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Singular Perturbations of Limit Points with Application to Tubular Chemical Reactors

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Abstract

Singular perturbation techniques are used to study solutions of certain nonlinear boundary value problems defined on domains with a circular hole of radius ε, in the limit $\varepsilon \to 0$. Asymptotic expansions are constructed to describe the behavior of solutions at and near simple and double limit points (cusps). In particular, the behavior of axisymmetric solutions in an annular domain at limit points is investigated. The results are applied to two model problems arising in chemical reactor theory. The asymptotic analysis predicts a surprisingly large sensitivity of limit points to the ε -domain perturbation considered here.

1 Introduction

Let $u(\underline{x}, \lambda)$ be the solution of a boundary value problem (BVP), where $\underline{x}$ is a point in a plane bounded domain $D \subseteq \Re^2$, depending on a real parameter λ. Of particular interest are certain critical values of λ where a solution branch becomes singular, such as limit points (turning points) λ_L or bifurcation points λ_B. Let D be modified by piercing a small circular hole of radius ε centered at $\underline{x}_0$, that is, the domain D_ε is obtained by subtracting from D a ball $\{\underline{x} \in D, \mid \underline{x} - \underline{x}_0 \mid < \varepsilon\}$. In addition, a boundary condition like $u = 0$ is imposed at the hole. This ε - domain perturbation will result in singular points $\lambda_L(\varepsilon)$, $\lambda_B(\varepsilon)$. In this paper we are interested in asymptotic expressions for $\lambda_L(\varepsilon) - \lambda_L$ for small ε. Bifurcation points are considered in a companion paper [1].

This problem is not only of intrinsic mathematical interest, also for $D \subseteq \Re^n, n > 2$, but there are applications in a number of different areas. One is the buckling of plates and shells with small holes. Both snapping (limit points) and buckling (bifurcation points) are known to occur in such structures, where it is usually simpler to calculate critical points for the structure without hole. Therefore, it is of interest to assess

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analytically, for small ε, the effect of the singular domain perturbation on the critical load. Another area of potential application is chemical reactor problems, in particular tubular reactors with both external and internal cooling. In these applications the domain D in question is usually two-dimensional.

We shall not address the problem in any generality in this paper. Rather, we shall show for a specific class of semilinear elliptic problems of the reaction–diffusion type how singular perturbation techniques can be applied to yield asymptotic expressions for $\lambda_L(\varepsilon)$. We restrict ourselves in the subsequent analysis to circular domains with a centrally located hole, that is, an annular domain, where $\varepsilon := a/b$ is small (a=inner radius, b=outer radius). In that case, positive radially symmetric solutions are of particular interest. We study the variation of limit points with ε for a class of nonlinear boundary value problems of second order in some detail. It is shown in [1] how the results can be extended to noncircular domains with one or several small holes.

We are not aware of any previous asymptotic treatment of limit point problems for domains with small holes. Considering problems of the form $\Delta u + \lambda f(\underline{x}, u) = 0$, in annular domains D_ε and assuming a linear boundary condition at the outer, Dirichlet boundary condition at the inner boundary of D_ε, we obtain an asymptotic expansion for simple (quadratic) limit points of the form

$$\lambda_L(\varepsilon) = \lambda_L + \lambda_1\delta + \lambda_2\delta^2 + 0(\delta^3) \qquad \delta \to 0 \tag{1.1}$$

where

$$\delta = \delta(\varepsilon) = \left(\log \frac{1}{\varepsilon}\right)^{-1} \tag{1.2}$$

For the special cases $f(u) = exp\, u$ and $f(u) = exp(u/1 + \mu u)$, which occur in the modelling of thermal ignition problems in tubular reactors (e.g. see [2]), the first two terms of the asymptotic expansion of the solution are worked out in detail. We also treat the asymptotic behavior as $\varepsilon \to 0$ at a double (cubic) limit point λ_D that results from the coalescence of two simple limit points. This situation occurs, for example, in the case $f(u) = exp(u/(1 + \mu u))$ at some value μ_D. Again we find

$$\begin{aligned} \lambda_D(\varepsilon) &= \lambda_D + \lambda_1\delta + \lambda_2\delta^2 + 0(\delta^3) \\ & \qquad\qquad\qquad\qquad\qquad \delta \to 0 \\ \mu_D(\varepsilon) &= \mu_D + \mu_1\delta + \mu_2\delta^2 + 0(\delta^3). \end{aligned} \tag{1.3}$$

The evaluation of the coefficients in (1.1) and (1.3) shows that a small hole can have a remarkably large influence on the location of the perturbed limit points.

Semilinear elliptic problems in annular domains $D_\varepsilon \subseteq I\!R^n$ have been studied in a series of recent papers by Bandle, Coffman, Marcus and others, including the limit situation $\varepsilon \to 0$ (e.g., see [3], where further references are given). However, the assumption is $n \geq 3$. The analytical techniques (reduction to a generalized Emden–Fowler equation and phase–plane methods) do not seem to be applicable to the case $n = 2$. Moreover, growth conditions on $f(\underline{x}, u)$ for $u \to \infty$ are assumed that exclude the tubular reactor examples given above.

The problems to be analyzed in this paper bear some resemblance to the following model problem in the asymptotic theory of incompressible flow at low Reynolds numbers, originally introduced by Kaplun and Lagerstrom in 1957. Let $y(r;\varepsilon)$ be defined by [4,5]

$$\frac{d^2y}{dr^2} + \frac{n-1}{r}\frac{dy}{dr} + y\frac{dy}{dr} = 0, \quad y = 0 \text{ at } r = \varepsilon, \quad y = 1 \text{ at } r = \infty. \tag{1.4}$$

The first two terms represent the Laplacian of an axisymmetric function in n dimensions, say the temperature, r being the radial variable (yy' may be considered as a heat loss). In the absence of the hole the equilibrium temperature is $y = 1$ everywhere. For $n = 2$, the introduction of a cylindrical cooling rod of radius ε constitutes a perturbation, which is expected to be small if $\varepsilon \ll 1$, except near the surface of the rod, as $y = 0$ at $r = \varepsilon$. Hence, the convergence of $y(r;\varepsilon)$ as ε tends to zero is nonuniform and we have a singular perturbation problem of the layer-type, where the boundary layer occurs at $r = \varepsilon$. An asymptotic solution of (1.4) was given in [4], it is described in more detail in [5] and [6]. The outer and inner expansions y_0 and y_i for $y(r;\varepsilon)$ are of the form (n=2)

$$\begin{aligned} y_0(r;\varepsilon) &= 1 - E(r)\delta + 0(\delta^2), & E(r) &= \textstyle\int_r^\infty t^{-1}e^{-t}dt \\ y_i(r;\varepsilon) &= \delta \log s + 0(\delta^2), & s &= r/\varepsilon. \end{aligned}$$

A different treatment of a generalization of (1.4) is found in [7]. Unlike the problems treated in what follows, the solution of BVP (1.4) is unique [8].

The paper is organized as follows. The formulation of the problem is given in Section 2, which also contains a brief review of the asymptotic expansion at regular points [1,9]. In Section 3, the simple limit point problem is treated. Asymptotic expansions of the solution are constructed at and near the limit point. In Section 4, the double limit point problem is treated for a class of BVPs depending on two parameters λ and μ. In Section 5, the results of Sections 2 and 3 are applied to the case $f(u) = expu$, which can be solved exactly. In Section 6, the asymptotic results of Section 2,3 and 4 are applied to the case $f(u) = exp(u/(1+\mu u))$, which is considered a more realistic model for certain exothermic reactions and for which an analytic solution is not known. The results of the perturbation method are compared with exact and accurate numerical solutions in order to confirm the validity of our asymptotic solutions.

2 Formulation of the class of problems

We propose to study the effect of the ε-domain perturbation defined in the introduction for certain nonlinear model problems involving the Laplacian and cylindrically symmetric solutions. We begin by describing the problem and the conditions which define limit points.

2.1 The basic equations

The problems we wish to consider can be viewed as originating from the two-dimensional BVP in the unit disk

$$\begin{aligned} \Delta U + \lambda f(r, U; \mu) &= 0, \qquad 0 \leq r < 1, \qquad 0 \leq \theta \leq 2\pi \\ \alpha \frac{\partial U}{\partial n} + U &= 0 \qquad \text{on} \qquad r = 1 \end{aligned} \tag{2.1}$$

where r is the radial variable, λ, μ, and α are parameters, and f is a smooth nonlinear function of r, U and μ. Cylindrically symmetric branches of solutions $U(r, \lambda)$ of (2.1), for fixed μ and α, are obtained from

$$BVP(I) \begin{cases} U'' + \frac{1}{r}U' + \lambda f(r, U; \mu) = 0 \qquad 0 < r < 1 \\ U'(0, \lambda) = 0 = \alpha U'(1) + U(1), \end{cases} \tag{2.2}$$

primes denoting differentiation with respect to r.

Associated with BVP(I) is a linear 'variational equation' obtained by differentiating BVP(I) with respect to λ. It is well-known (e.g. [10,11]) that a necessary condition for a solution branch of BVP(I) to have a simple (quadratic) limit point at $\lambda = \lambda_0$ is that the homogeneous variational equation has a nontrivial solution $V_0(r)$ satisfying the boundary conditions in (2.3). We assume that this is the case. Hence, for $\lambda = \lambda_0$ $\quad U_0(r) := U(r, \lambda_0)$ is a solution of BVP(I) and $V_0(r)$ is a nontrivial (smooth) solution of

$$Var(I) \quad L_0 V_0 = 0, \quad 0 < r < 1, \quad V_0'(0) = 0 = \alpha V_0'(1) + V_0(1) =: BV_0(1) \tag{2.3}$$

where

$$Lu = L(\lambda, U)u := u'' + \frac{u'}{r} + \lambda f_u(r, U; \mu)u, \quad L_0 := L(\lambda_0, U_0).$$

Without loss of generality we may assume that $V_0(0) = 1$.

Later we shall make use of the fact that the homogeneous differential equation $Lu = 0$, which has a smooth solution V satisfying $V'(0) = 0$, has a second, linear independent solution

$$S(r) = V(r) \int \frac{dr}{rV^2} \tag{2.4}$$

which, in view of the smoothness of $V(r)$, can be written in the form

$$S(r) = V(r)\log r + R(r), \qquad \text{with} \qquad LR = -\frac{2}{r}V'(r) \tag{2.5}$$

and $R(0) = R'(0) = 0$. With the normalization $V(0) = 1$, S satisfies the asymptotic relation

$$S(r) = \log r + o(1) \quad \text{as} \quad r \to 0+\,. \tag{2.6}$$

When a set of solution branches $U(r, \lambda; \mu)$, for different values of μ, has two or more simple limit points, a coalescence of two such limit points to one double limit point may occur at a critical value of μ. The condition that a solution branch of BVP(I) has a double limit point, also called a simple cubic limit point (cusp) at $(\lambda, \mu) = (\lambda_0, \mu_0)$ is that a 'second variational equation' has a solution $W(r)$ defined below [12,13]. Hence, for $\lambda = \lambda_0, \mu = \mu_0$ $\;U_0(r) := U(r, \lambda_0)$ is a solution of BVP(I) for $\mu = \mu_0$, $V_0(r)$ is a nontrivial (smooth) solution of Var(I) and $W(r)$ is a nontrivial (smooth) solution of

$$\begin{aligned} &L_0W = W'' + \tfrac{1}{r}W' + \lambda_0 f_u(r, U_0; \mu_0)W = -\lambda_0 f_{uu}(r, U_0; \mu_0)V_0^2 \\ &W'(0) = 0 = BW(1). \end{aligned} \tag{2.7}$$

Again without loss of generality we may assume that $W(0) = 1$.

The modified problem consists of deleting a small circular hole of radius ε from the center of the unit disk and imposing the condition $u = \overline{u}$ on the boundary $r = \varepsilon$. Let $u(r, \lambda; \varepsilon)$ denote the cylindrically symmetric solution of this modified problem. Then u satisfies

$$BVP(II) \begin{cases} u'' + \frac{1}{r}u' + \lambda f(r, u; \mu) = 0 & \varepsilon < r < 1 \\ \\ u(\varepsilon, \lambda; \varepsilon) = \overline{u}, \qquad Bu(1, \lambda; \varepsilon) = 0 \end{cases} \tag{2.8}$$

We want to study the behavior of the solution branches $u(r, \lambda; \varepsilon)$ as $\varepsilon \to 0$.

Let $U(r,\lambda)$ denote a solution branch of BVP(I) with a simple limit point at $\lambda = \lambda_0$, $U = U_0$. Under suitable conditions BVP(II) will have a nearby solution branch $u(r,\lambda;\varepsilon)$ satisfying

$$\lim_{\varepsilon \to 0} u(r,\lambda;\varepsilon) = U(r,\lambda), \qquad r \quad \text{fixed}, \tag{2.9}$$

for each $0 < r \leq 1$. Of course we don't expect the convergence in (2.9) to be uniform near $r = 0$ (unless $\overline{u} = U(0)$.) Moreover, for sufficiently small $\varepsilon > 0$, we expect the solution branch $u(r,\lambda;\varepsilon)$ to have a family of simple limit points

$$\lambda = \lambda_0(\varepsilon), \qquad u_0(r;\varepsilon) := u(r,\lambda_0(\varepsilon);\varepsilon), \tag{2.10}$$

satisfying

$$\lim_{\varepsilon \to 0} \lambda_0(\varepsilon) = \lambda_0, \qquad \lim_{\varepsilon \to 0} u_0(r;\varepsilon) = U_0(r), \tag{2.11}$$

for each r, $0 < r \leq 1$.

It is apparent that the existence of this family of limit points requires the existence of a family of nontrivial solutions $v_0(r;\varepsilon)$ of the variational problem

$$Var(II) \begin{cases} L_0 v_0 = v_0'' + \frac{1}{r} v_0' + \lambda_0(\varepsilon) f_u(r,u_0;\mu) v_0 = 0 \qquad \varepsilon < r < 1 \\ v_0(\varepsilon;\varepsilon) = 0 = B v_0(1;\varepsilon). \end{cases} \tag{2.12}$$

Consistent with the limits in (2.11) we expect the function $v_0(r;\varepsilon)$ to satisfy

$$\lim_{\varepsilon \to 0} v_0(r;\varepsilon) = V_0(r), \qquad r \quad \text{fixed}, \tag{2.13}$$

for each r, $0 < r \leq 1$.

