for $i = 1, \ldots, m+1$, denote the eigenvalues of C_m and let the row α_j be ordered in ascending order :

$$\alpha_1 < \alpha_2 < \cdots < \alpha_j < \alpha_{j+1} < \cdots < \alpha_m.$$

Property 3

$$\tilde{\alpha}_1 < \alpha_1 < \tilde{\alpha}_2 < \cdots < \alpha_j < \tilde{\alpha}_{j+1} < \alpha_{j+1} < \cdots < \alpha_m < \tilde{\alpha}_{m+1}.$$

Property 4

a. $$\sum_{j=1}^{m+1} \tilde{\alpha}_j = \alpha_{m+1} + \sum_{j=1}^{m} \alpha_j - \sum_{j=1}^{m} c_j^2(1-\alpha_j)$$

b. $$\prod_{j=1}^{m+1} \tilde{\alpha}_j = \alpha_{m+1} \prod_{j=1}^{m} \alpha_j - \sum_{i=1}^{m} c_i^2(1-\alpha_i) \prod_{\substack{j=1 \\ j\neq i}}^{m} \alpha_j.$$

Property 5

a. *If $\alpha_j < 1$, for $j = 1, \ldots, m$, then :*

$$\min(\alpha_{m+1}, 1) \leq \tilde{\alpha}_{m+1} \leq \max(\alpha_{m+1}, 1).$$

b. *If $\alpha_j > 1$, for $j = 1, \ldots, m$,then :*

$$\min(\alpha_{m+1}, 1) \leq \tilde{\alpha}_1 \leq \max(\alpha_{m+1}, 1).$$

These properties are special cases of the theorems derived in the previous section.

In [4], C. Börgers studies domain decomposed preconditioners for the Poisson equation on an L-shaped region. He proves that a combination of spectrally equivalent preconditioners on the subdomains with spectrally equivalent interface preconditioners does not necessarily give rise to a spectrally equivalent preconditioner for A. He also derives conditions under which the spectral equivalence is obtained. More specifically, the following are proven.

Theorem 8 *If the preconditioners B_{ii} are a constant times the discretisation matrices A_{ii},* i.e.,

$$B_{ii} = \frac{1}{\alpha_i} A_{ii}$$

with α_i independent of the grid size $h = h_x = h_y$, the following underbound holds for the spectral condition number

$$\kappa(M^{-1}A) \geq O(h^{-1}).$$

If the spectral radius of the iteration matrix is bounded by

$$\rho(I - B_{ii}^{-1} A_{ii}) \leq C\sqrt{h}, \tag{18}$$

then M is spectrally equivalent to A or equivalently

$$\kappa(M^{-1}A) = O(1).$$

For a rectangular domain, these results can be derived straightforwardly from the properties derived earlier. We consider spectrally equivalent preconditioners B_{ii}, *i.e.*,

$$\alpha_i^{j,k} = \mathrm{O}(1), \qquad \forall j,k.$$

We denote the eigenvalues of $\bar{\Delta}^k$, as defined in (17), by $\tilde{\alpha}^{j,k}$ with $j = 1, \ldots, m_1 + m_2 + 1$. Applying property 4 to $\bar{\Delta}^k$ gives

$$\begin{aligned}
\sum_{j=1}^{m_1+m_2+1} \tilde{\alpha}^{j,k} &= \sum_{i=1}^{2}\sum_{j=1}^{m_i} \alpha_i^{j,k} + \tilde{\sigma}^k \\
&\quad - (\mu_M^k)^{-1} \sum_{i=1}^{2} (\beta_3^i)^2 \sum_{j=1}^{m_i} V_{y_i}^2(m_i, j)(1 - \alpha_i^{j,k})\alpha_i^{j,k}(\lambda_i^{j,k})^{-1} \\
&= \sum_{i=1}^{2}\sum_{j=1}^{m_i} \alpha_i^{j,k} + \sigma^k \\
&\quad + (\mu_M^k)^{-1} \sum_{i=1}^{2} (\beta_3^i)^2 \sum_{j=1}^{m_i} V_{y_i}^2(m_i, j)(1 - \alpha_i^{j,k})^2(\lambda_i^{j,k})^{-1}, \qquad (19)
\end{aligned}$$

and

$$\prod_{j=1}^{m_1+m_2+1} \tilde{\alpha}^{j,k} = \sigma^k \prod_{i=1}^{2} \prod_{j=1}^{m_i} \alpha_i^{j,k}. \tag{20}$$

