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Singular Perturbation Analysis of Integral Equations—Part II

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Abstract. Singularly perturbed linear Volterra or Fredholm integral equations with kernels possessing jump discontinuities in a derivative are discussed within the framework of [5]. An intriguing and remarkable feature of such equations is that in general the leading order outer solution does not satisfy the unperturbed integral equation. Moreover the solution usually exhibits large amplitude boundary layer behavior at one or both endpoints. Our perturbation technique, which is based on an efficient asymptotic splitting of the integral equation, clearly reveals the rich asymptotic solution structure for this class of equations.

Key words. Singular perturbations, integral equation, jump-order, asymptotic splitting

AMS (MOS) subject classifications. 45B05, 45D05, 45M05

Section 1. Introduction

Singular perturbation methods have played an important role in the study of differential equations. However, the potential usefulness of these methods for other types of functional equations involving small parameters has not received the same attention. In Part I (throughout this paper [5] is referred to as Part I) we initiated a program to further demonstrate that singular perturbation methods do indeed provide a powerful analytical tool for the study of integral and integrodifferential equations. In the present Part II we continue this program by focusing on an important class of scalar integral equations with kernels possessing jump discontinuities in a derivative, of the form

$$\int_0^1 K(x,s)w(s)ds = h(x,\epsilon) + \epsilon w(x), \quad 0 \le x \le 1 \tag{1.1}$$

or

$$Kw = h + \epsilon w \tag{1.2}$$

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for small positive values of ϵ. The forcing function h is assumed to be smooth and to have an asymptotic expansion in powers of ϵ, while the kernel $K = K(x,s)$ is assumed to have jump-order ν across the diagonal $x = s$ for some positive integer ν. That is, K can be given as

$$K(x,s) = \begin{cases} K^-(x,s) & \text{for } 0 \le s < x \le 1 \\ K^+(x,s) & \text{for } 1 \ge s > x \ge 0 \end{cases} \tag{1.3}$$

for smooth functions K^- and K^+, and the jumps J of $\partial^i K/\partial x^i$ satisfy

$$J\Big[\frac{\partial^i K}{\partial x^i}\Big](x) \equiv 0 \quad \text{for all} \quad 0 \le x \le 1 \quad \text{and for} \quad i = 0, 1, \cdots, \nu - 1, \tag{1.4}$$

while

$$J\Big[\frac{\partial^\nu K}{\partial x^\nu}\Big](x) \ne 0 \quad \text{for} \quad 0 \le x \le 1, \tag{1.5}$$

where the *jump operator* J is defined for any suitable function $f = f(x,s)$ by the formula

$$\begin{aligned} J[f](x) &:= f(x, x^-) - f(x, x^+) \\ &= \lim_{\substack{s \to x \\ s < x}} f(x,s) - \lim_{\substack{s \to x \\ s > x}} f(x,s). \end{aligned} \tag{1.6}$$

It is possible to recast a scalar equation (1.1) of jump-order $\nu > 0$ as a $(\nu + 1)$-dimensional vector integral equation of jump-order 0 (cf. Section 7). Vector equations of jump-order 0 were studied in Part I. However, we prefer to treat (1.1) without recasting as a vector equation because our perturbation method is more direct and efficient for a scalar equation. Our approach clearly reveals the important role played by the jump-order in determining the asymptotic structure of the solution, particularly the scalings and amplitudes in the boundary layers and the generic form of the appropriate outer equation. Scalar equations of positive jump-order occur prominently in applications. For example, singularly perturbed boundary value problems for ODE's such as the beam problem (1.39) are often reformulated as in (1.1) using the Green function for the principal part of the differential operator.

We now discuss briefly several elementary examples to introduce the ideas. A simple scalar example with jump-order 0 is

$$\int_0^x w(s)ds + 2\int_x^1 w(s)ds = e^{\beta x} + \epsilon w(x) \tag{1.7}$$

with $h(x) = e^{\beta x}$ and

$$K(x,s) = \begin{cases} 1 & \text{for } s < x \\ 2 & \text{for } s > x, \end{cases} \tag{1.8}$$

so that K has the jump $J[K] = -1$. The equation (1.7) always has a unique solution (for every choice of the parameter β and for every $\epsilon > 0$), and the solution satisfies (see (1.11) of Part I)

$$w(x) = w(x, \epsilon) \sim \frac{1}{\epsilon}(2e^{\beta} - 1)e^{-x/\epsilon} - \beta e^{\beta x} \tag{1.9}$$

as $\epsilon \to 0^{+}$. The first term on the right side of (1.9) represents a boundary-layer correction with $O(1/\epsilon)$ amplitude at $x = 0$, while the last term represents a slowly-varying outer solution $w^*(x, \epsilon)$ with

$$w^*(x, 0) = -\beta e^{\beta x}. \tag{1.10}$$

An intriguing and remarkable feature of the present class of singularly perturbed integral equations is that in general this lowest-order outer solution $w^*(x, 0)$ does *not* satisfy the corresponding reduced equation

$$\int_0^x w(s)ds + 2\int_x^1 w(s)ds = e^{\beta x} \tag{1.11}$$

obtained by putting $\epsilon = 0$ in (1.7). One checks directly here that $w^*(x, 0)$ satisfies

$$\int_0^x w(s)ds + 2\int_x^1 w(s)ds = e^{\beta x} + (1 - 2e^{\beta}), \tag{1.12}$$

which is in agreement with the appropriate outer equation provided by our technique. This latter equation coincides with (1.11) if and only if $\beta = ln\,\frac{1}{2}$ which is also the condition of solvability of (1.11). In this latter case ($\beta = ln\,\frac{1}{2}$) the solution of (1.7) is uniformly $O(1)$ as $\epsilon \to 0^{+}$ since the $O(1/\epsilon)$ term vanishes in the boundary layer (see (1.9)).

This example illustrates an important feature of scalar equations of jump-order 0 for which $J[K](x) \neq 0$ for $0 \leq x \leq 1$, namely, there is a layer only at one endpoint. The following examples illustrate the point that there are generally layers at both endpoints for equations of jump-order $\nu \geq 1$.

A scalar example with jump-order 1 is

$$\int_0^x xw(s)ds + \int_x^1 sw(s)ds = e^{\beta x} + \epsilon w(x) \tag{1.13}$$

with $h(x) = e^{\beta x}$ and

$$K(x, s) = \begin{cases} x & \text{for } s < x \\ s & \text{for } s > x. \end{cases} \tag{1.14}$$

This kernel is continuous on $[0, 1] \times [0, 1]$, but the first derivative $\partial K/\partial x$ exhibits a jump

$$J[K_x] = 1. \tag{1.15}$$

The integral equation (1.13) can be differentiated twice to obtain a second order differential equation which implies that any solution w must be of the form

$$w(x,\epsilon) = \frac{\beta^2}{1-\epsilon\beta^2}e^{\beta x} + c_1 e^{x/\sqrt{\epsilon}} + c_2 e^{-x/\sqrt{\epsilon}} \tag{1.16}$$

for suitable constants c_1, c_2 which are determined uniquely by inserting (1.16) into (1.13). The resulting solution is easily seen to satisfy

$$w(x,\epsilon) \sim \frac{\beta}{\sqrt{\epsilon}}e^{-x/\sqrt{\epsilon}} + \frac{(1-\beta)e^{\beta}}{\sqrt{\epsilon}}e^{-(1-x)/\sqrt{\epsilon}} + \frac{\beta^2}{1-\epsilon\beta^2}e^{\beta x} \tag{1.17}$$

as $\epsilon \to 0^+$. The first two terms on the right side of (1.17) represent boundary-layer corrections with $O(1/\sqrt{\epsilon})$ amplitudes near the respective endpoints $x=0$ and $x=1$, while the last term represents a slowly-varying outer solution $w^*(x,\epsilon)$ with

$$w^*(x,0) = \beta^2 e^{\beta x}. \tag{1.18}$$

A direct calculation shows that this $w^*(x,0)$ satisfies

$$\int_0^x xw(s)ds + \int_x^1 sw(s)ds = e^{\beta x} + (\beta-1)e^{\beta} - \beta x, \tag{1.19}$$

which will be seen in Section 2 to agree with the appropriate outer equation provided by our technique. Again (1.19) does not coincide with the following reduced equation obtained by putting $\epsilon = 0$ in (1.13),

$$\int_0^x xw(s)ds + \int_x^1 sw(s)ds = e^{\beta x}, \tag{1.20}$$

and indeed (1.20) fails to have a solution because the function $e^{\beta x}$ appearing on the right side of (1.20) is not in the range of the integral operator represented by the left side, and this is true for every β. The amplitude of the layer at $x=0$ is generally large, of order $1/\sqrt{\epsilon}$, except in the special case $\beta = 0$ when the layer-amplitude is $O(1)$. Note that it follows directly upon differentiation of (1.13) that w satisfies $w'(0,\epsilon) = -\beta/\epsilon$ which foretells the special nature of the case $\beta = 0$ for the layer-amplitude at $x = 0$. An analogous situation holds for the amplitude of the layer at the right endpoint $x = 1$ in the case $\beta = 1$. As an aside we note that if the forcing function $e^{\beta x}$ in (1.13) is replaced by a general differentiable function $h = h(x)$, then the conditions

$$h'(0) = 0 \quad \text{and} \quad h'(1) - h(1) = 0 \tag{1.21}$$

are necessary and sufficent for the unique solvability of the resulting reduced equation $\int_0^1 K(x,s)w(s)ds = h(x)$. This further explains the special nature of the cases $\beta = 0$ and $\beta = 1$ for the particular forcing term $h(x) = e^{\beta x}$. Note that the actual outer equation (1.19) has a forcing term $h^*(x) = e^{\beta x} + (\beta - 1)e^{\beta} - \beta x$ that satisfies the conditions of (1.21).

An example with jump-order 2 is

$$\int_0^x x\left(\frac{x}{2} - s\right) w(s)ds - \int_x^1 \frac{s^2}{2} w(s)ds = e^{\beta x} + \epsilon w(x) \tag{1.22}$$

with

$$K(x,s) = \begin{cases} x\left(\frac{x}{2} - s\right) & \text{for } s < x \\ -\frac{s^2}{2} & \text{for } s > x. \end{cases} \tag{1.23}$$

Both K and K_x are continuous on $[0,1] \times [0,1]$ but the second derivative has the jump

$$J\left[K_{xx}\right] = 1. \tag{1.24}$$

Equation (1.22) can be solved by reducing the integral equation to an associated third order differential equation. A routine calculation shows that the problem has a unique solution $w(x) = w(x,\epsilon)$ which satisfies

$$\begin{aligned} w(x,\epsilon) \sim & \left[\left(\frac{\beta}{\epsilon^{2/3}} + \frac{\beta^2}{\epsilon^{1/3}}\right) cos\, \frac{\sqrt{3}x}{2\epsilon^{1/3}} + \frac{1}{\sqrt{3}}\left(-\frac{\beta}{\epsilon^{2/3}} + \frac{\beta^2}{\epsilon^{1/3}}\right) sin\, \frac{\sqrt{3}x}{2\epsilon^{1/3}}\right] e^{-x/(2\epsilon^{1/3})} \\ & + \left[\frac{\left(-2 + 2\beta - \beta^2\right) e^{\beta}}{2\epsilon - 2\epsilon^{2/3} + \epsilon^{1/3}}\right] e^{(1-x)/\epsilon^{1/3}} + \beta^3 e^{\beta x} + O\left(\epsilon^{1/3}\right) \end{aligned} \tag{1.25}$$

as $\epsilon \to 0^+$. The amplitude of the layer near $x = 1$ is $O(1/\epsilon^{1/3})$, while the amplitude is $O(1/\epsilon^{2/3})$ near $x = 0$ where the solution is a linear combination of two exponentially decaying sinusoidal oscillations. Away from the endpoints the solution is well-approximated by the leading-order outer solution

$$w^*(x,0) = \beta^3 e^{\beta x} \tag{1.26}$$

which satisfies the equation

$$\int_0^x x\left(\frac{x}{2} - s\right) w(s)ds - \int_x^1 \frac{s^2}{2} w(s)ds = e^{\beta x} + \left(-1 + \beta - \frac{1}{2}\beta^2\right) e^{\beta} - \beta x - \frac{1}{2}\beta^2 x^2. \tag{1.27}$$

It will be clear from the discussion in Section 4 that (1.27) coincides with the proper outer equation provided by our technique.