Similarly, if BVP(I) has a double limit point at $\lambda = \lambda_0$, $U = U_0(r)$, for $\mu = \mu_0$ then under suitable conditions on $f(r,u;\mu)$ BVP(II) will also have a nearby solution branch $u(r,\lambda;\varepsilon)$ satisfying (2.9), and for sufficiently small $\varepsilon > 0$, we expect this solution branch to have a family of double limit points

$$\lambda = \lambda_0(\varepsilon), \quad \mu = \mu_0(\varepsilon), \quad u_0(r;\varepsilon) := u(r,\lambda_0(\varepsilon);\varepsilon) \tag{2.14}$$

satisfying $\lambda_0(\varepsilon) \to \lambda_0$, $\mu_0(\varepsilon) \to \mu_0$, and $u_0(r;\varepsilon) \to U_0(r)$ as $\varepsilon \to 0$ for each r, $0 < r \leq 1$. The existence of these double limit points requires the existence of nontrivial solutions $v_0(r;\varepsilon)$ satisfying (2.12), and of solutions $w(r;\varepsilon)$ of the modified BVP (2.7), that is

$$L_0 w = -\lambda_0 f_{uu}(r, u_0; \mu_0) v_0^2, \qquad \varepsilon < r < 1,$$
$$w(\varepsilon; \varepsilon) = 0 = Bw(1; \varepsilon), \tag{2.15}$$

with the property $w(r;\varepsilon) \to W(r)$ for $\varepsilon \to 0$, r fixed, for each r, $0 < r \leq 1$.

2.2 Branches of regular solutions

Asymptotic expansions of solutions $u(r, \lambda; \varepsilon)$ of BVP(II) for fixed $\lambda \neq \lambda_0$ are obtained by following the procedure in [1]. A first approximation for the *outer* solution is, for $\varepsilon \ll r \leq 1$,

$$u(r;\varepsilon) \sim U(r) + \phi(\varepsilon)\tilde{u}(r) \tag{2.16}$$

where $U(r)$ satisfies (2.2). The dependence on λ (and μ) is dropped for convenience. $\phi(\varepsilon)$ is an asymptotic scale to be determined, with $\phi(\varepsilon) \to 0$ as $\varepsilon \to 0$. We don't expect the outer approximation to be valid near $r = \varepsilon$. Introducing an inner coordinate $s = r/\varepsilon$, the differential equation in (2.8) is transformed into

$$\frac{d^2u}{ds^2} + \frac{1}{s}\frac{du}{ds} + \varepsilon^2 \lambda f(\varepsilon s, u) = 0, \qquad s > 1 \tag{2.17}$$

Moreover, u must satisfy $u = \overline{u}$ for $s = 1$. Hence, for sufficiently smooth f the derivative terms dominate in (2.17) and we conclude that a first approximation for the *inner* solution is

$$u(s;\varepsilon) \sim \overline{u} + \nu(\varepsilon) \log s [C + o(1)], \qquad \varepsilon \to 0 \tag{2.18}$$

with $s > 1$ fixed. The order function $\nu(\varepsilon)$ will be found below by a leading order matching.

An equation for $\tilde{u}(r)$ is found by substituting (2.16) into (2.8) and carrying out the standard limit process. We obtain for $\varepsilon \to 0$

$$L\tilde{u} = 0, \qquad B\tilde{u}(1) = 0$$

$\tilde{u}$ is seen to be a solution of the variational equation, therefore

$$\tilde{u} = C_1 V(r) + C_2 S(r)$$

with V and S defined as in Section 2.1. The boundary condition at $r = 1$ requires

$$C_1 = -C_2 BS(1)/BV(1)$$

Since BVP (2.3) does not have a nontrivial solution at a regular point U, λ, it follows that $BV(1) \neq 0$. For small r we find, in view of (2.6) and $V(0) = 1$,

$$u(r;\varepsilon) = U(0) + o(1) + \phi(\varepsilon)C_2 \left[\log r - \frac{BS(1)}{BV(1)} + o(1)\right].$$

Matching this with the inner solution (2.18),

$$u(r;\varepsilon) = \overline{u} + \nu(\varepsilon)(\log r - \log\varepsilon)[C + o(1)]$$

we deduce that

$$\nu(\varepsilon) = -\frac{1}{\log\varepsilon} =: \delta(\varepsilon), \qquad C_2 = C = U(0) - \overline{u}, \quad \phi(\varepsilon) = \delta(\varepsilon).$$

Hence the leading term outer solution is

$$u(r,\varepsilon) \sim U(r) + \delta(\varepsilon)(U(0) - \overline{u}) \left[S(r) - \frac{BS(1)}{BV(1)}V(r)\right]. \tag{2.19}$$

Following the procedure in [1], we now assume an outer asymptotic expansion of the form

$$u(r,\varepsilon) \sim U(r) + \sum_{k=1}^{\infty} u_k(r)\delta^k(\varepsilon), \tag{2.20}$$

as $\varepsilon \longrightarrow 0$, for fixed r, $0 < r \leq 1$, and an inner asymptotic expansion

$$u(r,\varepsilon) \sim \overline{u} + \log s \sum_{k=1}^{\infty} a_k \delta^k(\varepsilon) \tag{2.21}$$

as $\varepsilon \longrightarrow 0$, with $s = r/\varepsilon > 1$ fixed. The functions $u_k(r)$ are obtained by substituting (2.20) into the differential equation (2.8) and equating coefficients of powers of δ to zero. The appropriate boundary conditions are $Bu_k(1) = 0$. The constants a_k in (2.21) and the constants of integration in $u_k(r)$ are determined by a straightforward matching.

3 Singular perturbation of simple limit points

When λ approaches a limit point λ_0, the asymptotic expansions (2.20), (2.21) break down, the solution branch $u(r,\lambda;\varepsilon)$ deviates appreciable from the unperturbed solution branch $U(r,\lambda)$, the difference exceeds the order $0(\delta)$. In this section we study the sensitivity of simple limit points with respect to the ε-domain perturbation for the class of problems defined by (2.2).

In constructing asymptotic approximations to the perturbed family of limit points $\lambda_0(\varepsilon)$, we must solve both parts of the extended system formed by BVP(II) and Var(II) simultaneously. Understanding the interplay between these two BVPs is crucial to obtaining the correct form of the inner and outer expansions for $u_0(r;\varepsilon)$ and $v_0(r;\varepsilon)$. To make this interplay more transparent we shall first consider simple (two terms) approximations to λ_0, u_0 and v_0, as in Section 2.2.

3.1 The leading terms of the asymptotic expansion

Based on the limits (2.11) and (2.13) we attempt simple outer approximations u_0, v_0 in the form

$$u_0(r;\varepsilon) \sim U_0(r) + \phi(\varepsilon)\tilde{u}(r), \qquad v_0(r;\varepsilon) \sim V_0(r) + \psi(\varepsilon)\tilde{v}(r) \tag{3.1}$$

for $\varepsilon \to 0$ with r fixed, $0 < r \leq 1$. An analogous expression for $\lambda_0(\varepsilon)$ is given by

$$\lambda_0(\varepsilon) \sim \lambda_0 + \chi(\varepsilon)\tilde{\lambda}, \qquad \varepsilon \to 0. \tag{3.2}$$

Here ϕ, ψ and χ are unknown $o(1)$ order functions; further unknown quantities are $\tilde{u}(r)$, $\tilde{v}(r)$ and the constant $\tilde{\lambda}$.

Near $r = \varepsilon$ the solution must be represented by an inner approximation. With the inner coordinate $s = \varepsilon/r$, the differential equation in (2.8) transforms into (2.17) for $u = u_0(r;\varepsilon)$. Hence the inner approximation has the form (2.18). By a similar argument based on the form of (2.12) we find that the inner approximation for v_0 satisfies

$$v_0(r;\varepsilon) = \nu(\varepsilon)\log s[1 + o(1)], \qquad \varepsilon \to 0, \tag{3.3}$$

with $s > 1$ fixed. In deriving (3.3) we have made use of $V_0(0) = 1$.

Equations for the correction terms $\tilde{u}$ and $\tilde{v}$ are obtained by substituting (3.1) and (3.2) into (2.8) and (2.12) and carrying out the usual limit process. We obtain

$$L_0\tilde{u} = -\left[\lim_{\varepsilon\to 0}\frac{\chi(\varepsilon)}{\phi(\varepsilon)}\right]\tilde{\lambda}f(r,U_0), \qquad B\tilde{u}(1) = 0 \tag{3.4}$$

and

$$\begin{aligned} L_0\tilde{v} &= -\left[\lim_{\varepsilon\to 0}\frac{\chi(\varepsilon)}{\psi(\varepsilon)}\right]\tilde{\lambda}f_u(r,U_0)V_0(r) \\ &\quad -\left[\lim_{\varepsilon\to 0}\frac{\phi(\varepsilon)}{\psi(\varepsilon)}\right]\lambda_0 f_{uu}(r,U_0)\tilde{u}(r)V_0(r), \quad B\tilde{v}(1) = 0. \end{aligned} \tag{3.5}$$

First we focus on (3.4). Since $f(r,U_0) \not\equiv 0$, we can rule out the possibility that $\chi \gg \phi$ as $\varepsilon \to 0$. Suppose $\chi = 0(\phi)$, say

$$\lim_{\varepsilon\to 0}\frac{\chi(\varepsilon)}{\phi(\varepsilon)} = 1,$$

Then (3.4) becomes

$$L_0\tilde{u} = -\tilde{\lambda}f(r,U_0), \qquad B\tilde{u}(1) = 0. \tag{3.6}$$

It proves convenient to express a particular solution of (3.6) in terms of the solution of the related initial-value problem

$$L_0 z = -f(r,U_0), \qquad r > 0, \qquad z(0) = z'(0) = 0. \tag{3.7}$$

For smooth f (3.7) has a smooth solution $z(r)$. We can write the solution of (3.6) as

$$\tilde{u}(r) = c_1 V_0(r) + \tilde{\lambda}\left[-Bz(1)\frac{S_0(r)}{BS_0(1)} + z(r)\right], \tag{3.8}$$

where S_0 is a solution of $L_0 S_0 = 0$ satisfying (2.6) and c_1 is an arbitrary constant. To gain further information we must carry out a matching with the inner expansion (2.18). For small r, the outer approximation in (3.1) satisfies

$$u_0(r;\varepsilon) = [U_0(0) + o(1)] + \phi(\varepsilon)\left[-\tilde{\lambda}\frac{Bz(1)}{BS_0(1)}(\log r) + O(1)\right], \tag{3.9}$$

as $\varepsilon \to 0$ with $\varepsilon < r \ll 1$. Matching of this expression with the inner approximation as in Section (2.2) requires $a_1 = U_0(0) - \overline{u}$,

$$\phi(\varepsilon) \sim \delta(\varepsilon), \qquad \nu(\varepsilon) \sim \delta(\varepsilon), \qquad \varepsilon \to 0 \tag{3.10}$$

and

$$\tilde{\lambda} = -\frac{BS_0(1)}{Bz(1)}\left[U_0(0) - \overline{u}\right]. \tag{3.11}$$

(If $Bz(1) = 0$, we are dealing with a higher order limit point.)

An interesting feature of this partial result is that it provides a first correction to the value of the limit point $\lambda_0(\varepsilon)$ without any input from the variational problem (3.5). However, in order to determine the constant c_1 in (3.8) we have to solve (3.5) for $\tilde{v}$.

There is a second, more subtle way in which the companion variational problem influences the form of the expansion for u_0. We have yet to consider the possibility that

$$\lim_{\varepsilon \to 0} \frac{\chi(\varepsilon)}{\phi(\varepsilon)} = 0. \tag{3.12}$$

In this event (3.4) becomes $L_0\tilde{u} = 0, \quad B\tilde{u}(1) = 0$, which has the solution $\tilde{u}(r) = d_1 V_0(r)$, with d_1 an arbitrary constant. The outer approximation (3.1) then becomes

$$u_0(r;\varepsilon) \sim U_0(r) + \phi(\varepsilon) d_1 V_0(r),$$

as $\varepsilon \to 0$ with $0 < r \leq 1$ fixed. Since $V_0(r)$ is smooth for all r this expression is consistent with the inner approximation for $u_0(r,\varepsilon)$ so long as

$$\delta(\varepsilon) \ll \phi(\varepsilon) \ll 1 \qquad \text{as} \qquad \varepsilon \to 0. \tag{3.13}$$

Without recourse to the variational problem we cannot say anything further about ϕ.

Since v_0 satisfies a homogeneous BVP we may assume that $\tilde{v}$ in (3.1) is not simply a multiple of V_0. Thus, we assume that

$$\psi(\varepsilon) \sim \phi(\varepsilon) \qquad \text{as} \qquad \varepsilon \to 0, \tag{3.14}$$

with the consequence that (3.5) becomes

$$L_0\tilde{v} = -\lambda_0 d_1 f_{uu}(r, U_0) V_0^2, \qquad B\tilde{v}(1) = 0. \tag{3.15}$$

At this point our argument proceeds as before for $\tilde{u}$. The inhomogeneous term in (3.15) forces $\tilde{v}$ to have $\log r$ behavior for small r. But with $\delta \ll \psi(\varepsilon) \ll 1$ as $\varepsilon \to 0$, we can't match the resulting outer expansion for v_0 in (3.1) with the inner expansion (3.3). Thus we conclude that the limit (3.12), and consequently (3.13), is not possible.

The only viable choice is (3.10) and (3.11). It then becomes apparent that we must take $\psi(\varepsilon) \sim \delta(\varepsilon)$ as $\varepsilon \to 0$, so that (3.5) becomes

$$L_0\tilde{v} = -\tilde{\lambda} f_u(r, U_0)V_0 - \lambda_0 f_{uu}(r, U_0)\tilde{u}V_0, \qquad B\tilde{v}(1) = 0. \tag{3.16}$$

Substituting (3.8) into (3.16), it is seen that the solution $\tilde{v}$ may be obtained in the form

$$\tilde{v} = k_1 V_0(r) + k_2 S_0(r) + c_1 z_1(r) + z_2(r) \tag{3.17}$$

where

$$\begin{aligned} L_0 z_1 &= -\lambda_0 f_{uu}(r, U_0)V_0^2 \\ L_0 z_2 &= -\tilde{\lambda} f_u(r, U_0)V_0 - \lambda_0 f_{uu}(r, U_0)V_0[(U_0(0) - \overline{u})S_0(r) + \tilde{\lambda} z], \end{aligned} \tag{3.18}$$

with $z_j(0) = z_j'(0) = 0$ $j = 1, 2$ and $z(r)$ defined in (3.7). Setting $k_1 = 0$ (see below), and matching to order $0(\delta)$ with the inner expansion (3.3), we obtain $k_2 = 1$, while the boundary condition at $r = 1$ yields

$$c_1 = -(Bz_1(1))^{-1}[BS_0(1) + Bz_2(1)] \tag{3.19}$$

Thus we have completed our determination of the leading-order corrections for the family of limit points.