Let $\alpha^{1,k}$ be the smallest of the eigenvalues $\alpha_i^{j,k}$, for $i = 1, 2$ and $j = 1, \ldots, m_i$, and analogously, let $\alpha^{m,k}$ be the largest eigenvalue. Let $\tilde{\alpha}^{1,k}$ and $\tilde{\alpha}^{m+1,k}$ be the smallest, and the largest eigenvalue of $\bar{\Delta}^k$ respectively. From (19) and from the interlacing property 3, we derive the following bounds for the extreme eigenvalues of $\bar{\Delta}^k$:

$$\alpha^{1,k} + \sigma^k + p^k < \tilde{\alpha}^{1,k} + \tilde{\alpha}^{m+1,k} \tag{21}$$

$$\tilde{\alpha}^{1,k} + \tilde{\alpha}^{m+1,k} < \alpha^{m,k} + \sigma^k + p^k, \tag{22}$$

with

$$p^k = (\mu_M^k)^{-1} \sum_{i=1}^{2} (\beta_3^i)^2 \sum_{j=1}^{m_i} V_{y_i}^2(m_i, j)(1 - \alpha_i^{j,k})^2 (\lambda_i^{j,k})^{-1}.$$

Let

$$\frac{\left(1 - \alpha_i^{j,k}\right)^2}{\mu_M^k} < \delta^k, \qquad \forall i, j.$$

The perturbation term p^k is bounded by

$$p < \delta^k \sum_{i=1}^{2} (\beta_3^i)^2 \sum_{j=1}^{m_i} V_{y_i}^2(m_i, j)(\lambda_i^{j,k})^{-1} = \delta^k (d_1^k + d_2^k),$$

with $d_i^k = \mathrm{O}(1)$.

Let M_* be spectrally equivalent to S or equivalently $\sigma^k = \mathrm{O}(1)$. From (21) and (22) follows that

$$\tilde{\alpha}^{m+1,k} = \mathrm{O}(1) + \mathrm{O}(\delta^k).$$

Let $(1 - \alpha_i^{j,k}) = \mathrm{O}(1)$, for all j and k. As $\mu_M^1 = \mathrm{O}(h)$, we have

$$\delta^1 = \mathrm{O}(h^{-1}).$$

We conclude that $\tilde{\alpha}^{m+1,1} = \mathrm{O}(h^{-1})$. According to (20), the product of the eigenvalues of $\bar{\Delta}^k$ must be of order $\mathrm{O}(1)$. So, the largest eigenvalue of order $\mathrm{O}(h^{-1})$ must be compensated by a small eigenvalue of order $\mathrm{O}(h)$. This implies that $\kappa(\bar{\Delta}^1) = \mathrm{O}(h^{-2})$ and thus $\kappa(M^{-1}A) \geq \mathrm{O}(h^{-2})$.

If

$$\delta^k \leq \mathrm{O}(1) \tag{23}$$

all eigenvalues of $\bar{\Delta}^k$ are of order 1, and M is spectrally equivalent to A. A sufficient condition for (23) is that

$$1 - \alpha_i^{j,k} = \mathrm{O}(\sqrt{h}).$$

Table 1

α_i	M_s	λ_m	λ_M	$\kappa(M^{-1}A)$	*iterations*
$1-\sqrt{h}$	S	0.544	1.61	2.95	13
$1-h$	$\tilde{S}$	0.771	1.00	1.30	7

We can follow a similar path when M_s is spectrally equivalent to $\tilde{S}$, or $\tilde{\sigma}^k = \mathrm{O}(1)$. ,he following relationship holds between the eigenvalues of S and $\tilde{S}$:

$$\lambda_{\tilde{S}}^k = \lambda_S^k + \sum_{i=1}^{2} (\beta_3^i)^2 \sum_{j=1}^{m_i} V_{y_i}^2(m_i, j)(1-\alpha_i^{j,k})(\lambda_i^{j,k})^{-1}. \tag{24}$$