The examples (1.7), (1.13), and (1.22) can all be solved directly by reduction to associated differential equations obtained by differentiation of the integral equations. Such reduction to a differential equation is generally not possible, as illustrated by (1.1) with such kernels as

$$K(x,s) = \begin{cases} 1/(1+x+s) & \text{for } s < x \\ 1/(1+2x) & \text{for } s > x, \end{cases} \tag{1.28}$$

and

$$K(x,s) = \begin{cases} 1/(1+xs-(1/2)s^2) & \text{for } s < x \\ 1/(1+(1/2)x^2) & \text{for } s > x. \end{cases} \tag{1.29}$$

The kernel (1.28) has jump-order 1 just as (1.14), while (1.29) has jump-order 2. In such cases as (1.28) and (1.29) we cannot obtain a simple differential equation from the integral equation (1.1) (cf. the integrodifferential equation (1.35)) and we do not have any simple exact expression for the solution of the integral equation in these cases. Moreover a direct numerical approach for the solution of such integral equations presents formidable difficulties in the case of small ϵ. We show below that the asymptotic techniques of Part I suffice for the study of a broad class of equations including integral equations with kernels such as (1.28) and (1.29). For such a class of equations we obtain existence and uniqueness of solutions of boundary-layer type and we obtain precise information on the resulting solutions. For the kernel (1.28) we find a solution of (1.1) exhibiting similar qualitative features as the explicitly known solution of (1.13). Similarly for (1.29) we find a solution exhibiting the properties of the solution of (1.22).

The width of the layers is generally $O(\epsilon)$ in the case of jump-order 0 as for (1.7)-(1.9), while the layers have the thicker width $O\left(\epsilon^{1/2}\right)$ for jump-order 1 as in (1.13)-(1.17). In the case of jump-order 2 the layers have the width $O\left(\epsilon^{1/3}\right)$ as occurs for (1.22)-(1.25). More generally the layers have width $O\left(\epsilon^{1/(\nu+1)}\right)$ for jump-order ν. Hence it is often convenient to replace $\epsilon = \epsilon_{old}$ in (1.1) by

$$\epsilon = \epsilon_{new} = \left(\epsilon_{old}\right)^{1/(\nu+1)}, \tag{1.30}$$

in which case, for example, (1.13) would be rewritten as (with $\nu = 1$)

$$\int_0^x xw(s)ds + \int_x^1 sw(s)ds = e^{\beta x} + \epsilon^2 w(x), \tag{1.31}$$

while (1.22) becomes

$$\int_0^x x\left(\frac{x}{2} - s\right) w(s)ds - \int_x^1 \frac{s^2}{2} w(s)ds = e^{\beta x} + \epsilon^3 w(x) \tag{1.32}$$

The general equation (1.1) or (1.2) with jump-order ν can be rewritten with (1.30) as

$$Kw = h + \epsilon^{\nu+1} w, \tag{1.33}$$

where we assume that $h = h(x, \epsilon)$ has an asymptotic expansion of the form

$$h(x, \epsilon) \sim \sum_{j=0}^{\infty} h_j(x)\epsilon^j \quad \text{for} \quad 0 \leq x \leq 1 \tag{1.34}$$

as $\epsilon \to 0^+$. (The case where the expansion for $h(x, \epsilon)$ involves a finite number of negative powers of ϵ can be reduced to (1.32)-(1.33) by multiplying the integral equation by a suitable power of ϵ and then relabeling or rescaling w.) The integral equation (1.33) can be differentiated $\nu + 1$ times to yield the integrodifferential equation (cf. (1.4)-(1.5))

$$\epsilon^{\nu+1}\frac{d^{\nu+1}w}{dx^{\nu+1}} - J\left[\frac{\partial^\nu K}{\partial x^\nu}\right](x)w = -h^{(\nu+1)}(x) + \int_0^1 \frac{\partial^{\nu+1}K(x,s)}{\partial x^{\nu+1}}\, w(s)\, ds. \tag{1.35}$$

We show that the leading-order layer-correction equation will be given by (1.35) with right side equal to zero and with the jump J evaluated on the left side at the appropriate endpoint (cf. (5.6), (5.10)). It will follow then that the boundary-layer width is generally $O(\epsilon)$ for the integral equation (1.33) of jump-order ν.

For simplicity we restrict consideration here to the special class of equations (1.33) (with jump-order ν) which satisfy the boundary-layer stability condition

$$|\text{Re}\ \mu(x)| \geq \kappa > 0 \tag{1.36}$$

for some fixed positive constant κ, uniformly for all $0 \leq x \leq 1$ and for all roots $\mu = \mu(x)$ of the equation

$$\mu^{\nu+1} = J\left[\partial^\nu K/\partial x^\nu\right](x). \tag{1.37}$$

In the vector case the right side of (1.37) is replaced by λ, and (1.36) must hold for all eigenvalues $\lambda = \lambda(x)$ of the jump matrix $J\left[\partial^\nu K/\partial x^\nu\right]$, for $0 \leq x \leq 1$. This condition (1.36) coincides with the condition of Part I in the case $\nu = 0$. For odd ν the condition (1.36) eliminates the case of purely oscillatory solutions as occurs, for example, for the kernel

$$K(x,s) = \begin{cases} s & \text{for } s < x \\ x & \text{for } s > x \end{cases} \tag{1.38}$$

which has jump-order 1 but with $J[K_x] = -1$ instead of $J[K_x] > 0$ as in the earlier examples (1.14) and (1.28). The uniformity in x of the condition (1.36) along with the assumed smoothness of the jump appearing on the right side of (1.37) also eliminate interior layers. The ideas and techniques of Part I can be more broadly applied, but we are able to obtain rigorous results in a clear and simple manner for a special class of equations satisfying the assumption (1.36). We shall discuss related problems involving interior layers in a separate work.

The general integral equation problem considered here includes important singularly perturbed boundary-value problems for differential equations that can be reformulated as integral equations of the type (1.1) with positive jump-order $\nu > 0$. For example the scalar boundary-value problem

$$\begin{gathered} \epsilon^4 y^{(4)} + b(x)y = f(x) \quad \text{for} \quad 0 < x < 1, \\ y(0) = \alpha_0,\ y'(0) = \alpha_1, \quad y(1) = \beta_0,\ y'(1) = \beta_1, \end{gathered} \tag{1.39}$$

which arises in linearized beam theory, can be replaced by an equivalent integral equation of the type (1.33) for $w = y$ with forcing function

$$\begin{aligned} h(x,\epsilon) := & -\epsilon^4 \left[[\alpha_0 + (2\alpha_0 + \alpha_1)x](1-x)^2 + [\beta_0 + (2\beta_0 - \beta_1)(1-x)]x^2 \right] \\ & + \int_0^1 G(x,s) f(s)\, ds, \end{aligned} \tag{1.40}$$

and with kernel

$$K(x,s) := G(x,s)b(s) \tag{1.41}$$

where $G = G(x,s)$ is the Green function for the differential operator d^4/dx^4 (the principal part of the operator on the left side of the differential equation of (1.39)) relative to the boundary conditions of (1.39),

$$G(x,s) := \begin{cases} G^-(x,s) = \frac{1}{6} s^2 (1-x)^2 \left[-3x + (1+2x)s \right] & \text{for} \quad s < x \\ G^+(x,s) = \frac{1}{6} x^2 (1-s)^2 \left[-3s + (1+2s)x \right] & \text{for} \quad s > x. \end{cases} \tag{1.42}$$

The terms multiplying ϵ^4 on the right side of (1.40) involving the boundary values can be written as $-\alpha_0 G_{sss}(x,0) + \alpha_1 G_{ss}(x,0) + \beta_0 G_{sss}(x,1) - \beta_1 G_{ss}(x,1)$. As function of s the Green function satisfies the differential equation $G_{ssss}(x,s) = 0$ for $s \neq x$ along with the homogeneous boundary conditions $G = G_s = 0$ for $s = 0$ and for $s = 1$, and it also satisfies the jump condition $J[G_{sss}](x) = +1$. The resulting kernel K of (1.41) satisfies $J[K] = J[K_x] = J[K_{xx}] = 0$ while the third derivative K_{xxx} possesses the nonzero jump $J[K_{xxx}] = -b(x)$, so the jump-order is 3 if the given function b is everywhere nonzero.

Olmstead & Angell [7] discuss several interesting examples of singularly perturbed integral equations of the type (1.1)-(1.3). However, their examples and resulting scalings for problems with layers at both endpoints correspond to very special cases, and the general structure of solutions for (1.1) is not clearly revealed. Their classification of (1.1) into either a *Case I* (where the layer amplitudes are $O(1/\epsilon)$) or *Case II* (where the layer amplitudes are of larger magnitude) has no inherent significance for the problem and can be misleading since the layer amplitudes are typically $o(1/\epsilon)$ for equations of positive jump-order. Moreover the crucial role played by the jump-order in the scalings and in the specific structure of the outer solution is not brought out in [7]. The asymptotic splitting procedure introduced in Part I and used here is simpler and more efficient for the present class of equations. We are able to provide some general insight into the widths and amplitudes of the layers. Moreover we give a generic equation for the outer solution, and we provide general criteria under which the integral equation (1.33) of jump-order ν has a solution of boundary-layer type.

Shubin [1] employs work of Èskin [2, 3] to study (1.1) for invertible operators K (with trivial null space) in what we call the standard case (cf. our discussion following (2.34)). Shubin obtains an existence theorem for (1.1), determines an asymptotic approximate solution to leading order, and obtains a corresponding error estimate in a certain Sobolev space. Our methods differ from those of [1]. We handle the standard and nonstandard cases, we include both invertible and noninvertible operators (the latter having nontrivial null spaces), and we obtain full asymptotic expansions for the solutions.

Our expansion procedure is introduced in Section 2 where it is noted that the formal reduced equation

$$Kw = h_0 \tag{1.43}$$

obtained by putting $\epsilon = 0$ in (1.33) is generally *not* the correct outer equation for such problems as considered here. The form of the correct outer equation is discussed briefly in Section 2 where it is seen that certain auxiliary terms must be added to (1.43) to provide the correct equation. Some of these additional terms involve $K^-(x, s)$ and certain of its derivatives at $s = 0$, while other of the additional terms involve $K^+(x, s)$ and certain of its derivatives at $s = 1$. And more generally there are yet further additional terms involving functions contained in the null space of K, as discussed in Section 6. The perturbation technique is illustrated in Section 3 and Section 4 for certain equations of respective jump-order 1 and 2, and then a general discussion is given in Section 5 for an important class of equations of jump-order ν for which the outer solution is of order unity. Related equations for which the outer solution becomes large for $\epsilon \to 0$ are discussed in Section 6 where the relationship of these results to problems for singularly perturbed differential equations is also discussed. Finally, questions of existence and uniqueness along with error estimates for (1.1) are discussed briefly in Section 7. Readers who are interested primarily in the examples can skip Section 2 and proceed directly to Section 3.