3.2 Asymptotic expansions at a limit point

Motivated by these preliminary results we assume that full expansions for the family of limit points have the form

$$\lambda_0(\varepsilon) \sim \lambda_0 + \sum_{k=1}^{\infty} \lambda_k \delta^k(\varepsilon) \qquad \text{as} \quad \varepsilon \to 0 \tag{3.20}$$

with outer and inner expansions for $u_0(r;\varepsilon)$ given in the form (2.20) and (2.21), respectively, $U(r)$ in (2.20) being replaced by $U_0(r)$. These expansions must be supplemented by outer and inner expansions for $v_0(r;\varepsilon)$, that is,

$$v_0(r;\varepsilon) \sim V_0(r) + \sum_{k=1}^{\infty} v_k(r)\delta^k(\varepsilon) \tag{3.21}$$

$$v_0(r;\varepsilon) \sim \log s \sum_{k=1}^{\infty} b_k \delta^k(\varepsilon) \tag{3.22}$$

as $\varepsilon \to 0$ with r fixed, $0 < r \le 1$ in (3.21) and with $s = r/\varepsilon > 1$ fixed in (3.22), and with $\lambda_1 = \tilde{\lambda}$, $u_1 = \tilde{u}$, and $v_1 = \tilde{v}$.

A few remarks are in order regarding the form of these expansions. An induction argument based on our preceding analysis can be employed to verify that no intermediate terms, e.g. $0(\delta^{3/2})$, have been missed. On the other hand, our expansions fail to account for $0(\varepsilon)$ effects because such terms are transcendentally small compared to $0(\delta)$ terms. As a consequence, the effect of the nonlinearity never appears in the inner expansions. The constants a_k and b_k are determined by a straightforward matching. It can be shown that the outer expansions contain the inner expansions.

In order to determine equations for the unknowns λ_k, u_k and v_k one simply substitutes (3.20) (2.20) and (3.21) into the system of differential equations (2.8) for $u = u_0$ and (2.12), and equates coefficients of powers of δ to zero. Assuming the implied smoothness of f holds we find

$$\begin{aligned} 0(\delta): \quad L_0 u_1 &= -\lambda_1 f(r, U_0), \\ L_0 v_1 &= -\lambda_1 V_0 f_u(r, U_0) - \lambda_0 u_1 V_0 f_{uu}(r, U_0), \end{aligned} \tag{3.23}$$

$$\begin{aligned} 0(\delta^2): \quad L_0 u_2 &= -\lambda_2 f(r, U_0) - \lambda_1 u_1 f_u(r, U_0) - \frac{1}{2}\lambda_0 u_1^2 f_{uu}(r, U_0), \\ L_0 v_2 &= -\lambda_2 V_0 f_u(r, U_0) - \lambda_1 [v_1 f_u(r, U_0) + u_1 V_0 f_{uu}(r, U_0)] \\ &\quad -\lambda_0 [(u_2 V_0 + u_1 v_1) f_{uu}(r, U_0) - \frac{1}{2} u_1^2 V_0 f_{uuu}(r, U_0)], \end{aligned} \tag{3.24}$$

$$\begin{aligned} 0(\delta^k): \quad L_0 u_k &= g_k(r, \lambda_1, \ldots, \lambda_k, u_1, \ldots, u_{k-1}), \\ L_0 v_k &= h_k(r, \lambda_1, \ldots, \lambda_k, u_1, \ldots, u_k, v_1, \ldots, v_{k-1}), \end{aligned} \tag{3.25}$$

where g_k and h_k are functions of the indicated arguments. The appropriate outer boundary conditions are

$$B u_k(1) = B v_k(1) = 0, \qquad k = 1, 2, \ldots. \tag{3.26}$$

This system of problems must be solved in sequence. As our preliminary results indicate, in order to determine λ_k, $k = 1, 2, \ldots$, one must first determine u_{k-1} *and* v_{k-1}. Then, solution of the equation for u_k, subject to (3.26) and subsequent matching with the inner expansion (2.21) for u_{k-1} yields λ_k. The function u_k will still involve an arbitrary constant c_k in the form

$$u_k(r) = c_k V_0(r) + \cdots.$$

To ascertain c_k one must solve the companion problem for v_k. Analogous arbitrary constants in v_k (like k_1 in (3.17)) may be set to zero since $v_0(r;\varepsilon)$ equals $V_0(r)$ to leading order and v_0 satisfies a homogeneous BVP (2.12). The solutions u_k, v_k may be obtained by solving appropriate linear initial value problems such as (3.7) and (3.18) for $k = 1$.

With a view towards later application, we list here the results of these calculations for u_2, λ_2

$$u_2(r) = c_2 V_0(r) + c_1 S_0(r) + \lambda_2 z(r) + z_3(r) \tag{3.27}$$

$$\lambda_2 = -\frac{1}{Bz(1)}\left[c_1 B S_0(1) + B z_3(1)\right] \tag{3.28}$$

where

$$L_0 z_3 = -\lambda_1 f_u u_1 - \frac{1}{2}\lambda_0 f_{uu} u_1^2, \qquad z_3(0) = z_3'(0) = 0 \tag{3.29}$$

$$c_2 = -\frac{1}{Bz_1(1)} B z_4(1), \qquad a_2 = c_1 \tag{3.30}$$

$$\begin{aligned} L_0 z_4 \quad = \quad & -\lambda_2 f_u V_0 - \lambda_1(v_1 f_u + u_1 V_0 f_{uu}) - \tfrac{1}{2}\lambda_0 f_{uuu} V_0 u_1^2 \\ & -\lambda_0 f_{uu}\left[u_1 v_1 + V_0(c_1 S_0 + \lambda_2 z + z_3)\right], \quad z_4(0) = z_4'(0) = 0. \end{aligned} \tag{3.31}$$

Note that $L_0 Z = \log^m r$, $m = 1, 2, \ldots$, has a particular solution that behaves like $r^2(\log r)^m$ as $r \to 0$. Hence the above representation of $u_2(r)$ is well defined. The same is true for $z_4(r)$. Finally, we have $v_2(r) = c_2 z_1(r) + z_4(r)$.

3.3 Asymptotic expansion near a limit point

The preceding analysis provides a means of constructing approximations to the limit point itself. Constructing approximations to $u(r,\lambda;\varepsilon)$ for values of λ near a limit point is also a relatively easy task. The expansion obtained in Section 2.2 is restricted to values of λ not too close to λ_0. For λ near λ_0, the deviation $u(r,\lambda;\varepsilon) - U(r,\lambda)$, except near $r = \varepsilon$, will no longer be $0(\delta)$. Due to the quadratic nature of the limit point, that deviation will be appreciably larger when λ is approaching λ_0.

To analyze the solution near the limit point, it is appropriate to set

$$\lambda = \lambda_0 + \alpha_1 \delta(\varepsilon) + \alpha_2 \delta(\varepsilon)^2 + \cdots, \qquad \varepsilon \to 0 \tag{3.32}$$

and to take the outer expansion for u in the form

$$u(r, \lambda; \varepsilon) \sim U_0(r) + \sum_{k=1}^{\infty} U_k(r) \delta^{k/2}(\varepsilon) \tag{3.33}$$

as $\varepsilon \to 0$ with r fixed, $0 < r \leq 1$. The inner expansion also proceeds in powers of $\delta^{1/2}$. The coefficients $U_k(r)$ are found in the usual manner, that is, by substituting (3.32) and (3.33) into BVP(II). The variational BVP Var(II) does not enter into this calculation. One easily finds $L_0 U_1 = 0$ and

$$LU_2 = -\alpha_1 f(r, U_0) - \frac{1}{2} \lambda_0 f_{uu}(r, U_0) U_1^2 \tag{3.34}$$

with $BU_1(1) = BU_2(1) = 0$. Regularity of $U_1(r)$ requires $U_1 = B_1 V_0(r)$. The right hand side of (3.34) shows that $U_2(r)$ can be obtained in terms of the functions $z(r)$ and $z_1(r)$ defined by (3.7) and (3.18), respectively. Thus we have

$$U_2(r) = B_2 V_0(r) + D_2 S_0(r) + \alpha_1 z(r) + \frac{1}{2} B_1^2 z_1(r). \tag{3.35}$$

The inner expansion consistent with (3.32) and (3.33) has the form

$$u(r, \lambda; \varepsilon) \sim \overline{u} + \log s(A_1 \delta + A_2 \delta^{3/2} + A_3 \delta^2 + \cdots) \tag{3.36}$$

Matching the $0(\delta^{1/2})$ and $0(\delta)$ terms with the outer expansion yields

$$\begin{aligned} A_1 &= U_0(0) - \overline{u}, \qquad & A_2 &= B_1 V_0(0) = B_1, \\ D_2 &= A_1, & A_3 &= B_2 \end{aligned} \tag{3.37}$$

Applying the boundary condition $BU_2(1) = 0$, we may solve (3.35) for B_1,

$$B_1^2 = -\frac{2}{Bz_1(1)} [(U_0(0) - \tilde{u}) BS_0(1) + \alpha_1 Bz(1)] = \frac{2Bz(1)}{Bz_1(1)} (\tilde{\lambda} - \alpha_1) \tag{3.38}$$

with $\tilde{\lambda}$ given by (3.11). Note that at the limit point $\alpha_1 = \tilde{\lambda}$, so that $B_1 = 0$, and the fractional powers of δ disappear in the expansions (3.33) and (3.36).

In order to discuss (3.38), two cases must be distinguished. Setting $q_0 := Bz(1)/(Bz_1(1))$, we have

$$(i) \qquad q_0 > 0 \qquad B_1 = \pm\sqrt{2q_0(\tilde{\lambda} - \alpha_1)}, \tag{3.39}$$

yielding two real solutions B_1 for $\alpha_1 < \tilde{\lambda}$, that is $\lambda < \lambda_0(\varepsilon)$ (subcritical case), and

$$(ii) \qquad q_0 < 0 \qquad B_1 = \pm\sqrt{-2q_0(\alpha_1 - \tilde{\lambda})} \tag{3.40}$$

with two real solutions B_1 for $\alpha_1 > \tilde{\lambda}$, that is, $\lambda > \lambda_0(\varepsilon)$ (supercritical case). Clearly $sgn\, q_0$ depends on the nonlinear function f and on the constant α in the boundary condition at $r = 1$. We have also computed a representation for $U_3(r)$ similar to (3.35) in terms of V_0, S_0, z_1 (defined in (3.18)) and one additional function $z_0(r)$.

4 Singular perturbation of double limit points

In this section we assume that $f = f(r, U; \mu)$ is such that BVP(I) has a double limit point at $(\lambda, \mu) = (\lambda_0, \mu_0)$. The conditions for the occurrence of this kind of singular point have been given in Section 2. Thus we assume that the system (2.2), (2.3) and (2.7) has a solution, which is denoted by U_0, V_0, W, with $V_0 \not\equiv 0$. Assuming that the modified problem (2.8), (2.12) and (2.15) also has a solution u_0, v_0, w, we propose to construct an asymptotic expansion for $u_0(r;\varepsilon), v_0(r,\varepsilon), w(r,\varepsilon)$ for the family of double limit points $\lambda_0(\varepsilon), \mu_0(\varepsilon)$ as $\varepsilon \to 0$. The method largely follows the one for simple limit points, so we merely outline the main steps.

4.1 The leading terms of the asymptotic expansion.

As in Section 3.1, we assume simple two-term outer approximations u_0, v_0, λ_0 in the form (3.1) and (3.2), supplemented by

$$w(r;\varepsilon) \sim W(r) + \nu(\varepsilon)\tilde{w}, \quad \mu_0(\varepsilon) \sim \mu_0 + \xi(\varepsilon)\tilde{\mu} \tag{4.1}$$

for $\varepsilon \to 0$. The leading term inner approximation will be of the form (2.18), (3.3), with the additional relation

$$w(r;\varepsilon) \sim \delta(\varepsilon)\log s(1 + o(1)) \tag{4.2}$$

Form the equations for u_0 and v_0, we now obtain

$$L_0\tilde{u} = -\left[\frac{\chi}{\phi}\right]_0 \tilde{\lambda} f - \left[\frac{\xi}{\phi}\right]_0 \lambda_0 \tilde{\mu} f_\mu \tag{4.3}$$

$$L_0\tilde{v} = -\left[\frac{\chi}{\psi}\right]_0 \tilde{\lambda} f_u V_0 - \left[\frac{\phi}{\psi}\right]_0 \lambda_0 f_{uu} V_0 \tilde{u} - \left[\frac{\xi}{\psi}\right]_0 \lambda_0 \tilde{\mu} f_{u\mu} V_0 \tag{4.4}$$

where the symbol $[A/B]_0$ means $\lim_{\varepsilon\to 0}[A(\varepsilon)/B(\varepsilon)]$, and where f and all derivatives of f with respect to u and μ are to be evaluated at r, U_0, μ_0, for example $f = f(r, U_0; \mu_0)$. From the equation for w we obtain

$$\begin{aligned} L_0\tilde{w} = & -\left[\frac{\phi}{\nu}\right]_0 \left(\lambda_0 f_{uuu} V_0^2 + \lambda_0 f_{uu} W\right) \tilde{u} \\ & -\left[\frac{\chi}{\nu}\right]_0 \left(f_{uu} V_0^2 + f_u W\right) \tilde{\lambda} \\ & -\left[\frac{\xi}{\nu}\right]_0 \left(f_{uu\mu} V_0^2 + f_{u\mu} W\right) \lambda_0 \tilde{\mu} - \left[\frac{\psi}{\nu}\right]_0 2\lambda_0 f_{uu} V_0 \tilde{v} \end{aligned} \tag{4.5}$$

Consider first equation (4.3), which is analogous to (3.4). As in Section 3, several possibilities for the $o(1)$ order functions must be distingnished. Here we have the following four cases

(1) $\left[\frac{\chi}{\phi}\right]_0 = 1 = \left[\frac{\xi}{\phi}\right]_0$, (2) $\left[\frac{\chi}{\phi}\right]_0 = 1 \quad \left[\frac{\xi}{\phi}\right]_0 = 0$,

(3) $\left[\frac{\chi}{\phi}\right]_0 = 0 \quad \left[\frac{\xi}{\phi}\right]_0 = 1$, (4) $\left[\frac{\chi}{\phi}\right]_0 = 0 = \left[\frac{\xi}{\phi}\right]_0$.

The simplest case is (1). Case (4) can be ruled out by arguments parallel to those of Section 3, where $[\chi/\phi]_o = 0$ was ruled out. But the possibilities of (2) and (3) have yet to be considered. Further cases of order relations arise from equations (4.4) and (4.5). A careful analysis of all possible cases has been carried out by J. Narr [14]. The calculations are not difficult but very tedious, the result being that (1) is the only viable choice in the present analysis. Moreover, all order functions $\phi, \psi, \chi, \nu, \xi$ must be $0(\delta)$ as $\varepsilon \to 0$ to render matching of outer and inner approximations [14].