If $(1-\alpha_i^{j,k}) = \mathrm{O}(1)$, (24) gives that

$$\lambda_{\tilde{S}}^k = \lambda_S^k + \mathrm{O}(1).$$

The perturbation p^k from (21) and (22) is also O(1). For k equal to 1, $\lambda_{\tilde{S}}^1$ is of order O(1), while λ_S^k only is of order $\mathrm{O}(h)$. The product (20), must be of order $\mathrm{O}(h)$ while the sum of the smallest and largest eigenvalue is of order O(1). This gives

$$\begin{aligned} \tilde{\alpha}^{1,1} &= \mathrm{O}(h) \\ \tilde{\alpha}^{m+1,1} &= \mathrm{O}(1), \end{aligned}$$

and $\kappa(\bar{\Delta}^k) = \mathrm{O}(h^{-1})$. If however $(1-\alpha_i^{j,k}) = \mathrm{O}(h)$, λ_S^k and $\lambda_{\tilde{S}}^k$ are of the same order and $\kappa(M^{-1}A) = \mathrm{O}(1)$.

We illustrate these theoretical results with some numerical examples for the Laplace operator. Ω is the unit square divided in two vertical strips with the interface at $x = 0.5$. The preconditioners on the subdomains are $B_{ii} = 1/\alpha_i A_{ii}$, with $\alpha_i = 0.8$. In figure 1 and 2, we plot on logarithmical scales, λ_m, λ_M and $\kappa(M^{-1}A)$ versus n ($h_x = h_y = n^{-1}$). For figure 1, the interface preconditioner M_s is S and for figure 2, $M_s = \tilde{S}$. These quantities have the behaviour as predicted by the theory. Figure 3 gives the number of iteration steps to reduce the original residual by a factor of 10^{-10} for the Poisson equation. The right hand side corresponds to the exact continuous solution

$$u(x,y) = 0.75\mathrm{e}^{xy}\sin(\pi x)\sin(\pi y).$$

When we replace the constant α_i by $(1-\sqrt{h})$, respectively $(1-h)$, the extreme eigenvalues become independent of the grid. The values are given in table 1.

This analysis shows that the combination of spectrally equivalent preconditioners on the subdomains and on the interface does not necessarily result in