Section 2. Description of Perturbation Method

For equation (cf. (1.33))

$$Kw = h + \epsilon^{\nu+1} w \tag{2.1}$$

of jump-order ν we represent a boundary-layer solution $w(x) = w(x, \epsilon)$ in the form (see (3.3) of Part I)

$$w(x, \epsilon) \sim \widetilde{\phi}(\epsilon)\widetilde{w}(\widetilde{x}, \epsilon) + w^*(x, \epsilon) + \widehat{\phi}(\epsilon)\widehat{w}(\widehat{x}, \epsilon) \tag{2.2}$$

with boundary-layer variables

$$\widetilde{x} := \frac{x}{\epsilon} \quad \text{and} \quad \widehat{x} := \frac{1 - x}{\epsilon}, \tag{2.3}$$

where $w^* = w^*(x,\epsilon)$ is a suitable outer solution, $\widetilde{\phi}(\epsilon)\widetilde{w}(\widetilde{x},\epsilon)$ is a left boundary-layer correction of amplitude $\widetilde{\phi}(\epsilon)$ at the left endpoint $x = 0$, and $\widehat{\phi}(\epsilon)\widehat{w}(\widehat{x},\epsilon)$ is a right boundary-layer correction of amplitude $\widehat{\phi}(\epsilon)$ at the right endpoint $x = 1$. The boundary-layer correction functions are expected to decay for large values of their scaled variables,

$$\widetilde{w}(\widetilde{x},\epsilon) \to 0 \quad \text{as} \quad \widetilde{x} \to \infty, \quad \text{and} \quad \widehat{w}(\widehat{x},\epsilon) \to 0 \quad \text{as} \quad \widehat{x} \to \infty. \tag{2.4}$$

It is natural to inquire into the relationship, if any, between the outer solution w^* and the reduced equation

$$Kw = h_0 \tag{2.5}$$

obtained by putting $\epsilon = 0$ in (2.1), where $h_0 = h_0(x) = h(x,0)$ is the leading term in the expansion (1.34). Indeed a central challenge for this class of singularly perturbed integral equations is to determine the structure of the solution and to find an equation that characterizes the leading term in a suitable expansion of the outer solution. In fact the formal reduced equation (2.5) is generally not the correct outer equation which is given instead by a modified equation of the type (2.13) below. This is in contrast with many singularly perturbed problems for differential equations where the leading outer solution generally satisfies the reduced equation.

Existence and/or uniqueness can typically fail for a Fredholm integral equation of the first kind such as (2.5). Indeed existence will generally fail for typical problems of interest because the function h_0 need not be in the range of the integral operator K, as illustrated by the earlier examples (1.11) and (1.20). Uniqueness can fail for (2.5) because the reduced homogeneous equation

$$Kw = 0 \tag{2.6}$$

may have nontrivial solutions [7] as occur, for example, in analogous initial value problems and boundary value problems for *singular* singularly perturbed differential equations (cf. [6]). Such nonuniqueness is discussed below in Section 6. Here we focus principally on a class of *regular* (nonsingular) singularly perturbed integral equations for which the outer solution $w^*(x,\epsilon)$ of (2.2) is of order unity with an expansion of the type

$$w^*(x,\epsilon) \sim \sum_{j=0}^{\infty} w_j^*(x)\epsilon^j, \tag{2.7}$$

where (2.7) eliminates certain (singular) problems for which the expansion of the outer solution contains additional terms of $O(1/\epsilon^{\nu+1})$, as discussed in Section 6. Similarly the boundary-layer correction functions $\widetilde{w}$ and $\widehat{w}$ are assumed to have expansions of the type

$$\widetilde{w}(\widetilde{x},\epsilon) \sim \sum_{j=0}^{\infty} \widetilde{w}_j(\widetilde{x})\epsilon^j \quad \text{and} \quad \widehat{w}(\widehat{x},\epsilon) \sim \sum_{j=0}^{\infty} \widehat{w}_j(\widehat{x})\epsilon^j, \tag{2.8}$$

where the boundary-layer amplitudes $\widetilde{\phi}(\epsilon)$ and $\widehat{\phi}(\epsilon)$ can be chosen, without loss of generality, to achieve the normalizations

$$\widetilde{w}_0(\widetilde{x}) \not\equiv 0 \quad \text{and} \quad \widehat{w}_0(\widehat{x}) \not\equiv 0. \tag{2.9}$$

The following matching conditions are imposed on the boundary-layer coefficient functions (cf. (2.4))

$$\widetilde{w}_j(\widetilde{x}) \to 0 \quad \text{as} \quad \widetilde{x} \to \infty \quad \text{and} \quad \widehat{w}_j(\widehat{x}) \to 0 \quad \text{as} \quad \widehat{x} \to \infty \tag{2.10}$$

for $j = 0, 1, \cdots$.

As will be shown, the number of independent exponentially decaying solution-components of boundary-layer type is related to the roots of (1.37). The assumption (1.5) implies that the roots μ of (1.37) are distributed in the complex plane like the appropriate roots of (plus or minus) unity. If n and p denote the number of roots of (1.37) respectively with negative and positive real parts,

$$\left.\begin{matrix} n \\ p \end{matrix}\right\} := \text{number of roots of (1.37) with} \left\{\begin{matrix} \text{negative} \\ \text{positive} \end{matrix}\right. \text{real part,} \tag{2.11}$$

then (1.36) and the assumed smoothness of $J\left[\partial^\nu K/\partial x^\nu\right]$ imply that n and p have fixed values, the same for all $0 \leq x \leq 1$, given as follows:

$$\begin{array}{ll} & \begin{array}{cc} \scriptstyle J[\partial^\nu K/\partial x^\nu]>0 & \scriptstyle J[\partial^\nu K/\partial x^\nu]<0 \end{array} \\ \begin{array}{c} \nu \text{ even, with } \frac{1}{2}\nu \text{ even} \\ \\ \\ \nu \text{ even, with } \frac{1}{2}\nu \text{ odd} \\ \\ \\ \nu \text{ odd} \end{array} & \left(\begin{array}{cc} \begin{array}{c} n = \frac{1}{2}\nu \\ p = \frac{1}{2}\nu + 1 \end{array} & \begin{array}{c} n = \frac{1}{2}\nu + 1 \\ p = \frac{1}{2}\nu \end{array} \\ & \\ \begin{array}{c} n = \frac{1}{2}\nu + 1 \\ p = \frac{1}{2}\nu \end{array} & \begin{array}{c} n = \frac{1}{2}\nu \\ p = \frac{1}{2}\nu + 1 \end{array} \\ & \\ n = p = \frac{1}{2}(\nu+1) & n = p = \frac{1}{2}(\nu+1) \end{array}\right) \end{array} \tag{2.12}$$

The integers n and p give the respective numbers of independent exponentially decaying solution-components of boundary-layer type at the respective endpoints $x = 0$ and $x = 1$.

For the present class of problems our technique replaces the reduced equation (2.5) with the following modified equation for the leading outer solution $w_0^*(x)$,

$$\int_0^1 K(x,s) w_0^*(s) ds = h_0(x) - \sum_{k=1}^{N} c_k g_k(x) \tag{2.13}$$

for a suitable positive integer N, for suitable constants $c_1, c_2, \cdots, c_N$, and where $g_1(x), g_2(x), \cdots, g_N(x)$ are certain linearly independent functions depending on $K^-(x,s)$ and certain of its derivatives at $s=0$ along with $K^+(x,s)$ and certain of its derivatives at $s=1$. The terms involving the functions g_k on the right side of (2.13) are generated by integrals of the layer-correction terms. Specifically there will hold

$$\sum_{k=1}^{N} c_k g_k(x) = \underset{\epsilon\to 0+}{layer\ lim}\left[\epsilon\widetilde{\phi}(\epsilon)\int_0^{1/\epsilon} K^-(x,\epsilon\tilde{s})\widetilde{w}(\tilde{s},\epsilon)d\tilde{s} + \epsilon\widehat{\phi}(\epsilon)\int_0^{1/\epsilon} K^+(x,1-\epsilon\hat{s})\widehat{w}(\hat{s},\epsilon)d\hat{s}\right] \tag{2.14}$$

where *layer lim* denotes a certain *"layer-limit"* discussed in the following sections. This layer-limit determines the appropriate functions $g_k = g_k(x)$ for use in (2.13) and it also fixes the orders of the boundary-layer amplitudes $\widetilde{\phi}(\epsilon)$ and $\widehat{\phi}(\epsilon)$. The added terms on the right side of the modified outer equation (2.13) given by (2.14) acccount for the interplay between the outer solution w_0^* and the layer corrections, where our assumptions will imply exponential decay for the boundary-layer correction terms.

The number N of terms appearing in the summation on the right side of (2.13) is related to the codimension of the range of the operator K, and generally satisfies

$$N \geq \nu + 1 \tag{2.15}$$

for a kernel K of jump-order ν as characterized by (1.4)-(1.5). As discussed in Section 6, there generally holds $N > \nu + 1$ when K has a nontrivial null space. However for a wide class of equations one has

$$N = n + p = \nu + 1 \tag{2.16}$$

where n and p are the respective numbers of roots of (1.37) with negative and positive real parts as in (2.11)-(2.12). For example the integral equation (1.13) with jump-order 1 has the leading outer equation (1.19) which is of the form (2.13) with $N = 2 = \nu + 1$ ($n = p = 1$) and with the two functions

$$g_1(x) = x, \quad g_2(x) = 1, \tag{2.17}$$

and with constants $c_1 = -\beta, c_2 = (\beta - 1)\,e^{\beta}$. Similarly the equation (1.22) of jump-order 2 has the leading outer equation (1.27) of the form (2.13) with $N = 3 = \nu + 1$ ($n = 2$, $p = 1$), with functions

$$g_1(x) = x^2, \quad g_2(x) = x, \quad g_3(x) = 1, \tag{2.18}$$

and constants $c_1 = -\frac{1}{2}\beta^2, c_2 = -\beta, c_3 = \left(-1 + \beta - \frac{1}{2}\beta^2\right)e^{\beta}$.

It is not surprising that an important class of problems corresponds to the choice $N = \nu + 1$ in (2.13),

$$\int_0^1 K(x,s)w_0^*(s)ds = h_0(x) - \sum_{k=1}^{\nu+1} c_k g_k(x) \tag{2.19}$$

for a kernel of jump-order ν. Indeed for such a kernel, the equation (2.19) of first kind can be differentiated repeatedly $\nu+1$ times, and the constants c_k $(k = 1, 2, \cdots, \nu+1)$ can be eliminated from the resulting collection of differentiated equations to obtain an associated, more tractable equation of *second* kind (if the functions $g_1, \ldots, g_{\nu+1}$ are linearly independent). For example in the case $\nu = 1$, the equation (2.19) and its first differentiated equation provide the following system of 2 equations

$$\begin{aligned} \int_0^1 K(x,s)w_0^*(s)ds &= h_0(x) - [c_1 g_1(x) + c_2 g_2(x)] \\ \int_0^1 K_x(x,s)w_0^*(s)ds &= h_0'(x) - [c_1 g_1'(x) + c_2 g_2'(x)], \end{aligned} \tag{2.20}$$

where the last equation can be differentiated again to yield

$$J[K_x](x)w_0^*(x) + \int_0^1 K_{xx}(x,s)w_0^*(s)ds = h_0''(x) - [c_1 g_1''(x) + c_2 g_2''(x)] \tag{2.21}$$

with $J[K](x) \equiv 0$ and $J[K_x](x) \neq 0$ for jump-order 1. The system (2.20) can be solved for the two constants c_1, c_2 to give

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = W^{-1}(x) \left[\begin{pmatrix} h_0(x) \\ h_0'(x) \end{pmatrix} - \int_0^1 \begin{pmatrix} K(x,s) \\ K_x(x,s) \end{pmatrix} w_0^*(s)ds \right] \tag{2.22}$$

provided that the functions g_1, g_2 are linearly independent with invertible Wronskian matrix

$$W(x) \equiv W[g_1, g_2](x) = \begin{pmatrix} g_1(x) & g_2(x) \\ g_1'(x) & g_2'(x) \end{pmatrix} \tag{2.23}$$

for $0 \leq x \leq 1$. In this case (2.22) can be used to eliminate the constants in (2.21) and we find the equation of second kind,

$$J[K_x](x)w_0^*(x) + \int_0^1 \mathcal{K}(x,s)w_0^*(s)ds = h^*(x), \tag{2.24}$$

with modified kernel $\mathcal{K}$ and forcing function h^* given as

$$\mathcal{K}(x,s) := K_{xx}(x,s) - (\, g_1''(x), \quad g_2''(x)\,)\, W^{-1}(x) \begin{pmatrix} K(x,s) \\ K_x(x,s) \end{pmatrix} \tag{2.25}$$

and

$$h^*(x) := h_0''(x) - (\, g_1''(x), \quad g_2''(x)\,)\, W^{-1}(x) \begin{pmatrix} h_0(x) \\ h_0'(x) \end{pmatrix}. \tag{2.26}$$