The differential equations for $\tilde{u}, \tilde{v}$, and $\tilde{w}$ are now obtained from (4.3)–(4.5) by setting all $[A/B]_o$-limits to unity. In addition, $\tilde{u}, \tilde{v}$, and $\tilde{w}$ must satisfy the boundary conditions at $r = 1$. The solution of (4.3) can be written as

$$\tilde{u}(r) = c_1 V_0(r) + d_1 S_0(r) + \tilde{\lambda} z(r) + \tilde{\mu}\zeta(r) \tag{4.6}$$

where $z(r)$ is defined by (3.7) and $\zeta(r)$ is the solution of the initial value problem

$$L_0\zeta = -\lambda_0 f_\mu(r, U_0; \mu_0), \quad r > 0, \quad \zeta(0) = \zeta'(0) = 0 \tag{4.7}$$

Applying $Bu_1(1) = 0$ and matching with the terms of orders $\delta \log r$ and δ^2 of the inner expansion, we obtain $d_1 = U_0(0) - \overline{u}$, $a_2 = c_1$, and a first equation for the determination of $\tilde{\lambda}, \tilde{\mu}, viz,$

$$\tilde{\lambda} Bz(1) + \tilde{\mu} B\zeta(1) = -(U_0(0) - \overline{u}) BS_0(1) \tag{4.8}$$

Similarly, the solutions of (4.4) and (4.5) can be constructed by extending the scheme used for the calculation of $\tilde{v}$ at the end of Section 3.1, which yield a second equation for calculating $\tilde{\lambda}$ and $\tilde{\mu}$ and the yet unknown coefficient c_1. Inserting (4.6) into (4.4), we have

$$L_0\tilde{v} = -c_1\lambda_0 f_{uu} V_0^2 - \tilde{\lambda} V_0(f_u + \lambda_0 f_{uu} z) - \tilde{\mu}\lambda_0 V_0(f_{uu}\zeta + f_{u\mu}) - \lambda_0 d_1 f_{uu} V_0 S_0 \tag{4.9}$$

Hence, the solution can be obtained in the form

$$\tilde{v}(r) = k_2 S_0(r) + c_1 W(r) + \tilde{\lambda} z_1(r) + \tilde{\mu}\zeta_1(r) + z_2(r) \tag{4.10}$$

where

$$L_0 z_1 = -V_0(f_u + \lambda_0 f_{uu} z), \qquad z_1(0) = z_1'(0) = 0, \tag{4.11}$$

$$L_0 \zeta_1 = -\lambda_0 V_0(f_{uu}\zeta + f_{u\mu}), \qquad \zeta_1(0) = \zeta_1'(0) = 0, \tag{4.12}$$

$$L_0 z_2 = -\lambda_0 d_1 f_{uu} V_0 S_0, \qquad z_2(0) = z_2'(0) = 0, \tag{4.13}$$

and use has been made of (2.7). Matching with the inner expansion for $\tilde{v}$ and applying the boundary condition $B\tilde{v}(1) = 0$ leads to $k_2 = 1$, $b_2 = c_1 W(0) = a_1$, and

$$\tilde{\lambda} Bz_1(1) + \tilde{\mu} B\zeta_1(1) = -Bz_2(1) - BS_0(1). \tag{4.14}$$

The coefficients $\tilde{\lambda}$ and $\tilde{\mu}$ can now be determined by solving (4.8) and (4.14). In order to find c_1, equation (4.5) must be solved for $\tilde{w}$. Inserting (4.6) and (4.10) into (4.5), an equation of similar structure as (4.9) is obtained, from which it is apparent that the solution can be written in the form

$$\tilde{w}(r) = k_2 S_0(r) + (\tilde{\lambda}/\lambda_0) W(r) + c_1 z_3(r) + z_4(r) \tag{4.15}$$

where

$$L_0 z_3 = -\lambda_0 V_0 (3 f_{uu} W + f_{uuu} V_0^2) \tag{4.16}$$

$$\begin{aligned} L_0 z_4 = & -\tilde{\lambda} f_u W - \lambda_0 f_{uu}[W(d_1 S_0 + \tilde{\lambda} z + \tilde{\mu}\zeta) + 2V_0(S_0 + \tilde{\lambda} z_1 + \tilde{\mu}\zeta_1 + z_2)] \\ & -\lambda_0 \mu_1 (f_{u\mu} W + f_{uu\mu} V_0^2) - \lambda_0 f_{uuu} V_0^2 (d_1 S_0 + \tilde{\lambda} z + \tilde{\mu}\zeta) \end{aligned} \tag{4.17}$$

with $z_i(0) = z_i'(0) = 0$, $i = 3, 4$. Matching with (4.2) and applying $B\tilde{w}(1) = 0$ one finds $k_2 = 1$, and

$$c_1 = -\frac{1}{Bz_3(1)}[BS_0(1) + Bz_4(1)]. \tag{4.18}$$

This completes the determination of the leading-order corrections for the set of double limit points.

4.2 Asymptotic expansions at and near a double limit point

It is obvious that the simple approximations given in the preceding section can be extended to full asymptotic expansions for the family of double limit points in the form

$$\lambda_0(\varepsilon) \sim \lambda_0 + \sum_{k=1}^{\infty} \lambda_k \delta^k(\varepsilon), \qquad \mu_0(\varepsilon) \sim \mu_0 + \sum_{k=1}^{\infty} \mu_k \delta^k(\varepsilon),$$

with outer and inner expansions for $u_0(r;\varepsilon)$, $v_0(r;\varepsilon)$ of the form (2.20), (2.21), (3.21) and (3.22), supplemented by corresponding expansions for $w(r;\varepsilon)$. The coefficients functions $u_k(r)$, $v_k(r)$ and $w_k(r)$, $k = 1, 2, \ldots$ in the outer expansions are obtained from differential equations and boundary conditions analogous to (3.23)–(3.26). The remarks near the end of Section 3.2 concerning the sequential solution of these equations and the determination of the constants of integration also apply here.

We list here the results of the straightforward but lengthy calculations of the second order correction terms for later application.

$$u_2(r) = c_2 V_0(r) + c_1 S_0(r) + \lambda_2 z + \mu_2 \zeta + z_5 \tag{4.19}$$

$$\begin{aligned} \lambda_2 Bz(1) + \mu_2 B\zeta(1) &= -Bz_5(1) - c_1 BS_0(1) \\ \lambda_2 Bz_1(1) + \mu_2 B\zeta_1(1) &= -Bz_6(1) - c_1 BS_0(1) \end{aligned} \tag{4.20}$$

where the initial value problems defining z_5, z_6 and z_7 (see (4.21)) uniquely are given in the Appendix. The coefficients λ_2 and μ_2 can be determined from (4.20). In order to compute c_2, the BVP for w_2 must be solved. The result is

$$c_2 = -\frac{1}{Bz_3(1)}\left[\frac{\lambda_1}{\lambda_0}BS_0(1) + Bz_7(1)\right]. \tag{4.21}$$

As in the case of simple limit points, expansions for values λ, μ near λ_0, μ_0 can be constructed along the lines given in Section 3.3. In view of the cubic nature of the double limit point, the appropriate form for the outer expansion for u will be

$$u(r,\lambda,\mu;\varepsilon) \sim U_0(r) + \sum_{k=1}^{\infty} U_k(r)\delta^{k/3}(\varepsilon)$$

as $\varepsilon \to 0$, with r fixed, $0 < r \leq 1$. The inner expansion will then also proceed in powers of $\delta^{1/3}$. One finds $U_1 = B_1V_0(r)$ and $U_2 = B_2V_0(r) + B_1^2V_1$ where V_1 is a solution of

$$L_0V_1 = -(\lambda_0/2)f_{uu}(r,U_0;\mu_0)V_0^2, \quad BV_1(1) = 0.$$

The constant α_1 in (3.32) enters first in the differential equation for $U_3(r)$. We omit further discussion of the asymptotic solution, as it closely parallels the solution near a simple limit point.

5 Application to a simple thermal ignition model

Limit points play a prominent role in the study of reactors involving a selfheating chemical whose reaction velocity follows the Arrhenius law and which dissipates energy by conduction only. When the exothermicity of the reactant mixture, measured by a parameter λ, reaches a limit point λ_0 a thermal explosion can occur. A reactor must be designed such as to avoid reactor runaways. Exothermic reactions in tubular reactors of finite length have recently been considered by Hagan et al [15]–[17].

In this and the following section we address the following problem. A reacting material is confined to an 'infinite' circular cylinder of radius normalized to one, the heat flux across the boundary is governed by Newtonian cooling. Such a situation would determine a limit point λ_0. Now suppose one were to place a 'cooling' rod of radius ε along the axis of the reactor and to maintain the temperature at a constant value on this inner cylinder. Clearly the presence of such a rod will have an effect on the limit point. The problem is to estimate this effect for small ε. We shall show that it is quite dramatic.

We first take up the limiting situation of infinite 'activation energy', the resulting model is usually called the Frank-Kamenetzkii approximation (see [18] for a derivation of the basic equations). In nondimensional form the BVP for this model is given by

$$\Delta U + \lambda e^U = 0 \quad \text{for} \quad 0 \le r < 1 \quad 0 \le \theta \le 2\pi \tag{5.1}$$

$$\alpha \frac{\partial U}{\partial n} + U = 0 \quad \text{on} \quad r = 1 \tag{5.2}$$

where U is proportional to the deviation of the temperature in the vessel from the temperature on the boundary, $\alpha = 1/Bi$ and Bi is the Biot number. This BVP is of the form (2.1) with $f(r, U) = e^U$. Radially symmetric solutions $U(r, \lambda)$ satisfy

$$BVP(I) \begin{cases} U'' + \frac{1}{r}U' + \lambda e^U = 0 & 0 < r < 1 \\ U'(0, \lambda) = 0 = \alpha U'(1, \lambda) + U(1, \lambda). \end{cases} \tag{5.3}$$

If $\alpha = 0$, this BVP has the well-known exact solution

$$U(r, \lambda) = \log\left(\frac{1+\gamma}{r^2+\gamma}\right)^2, \quad \gamma = -1 + \frac{4}{\lambda}\left(1 \pm \sqrt{1 - \frac{\lambda}{2}}\right). \tag{5.4}$$

For $0 < \lambda < 2$ (5.3) has exactly two solutions. Clearly

$$\lambda = \lambda_0 = 2, \qquad U(r, 2) = U_0(r) = \log\left(\frac{2}{r^2+1}\right)^2 \tag{5.5}$$

defines the limit point. If $\alpha \neq 0$, the solution of (5.3) is [19]

$$U(r, \lambda) = \log\left(\frac{1+\gamma}{r^2+\gamma}\right)^2 + \frac{4\alpha}{1+\gamma} \tag{5.6}$$

where $\gamma = \gamma(\lambda)$ is a solution of the transcendental equation

$$\lambda = \frac{8\gamma}{(\gamma+1)^2} e^{-4\alpha/(\gamma+1)}. \tag{5.7}$$

It is easily shown that (5.7) also has exactly two solutions for $0 < \lambda < \lambda_0$ which coincide at the limit point $\lambda = \lambda_0$ given by (5.7) with

$$\gamma = \gamma_0 = 2\alpha + \sqrt{1 + 4\alpha^2} \tag{5.8}$$

and $U_0(r) = U(r, \lambda_0)$ as in (5.6) with $\gamma = \gamma_0$.

The homogeneous variational problem (2.3) for $\lambda = \lambda_0$ is

$$Var(I) \begin{cases} L_0 V_0 := V_0'' + \frac{1}{r} V_0' + \dfrac{8\gamma_0}{(r^2+\gamma_0)^2} V_0 = 0 \qquad 0 < r < 1 \\ V_0'(0) = 0 = \alpha V_0'(1) + V_0(1). \end{cases} \tag{5.9}$$

A smooth solution of Var(I) normalized to one at $r = 0$ is given by

$$V_0(r) = \frac{\gamma_0 - r^2}{\gamma_0 + r^2} \tag{5.10}$$

with a second, linearly independent solution having $\log r$ behavior given by

$$S_0(r) = \frac{2r^2}{\gamma_0 + r^2} + \frac{\gamma_0 - r^2}{\gamma_0 + r^2} \log r \qquad 0 < r \le 1. \tag{5.11}$$

Introducing the 'cooling rod' of radius ε into the reactor leads to the following perturbed problem for $u(r, \lambda; \varepsilon)$

$$BVP(II) \begin{cases} u'' + \frac{1}{r} u' + \lambda e^u = 0 \qquad \varepsilon < r < 1 \\ u(\varepsilon, \lambda; \varepsilon) = \overline{u} \\ \alpha u'(1, \lambda; \varepsilon) + u(1, \lambda; \varepsilon) = 0. \end{cases} \tag{5.12}$$

It turns out that this BVP also has an explicit solution which can be found by the same substitution $u(r) = c \log w(r)$ that leads to the analytic solutions (5.4) and (5.6). With $c = -2$, the differential equation for w is

$$-ww'' + (w')^2 - \frac{1}{r} ww' + \frac{\lambda}{2} = 0. \tag{5.13}$$

We attempt to find a solution of the form

$$w = Ar^a + Br^b \tag{5.14}$$

where A, B, a, b are constants. A straightforward calculation shows that (5.14) is a solution of (5.13) if we set

$$a = -\beta, \quad b = 2 + \beta, \quad \lambda = 8(1+\beta)^2 AB \tag{5.15}$$

with A, B arbitrary and β determined by (5.15) in terms of λ, A and B. The constants A, B can be determined from the boundary conditions in (5.12). One finds

$$A = -B\varepsilon^{2+2\beta} + \varepsilon^{\beta} e^{-\overline{u}/2}$$

$$2\alpha[\beta A - (2+\beta)B] + 2(A+B)\log(A+B) = 0.$$

Together with $\lambda = 8(1+\beta)^2 AB$, we thus have three nonlinear equations that determine the constants β, A and B.

In the special case $\alpha = 0$ more explicit results can be obtained. Here $A + B = 1$, and the solution of (5.12) is given by

$$u(r) = -2\log\left[Ar^{-\beta} + (1-A)r^{2+\beta}\right] \tag{5.16}$$

$$A = \varepsilon^{\beta}\,\frac{e^{-\overline{u}/2} - \varepsilon^{2+\beta}}{1 - \varepsilon^{2+2\beta}}, \quad \lambda = 8(1+\beta)^2 A(1-A). \tag{5.17}$$

Elementary considerations show that $\lambda = \lambda(\beta) \to 0$ as $\beta \to 0$ and as $\beta \to \infty$, and that $\lambda(\beta)$ has a maximum λ_m at some value $\beta = \beta_m$. This maximum represents the limit point $\lambda_0(\varepsilon)$. A direct numerical solution of the equation $\partial\lambda/\partial\beta = 0$ is inconvenient, but it is a simple matter to find β_m by calculating $\lambda = \lambda(\beta)$ near β_m from (5.17). Some numerical values of $\lambda_0(\varepsilon)$ for $\overline{u} = 0$ are contained in Table 1 below.