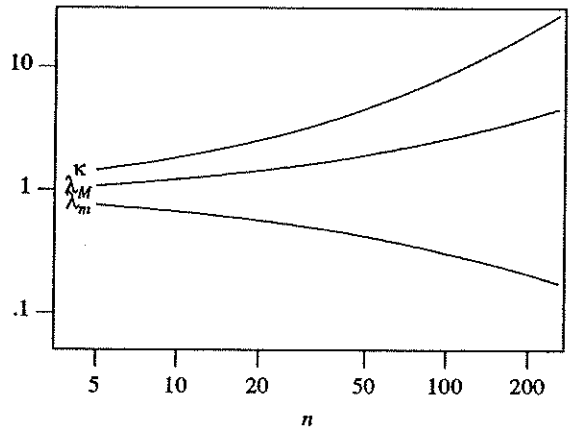

Figure 1 $M_* = S$

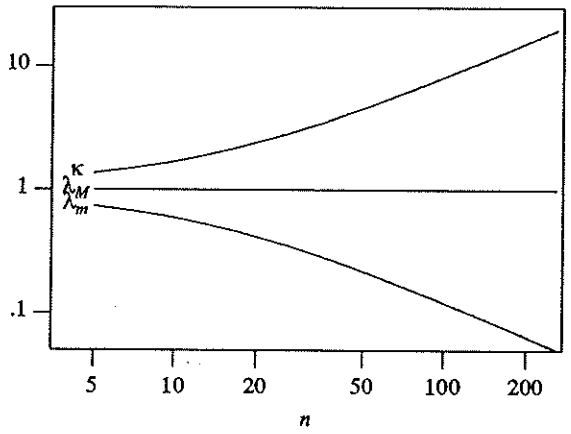

Figure 2 $M_* = \tilde{S}$

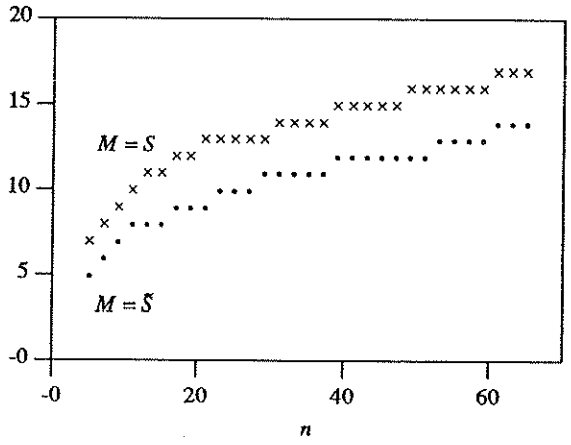

Figure 3

a spectrally equivalent preconditioner for the original problem. The condition number can even grow with h^{-2}. A sufficient condition to obtain spectral equivalence is given by (18). Börgers noted that when the B_{ii} represent a number of multigrid cycles, the number of iteration steps to reduce the original starting error by a certain factor is independent of the grid size. So, M seems to be spectrally equivalent to A even though (18) is not satisfied for these multigrid cycles. Though for multigrid cycles, the assumption that the eigenvectors of B_{ii} are the same as those of A_{ii} does not hold, the qualitative results derived may possibly give an explanation. The condition (18) is only a sufficient condition. A less stringent condition is given by (23). It suffices that $(1 - \alpha_i^{j,k}) = \mathrm{O}(\sqrt{h})$ for those components for which $\mu_M^k = \mathrm{O}(h)$. This can be satisfied even when $\rho(I - B_{ii}^{-1}A_{ii})$ is O(1). The multigrid cycles giving a spectrally equivalent preconditioner is possibly explained by such an effect of favourite alignment of eigenvectors.

These theoretical results indicate that the eigenspectra of the preconditioned systems on the subdomains and on the separator set should all be clustered around the same value. This asks for a possible scaling of the preconditioners. This however can introduce an imbalance between the subdomain preconditioners and the unchanged off-diagonal blocks, resulting in the extreme eigenvalues for $M^{-1}A$. Another approach, appropriately replacing also the off-diagonal blocks as proposed in [5] does not suffer from this drawback.

5 Domain decomposed preconditioners with incomplete factorisations

In this section, we study using some numerical examples, the eigenstructure of the preconditioned system when the subdomain preconditioners are incomplete factorisations as *e.g.*, the ILU or $MILU$ preconditioners [20, 15]. These preconditioners yield condition numbers of $O(h)$, and $O(\sqrt{h})$ respectively. Given a good preconditioner for the interface system, the eigenspectrum of $M^{-1}A$ will be determined dominantly by the eigenspectra of the blocks $B_{ii}^{-1/2}A_{ii}B_{ii}^{-1/2}$. As the subdomain preconditioners can adapt better to the local problems, we can expect that the overall preconditioner will perform better than applying the preconditioning method directly to the whole domain. As example we take the following problem

$$\frac{\partial}{\partial x}\left(-e^{5xy}\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(-e^{-5xy}\frac{\partial u}{\partial y}\right)+\frac{u}{1+x+y}=f(x,y)$$
$$(x,y)\in[0,1]\otimes[0,1]$$

The right hand side and the boundary conditions are taken such that the exact continuous solution equals $0.75e^{xy}\sin(\pi x)\sin(\pi y)$. For the subdomain preconditioners, we take the modified ILU preconditioner. Figure 4 gives λ_m, λ_M and $\kappa(M^{-1}A)$ as a function of n when applying the preconditioning method to the original problem. It is well known that λ_m is bounded by below by a constant, while λ_M grow linearly with n.

We consider first a splitting of the square in two vertical strips $\Omega_1=[0,0.5]\otimes[0,1]$ and $\Omega_2=[0.5,1]\otimes[0,1]$. As preconditioner on the interface we use a tridiagonal approximation to $\tilde{S}$ constructed by probing $\tilde{S}$ with a set of two test vectors [9, 19]. This is a very flexible and general applicable technique yielding preconditioners that use information from the original operator and from the original geometry. The resulting domain decomposed preconditioner has a condition number that is smaller by a factor of 1.4 than for the full domain version. In figure 8, we also plot the number of iteration steps needed to reduce the original residual by a factor of 10^{-10} starting with a random guess. This shows an improvement of 15% in iteration steps. In the two subdomain case, using a left to right ordering of the grid lines in subdomain Ω_1 and a right to left ordering in Ω_2, the arithmetic complexity of inverting the preconditioner is the same as in the one domain situation [8]. So besides the natural parallelism, domain decomposition yields here an overall faster method.