One sees that the procedure is effective in reducing (2.19) to the associated second-kind equation (2.24) *provided that the functions g_1, g_2 are smooth and linearly independent.*

The resulting equation (2.24) of second kind will be solvable for an important class of problems, and one finds that a solution to (2.24) will also provide a solution to the equation (2.19) of first kind. Indeed from (2.24)-(2.26) there follows

$$\begin{aligned} J[K_x](x)w_0^*(x) &+ \int_0^1 K_{xx}(x,s)w_0^*(s)ds - h_0''(x) \\ &= (\, g_1''(x), \quad g_2''(x)\,)\, W^{-1}(x) \left[\int_0^1 \begin{pmatrix} K(x,s) \\ K_x(x,s) \end{pmatrix} w_0^*(s)ds - \begin{pmatrix} h_0(x) \\ h_0'(x) \end{pmatrix} \right], \end{aligned} \tag{2.27}$$

while the definition of W yields the matrix result $(\, g_1'(x), \quad g_2'(x)\,)\, W^{-1}(x) = (\, 0, \quad 1\,)$. Using these results along with the identity $(d/dx)W^{-1}(x) = -W^{-1}(x)W'(x)W^{-1}(x)$, one sees easily by a direct differentiation that the following vector function

$$W^{-1}(x) \left[\int_0^1 \begin{pmatrix} K(x,s) \\ K_x(x,s) \end{pmatrix} w_0^*(s)ds - \begin{pmatrix} h_0(x) \\ h_0'(x) \end{pmatrix} \right] \tag{2.28}$$

has a vanishing first derivative for $0 < x < 1$, so that this vector function must coincide with a constant vector, say

$$W^{-1}(x) \left[\int_0^1 \begin{pmatrix} K(x,s) \\ K_x(x,s) \end{pmatrix} w_0^*(s)ds - \begin{pmatrix} h_0(x) \\ h_0'(x) \end{pmatrix} \right] = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \quad \text{for} \quad 0 \le x \le 1 \tag{2.29}$$

for suitable constants c_1, c_2. Multiplying both sides of (2.29) on the left by $W(x)$, one sees that the first component of the resulting vector equation is

$$\int_0^1 K(x,s)w_0^*(s)ds - h_0(x) = c_1 g_1(x) + c_2 g_2(x), \tag{2.30}$$

which coincides with (2.19) in the present case $\nu = 1$.

This same procedure handles the equation (2.19) with general jump-order $\nu \ge 0$ if the functions $g_1, \ldots, g_{\nu+1}$ are of class $C^{\nu+2}$ and if these functions are linearly independent with invertible Wronskian matrix

$$W(x) \equiv W[g_1, g_2, \cdots, g_{\nu+1}](x) = \begin{pmatrix} g_1 & g_2 & \cdots & g_{\nu+1} \\ g_1' & g_2' & \cdots & g_{\nu+1}' \\ \vdots & \vdots & & \vdots \\ g_1^{(\nu)} & g_2^{(\nu)} & \cdots & g_{\nu+1}^{(\nu)} \end{pmatrix}(x). \tag{2.31}$$

For the case of jump-order ν one sees that equation (2.19) must be differentiated a total of ν times in order to obtain a $(\nu+1)$-dimensional linear system with coefficient matrix given by (2.31), analogous to (2.20). This latter system can be solved for the constants c_k (cf. (2.22)), and the resulting constants are then inserted into the corresponding second-kind equation obtained by differentiating (2.19) $\nu+1$ times using $J[\partial^\nu K/\partial x^\nu] \neq 0$. The procedure is effective in reducing (2.19) to a corresponding second-kind equation analogous to (2.24)-(2.26). Again one sees that any solution of this latter equation of second kind is also a solution of (2.19) for suitable constants c_k.

For the present class of equations we find a simplest case in which (2.16) holds and the layer-limit (2.14) produces the particular layer-amplitudes

$$\widetilde{\phi}(\epsilon) = \frac{1}{\epsilon^n} \quad \text{and} \quad \widehat{\phi}(\epsilon) = \frac{1}{\epsilon^p}, \tag{2.32}$$

where n and p are the integers introduced in (2.11), and with a resulting decomposition of the summation (2.14) as

$$\sum_{k=1}^{N} c_k g_k(x) = \sum_{k=1}^{n} c_k g_k(x) + \sum_{k=n+1}^{n+p} c_k g_k(x) \tag{2.33}$$

where the first n terms in the summation are associated with the left-layer at $x=0$ while the last p terms are associated with the right-layer at $x=1$. Indeed in this simplest case there holds

$$g_k(x) = \begin{cases} \left.\dfrac{\partial^{k-1} K^-(x,s)}{\partial s^{k-1}}\right|_{s=0} & \text{for } k=1,\cdots,n \\ \left.\dfrac{\partial^{k-n-1} K^+(x,s)}{\partial s^{k-n-1}}\right|_{s=1} & \text{for } k=n+1,\cdots,n+p. \end{cases} \tag{2.34}$$

Given our assumed smoothness on the data, this simplest case occurs when the $n+p=\nu+1$ functions of (2.34) are linearly independent. On the other hand when these functions are *not* linearly independent, the present technique still succeeds in handling such problems that actually possess solutions of boundary-layer type—but in this latter case the layer-limit (2.14) produces layer-amplitudes that are generally of larger orders than indicated by (2.32). The essential requirement is that the layer-limit (2.14) should produce an appropriate collection of independent functions for use in (2.13). Moreover the technique alerts us that a solution of boundary-layer type does *not* exist when such is the case (cf. Example 4.4) because the layer-limit will not be capable of providing a suitable collection of functions in such a case. These and related matters are illustrated with several examples in the following sections where it is convenient to refer to the simplest case as the *standard case*.

Note that it follows in the standard case from (2.12) and (2.32) that the order of either $\widetilde{\phi}(\epsilon)$ or $\widehat{\phi}(\epsilon)$ must exceed that of the other by 1 if the jump-order ν of the kernel is even—in this case the layer amplitude must be larger at one of the endpoints than at the other. In particular there will be a layer at only one endpoint for such an equation of jump-order 0.

The present development for the original equation (cf. (1.1))

$$\int_0^1 K(x,s)w(s)ds = h(x,\epsilon) + \epsilon w(x) \tag{2.35}$$

of jump-order ν is based on the assumption (cf. (1.36))

$$|\mathrm{Re}\ \mu(x)| \geq \kappa > 0 \tag{2.36}$$

for some fixed positive constant κ, uniformly for all $0 \leq x \leq 1$ and for all roots $\mu = \mu(x)$ of the equation (1.37). This assumption serves to eliminate interior layers and it also fixes the values of the boundary layer widths for (2.35) as (cf. (1.30), (2.3))

$$\text{layer widths} = O\left(\epsilon^{1/(\nu+1)}\right). \tag{2.37}$$

The procedure must be modified if the earlier jump condition (cf (1.5))

$$J\left[\frac{\partial^\nu K}{\partial x^\nu}\right](x) \neq 0 \quad \text{for} \quad 0 \leq x \leq 1, \tag{2.38}$$

is weakened to permit, say, isolated zeros for this jump. There is the possibilty of an interior layer at $x = x_0$ if this jump $J[\partial^\nu K/\partial x^\nu](x)$ vanishes at an interior point x_0. The equation (2.35) can be differentiated $\nu+1$ times to yield (cf. (1.35))

$$\epsilon\frac{d^{\nu+1}w}{dx^{\nu+1}} - J\left[\frac{\partial^\nu K}{\partial x^\nu}\right](x)w = -h^{(\nu+1)}(x) + \int_0^1 \frac{\partial^{\nu+1}K(x,s)}{\partial x^{\nu+1}}\,w(s)\,ds \tag{2.39}$$

where we are retaining the original small parameter here as in (1.1). If there is an isolated zero of order m at the interior point x_0, then we conjecture that the layer-correction equation at x_0 is obtained by differentiating (2.39) an additional m times. For example in the case of a simple zero ($m = 1$) with

$$J\left[\frac{\partial^\nu K}{\partial x^\nu}\right](x_0) = 0, \quad J\left[\frac{\partial^\nu K}{\partial x^\nu}\right]'(x_0) \neq 0, \tag{2.40}$$

and with (2.38) holding for $x \neq x_0$, then the layer-correction equation is conjectured to be given by differentiating (2.39) one more time to give

$$\begin{aligned}\epsilon\frac{d^{\nu+2}w}{dx^{\nu+2}} - J\left[\frac{\partial^\nu K}{\partial x^\nu}\right](x)w'(x) - \left[J\left[\frac{\partial^\nu K}{\partial x^\nu}\right]'(x) + J\left[\frac{\partial^{\nu+1} K}{\partial x^{\nu+1}}\right](x)\right]w \\ = -h^{(\nu+2)}(x) + \int_0^1 \frac{\partial^{\nu+2}K(x,s)}{\partial x^{\nu+2}}\,w(s)\,ds.\end{aligned} \tag{2.41}$$

If w is large in the layer, then the leading-order layer-correction equation will be given by (2.41) with right side equal to zero and with the jump and its derivatives evaluated on the left side with appropriate Taylor expansions about x_0. In such a case the layer width is $O\left(\epsilon^{1/(\nu+2)}\right)$ which is thicker than (2.37) in the standard case. Of course these comments apply equally well to the situation where the jump vanishes at an endpoint.

Such zeros of the jump at interior or boundary points can lead to considerable complications including exponential growth as illustrated in Section 2 of Part I. The case of an isolated zero of the jump at an interior point will be considered in a future work.

Section 3. Jump-Order One

In this section we illustrate our procedure for jump-order 1, first with an example of the standard case and then with several examples of nonstandard cases. For jump-order 1 (cf. (2.11)-(2.12))

$$n = p = 1, \tag{3.1}$$

and the standard case occurs when the two functions (cf. (2.34))

$$K^-(x,0), \quad K^+(x,1) \tag{3.2}$$

consisting of $n = 1$ function $K^-(s,0)$ at the left endpoint and $p = 1$ function $K^+(s,1)$ at the right endpoint are linearly independent. In such a standard case the leading-order outer equation is (cf. (2.13), (2.16))

$$\int_0^1 K(x,s)w(s)ds = h_0(x) - [c_1 g_1(x) + c_2 g_2(x)] \tag{3.3}$$

for suitable constants c_1, c_2, with the functions g_1, g_2 given as

$$g_1(x) = K^-(x,0), \quad g_2(x) = K^+(x,1), \tag{3.4}$$

and where the solution amplitudes $\widetilde{\phi}, \widehat{\phi}$ at the endpoints (cf. (2.2)-(2.3)) will be given in this case by (2.32) as

$$\widetilde{\phi}(\epsilon) = \widehat{\phi}(\epsilon) = \frac{1}{\epsilon}. \tag{3.5}$$

On the other hand, if one or both of the functions of (3.2) vanish, or if those two functions are otherwise linearly dependent, then the endpoint-amplitudes $\widetilde{\phi}, \widehat{\phi}$ are of larger order than indicated by (3.5).

Example 3.1: *A Standard Equation of Jump-Order 1.* To illustrate our perturbation technique we discuss in some detail the equation

$$\int_0^x \frac{w(s)}{1+x+s} ds + \int_x^1 \frac{w(s)}{1+2x} ds = h(x,\epsilon) + \epsilon^2 w(x), \quad 0 \le x \le 1, \tag{3.6}$$

which corresponds to equation (2.1) with kernel K given by (1.28),

$$K(x,s)=\begin{cases} K^-(x,s)=\frac{1}{1+x+s} & \text{for } s<x,\\ K^+(x,s)=\frac{1}{1+2x} & \text{for } s>x, \end{cases} \tag{3.7}$$

with $J[K](x)\equiv 0$ and $J\left[\partial K/\partial x\right](x)=1/(1+2x)^2\geq 1/9$, and where the two functions $g_1(x)=1/(1+x)$, $g_2(x)=1/(1+2x)$ of (3.4) are linearly independent. Note that there seems to be no way to reduce the integral equation (3.6) to any associated differential equation of any fixed order.