For $\overline{u} = 0$ and sufficiently small ε, we may use the approximation $A \sim \varepsilon^{\beta}$. Then we still have $u(1) = 0$ but

$$u(\varepsilon) \sim -2\log\left[1 + (1-\varepsilon^{\beta})\varepsilon^{2+\beta}\right] = 0(\varepsilon^2) \qquad \varepsilon \to 0 \tag{5.18}$$

provided $\beta > 0$. The equation for λ becomes $\lambda = 8\varepsilon^{\beta}(1-\varepsilon^{\beta})(1+\beta)^2$. Setting $\beta = -\delta\log\eta$, with δ defined as in (1.2), we have

$$\lambda = 8\eta(1-\eta)(1-\delta\log\eta)^2 \qquad 0 < \delta \ll 1 \tag{5.19}$$

Clearly, the limit point (maximum) λ_m is near $\eta = 1/2$. For such an η β is positive and $\beta = 0(\delta)$. Thus (5.18) is a consistent estimate. Setting $\partial\lambda/\partial\eta = 0$ we find

$$1 - 2\eta + \delta[(2\eta-1)\log\eta + 2(\eta-1)] = 0.$$

Assuming $\eta \sim (1/2) + \delta\eta_1 + \delta^2\eta_2 + \cdots$, we obtain $\eta_1 = -1/2$, $\eta_2 = (\log 2 - 1)/2$. Substituting these values into (5.19), the result is

$$\lambda_m(\varepsilon) = 2 + 4\delta\log 2 + 2\delta^2(1+\log^2 2) + 0(\delta^3) \qquad \text{as} \quad \delta \to 0. \tag{5.20}$$

A similar calculation for $\overline{u} \neq 0$ yields, with $A \sim e^{-\overline{u}/2}\varepsilon^{\beta}$

$$\lambda_m(\varepsilon) = 2 - 4\delta \log\left(\frac{1}{2}e^{\overline{u}/2}\right) + 2\delta^2\left[1 + \log^2\left(\frac{1}{2}e^{\overline{u}/2}\right)\right] + 0(\delta^3) \quad \text{as} \quad \delta \to 0. \qquad (5.21)$$

We now compare these results with our asymptotic expansions of Section 3. For simplicity we set $\alpha = \overline{u} = 0$. Then the solution of (3.7) with $f = e^u$ and $U_0(r)$ according to (5.5) is $z(r) = -r^2/(1+r^2)$. Making use of (5.10) and (5.11) with $\gamma_0 = 1$, the leading order correction (3.11) becomes

$$\tilde{\lambda} = \lambda_1 = 2\log 4 \qquad (5.22)$$

while the outer correction $\tilde{u}$ is obtained from (3.8) as

$$\tilde{u}(r) = u_1(r) = \frac{1-r^2}{1+r^2}\left(c_1 + \frac{1}{2}\lambda_1 \log r\right). \qquad (5.23)$$

In order to determine the constant c_1 we must solve for $\tilde{v}$, which amounts to computing z_1 and z_2 from (3.18). It is easily verified that $z_1 = -2r^2/(1+r^2)^2$ and that the equation for z_2 becomes

$$L_0 z_2 = -4\lambda_1 \cdot \frac{1-r^2}{(1+r^2)^3}\left(1 + \frac{1-r^2}{1+r^2}\log r\right) := H_2, \quad z_2(0) = z_2'(0) = 0. \qquad (5.24)$$

We note that the solution of the equation $L_0 Z = H(r)$ satisfying $Z(0) = Z'(0) = 0$ is given by the formula

$$Z(r) = \int_0^r G(r,\rho)H(\rho)d\rho \qquad (5.25)$$

where

$$G(r,\rho) = \frac{\rho}{(1+r^2)(1+\rho^2)}\left[2(r^2-\rho^2) + (1-r^2)(1-\rho^2)\log\frac{r}{\rho}\right]. \qquad (5.26)$$

It turns out that for all functions $H(r)$ to follow below the integration in (5.25) can be performed explicitly, but the algebra may be quite lengthy. Alternatively, observe that

$$\begin{aligned} L_0(Z(r)\log r) &= (\log r)L_0 Z + \tfrac{2}{r}Z' \\ L_0(Z(r)\log^2 r) &= (\log^2 r)L_0 Z + \tfrac{4}{r}Z'\log r + \tfrac{2}{r^2}Z \end{aligned}$$

These relations are useful for calculating z_2 and z_3. The solution of (5.24) is $z_2 = -\lambda_1 r^2(1+r^2)^{-2}\log r$. Substitution of $z_1(1)$ and $z_2(1)$ into (3.19) yields $c_1 = 2$.

To find λ_2 and $u_2(r)$ from (3.27)-(3.31), z_3 and z_4 must be calculated. Writing $L_0 z_3 = H_3$ we find

$$H_3(r) = -\frac{\lambda_1^2}{4}\left[H_1(r)\log^2 r + 8\frac{1-r^2}{(1+r^2)^3}\log r\right] + 2H_2(r) - 4H_1(r)$$

where $H_1 = 4(1-r^2)^2/(1+r^2)^4$ and H_2 was defined in (5.24). From this z_3 is found to be

$$z_3(r) = -(\log^2 4)\frac{r^2}{(1+r^2)^2}\left[\log^2 r - \frac{1}{2}(1+r^2)\right] + 2z_1(r) + 2z_2(r) \tag{5.27}$$

Next λ_2 can be calculated from (3.28) to give $\lambda_2 = 2 + 2\log^2 2 \doteq 2.9609$. Finally, solving (3.31) for $z_4(r)$, we obtain $c_2 \doteq 1.614$ from (3.30).

Summarizing the results to this point for the limit-point expansions, we have

$$\lambda_0(\varepsilon) = 2 + \delta(\varepsilon)2\log 4 + \delta^2(\varepsilon)2(1+\log^2 2) + 0(\delta^3), \quad \varepsilon \to 0, \tag{5.28}$$

with the inner expansion for u_0 given by

$$u_0(r;\varepsilon) = \log s\left[\delta(\varepsilon)\log 4 + 2\delta^2(\varepsilon) + 0(\delta^3)\right], \quad \varepsilon \to 0 \tag{5.29}$$

with $s = r/\varepsilon \geq 1$; and with the outer expansion given by

$$\begin{aligned} u_0(r;\varepsilon) \;=\; & \log\frac{4}{(r^2+1)^2} + \delta(\varepsilon)\frac{1-r^2}{1+r^2}\left[2 + (\log 4)\log r\right] \\ & +\delta^2(\varepsilon)\left\{\frac{1-r^2}{1+r^2}(c_2 + 2\log r) + \frac{2r^2}{1+r^2}\right. \\ & \left. -\frac{r^2}{(1+r^2)^2}[2 + (\log 4)\log r]^2\right\} + 0(\delta^3). \end{aligned} \tag{5.30}$$

The approximation to $\lambda_0(\varepsilon)$ provided by (5.28) represents the main result of this section. It agrees with the approximate result (5.20) based on the exact solution (5.16). The slow convergence to zero of $1/\log(1/\varepsilon)$ as $\varepsilon \to 0$ implies that even for very thin cooling rods we can expect significant increases in the safe operating range of λ beyond the unperturbed value of $\lambda_0 = 2$. For example, for $\varepsilon = 0.001$ our formula predicts an increase in λ_0 of more than twenty percent. In Table 1 we compare the predictions of both two and three terms of formula (5.28) with numerical calculations of $\lambda_0(\varepsilon)$, based on the exact solution (5.17), for several values of ε. In each instance three terms of (5.28) provide a better approximation than do just two terms. Even for $\varepsilon = 0.1$ ($1/\log 10 \doteq 0.4343$) the accuracy is impressive.

ε	exact analytic solution for $\lambda_0(\varepsilon)$	two–term asymptotic approximation $\lambda_1 = 2.77259$	three–term asymptotic approximation $\lambda_2 = 2.96091$
0.1	3.89151	3.20412	3.76258
0.04	3.20030	2.86135	3.14712
0.01	2.76102	2.60206	2.74168
0.001	2.46934	2.40137	2.46342
0.0001	2.33846	2.30103	2.33593

Table 1 Comparison of exact solution (5.17) and asymptotic approximations (5.28) to the limit point $\lambda_0(\varepsilon)$ for BVP (II), given by (5.12),$\bar{u}$=0

A more detailed picture of the effect of introducing a cooling rod is provided in the bifurcation diagram in Fig. 1, where

$$\|u\| := \left[\frac{1}{1-\varepsilon}\int_{\varepsilon}^{1} u^2(r,\lambda;\varepsilon)dr\right]^{1/2} .$$

The curves were obtained directly from the exact solution (5.16)–(5.17), with $\overline{u} = 0$. Qualitatively these curves are very similar, but the location of the limit point λ_0 (at which a thermal explosion can occur) is very sensitive to the size of the cooling rod. It would be interesting to find out how the range of stable reactor performance analysed in [16,17] can be improved by introducing a cooling rod.

The inner and outer approximations to $u_0(r;\varepsilon)$ provided by (5.29) and (5.30), respectively, are also quite accurate. For example, in Fig. 2 we compare the exact solution with both the two-term inner and the two-term outer approximations, which include all terms of order δ^2, for $\varepsilon = 1/25$. The approximations are both very good in their assumed regions of validity. As ε is decreased the accuracy improves, in agreement with the order of magnitude error estimates in (5.29) and (5.30).

It is also possible to combine the inner and outer approximations into uniformly valid representations. Adding one term of (5.29) to one term of (5.30) and subtracting out the common (overlapping) part leads to the one term composite approximation

$$u(r;\varepsilon) = \log\left(\frac{2}{r^2+1}\right)^2 + (\log 4)\frac{\log r}{\log 1/\varepsilon} + o\left(\frac{1}{\log 1/\varepsilon}\right),$$

as $\varepsilon \to 0$ uniformly on $\varepsilon \le r \le 1$. The same process, except taking two terms from both (5.29) and (5.30), yields the two term composite approximation

$$\begin{aligned} u(r;\varepsilon) &= \log\left(\frac{2}{r^2+1}\right)^2 + \frac{1-r^2}{1+r^2}\frac{1}{\log 1/\varepsilon}(2+(\log 4)(\log r)) \\ &\quad + \frac{2\log r}{(\log 1/\varepsilon)^2} + o\left(\frac{1}{\log^2 1/\varepsilon}\right), \end{aligned}$$

as $\varepsilon \to 0$ uniformly on $\varepsilon \le r \le 1$.

In Fig.3 and 4 we compare both the one term and the two term composite approximations with the exact solutions of $u_0(r;\varepsilon)$ for $\varepsilon = .04$ and $.001$, respectively. The improvement in the accuracy of the two term approximation over the one term approximation is readily apparent, as is the improvement in each approximation for smaller values of ε. On the other hand, the two term composite approximation is not as accurate as either the corresponding inner or outer approximations in their regions of validity (compare Fig. 2 and 3). This situation is common in problems where the boundary layer terms do not decay exponentially.

6 Application to a more accurate thermal ignition model

We now consider the thermal ignition problem under the more realistic assumption of a finite 'activation energy' E. The resulting equation is ofter called the Arrhenius equation [2] or the Bratu problem [20]. It contains an additional parameter $\mu = RT_s/E$, where T_s is the surface temperature. In dimensionless variables the Arrhenius BVP for an infinite circular cylinder can be written in the form [20]

$$\begin{aligned} &\Delta U + \lambda e^{U/(1+\mu U)} = 0 \qquad \text{for} \qquad 0 \le r < 1, \quad 0 \le \theta < 2\pi \\ &\alpha\frac{\partial U}{\partial n} + U = 0 \qquad \text{on} \qquad r = 1, \end{aligned} \tag{6.1}$$

where U, λ and α have the same meaning as in equation (5.1). In contrast to the case $\mu = 0$ (Section 5) there is no finite λ_0 beyond which solutions no longer exist. Rather, positive radially symmetric solutions of (6.1) exist for all positive λ and μ. For $\alpha = 0$, this is proved in [21]. For small $\mu > 0$, there exists a critical value $\lambda_c(\mu)$ for which the smallest positive solution of (6.1) approaches an infinite change of temperature with respect to λ. This value $\lambda_c(\mu)$ is defined as the critical ignition parameter for given μ. Mathematically, λ_c is a simple limit point $\lambda_0(\mu)$. There is a second such limit point $\lambda_s < \lambda_c$, connecting the unstable branch bending backwards for $\lambda < \lambda_c$ with an upper branch of stable solutions (see Fig. 5). For λ sufficiently large, the solution of (6.1) is unique.

The largest value of μ for which ignition can occur is another critical value μ_c for the system, because the criterion for ignition disappears for $\mu > \mu_c$. This has considerable practical significance, in particular for low activation energy. Mathematically, the solution branch has a double limit point at $\mu = \mu_c = \mu_0$ as defined in Section (2.1): the two limit points $\lambda_c(\mu)$ and $\lambda_s(\mu)$ coalesce at $\mu = \mu_c$. For $\mu > \mu_c$, (6.1) has a unique solution for every $\lambda > 0$. Because of its principal importance, much effort has been devoted to the accurate determination of the critical values λ_c and μ_c for various geometrics of the reactor vessel (e.g., see [2], [20], [22]). As in Section 5, we consider here the effect of introducing a central cooling rod into a tubular reactor.

6.1 Numerical solutions

The cylindrically symmetric solutions $U(r, \lambda, \mu)$ of (6.1) are defined by

$$BVP(I) \qquad \begin{aligned} &U'' + \tfrac{1}{r}U' + \lambda e^{U/(1+\mu U)} = 0 \qquad 0 < r < 1 \\ &U'(0, \lambda, \mu) = 0 = BU(1, \lambda, \mu) \end{aligned} \tag{6.2}$$

where B is the boundary operator defined in Section 5. This BVP does not appear to have an exact solution. At a simple limit point $\lambda_0(\mu)$, the variational problem (2.3) becomes

$$Var(I) \qquad \begin{aligned} &LV_0 := V_0'' + \tfrac{1}{r}V_0' + \lambda_0(\mu)\frac{1}{(1+\mu U_0)^2}e^{U_0/(1+\mu U_0)}V_0 = 0 \\ &V_0'(0, \lambda_0, \mu) = 0 = BV_0(1, \lambda_0, \mu) \end{aligned} \tag{6.3}$$

where $U_0 = U(r, \lambda_0, \mu)$ is the solution of (6.2) for $\lambda = \lambda_0(\mu)$. $V_0(r)$ is a smooth nontrivial solution with the normalization $V_0(0) = 1$. To find the double limit point λ_0, μ_0, the additional BVP (2.7), which in the present example becomes

$$\begin{aligned} &L_0 W = -\lambda_0 V_0^2 e^{U_0/(1+\mu_0 U_0)} \frac{1 - 2\mu_0(1 + \mu_0 U_0)}{(1 + \mu_0 U_0)^4} \\ &W'(0, \lambda_0, \mu_0) = 0 = BW(1, \lambda_0, \mu_0) \end{aligned} \tag{6.4}$$

must be solved, with $W(0) = 1$.