The next example (figure 6) shows the results in the case of four vertical strips. The Schur complement matrix S is a block tridiagonal matrix. The diagonal blocks are the Schur complements for the interfaces as internal edges between two neighbouring subdomains. As preconditioner for S, we take a block diagonal matrix where the diagonal blocks are the probing preconditioners preconditioners for the two subdomain case. The smallest eigenvalue is constant as

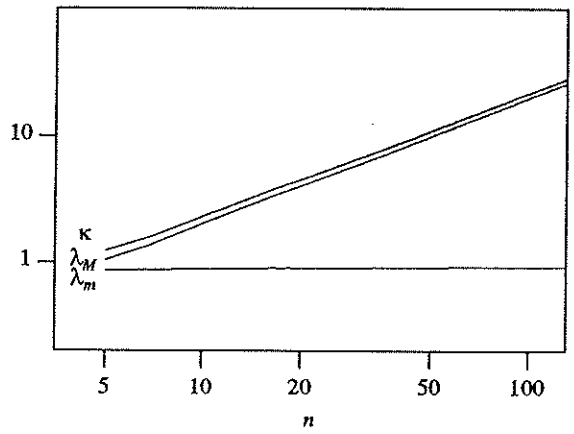

Figure 4 1 subdomain

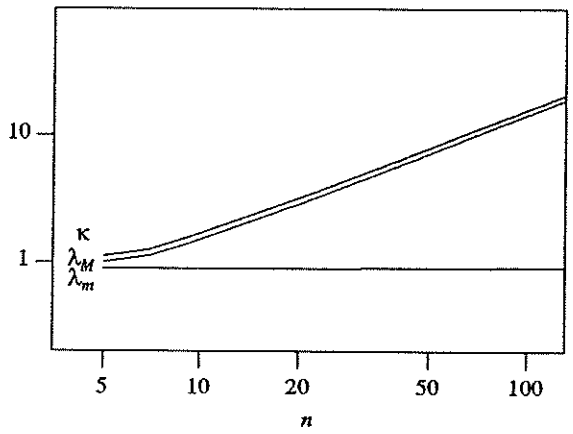

Figure 5 2 strips

function of the grid size, but is smaller than with only one subdomain. This has its origin in neglecting the coupling between the interfaces, [12, 13]. The largest eigenvalue is again determined by the largest eigenvalues on the subdomains and grows linearly with n. It is smaller than for the full domain version. The condition number now is larger due only to the smaller λ_m. As preconditioned conjugate gradient iteration quickly filters out components of the error corresponding to isolated eigenvalues, the iteration count is comparable to the two subdomain version.

In the last example, the domain is split in four square boxes. The separator set consists of four edges and one corner. The Schur complement S for the general case of a decomposition into boxes, has the following structure :

$$S = \begin{pmatrix} S_e & A_{ec} \\ A_{ec}^T & \tilde{S}_c \end{pmatrix}$$

The matrix S_e, is a block matrix with :

$$(S_e)_{jj} = (A_{ee})_{jj} - \sum_i \left(A_{de}^T\right)_{ji} (A_{dd})_{ii}^{-1} (A_{de})_{ij} \tag{25}$$

$$(S_e)_{jk} = -\left(A_{de}^T\right)_{ji} (A_{dd})_{ii}^{-1} (A_{de})_{ik} \tag{26}$$

$$\tilde{S}_c = A_{cc} - A_{dc}^T A_{dd}^{-1} A_{dc}.$$

In (25), the summation is done over the two subdomains that are adjacent to edge j. In (26), i is the number of the subdomain for which edge j and edge k both are internal boundaries. S itself can be factorised in a block LU factorization :

$$S = \begin{pmatrix} S_e & \\ A_{ec}^T & S_c \end{pmatrix} \begin{pmatrix} I & S_e^{-1} A_{ec} \\ & I \end{pmatrix} \quad , \tag{27}$$