The decomposition (2.2)-(2.3) of w into the sum of an outer solution plus boundary-layer corrections is inserted into (3.6) and the resulting equation can be rewritten in the following form,

$$\begin{aligned} &h(x,\epsilon)+\epsilon^2 w^*(x,\epsilon)\\ &\quad=\int_0^1 K(x,s)w^*(s,\epsilon)ds+\widetilde{\phi}(\epsilon)\int_0^1\frac{\widetilde{w}(s/\epsilon,\epsilon)}{1+x+s}ds+\widehat{\phi}(\epsilon)\int_0^1\frac{\widehat{w}((1-s)/\epsilon,\epsilon)}{1+2x}ds\\ &\quad+\widetilde{L}(x,\epsilon)+\widehat{L}(x,\epsilon) \end{aligned} \tag{3.8}$$

where $\widetilde{L}$ and $\widehat{L}$ are decaying layer terms given as

$$\begin{aligned} \widetilde{L}(x,\epsilon)&:=\widetilde{\phi}(\epsilon)\left[-\epsilon^2\widetilde{w}(x/\epsilon,\epsilon)+\int_x^1\left[\frac{1}{1+2x}-\frac{1}{1+x+s}\right]\widetilde{w}(s/\epsilon,\epsilon)ds\right]\\ \widehat{L}(x,\epsilon)&:=-\widehat{\phi}(\epsilon)\left[\epsilon^2\widehat{w}((1-x)/\epsilon,\epsilon)+\int_0^x\left[\frac{1}{1+2x}-\frac{1}{1+x+s}\right]\widehat{w}((1-s)/\epsilon,\epsilon)ds\right], \end{aligned} \tag{3.9}$$

and where certain terms have been added and subtracted in (3.8)-(3.9) so as to obtain a form that allows an *asymptotic splitting* as discussed in the next paragraph. We anticipate that $\widetilde{L}(x,\epsilon)$ will decay exponentially as $\epsilon\to 0^+$ for fixed $0<x\leq 1$, while $\widehat{L}(x,\epsilon)$ is expected to decay similarly for fixed $0\leq x<1$. The formulas for $K^-(x,s)$ $(s<x)$ and $K^+(x,s)$ $(s>x)$ given by (3.7) have been taken to be extended naturally for all relevant values of x and s in (3.8) and (3.9). For other examples where it may not be as natural to extend K^- and K^+ as here, we employ the related splitting procedure involving suitable Taylor expansions described in Part I.

Based on our assumptions, the layer-correction terms are expected to exhibit exponential decay and so it is anticipated that the expression $\widetilde{L}(x,\epsilon)+\widehat{L}(x,\epsilon)$ is unimportant in (3.8) in the outer limiting process $\epsilon\to 0^+$ with fixed $0<x<1$. For this reason (3.8) prompts us to take the following as the *outer equation*,

$$\begin{aligned} &h(x,\epsilon)+\epsilon^2 w^*(x,\epsilon)\\ &\quad=\int_0^1 K(x,s)w^*(s,\epsilon)ds+\epsilon\left[\widetilde{\phi}(\epsilon)\int_0^{1/\epsilon}\frac{\widetilde{w}(\sigma,\epsilon)}{1+x+\epsilon\sigma}d\sigma+\widehat{\phi}(\epsilon)\int_0^{1/\epsilon}\frac{\widehat{w}(\sigma,\epsilon)}{1+2x}d\sigma\right], \end{aligned} \tag{3.10}$$

where the changes of integration variables $s = \epsilon\sigma$ and $1 - s = \epsilon\sigma$ have been made in the respective integrals involving $\widetilde{w}$ and $\widehat{w}$. It follows from (3.8) and (3.10) that the layer-sum $\widetilde{L}(x,\epsilon) + \widehat{L}(x,\epsilon)$ should vanish, where it is anticipated that $\widehat{L}$ is negligible near the left endpoint $x = 0$ in the limiting process $\epsilon \to 0^+$ with fixed $\widetilde{x} = x/\epsilon > 0$ (and with $\widehat{x} = \frac{1}{\epsilon} - \widetilde{x} \to \infty$) while $\widetilde{L}$ is similarly negligible near the right endpoint in the limiting process $\epsilon \to 0^+$ with fixed $\widehat{x} > 0$. Hence it is natural to split the null requirement on the layer-sum $\widetilde{L}(x,\epsilon) + \widehat{L}(x,\epsilon)$ into the two separate boundary-layer conditions $\widetilde{L} = 0$ and $\widehat{L} = 0$, or with (3.9),

$$\epsilon\widetilde{w}(\widetilde{x},\epsilon) = \int_{\widetilde{x}}^{\infty} \left[\frac{1}{1+2\epsilon\widetilde{x}} - \frac{1}{1+\epsilon\widetilde{x}+\epsilon\sigma}\right] \widetilde{w}(\sigma,\epsilon)d\sigma \tag{3.11}$$

and

$$\epsilon\widehat{w}(\widehat{x},\epsilon) = -\int_{\widehat{x}}^{\infty} \left[\frac{1}{3-2\epsilon\widehat{x}} - \frac{1}{3-\epsilon\widehat{x}-\epsilon\sigma}\right] \widehat{w}(\sigma,\epsilon)d\sigma, \tag{3.12}$$

where the common factors $\widetilde{\phi}$ and $\widehat{\phi}$ from (3.9) have been cancelled in (3.11) and (3.12) respectively, and all quantities have been rewritten in terms of the appropriate scaled variables $\widetilde{x}$ and $\widehat{x}$ of (2.3). Moreover it has been convenient to replace $1/\epsilon$ with ∞ in the upper limits of integration on the right sides of (3.11) and (3.12), where this replacement anticipates a suitable exponential decay for $\widetilde{w}$ and $\widehat{w}$ which will imply that the typical integrals $\int_{1/\epsilon}^{\infty}[\cdots]d\sigma$ involved are exponentially small and hence negligible.

We now indicate how the equations (3.10), (3.11) and (3.12) can be used to determine the boundary-layer amplitudes $\widetilde{\phi}(\epsilon), \widehat{\phi}(\epsilon)$ along with the coefficients $w_j^*(x), \widetilde{w}_j(\widetilde{x}), \widehat{w}_j(\widehat{x})$ in the solution expansions of (2.7)-(2.8), subject to the matching conditions of (2.10). Note that the anticipated exponential decay of the layer-corrections has allowed an *asymptotic splitting* of the equation (3.8) into three separate equations consisting of the outer equation (3.10) and the two boundary-layer equations (3.11) and (3.12), where these three equations are still coupled through the boundary-layer integrals appearing on the right side of the outer equation (3.10). It is this coupling which leads to a determination of the layer-amplitudes $\widetilde{\phi}$ and $\widehat{\phi}$.

We first differentiate (3.11) $\nu + 1 = 2$ times with respect to $\widetilde{x}$ to yield the integrodifferential equation

$$\widetilde{w}''(\widetilde{x},\epsilon) - \frac{1}{(1+2\epsilon\widetilde{x})^2}\widetilde{w}(\widetilde{x},\epsilon) = \epsilon\int_{\widetilde{x}}^{\infty} \left[\frac{8}{(1+2\epsilon\widetilde{x})^3} - \frac{2}{(1+\epsilon\widetilde{x}+\epsilon\sigma)^3}\right] \widetilde{w}(\sigma,\epsilon)d\sigma, \tag{3.13}$$

and a similar integrodifferential equation is obtained for $\widehat{w}$ upon differentiation of (3.12). The expansion for $\widetilde{w}$ from (2.8) is inserted into (3.13) and we find (using appropriate Taylor expansions for $(1+2\epsilon\widetilde{x})^{-2}, (1+2\epsilon\widetilde{x})^{-3}$, and $(1+\epsilon\widetilde{x}+\epsilon\sigma)^{-3}$)

$$\begin{aligned}&\sum_{j=0}^{\infty}\left[\widetilde{w}_j''(\widetilde{x}) - \sum_{k=0}^{j}(k+1)(-2\widetilde{x})^k\widetilde{w}_{j-k}(\widetilde{x})\right]\epsilon^j \\ &\sim \sum_{j=1}^{\infty}\left[\int_{\widetilde{x}}^{\infty}\sum_{k=0}^{j-1}(-1)^k(k+1)(k+2)\left[4(2\widetilde{x})^k - (\widetilde{x}+\sigma)^k\right]\widetilde{w}_{j-1-k}(\sigma)d\sigma\right]\epsilon^j,\end{aligned} \tag{3.14}$$

from which we obtain the equations

$$\widetilde{w}_0''(\widetilde{x}) - \widetilde{w}_0(\widetilde{x}) = 0 \quad \text{for} \quad \widetilde{x} > 0, \tag{3.15$_0$}$$

and

$$\begin{aligned} &\widetilde{w}_j''(\widetilde{x}) - \widetilde{w}_j(\widetilde{x}) \\ &\quad = \sum_{k=1}^{j} (-1)^k (k+1) \left[(2\widetilde{x})^k \widetilde{w}_{j-k}(\widetilde{x}) - k \int_{\widetilde{x}}^{\infty} \left[4(2\widetilde{x})^{k-1} - (\widetilde{x}+\sigma)^{k-1} \right] \widetilde{w}_{j-k}(\sigma) d\sigma \right] \end{aligned} \tag{3.15$_j$}$$

for $j = 1, 2, \cdots$. The general solution of $(3.15)_0$ is given as a linear combination of $e^{-\widetilde{x}}$ and $e^{+\widetilde{x}}$, but the appropriate matching condition from (2.10) eliminates $e^{+\widetilde{x}}$ and so our boundary-layer solution is

$$\widetilde{w}_0(\widetilde{x}) = a_0 e^{-\widetilde{x}} \tag{3.16$_0$}$$

for a suitable constant of integration a_0. The equation $(3.15)_1$ then becomes

$$\widetilde{w}_1''(\widetilde{x}) - \widetilde{w}_1(\widetilde{x}) = -4\widetilde{x}\widetilde{w}_0(\widetilde{x}) + 6 \int_{\widetilde{x}}^{\infty} \widetilde{w}_0(\sigma) d\sigma, \tag{3.17}$$

which can be solved subject to the matching condition of (2.10) to give

$$\widetilde{w}_1(\widetilde{x}) = \left[\left(\widetilde{x}^2 - 2\widetilde{x} \right) a_0 + a_1 \right] e^{-\widetilde{x}} \tag{3.16$_1$}$$

for a suitable integration constant a_1. Similarly we can construct as many of the layer-coefficients $\widetilde{w}_j(\widetilde{x})$ (for $j = 0, 1, \cdots$) as might be required for accuracy in the expansion (2.8) for the layer-correction $\widetilde{w}(\widetilde{x}, \epsilon)$ near $x = 0$, where each additional $\widetilde{w}_j$ introduces an additional constant of integration a_j, and where all of these layer-correction functions exhibit the expected exponential decay as $\widetilde{x} \to \infty$.