The perturbed problem is now obtained by introducing the 'cooling rod' of radius ε. This leads to BVP (II) in the same way as in the previous section. The solution is denoted by $u(r,\lambda,\mu;\varepsilon)$, obtained from (6.2) after replacing the interval $0<r<1$ by $\varepsilon<r<1$ and $U'(0,\lambda,\mu)=0$ by the inner boundary condition

$$u(\varepsilon,\lambda,\mu;\varepsilon)=\overline{u}.$$

In other words, $u(r,\lambda,\mu;\varepsilon)$ is the solution of (5.12) where the function e^u is replaced by $exp(u/(1+\mu u))$. Similarly, the solutions of the perturbed BVPs corresponding to (6.3) and (6.4), needed for the calculation of the limit points $\lambda_0(\mu,\varepsilon)$ and $\mu_0(\varepsilon)$ are denoted by $v_0(r,\lambda_0,\mu;\varepsilon)$ and $w(r,\lambda_0,\mu_0;\varepsilon)$, respectively.

Regular solutions were found by solving (6.2) numerically by the general BVP solver COLPAR [23,24,25]. The limit points $\lambda_0(\mu)$ and μ_0 were determined by solving the extended system (6.2), (6.3) and (6.2)–(6.4), respectively, also using COLPAR. Similarly, the corresponding perturbed BVPs for $\varepsilon>0$ were solved numerically. It appears that the double limit points μ_0 and $\mu_0(\varepsilon)$ have not been calculated previously by solving an appropriate extended system. Numerical difficulties were encountered for very small values of ε, because COLPAR is not designed to handle stiff BVPs. A bifurcation diagram of solutions of BVP (I) is contained in Fig. 5. Similarly, for a fixed value of μ, solution branches for BVP (II) for various values of ε are given in Fig. 6. These numerical solutions were generated by the continuation program COLCON developed by Bader and Kunkel [25]. However, as the accuracy of the solutions decreases considerably with decreasing ε, solutions for $\varepsilon=0.001$ are not displayed in Fig. 6

6.2 Asymptotic solutions for simple limit points

We proceed to construct asymptotic expansions for small ε to the same order of δ as in the previous section. This means we have to find u_1, v_1 and u_2 in order to obtain a three term approximation for $\lambda_0(\mu,\varepsilon)$ and $u_0(r,\lambda_0,\mu;\varepsilon)$ with μ fixed.

From the unperturbed limit point solution we have numerical solutions for $U_0(r)$ and $V_0(r)$ defined by BVPs (6.2) and (6.3), respectively, and the value $\lambda_0=\lambda_0(\mu,0)$. A second solution $S_0(r)$ of $L_0V=0$ is computed following (2.5), that is, by numerically solving an initial value problem for $R_0(r)$. Similarly, calculating $z(r)$ from (3.7), we have $\tilde{\lambda}=\lambda_1$ from the formula (3.11) while $u_1(r)$ is given by (3.8), up to the constant c_1. In the special case $\overline{u}=\alpha=0$ $\lambda_1=-U_0(0)S_0(1)/z(1)$. Next we substitute

$$f_u=\frac{1}{(1+\mu u)^2}e^{u/(1+\mu u)},\qquad f_{uu}=\frac{1-2\mu(1+\mu u)}{(1+\mu u)^4}e^{u/(1+\mu u)}$$

with $u = U_0(r)$ into (3.18) and solve numerically for $z_1(r)$ and $z_2(r)$, whereupon c_1 is calculated from (3.19). Finally, we calculate λ_2 and u_2 numerically from (3.27)-(3.31). Note that $S_0(r)$, $z(r), \ldots, z_4(r)$ are all regular solutions of their defining initial value problems, thus the numerical solutions present no difficulties and are computationally inexpensive compared to a direct numerical solution of the perturbed BVP (II) for small ε.

μ	0.1	0.2	0.24	0.242
$\lambda_0(\mu,0)$	2.26128	2.68135	2.98196	3.00500
$\lambda_1(\mu)$	3.08573	3.56891	3.87262	3.88038
$\lambda_2(\mu)$	3.33386	3.90763	4.29766	4.44751

	μ=0.1			μ=0.2		
ε	numerical solution for $\lambda_0(\mu,\varepsilon)$	2–term asymptotic expansion	3–term asymptotic expansion	numerical solution for $\lambda_0(\mu,\varepsilon)$	2–term asymptotic expansion	3–term asymptotic expansion
0.1	4.3806	3.6014	4.2302	5.1571	4.2313	4.9683
0.04	3.6041	3.2199	3.5417	4.2463	3.7901	4.1672
0.01	3.1114	2.9313	3.0885	3.6697	3.4563	3.6406
0.001	2.7848	2.7080	2.7778	3.2888	3.1980	3.2799

Table 2 Comparison of numerical and asymptotic approximations to the simple limit points $\lambda_0(\mu,\varepsilon)$ of BVP(II), given by (2.8) with $f(r,u;\mu) = \exp(u/(1+\mu u))$ and $\alpha = \bar{u} = 0$.

Some results are given in Table 2, which shows the exact numerical limit points $\lambda_0 = \lambda_0(\mu, 0)$ for various values of μ, together with the coefficients λ_1 and λ_2 of the asymptotic expansion (3.20). Furthermore, predictions based on two–term and three–term approximations for several values of ε are given for $\mu = 0.1$ and $\mu = 0.2$ and compared with numerically computed values of $\lambda_0(\mu,\varepsilon)$. Only the limit points connecting a lower solution branch with the unstable portion have been computed. In Fig. 7, an accurate numerical solution is compared with one and two term composite asymptotic approximations for $\mu = 0.2$ and $\varepsilon = 0.01$.

6.3 Asymptotic solutions for double limit points

The double limit point μ_0 for the unperturbed BVP (6.2) must be calculated numerically, which yields the functions U_0, V_0 and W. The function S_0 also needed below is obtained from (2.5). Again, a direct calculation of $\mu_0(\varepsilon)$ for small ε runs into serious difficulties, while an asymptotic solution is easily performed by solving a series of initial value problems as described in Section 4. In particular, the leading term coefficients $\tilde{\lambda}$ and $\tilde{\mu}$ defined in (3.2) and (4.1) are obtained from equations (4.8) and (4.14) upon numerically solving (3.7) and (4.7) for $z(r)$ and $\zeta(r)$, and (4.11)–(4.13) for $z_1(r)$, $\zeta_1(r)$ and $z_2(r)$. In addition, $z_3(r)$ and $z_4(r)$ must be computed from (4.16), (4.17) if the function $\tilde{u}(r)$ is to be determined according to (4.6). Having c_1 from (4.18), the functions $\tilde{v}$ and $\tilde{w}$ are also determined. Finally, the terms involving λ_2, μ_2 and $u_2(r)$, which are needed for three–term asymptotic approximations, are obtained from (4.19) and (4.21). To this end, three more initial value problems for z_5, z_6 and z_7 have to be solved.

	numerical solution		2–term asymptotic approximation λ_1=4.0297 μ_1=0.0118		3–term asymptotic approximation λ_2=4.4250 μ_2=-0.0023	
ε	$\lambda_0(\varepsilon)$	$\mu_0(\varepsilon)$	$\lambda_0(\varepsilon)$	$\mu_0(\varepsilon)$	$\lambda_0(\varepsilon)$	$\mu_0(\varepsilon)$
0.0	3.0063	0.2421				
0.1	5.7982	0.2451	4.7564	0.2472	5.5912	0.2468
0.05	4.9499	0.2448	4.3514	0.2460	4.8446	0.2458
0.01	4.1221	0.2442	3.8813	0.2447	4.0900	0.2445
0.005	3.9460	0.2440	3.7669	0.2443	3.9245	0.2442
0.001	3.6923	0.2437	3.5897	0.2438	3.6824	0.2437

Table 3 Comparison of numerical and asymptotic approximations to the double limit point $\lambda_0(\varepsilon)$, $\mu_0(\varepsilon)$ of BVP (II) given by (2.8) with $f(r, u; \mu) = exp(u/(1 + \mu u))$ and $\overline{u} = \alpha = 0$.

The results of these calculations are given in Table 3. Two–term and three–term asymptotic approximations are compared with numerical solutions for small ε. We note that the critical parameter $\mu_0(\varepsilon)$ is relatively insensitive to the presence of a cooling rod, while the operating range given by $\lambda_0(\varepsilon)$ shows a strong dependence on ε, which is also displayed in Fig. 8. Comparing a numerical solution for ε not too small with a two–term asymptotic solution at $\lambda = \lambda_0(\varepsilon)$, $\mu = \mu_0(\varepsilon)$, a similar behavior as in Fig. 7 is found.

7 Concluding remarks

We conclude that our asymptotic solutions based on a three–term approximation can safely be used for the calculation of simple and double limit points for sufficiently small ε, say $\varepsilon \leq 0.02$ (see Tables 1–3). For larger values of ε, numerical solutions of acceptable accuracy can usually be obtained in the type of model problems considered in this paper, while serious difficulties are likely to be encountered for $\varepsilon < 0.01$, even with a relatively robust code such as COLPAR.

It would also be of interest to study the dependence of the limit points on the Biot number [20], that is, on α in the boundary condition (2.3). Furthermore, a temperature condition $u = \bar{u} > 0$ at the surface of the cooling rod rather than $u = 0$ is probably more realistic in reactor applications. We have obtained numerical and asymptotic solutions for the model problem in Section 6 for $\bar{u} \neq 0$ and for an extended range of α. We expect to publish the results elsewhere.

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Appendix

The initial value problems for z_5 and z_6 that occur in the calculation of a three term asymptotic approximation to the double limit points $\lambda_0(\varepsilon)$, $\mu_0(\varepsilon)$ are

$$\begin{aligned}
Lz_5 &= -\lambda_1(u_1 f_u + \mu_1 f_\mu) - [f] \\
Lz_6 &= -\lambda_1 v_1 f_u - f_{uu}[V_0(\lambda_1 u_1 + c_1\lambda_0 S_0 + \lambda_0 z_5) + \lambda_0 u_1 v_1] \\
&\quad -\mu_1 f_{u\mu}(\lambda_1 V_0 + \lambda_0 v_1) - [f_u] \\
Lz_7 &= -\lambda_0(W f_{uu} + V_0^2 f_{uuu})(c_1 S_0 + \lambda_2 z + \mu_2\zeta + z_5) \\
&\quad -2\lambda_0 V_0 f_{uu}(c_1 S_0 + \lambda_2 z_1 + \mu_2 \zeta_1 + z_6) - T \\
z_i(0) &= z_i'(0) \;=\; 0 \quad i = 5, 6, 7
\end{aligned}$$

where

$$\begin{aligned}
[f] &:= \frac{1}{2}\lambda_0(u_1^2 f_{uu} + 2\mu_1 u_1 f_{u\mu} + \mu_1^2 f_{\mu\mu}) \\
T &:= W\{\lambda_2 f_u + \lambda_1(u_1 f_{uu} + \mu_1 f_{u\mu}) + [f_u] + \lambda_0\mu_2 f_{u\mu}\} \\
&\quad +V_0^2\{\lambda_2 f_{uu} + \lambda_1(u_1 f_{uuu} + \mu_1 f_{uu\mu}) + [f_{uu}] + \lambda_0\mu_2 f_{uu\mu}\}
\end{aligned}$$

$$+2v_1V_0(\lambda_1 f_{uu} + \lambda_0 u_1 f_{uuu} + \lambda_0 \mu_1 f_{uu\mu})$$

$$+w_1(\lambda_1 f_u + \lambda_0 u_1 f_{uu} + \lambda_0 \mu_1 f_{u\mu}) + \lambda_0 v_1^2 f_{uu}.$$

Note that $u_1 = \tilde{u},\ v_1 = \tilde{v},\ w_1 = \tilde{w}$ were defined by (4.6),(4.10),(4.15).

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Fig. 1 Bifurcation diagram $\|u\|$ versus λ for exact solutions of BVP(I) given by (5.3) and BVP (II) given by (5.12). Parameters $\alpha = \overline{u} = 0$, $0 \le \epsilon \le 0.1$.

Fig. 2 Comparison of two term inner and outer asymptotic expansions of $u_0(r;\epsilon)$ with exact solution of (5.12) for $\epsilon = 0.04$ at the limit point $\lambda_0(\epsilon) = 3.2003$.

Fig. 3 Comparison of one and two term composite asymptotic approximations of $u_0(r;\epsilon)$ with exact solution of (5.12) at the limit point $\epsilon = 0.04$, $\lambda_0(\epsilon) = 3.2003$.

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Fig. 7 Comparison of one and two term composite asymptotic approximations of $u_0(r,\lambda_0,\mu;\epsilon)$ with numerical solution of BVP(II) for $f(r,u,\mu) = \exp(u/(1+\mu u))$, $\alpha = \overline{u} = 0$. Parameters $\mu = 0.2$, $\epsilon = 0.01$, $\lambda_0(\epsilon) = 3.6697$.