with $S_c = \tilde{S}_c - A_{ec}^T S_e^{-1} A_{ec}$. Note that S_c is also the Schur complement matrix for the corner system in the original matrix A. We construct a preconditioner for S in an analogous way to domain decomposed preconditioners by replacing S_c and S_e in (27) by approximations. We replace S_e by a block diagonal matrix as in the multiple strip case. A possible choice for a preconditioner for S_c is the discretisation matrix on the coarse grid defined by the corner points. This approximation is also used in the domain decomposed preconditioner as proposed by Bramble, Pasciak and Schatz, [5], and as a way of global information transport in the additive Schwarz method, [14]. An appropriate scaling of this coarse grid approximation is here also very important. In the four box case, this coarse grid approximation is just a constant.

Figure 7 shows that, in contrast to the multiple strip case, λ_m as well is a function of n. This is probably due to a non optimal scaling of the corner preconditioner. The iteration count however is comparable to applying the $MILU$ preconditioner on the whole domain.

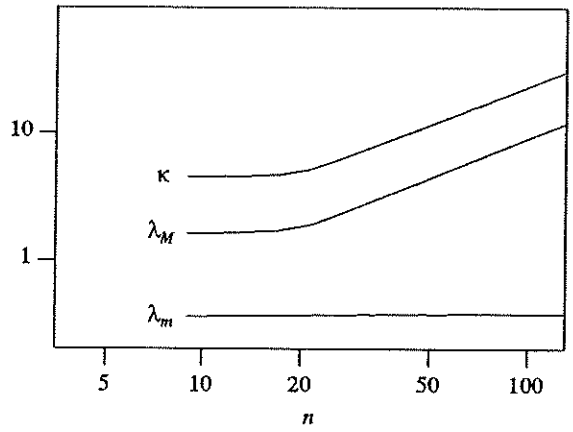

Figure 6 4 strips

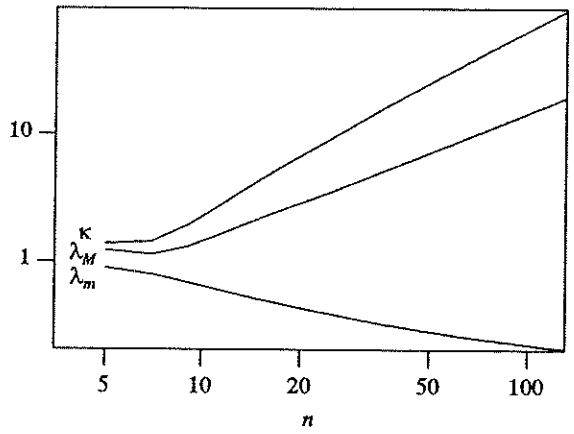

Figure 7 4 boxes

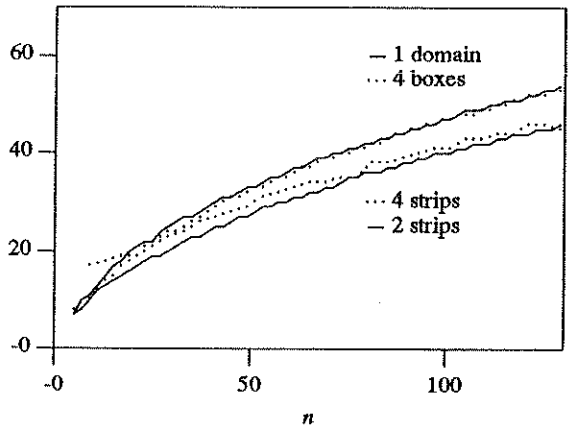

Figure 8 iteration count

6 Conclusions

We have shown that the eigenvalue spectrum of domain decomposed preconditioners of the form (5) is a perturbation of the union of the spectra of the preconditioned systems on the subdomains and on the interfaces. In case of bad scaling of the preconditioners to eachother and of the preconditioners to the unchanged off-diagonal blocks, some extreme eigenvalues can be introduced. When combining suboptimal subdomain preconditioners, such as *e.g.*, incomplete factorisations, we can obtain better preconditioners than applying the method directly on the full domain.

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