Beginning instead with the integrodifferential equation for $\widehat{w}$ analogous to (3.13) but obtained from (3.12), we similarly find equations for the layer-corrections at $x = 1$, such as

$$\begin{aligned} &\widehat{w}_0''(\widehat{x}) - \frac{1}{9}\widehat{w}_0(\widehat{x}) = 0 \quad \text{and} \\ &\widehat{w}_1''(\widehat{x}) - \frac{1}{9}\widehat{w}_1(\widehat{x}) = \frac{1}{27} \left[4\widehat{x}\widehat{w}_0(\widehat{x}) - 6 \int_{\widehat{x}}^{\infty} \widehat{w}_0(\sigma) d\sigma \right], \end{aligned} \tag{3.18}$$

and so forth. These equations can be solved with the appropriate matching conditions of (2.10), giving

$$\widehat{w}_0(\widehat{x}) = b_0 e^{-\widehat{x}/3}, \quad \widehat{w}_1(\widehat{x}) = \left[\frac{1}{9} \left(-\widehat{x}^2 + 6\widehat{x} \right) b_0 + b_1 \right] e^{-\widehat{x}/3} \tag{3.19}$$

for suitable integration constants b_0, b_1. The boundary-layer integration constants a_j, b_j and the boundary amplitudes $\widetilde{\phi}, \widehat{\phi}$ along with the outer solution will be determined now using the outer equation (3.10). Note that (2.9) implies the normalizations

$$a_0 \neq 0 \quad \text{and} \quad b_0 \neq 0. \tag{3.20}$$

The expansion (2.7) for w^* and the expansions of (2.8) for $\widetilde{w}$ and $\widehat{w}$ are inserted into (3.10) and we find (using the Taylor expansion for $(1+x+\epsilon\sigma)^{-1}$)

$$\begin{aligned}\sum_{j=0}^{\infty}\left[h_j(x) + w_{j-2}^*(x) - \int_0^1 K(x,s)w_j^*(s)ds\right]\epsilon^j \\ \sim \epsilon\widetilde{\phi}(\epsilon)\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\frac{1}{(1+x)^{k+1}}\int_0^{1/\epsilon}(-\sigma)^k\widetilde{w}_{j-k}(\sigma)d\sigma\right]\epsilon^j \\ +\epsilon\widehat{\phi}(\epsilon)\frac{1}{1+2x}\sum_{j=0}^{\infty}\left[\int_0^{1/\epsilon}\widehat{w}_j(\sigma)d\sigma\right]\epsilon^j,\end{aligned} \tag{3.21}$$

where we set $w_j^* \equiv 0$ for negative $j < 0$. Exhibiting the dominant terms in (3.21) we have

$$\begin{aligned}h_0(x) - \int_0^1 K(x,s)w_0^*(s)ds + \cdots \\ \sim \epsilon\widetilde{\phi}(\epsilon)\left[\frac{1}{1+x}\int_0^{1/\epsilon}\widetilde{w}_0(\sigma)d\sigma + \cdots\right] \\ + \epsilon\widehat{\phi}(\epsilon)\left[\frac{1}{1+2x}\int_0^{1/\epsilon}\widehat{w}_0(\sigma)d\sigma + \cdots\right],\end{aligned} \tag{3.22}$$

where we anticipate, upon letting $\epsilon \to 0$, that the right side should produce a linear combination of 2 independent functions as in (2.13) with $N = \nu + 1 = 2$ and with

$$g_1(x) = K^-(x,0) = \frac{1}{1+x} \quad \text{and} \quad g_2(x) = K^+(x,1) = \frac{1}{1+2x}. \tag{3.23}$$

If we try boundary-layer amplitudes of the form

$$\widetilde{\phi}(\epsilon) = \frac{1}{\epsilon^\alpha} \quad \text{and} \quad \widehat{\phi}(\epsilon) = \frac{1}{\epsilon^\beta} \tag{3.24}$$

for fixed constants α and β, then either of the choices $\alpha > 1$ or $\beta > 1$ is seen to be incompatible with the normalizations of (3.20), while either of the choices $\alpha < 1$ or $\beta < 1$ is seen to eliminate the occurrence of one or the other of the two possible functions of (3.23) resulting in either $c_1 = 0$ or $c_2 = 0$ in (2.13). The choices $\alpha = 1$ and $\beta = 1$ provide the maximum number of independent functions, resulting in the maximum flexibility. We anticipate that we shall need this flexibility and so we take $\alpha = \beta = 1$ with

$$\widetilde{\phi}(\epsilon) = \widehat{\phi}(\epsilon) = \frac{1}{\epsilon}, \tag{3.25}$$

yielding the layer-limit (cf. (2.14))

$$\begin{aligned} \underset{\epsilon\to 0+}{layer\ lim}\Bigg(\epsilon\widetilde{\phi}(\epsilon)\sum_{j=0}^{\infty}\left[\sum_{k=0}^{j}\frac{1}{(1+x)^{k+1}}\int_0^{1/\epsilon}(-\sigma)^k\widetilde{w}_{j-k}(\sigma)d\sigma\right]\epsilon^j \\ +\epsilon\widehat{\phi}(\epsilon)\frac{1}{1+2x}\sum_{j=0}^{\infty}\left[\int_0^{1/\epsilon}\widehat{w}_j(\sigma)d\sigma\right]\epsilon^j\Bigg) \\ =\frac{c_1}{1+x}+\frac{c_2}{1+2x} \end{aligned} \tag{3.26}$$

with

$$c_1=\int_0^{\infty}\widetilde{w}_0(\sigma)d\sigma \overset{(3.16)}{=} a_0 \quad\text{and}\quad c_2=\int_0^{\infty}\widehat{w}_0(\sigma)d\sigma \overset{(3.19)}{=} 3b_0. \tag{3.27}$$

Hence, upon letting $\epsilon\to 0^+$ in (3.22), we find with (3.25)-(3.27) the following leading-order outer equation,

$$\int_0^x \frac{w_0^*(s)}{1+x+s}ds+\int_x^1\frac{w_0^*(s)}{1+2x}ds=h_0(x)-\left[\frac{c_1}{1+x}+\frac{c_2}{1+2x}\right] \tag{3.28}$$

where the constants c_1 and c_2 can be evaluated directly from (3.28). Indeed if we multiply (3.28) by $1+2x$ and then differentiate with respect to x and set $x=0$ in the resulting equation, we find

$$c_1=h_0'(0)+2h_0(0), \tag{3.29}$$

while similarly if we multiply (3.28) by $1+x$ and differentiate and set $x=0$ we find

$$c_2=-\left[h_0'(0)+h_0(0)\right]-\int_0^1 w_0^*(s)ds. \tag{3.30}$$

These last two results can be used to eliminate c_1 and c_2 in (3.28), and we find the (first-kind Volterra) equation

$$\int_0^x\frac{x-s}{(1+2x)(1+x+s)}w_0^*(s)ds=h_0(x)-\frac{h_0'(0)+2h_0(0)}{1+x}+\frac{h_0'(0)+h_0(0)}{1+2x} \tag{3.31}$$

which uniquely determines $w_0^*(x)$ for $0\le x\le 1$. Indeed, two differentiations of (3.31) lead to a Volterra equation of second kind which is uniquely solvable. The equation (3.31), which is regular with no ϵ-dependency and which has a unique solution that is smooth, will generally be solved numerically for w_0^*. Then (3.27), (3.29) and (3.30) fix the values of c_1, c_2 and a_0, b_0, thereby completing the specification of the leading-order layer corrections $\widetilde{w}_0, \widehat{w}_0$ in (3.16) and (3.19).

The choices of (3.25) for the layer amplitudes can now be inserted back into (3.21) and we are led directly to the j^{th}-order outer equation

$$\int_0^1 K(x,s)w_j^*(s)ds = h_j^*(x) - \left[\frac{c_{j,1}}{1+x} + \frac{c_{j,2}}{1+2x}\right] \tag{3.32$_j$}$$

for $j = 1, 2, \cdots$, where the function h_j^* is defined as

$$h_j^*(x) := h_j(x) + w_{j-2}^*(x) - \sum_{k=1}^{j} \frac{1}{(1+x)^{k+1}} \int_0^\infty (-\sigma)^k \widetilde{w}_{j-k}(\sigma)d\sigma \tag{3.33$_j$}$$

and where the constants $c_{j,1}, c_{j,2}$ are given as

$$c_{j,1} := \int_0^\infty \widetilde{w}_j(\sigma)d\sigma \quad \text{and} \quad c_{j,2} := \int_0^\infty \widehat{w}_j(\sigma)d\sigma. \tag{3.34$_j$}$$

Note that the function h_j^* as given by $(3.33)_j$ is defined in terms of earlier coefficient functions $w_k^*, \widetilde{w}_k$ for $k < j$. Hence the above procedure that led to the determination of the leading-order terms $w_0^*, \widetilde{w}_0, \widehat{w}_0$ can be continued recursively to obtain further terms in the expansions of (2.7)-(2.8) if desired, using the appropriate equations given above for $j = 1, 2, \cdots$.

In particular we find for the solution of the original integral equation (3.6),

$$w(x,\epsilon) = \frac{1}{\epsilon}\left[c_1 + O(\epsilon)\right] e^{-x/\epsilon} + w_0^*(x) + O(\epsilon) + \frac{1}{\epsilon}\left[\frac{1}{3}c_2 + O(\epsilon)\right] e^{-(1-x)/(3\epsilon)} \tag{3.35}$$

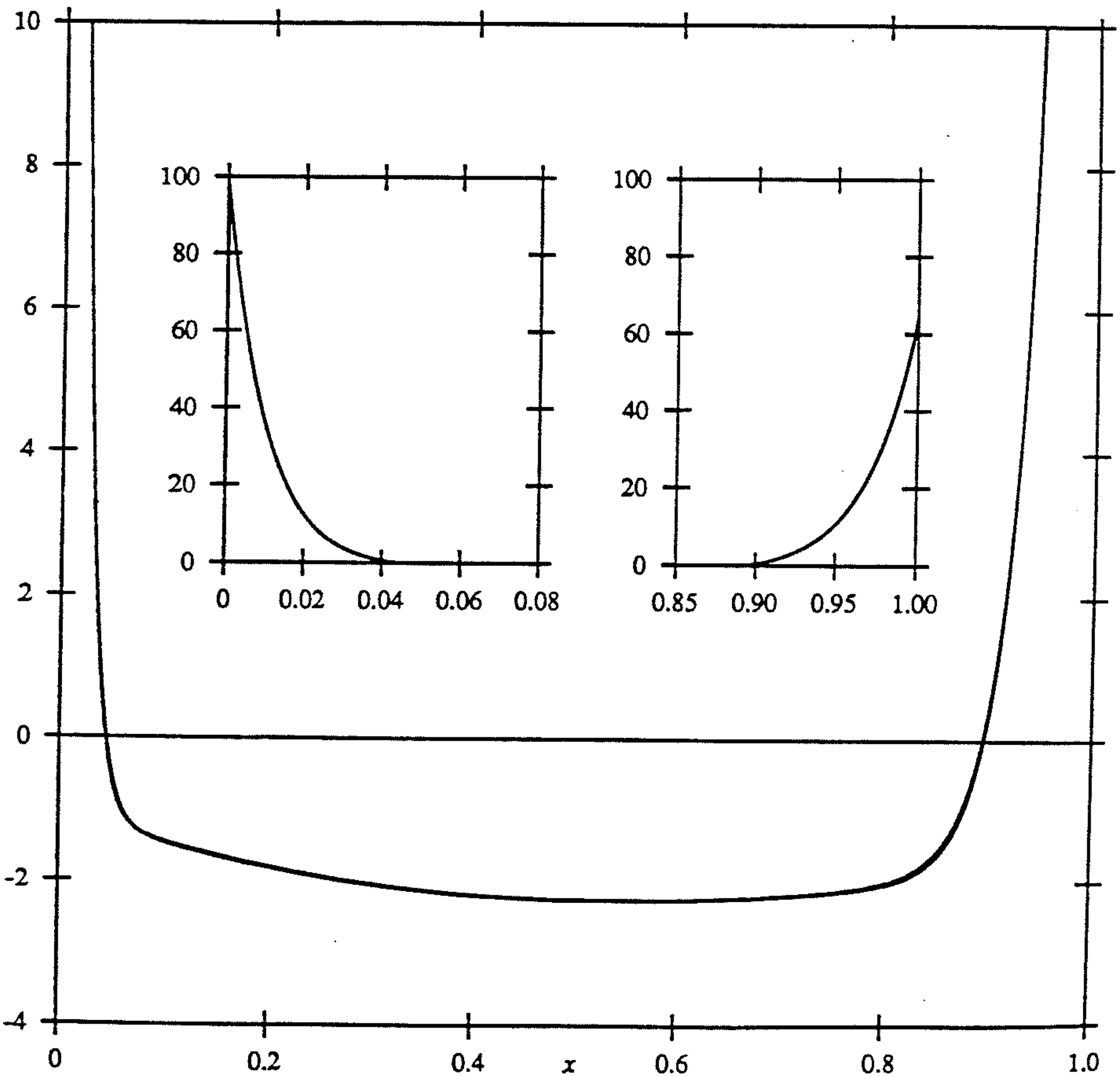
as $\epsilon \to 0^+$, uniformly for $0 \le x \le 1$, where $w_0^*(x)$ is determined by (3.31) and the constants c_1, c_2 are given by (3.29), (3.30). The graph of the solution is indicated in *Figure 1* for the case $\epsilon = 0.01$ and $h(x,\epsilon) = e^{-x}$ with $c_1 = 1, c_2 \doteq 1.9677$. The asymptotic solution (3.35) and the numerical solution obtained by a direct numerical solution of (3.6) differ by less than the precision of the graph. The solution values at the endpoints are found to be $w(0) \doteq 98.9$ and $w(1) \doteq 64.3$.