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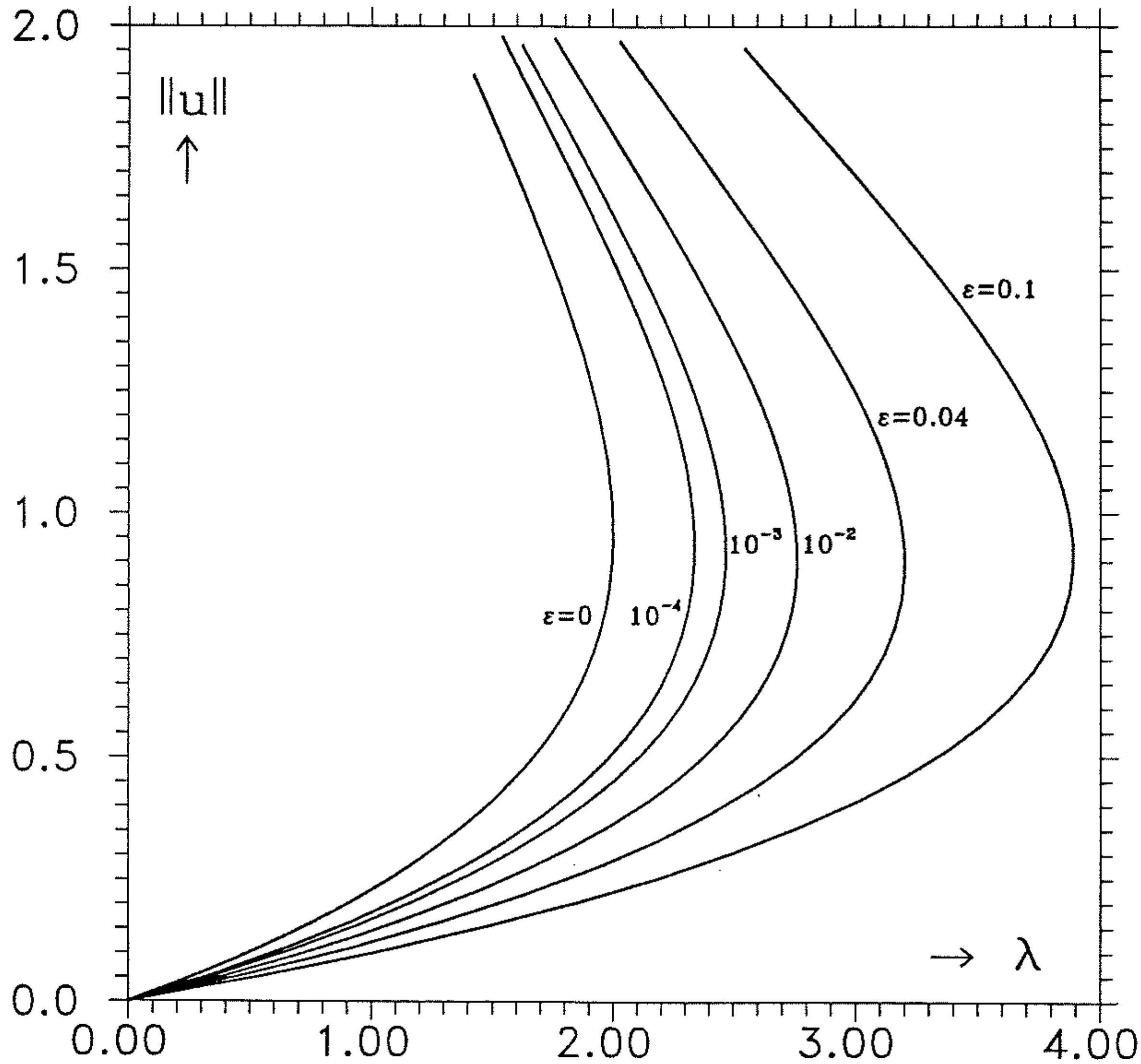
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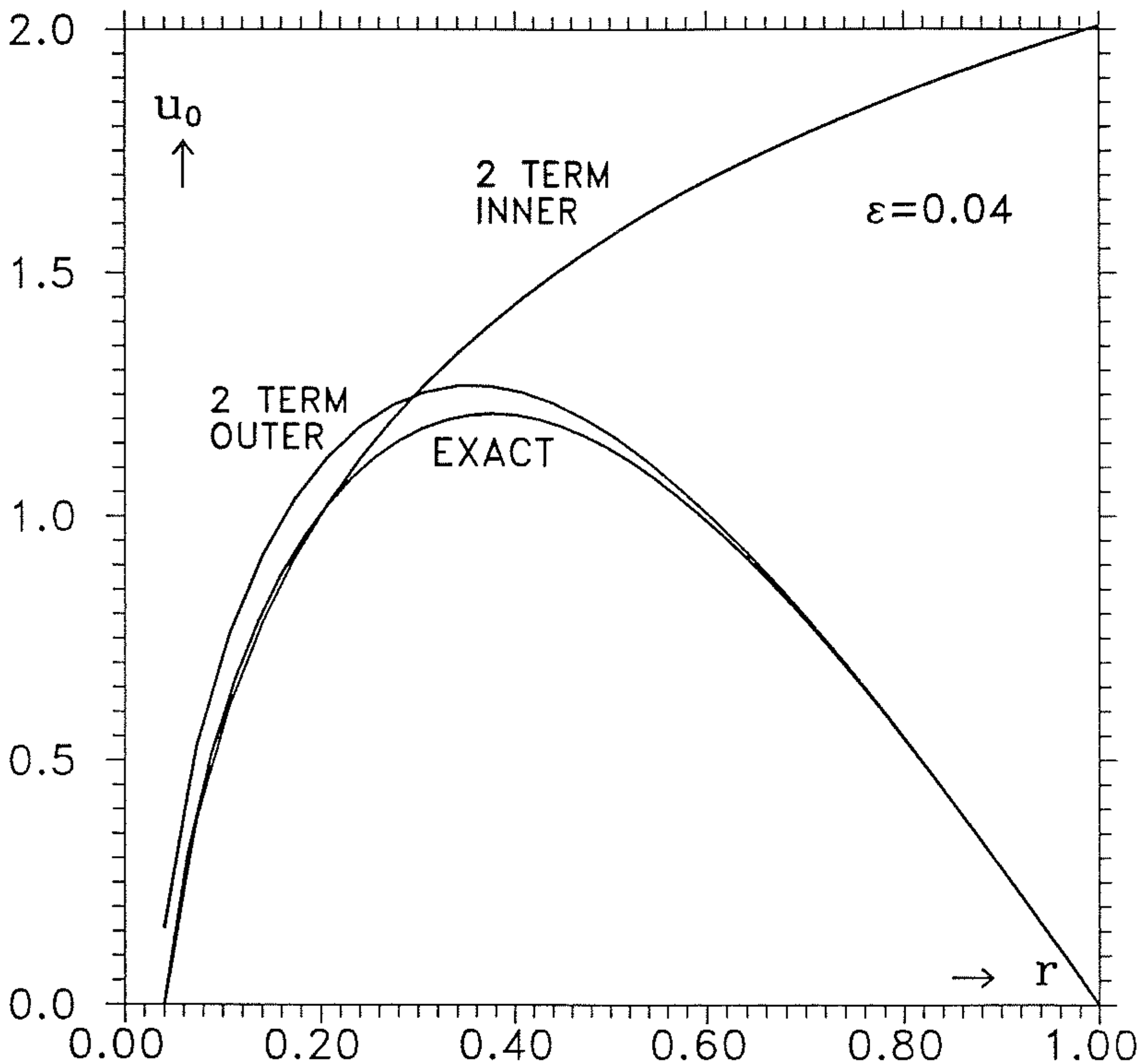
1.5
1.0
0.5
0.0
‖u‖ ↑
ε=0.1
ε=0.04
10^-3
10^-2
ε=0
10^-4
→ λ
0.00
1.00
2.00
3.00
4.00



2.0
1.5
1.0
0.5
0.0
0.00
0.20
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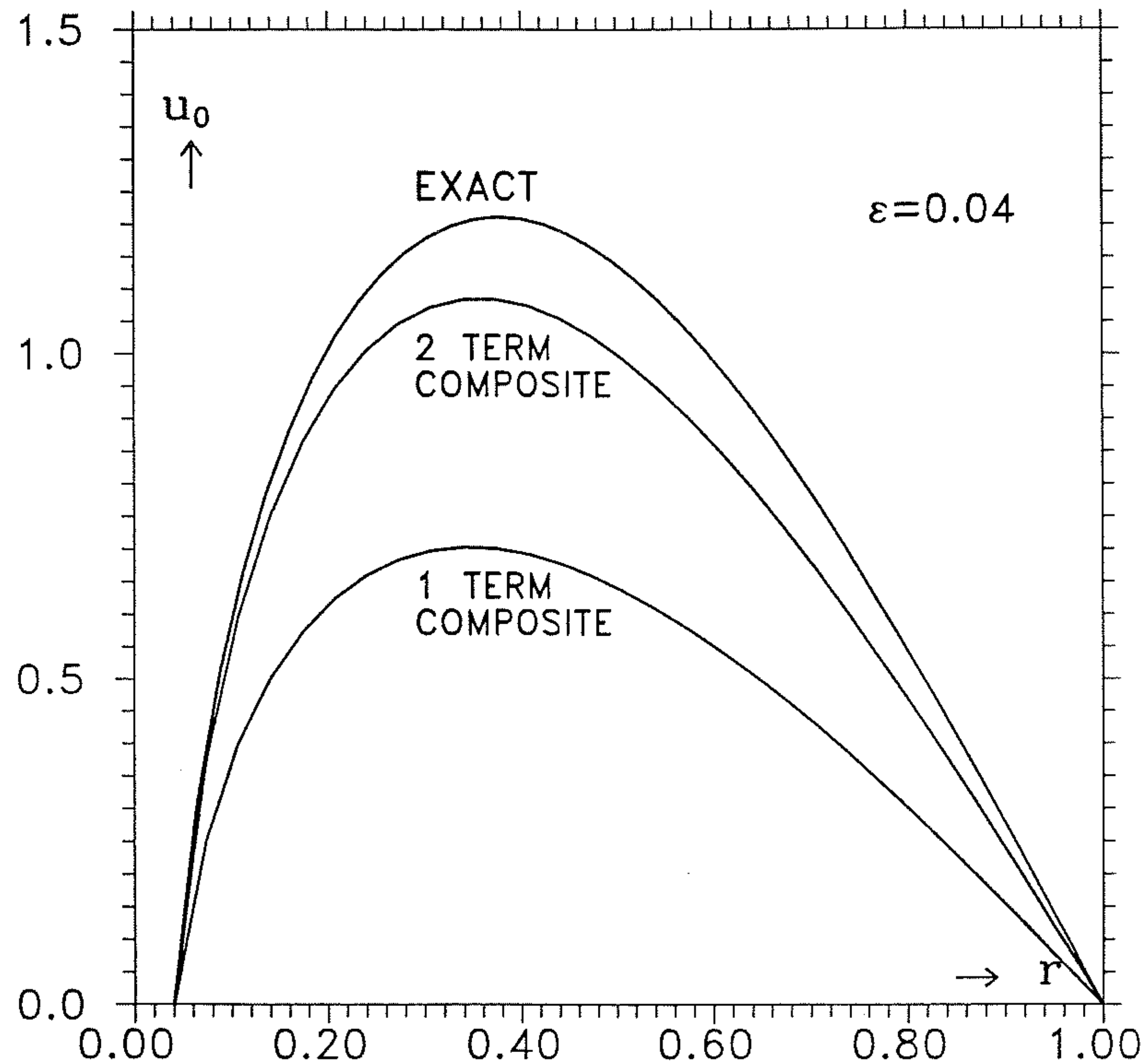
0.60
0.80
1.00
u_0 ↑
→ r
2 TERM INNER
2 TERM OUTER
EXACT
ε=0.04



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1.0
0.5
0.0
u0
EXACT
ε=0.04
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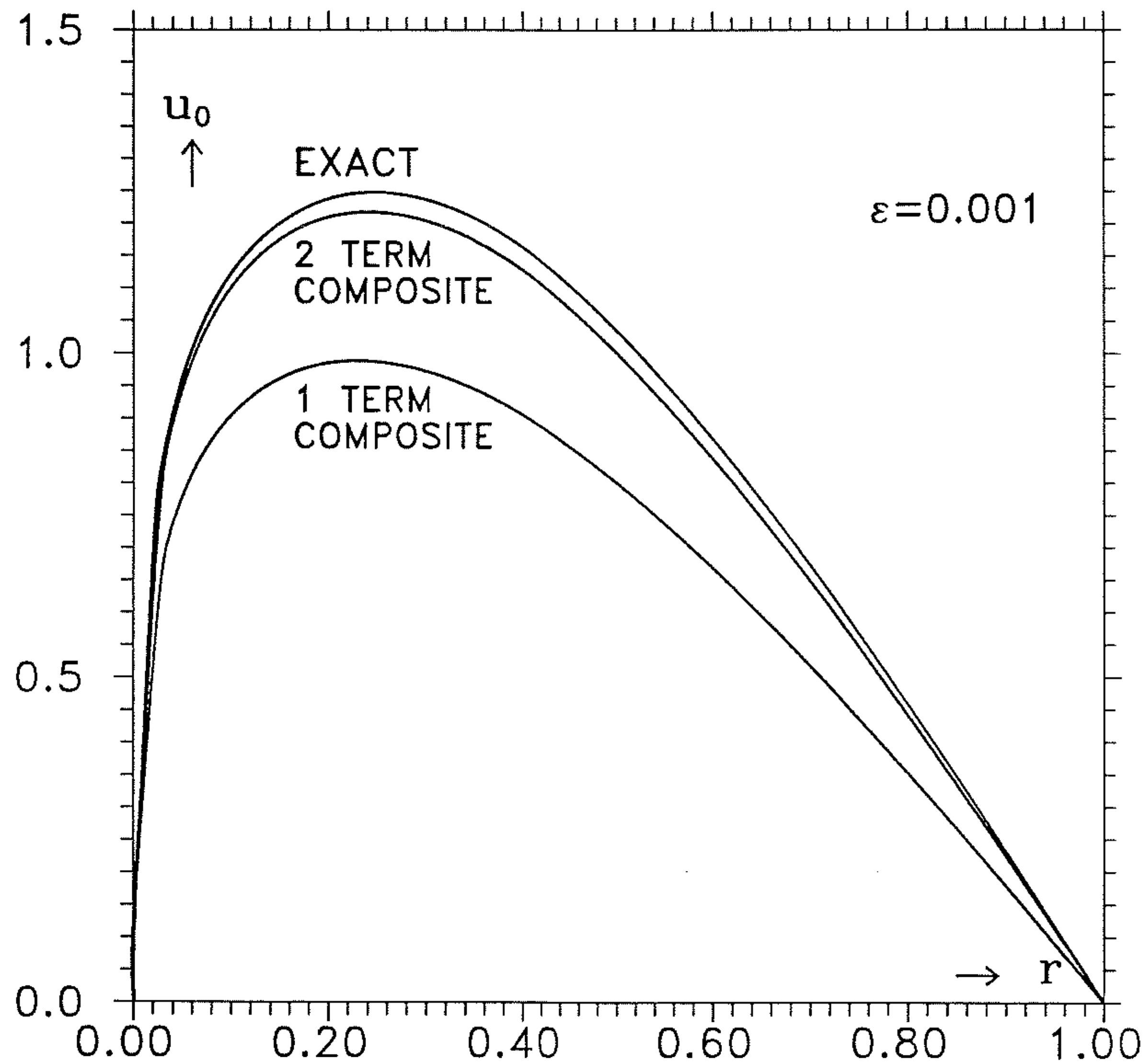
COMPOSITE
1 TERM
COMPOSITE
r
0.00
0.20
0.40
0.60
0.80
1.00



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0.5
0.0
0.00
0.20
0.40
0.60
0.80
1.00
u_0 ↑
EXACT
2 TERM COMPOSITE
1 TERM COMPOSITE
ε=0.001
→ r