— Figure 1 Here —

Fig. 1. Solution of (3.6) with $h(x,\epsilon) = e^{-x}$ and $\epsilon = 0.01$.

We shall now consider several nonstandard examples for which at least one of the two functions $K^-(x,0), K^+(x,1)$ of (3.2) vanishes or for which these functions are otherwise linearly dependent. In such cases the layer-amplitudes $\widetilde{\phi}(\epsilon)$ and/or $\widehat{\phi}(\epsilon)$ are of larger order than indicated by (3.25) for the standard case.

Figure 1



10
8
6
4
2
0
-2
-4
0
0.2
0.4
x
0.6
0.8
1.0
100
80
60
40
20
0
0
0.02
0.04
0.06
0.08
100
80
60
40
20
0
0.85
0.90
0.95
1.00

Example 3.2: *A Nonstandard Equation* ($K^-(x,0)$ and $K^+(x,1)$ linearly dependent). The equation

$$-\int_0^x (x+s)w(s)ds - 2x\int_x^1 w(s)ds = h(x,\epsilon) + \epsilon^2 w(x) \quad \text{for} \quad 0 \le x \le 1, \tag{3.36}$$

corresponds to (2.1) with kernel

$$K(x,s) = \begin{cases} K^-(x,s) = -(x+s) & \text{for } s < x, \\ K^+(x,s) = -2x & \text{for } s > x, \end{cases} \tag{3.37}$$

with $J[K](x) \equiv 0$ and $J[K_x](x) \equiv 1$. This is a nonstandard case because of the linear dependence of $K^-(x,0) = -x$ and $K^+(x,1) = -2x$.

The representation (2.2)-(2.3) for the solution $w(x,\epsilon)$ as a sum of an outer solution plus boundary-layer corrections is inserted into (3.36), and the earlier procedure that led to (3.10)-(3.12) yields an asymptotic splitting of the problem into the outer equation

$$\begin{aligned} h(x,\epsilon) + \epsilon^2 w^*(x,\epsilon) &= \int_0^1 K(x,s)w^*(s,\epsilon)ds \\ &- \epsilon\left[\widetilde{\phi}(\epsilon)\int_0^{1/\epsilon}(x+\epsilon\sigma)\widetilde{w}(\sigma,\epsilon)d\sigma + 2\widehat{\phi}(\epsilon)x\int_0^{1/\epsilon}\widehat{w}(\sigma,\epsilon)d\sigma\right] \quad \text{for} \quad 0 \le x \le 1, \end{aligned} \tag{3.38}$$

along with the boundary-layer equation for $\widetilde{w}$,

$$\widetilde{w}(\widetilde{x},\epsilon) = -\int_{\widetilde{x}}^{\infty}(\widetilde{x}-\sigma)\widetilde{w}(\sigma,\epsilon)d\sigma \quad \text{for} \quad \widetilde{x} \ge 0, \tag{3.39}$$

and the same equation for $\widehat{w}(\widehat{x},\epsilon)$ with $\widetilde{x}$ and $\widetilde{w}$ replaced by $\widehat{x}$ and $\widehat{w}$ in (3.39). These boundary-layer equations are easily solved for decaying solutions to give

$$\widetilde{w}_j(\widetilde{x}) = a_j e^{-\widetilde{x}} \quad \text{and} \quad \widehat{w}_j(\widehat{x}) = b_j e^{-\widehat{x}} \tag{3.40}$$

for the coefficients of the layer expansions in (2.8), for suitable constants of integration a_j and b_j.

The expansions of (2.7)-(2.8) are inserted into the outer equation (3.38) and we find with (3.40),

$$\begin{aligned} &\sum_{j=0}^{\infty}\left[\int_0^1 K(x,s)w_j^*(s)ds - h_j(x) - w_{j-2}^*(x)\right]\epsilon^j \\ &\qquad \sim \sum_{j=0}^{\infty}\left[\epsilon\widetilde{\phi}(\epsilon)\left(xa_j + a_{j-1}\right) + 2\epsilon\widehat{\phi}(\epsilon)xb_j\right]\epsilon^j \end{aligned} \tag{3.41}$$

with $w_j^* \equiv 0$ for $j < 0$, $a_{-1} = 0$, and where exponentially small terms have been neglected on the right side. Exhibiting the dominant terms we have

$$\begin{aligned} -h_0(x) + \int_0^1 K(x,s) w_0^*(s) ds + \cdots \\ \sim \epsilon \left[\widetilde{\phi}(\epsilon) a_0 + 2\widehat{\phi}(\epsilon) b_0\right] x + \epsilon^2 \left[\widetilde{\phi}(\epsilon)\left(a_0 + a_1 x\right) + 2\widehat{\phi}(\epsilon) b_1 x\right] + \cdots, \end{aligned} \tag{3.42}$$

where a linear combination of 2 independent functions must be produced by the order-unity terms on the right side, upon letting $\epsilon \to 0$. It is not possible to obtain 2 such independent functions from the first term on the right side (because of the linear dependence of $K^-(x,0) = -x$ and $K^+(x,1) = -2x$) and so this term must be set to zero,

$$\widetilde{\phi}(\epsilon) a_0 + 2\widehat{\phi}(\epsilon) b_0 = 0. \tag{3.43}$$

Without loss we take (note that (2.9) implies $a_0 \neq 0, b_0 \neq 0$)

$$\widetilde{\phi}(\epsilon) = \widehat{\phi}(\epsilon), \tag{3.44}$$

and then (3.43) leads to the necessary condition

$$a_0 + 2b_0 = 0. \tag{3.45}$$

Using (3.44)-(3.45) back in (3.42) we are led first to the choice

$$\widetilde{\phi}(\epsilon) = \widehat{\phi}(\epsilon) = \frac{1}{\epsilon^2}, \tag{3.46}$$

and further we are led to the lowest order outer equation

$$\int_0^1 K(x,s) w_0^*(s) ds = h_0(x) - [c_1 + c_2 x] \tag{3.47}$$

with

$$c_1 = -a_0, \quad c_2 = -\left(a_1 + 2b_1\right). \tag{3.48}$$

The constant c_2 can be evaluated from (3.47) by differentiating the equation with respect to x and setting $x = 0$, and similarly c_1 can be evaluated by first multiplying the equation by x^{-1} and then differentiating with respect to x and setting $x = 1$. In this way we find

$$\begin{aligned} c_1 &= -h_0'(1) + h_0(1) + \int_0^1 s w_0^*(s) ds \\ c_2 &= h_0'(0) + 2\int_0^1 w_0^*(s) ds, \end{aligned} \tag{3.49}$$

which can be used to eliminate c_1 and c_2 in (3.47) to yield the equation

$$\int_0^x x w_0^*(s)ds + \int_x^1 s w_0^*(s)ds = h_0(x) + h_0'(1) - h_0(1) - h_0'(0)x. \tag{3.50}$$

This equation determines $w_0^*(x)$ uniquely as

$$w_0^*(x) = h_0''(x), \tag{3.51}$$

and now (3.49) and (3.51) yield the values

$$c_1 = h_0(0) \quad \text{and} \quad c_2 = 2h_0'(1) - h_0(0). \tag{3.52}$$

The constants a_0, b_0 are now determined by (3.45), (3.48) and (3.52) as

$$a_0 = -h_0(0) \quad \text{and} \quad b_0 = \frac{1}{2}h_0(0), \tag{3.53}$$

and (3.48) and (3.52) yield also the relation (cf. (3.45))

$$a_1 + 2b_1 = -2h_0'(1) + h_0(0) \tag{3.54}$$

which would be used at the next level in determining a_1, b_1.

We are content here to stop with the lowest order terms in the expansion for the solution of the integral equation (3.36), given as

$$\begin{aligned} w(x,\epsilon) = h_0''(x) + O(\epsilon) + \frac{1}{\epsilon^2}\left[-h_0(0) + O(\epsilon)\right] e^{-x/\epsilon} \\ + \frac{1}{\epsilon^2}\left[\frac{1}{2}h_0(0) + O(\epsilon)\right] e^{-(1-x)/\epsilon} \end{aligned} \tag{3.55}$$

as $\epsilon \to 0^+$, uniformly for $0 \le x \le 1$. Due to the nonstandard nature nature of this example, both boundary-layer amplitudes are larger, of order $1/\epsilon^2$ rather than order $1/\epsilon$ as in the standard case of jump-order 1. Note that the single condition $h_0(0) = 0$ would suffice to reduce the layer-amplitudes by an order at both $x = 0$ and $x = 1$.

The procedure can be continued to obtain further terms in the expansion, as allowed by the smoothness of the data. For example, proceeding through terms of third order, we find

$$\begin{aligned} w(x,\epsilon) = {} & h_0^{(2)}(x) + \epsilon h_1^{(2)}(x) + \epsilon^2\left[h_0^{(4)}(x) + h_2^{(2)}\right] + O\left(\epsilon^3\right) \\ & - \frac{1}{\epsilon^2}\left[h_0(0) + \epsilon h_1(0) + \epsilon^2\left(h_0^{(2)}(0) + h_2(0)\right) + O\left(\epsilon^3\right)\right] e^{-x/\epsilon} \\ & + \frac{1}{2\epsilon^2}\left[h_0(0) + \epsilon\left(h_0'(0) - 2h_0'(1) + h_1(0)\right) + \epsilon^2\left(h_0^{(2)}(0) + h_1'(0) - 2h_1'(1) + h_2(0)\right)\right. \\ & \qquad \left. + O\left(\epsilon^3\right)\right] e^{-(1-x)/\epsilon}. \end{aligned} \tag{3.56}$$

Example 3.3: *A Nonstandard Equation* ($K^-(x,0)=0$). As another example consider the equation

$$-\int_0^x sw(s)ds - \int_x^1 xw(s)ds = h(x,\epsilon) + \epsilon^2 w(x) \quad \text{for} \quad 0 \le x \le 1, \tag{3.57}$$

which is (2.1) with kernel

$$K(x,s) = \begin{cases} K^-(x,s) = -s & \text{for } s < x \\ K^+(x,s) = -x & \text{for } s > x, \end{cases} \tag{3.58}$$

with $J[K] \equiv 0$ and $J[K_x] \equiv 1$. This is a nonstandard case because of the vanishing of $K^-(x,0)=0$. The layer amplitude at $x=1$ will follow that of the standard case, while the amplitude will be larger at $x=0$. In this case the asymptotic splitting produces the outer equation

$$\begin{aligned} h(x,\epsilon) + \epsilon^2 w^*(x,\epsilon) = &\int_0^1 K(x,s) w^*(s,\epsilon) ds \\ &- \epsilon \left[\epsilon \widetilde{\phi}(\epsilon) \int_0^{1/\epsilon} \sigma \widetilde{w}(\sigma,\epsilon) d\sigma + \widehat{\phi}(\epsilon) x \int_0^{1/\epsilon} \widehat{w}(\sigma,\epsilon) d\sigma \right] \end{aligned} \tag{3.59}$$

along with the same boundary-layer equations as in example (3.36) with the boundary-layer coefficients $\widetilde{w}_j$ and $\widehat{w}_j$ given again by (3.40). Note the extra factor of ϵ appearing with the $\widetilde{\phi}$-term in (3.59) generated by the multiplicative factor of s in $K^-(x,s)$.