Fig. 4

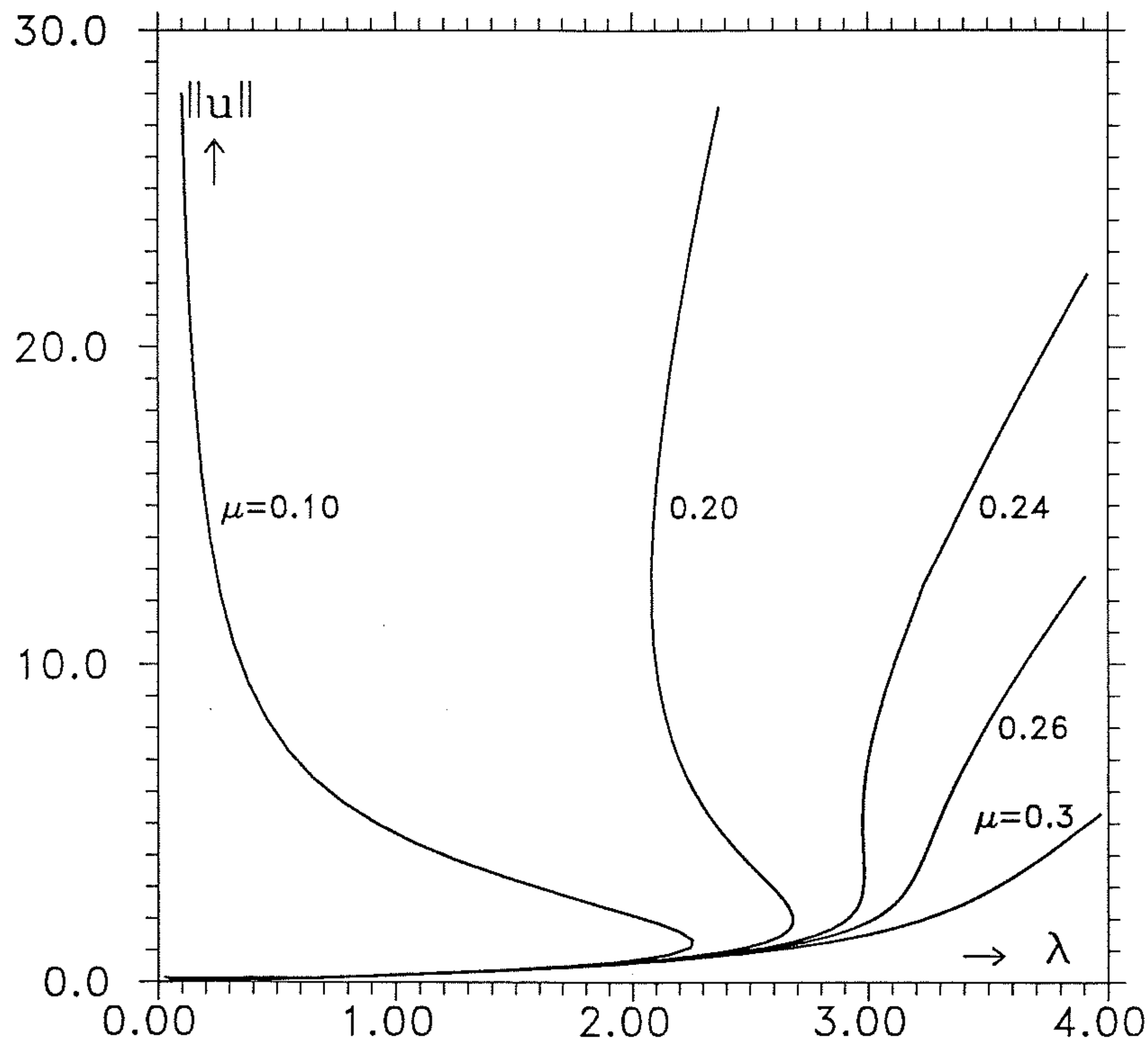
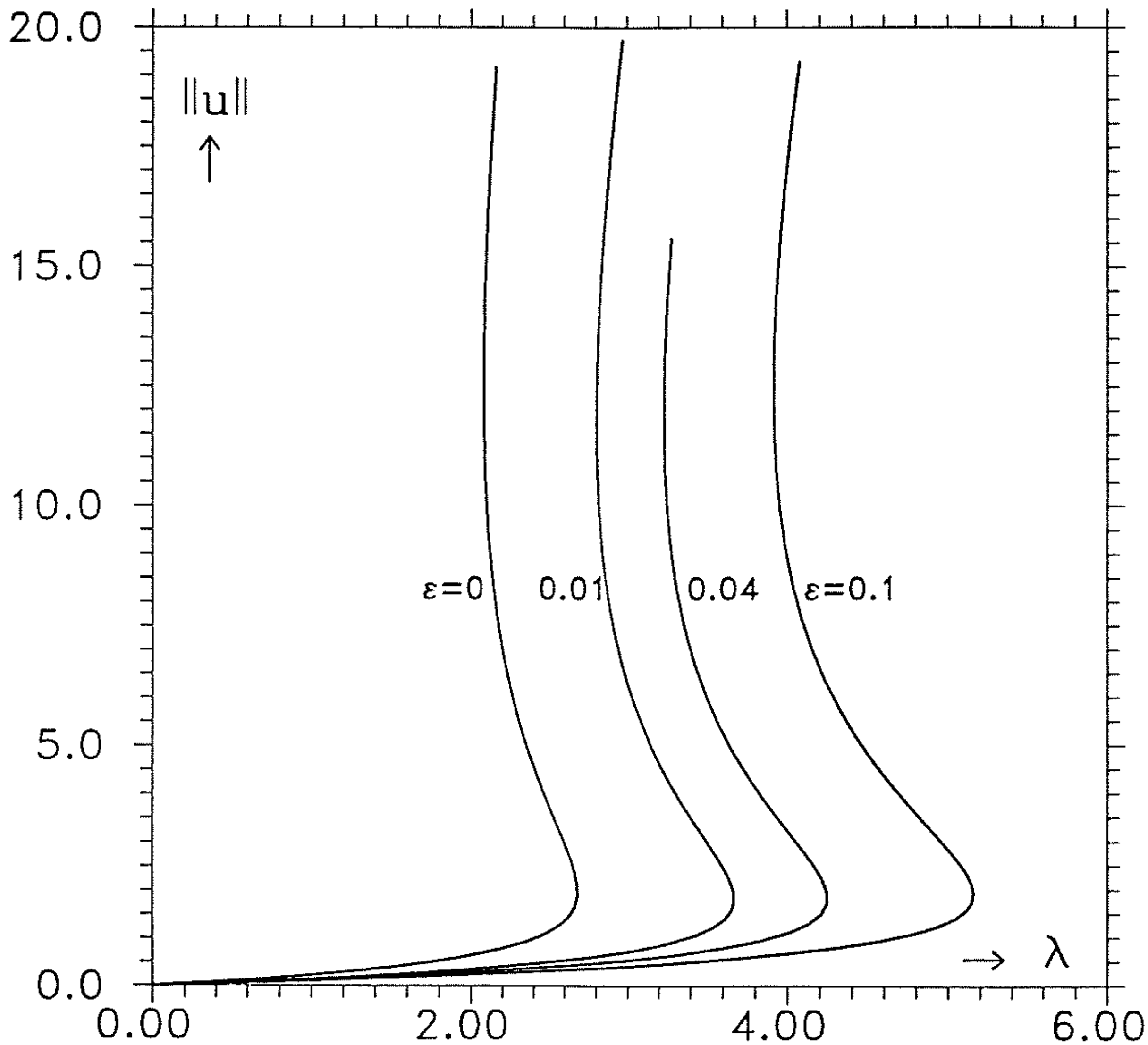


Fig. 5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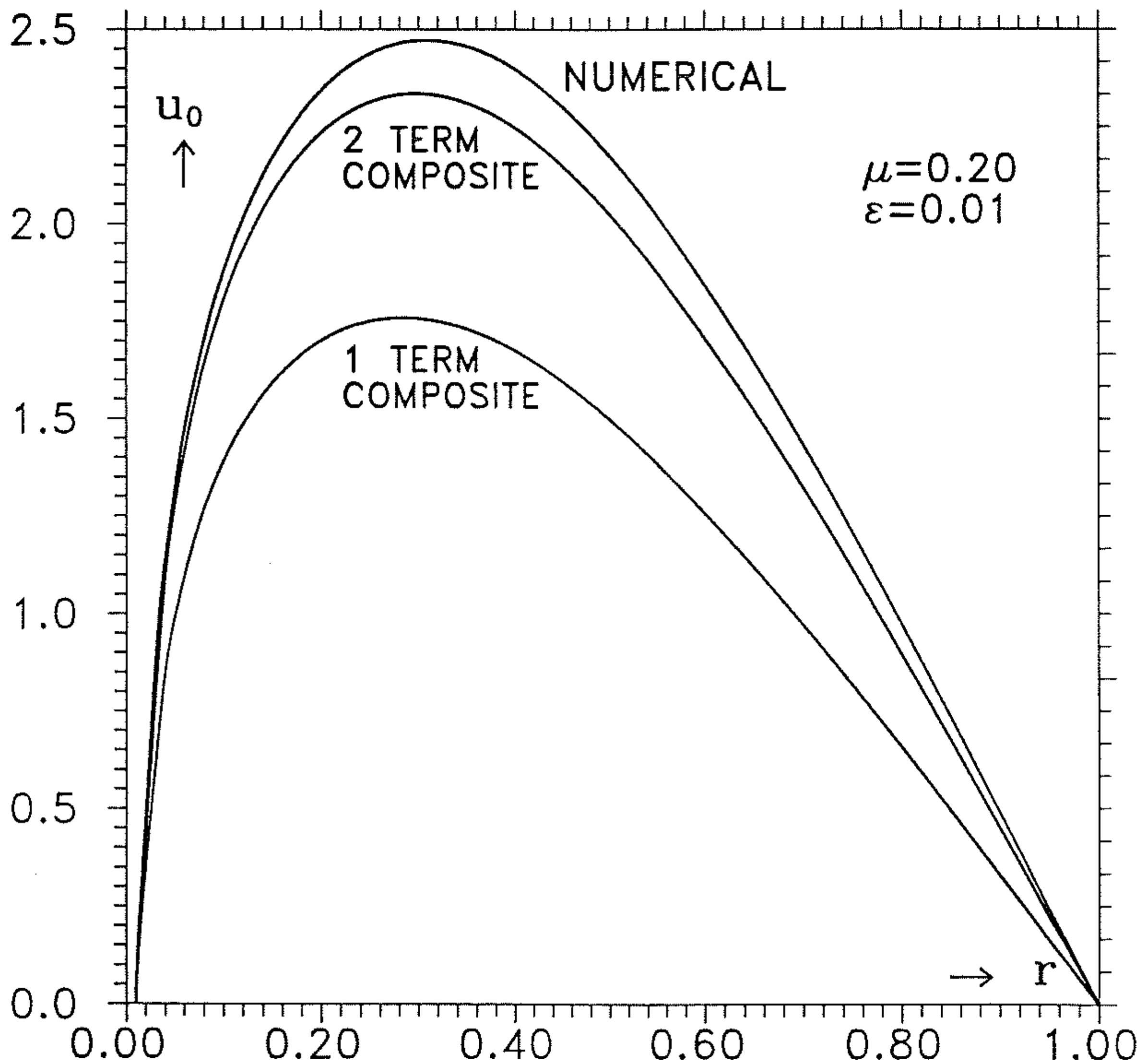
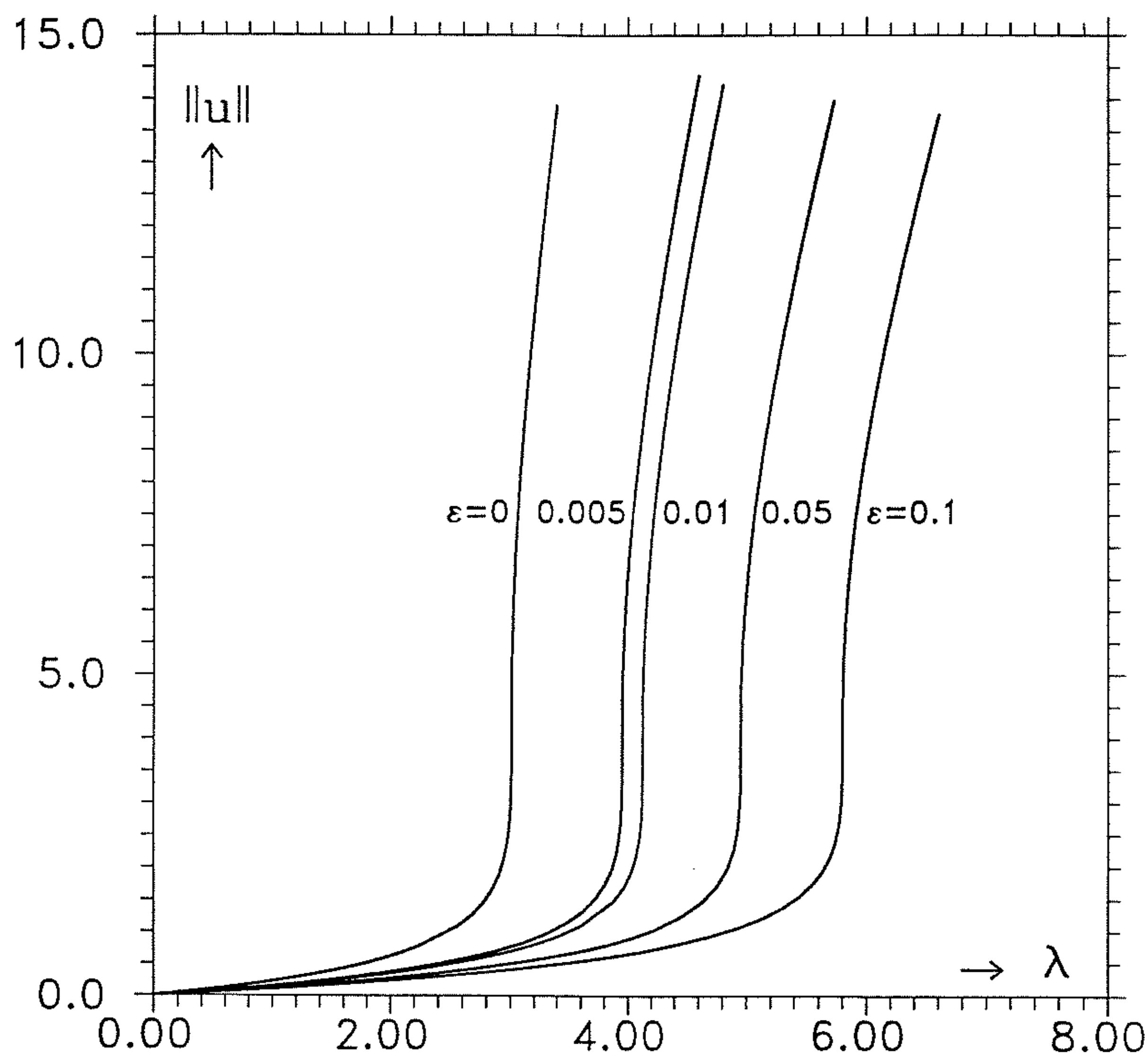
15.0
10.0
5.0
0.0
||u||
0.00
2.00
4.00
6.00
λ
ε=0
0.01
0.04
ε=0.1



2.5
2.0
1.5
1.0
0.5
0.0
0.00
0.20
0.40
0.60
0.80
1.00
u_0 ↑
NUMERICAL
2 TERM COMPOSITE
1 TERM COMPOSITE
μ=0.20
ε=0.01
→ r

Fig. 7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5.0
0.0
||u||
↑
ε=0
0.005
0.01
0.05
ε=0.1
→ λ
0.00
2.00
4.00
6.00
8.00