Inserting the expansions of (2.7)-(2.8) into (3.59) we find with (3.40),

$$\begin{aligned} &\left[-h_0(x) + \int_0^1 K(x,s) w_0^*(s) ds\right] + \epsilon\left[-h_1(x) + \int_0^1 K(x,s) w_1^*(s) ds\right] + \cdots \\ &\quad \sim \epsilon^2 \widetilde{\phi}(\epsilon) \left[a_0 + a_1\epsilon + \cdots\right] + \epsilon \widehat{\phi}(\epsilon) x \left[b_0 + b_1 \epsilon + \cdots\right], \end{aligned} \tag{3.60}$$

where the order-unity terms on the right side will produce a linear combination of 2 independent functions with the selections

$$\widetilde{\phi}(\epsilon) = \frac{1}{\epsilon^2} \quad \text{and} \quad \widehat{\phi}(\epsilon) = \frac{1}{\epsilon}. \tag{3.61}$$

The procedure then goes as expected. The solution has the structure

$$w(x,\epsilon) = \frac{1}{\epsilon^2}\left[-h_0(0) + O(\epsilon)\right] e^{-x/\epsilon} + \frac{1}{\epsilon}\left[-h_0'(1) + O(\epsilon)\right] e^{-(1-x)/\epsilon} + h_0''(x) + O(\epsilon) \tag{3.62}$$

with $w_0^*(x) = h_0''(x)$ and $a_0 = -h_0(0)$, $b_0 = -h_0'(1)$; details are omitted.

Example 3.4: *A Nonstandard Equation* (rapidly oscillating solution). As a final example in this section consider the equation

$$\int_0^x sw(s)ds + \int_x^1 xw(s)ds = h(x,\epsilon) + \epsilon^2 w(x) \tag{3.63}$$

which is (2.1) with kernel (= negative of (3.58); cf. (1.38))

$$K(x,s) = \begin{cases} K^-(x,s) = s & \text{for } s < x \\ K^+(x,s) = x & \text{for } s > x, \end{cases} \tag{3.64}$$

with $J[K] \equiv 0$ and

$$J\left[K_x\right](x) \equiv -1. \tag{3.65}$$

The formal procedure analogous to (3.8) and (3.10)-(3.12) can be applied to this problem, and the resulting "boundary-layer" differential equation at $x = 0$ is seen to be (cf. $(3.15)_0$)

$$\widetilde{w}_0''(\widetilde{x}) + \widetilde{w}_0(\widetilde{x}) = 0 \quad \text{for} \quad \widetilde{x} > 0, \tag{3.66}$$

with general solution (with $\widetilde{x} = x/\epsilon$)

$$\widetilde{w}_0(x/\epsilon) = a_0 \, sin\,(x/\epsilon) + b_0 \, cos\,(x/\epsilon). \tag{3.67}$$

Hence $\widetilde{w}_0$ leads to a rapid oscillation with no decay, and in fact the original problem (3.63) is seen to be equivalent to the boundary-value problem

$$\begin{gathered} \epsilon^2 w''(x,\epsilon) + w(x,\epsilon) = -\,h''(x,\epsilon) \\ w(0,\epsilon) = -\frac{h''(0,\epsilon)}{\epsilon^2},\ w'(1,\epsilon) = -\frac{h'(1,\epsilon)}{\epsilon^2}, \end{gathered} \tag{3.68}$$

whose solution oscillates rapidly as $\epsilon \to 0^+$. Moreover the solution is unbounded for the values

$$\epsilon = \frac{1}{\left(\left(n+\frac{1}{2}\right)\pi\right)^2}, \quad n = 0,1,2,\cdots \tag{3.69}$$

which are eigenvalues of the homogeneous boundary-value problem. The problem is not of boundary-layer type, and the assumption (1.36)-(1.37) excludes such oscillatory problems here.

Section 4. Jump-Order Two

In this section we illustrate our procedure for jump-order 2, first with an example of the standard case and then with several examples of nonstandard cases. The jump-order 2 case is the first instance in which oscillatory behavior can occur in a layer region. Moreover the layer-limit procedure for the outer equation is more subtle than for the jump-order 1 case.

For jump-order 2 either $n=2, p=1$ or $n=1, p=2$ depending on whether $J[K_{xx}]>0$ or $J[K_{xx}]<0$ (cf. (2.12)). The latter case can be transformed into the former by replacing x with $1-x$ in the integral equation, and so without loss we shall restrict consideration to the case

$$J[K_{xx}]>0, \quad \text{with} \quad n=2, p=1. \tag{4.1}$$

The standard (and simplest) case then occurs when the three functions

$$K^-(x,0), \quad K_s^-(x,0), \quad K^+(x,1) \tag{4.2}$$

consisting of $n=2$ functions $K^-(x,0), K_s^-(x,0)$ at the left endpoint and $p=1$ function $K^+(x,1)$ at the right endpoint are linearly independent. The resulting leading-order outer equation is (cf. (2.13), (2.16))

$$\int_0^1 K(x,s)w(s)ds = h_0(x) - [c_1 g_1(x) + c_2 g_2(x) + c_3 g_3(x)] \tag{4.3}$$

for suitable constants c_1, c_2, c_3 with the functions $g_1(x), g_2(x), g_3(x)$ given as

$$g_1(x) = K^-(x,0), \quad g_2(x) = K_s^-(x,0), \quad g_3(x) = K^+(x,1), \tag{4.4}$$

and where the solution amplitudes $\widetilde{\phi}(\epsilon), \widehat{\phi}(\epsilon)$ at the endpoints will be given by (2.32) as

$$\widetilde{\phi}(\epsilon) = \frac{1}{\epsilon^2} \quad \text{and} \quad \widehat{\phi}(\epsilon) = \frac{1}{\epsilon}. \tag{4.5}$$

If one or more of the functions of (4.2) vanish or if these three functions are otherwise linearly dependent, then the results (4.4) must be modified for $g_j(x)$ ($j=1,2,3$) and the endpoint-amplitudes $\widetilde{\phi}(\epsilon)$, $\widehat{\phi}(\epsilon)$ are of larger order than indicated by (4.5).

Example 4.1: *A Standard Equation of Jump-Order 2.* For the standard case we consider the equation

$$\int_0^x \frac{w(s)}{1+xs-\frac{1}{2}s^2}ds + \int_x^1 \frac{w(s)}{1+\frac{1}{2}x^2}ds = h(x,\epsilon) + \epsilon^3 w(x) \quad \text{for} \quad 0 \le x \le 1, \tag{4.6}$$

which is (2.1) with kernel (cf. (1.29))

$$K(x,s) = \begin{cases} \left(1+xs-\frac{1}{2}s^2\right)^{-1} & \text{for } s<x \\ \left(1+\frac{1}{2}x^2\right)^{-1} & \text{for } s>x, \end{cases} \tag{4.7}$$

with jump-order 2 and $J[K_{xx}] = \left(1+\frac{1}{2}x^2\right)^{-2} \ge \frac{4}{9}$ so that (4.1) holds. Moreover the three functions of (4.4),

$$g_1(x) = 1, \quad g_2(x) = -x, \quad \text{and} \quad g_3(x) = \left(1+\frac{1}{2}x^2\right)^{-1} \tag{4.8}$$

are linearly independent.

We again employ the decomposition (2.2)-(2.3) of w into the sum of an outer solution plus boundary-layer corrections. Following the approach described in Section 3, we find an asymptotic splitting that consists of the outer equation

$$\begin{aligned} h(x,\epsilon) + \epsilon^3 w^*(x,\epsilon) - \int_0^1 K(x,s) w^*(s,\epsilon) ds \\ = \epsilon \left[\widetilde{\phi}(\epsilon) \int_0^{1/\epsilon} \frac{\widetilde{w}(\sigma,\epsilon)}{1+\epsilon x\sigma - \frac{1}{2}\epsilon^2\sigma^2} d\sigma + \widehat{\phi}(\epsilon) \int_0^{1/\epsilon} \frac{\widehat{w}(\sigma,\epsilon)}{1+\frac{1}{2}x^2} d\sigma \right] \end{aligned} \tag{4.9}$$

for $0 \le x \le 1$, and the boundary-layer correction equations

$$\epsilon^2 \widetilde{w}(\widetilde{x},\epsilon) = \int_{\widetilde{x}}^{\infty} \left[\frac{1}{1+\frac{1}{2}\epsilon^2\widetilde{x}^2} - \frac{1}{1+\epsilon^2\widetilde{x}\sigma - \frac{1}{2}\epsilon^2\sigma^2} \right] \widetilde{w}(\sigma,\epsilon) d\sigma \tag{4.10}$$

for $\widetilde{x} \ge 0$, and

$$\epsilon^2 \widehat{w}(\widehat{x},\epsilon) = \int_{\widehat{x}}^{\infty} \left[\frac{1}{1+(1-\epsilon\widehat{x})(1-\epsilon\sigma) - \frac{1}{2}(1-\epsilon\sigma)^2} - \frac{1}{1+\frac{1}{2}(1-\epsilon\widehat{x})^2} \right] \widehat{w}(\sigma,\epsilon) d\sigma \tag{4.11}$$

for $\widehat{x} \ge 0$. The equations (4.10) and (4.11) are differentiated 3 times to yield the corresponding integrodifferential equations

$$\widetilde{w}'''(\widetilde{x},\epsilon) = \frac{1}{\left(1+\frac{1}{2}\epsilon^2\widetilde{x}^2\right)^2} \widetilde{w}(\widetilde{x},\epsilon) + \epsilon^2 \int_{\widetilde{x}}^{\infty} \left[\frac{3\widetilde{x}\left(2-\epsilon^2\widetilde{x}^2\right)}{\left(1+\frac{1}{2}\epsilon^2\widetilde{x}^2\right)^4} + \frac{6\epsilon^2\sigma^3}{\left(1+\epsilon^2\widetilde{x}\sigma - \frac{1}{2}\epsilon^2\sigma^2\right)^4} \right] \widetilde{w}(\sigma,\epsilon) d\sigma \tag{4.12}$$

and

$$\begin{aligned} \widehat{w}'''(\widehat{x},\epsilon) = -\frac{1}{\left(\frac{3}{2} - \epsilon\widehat{x} + \frac{1}{2}\epsilon^2\widehat{x}^2\right)^2} \widehat{w}(\widehat{x},\epsilon) \\ + \epsilon \int_{\widehat{x}}^{\infty} \left[\frac{6(1-\epsilon\sigma)^3}{\left(\frac{3}{2} - \epsilon\widehat{x} + \epsilon^2\widehat{x}\sigma - \frac{1}{2}\epsilon^2\sigma^2\right)^4} + \frac{3(1-\epsilon\widehat{x})\left(1+2\epsilon\widehat{x} - \epsilon^2\widehat{x}^2\right)}{\left(1+\frac{1}{2}(1-\epsilon\widehat{x})^2\right)^4} \right] \widehat{w}(\sigma,\epsilon) d\sigma. \end{aligned} \tag{4.13}$$

The expansion (2.8) for $\widetilde{w}$ is inserted into (4.12) and we find a collection of differential equations for the coefficient functions $\widetilde{w}_0, \widetilde{w}_1, \ldots$ of the form (cf. (3.15) or (5.10)-(5.11))

$$\widetilde{w}_j''' - \widetilde{w}_j = \begin{cases} 0 & \text{for } j = 0, \\ \widetilde{P}_j(\widetilde{x}) & \text{for } j \ge 1, \end{cases} \tag*{(4.14)$_j$}$$

for suitable functions $\widetilde{P}_j$ which are determined by the data recursively in terms of $\widetilde{w}_k$ for $k \leq j-1$. For example in the case $j=1$ the function $\widetilde{P}_1$ is given as (cf. (5.11))

$$\widetilde{P}_1(\widetilde{x}) = \widetilde{x} J[K_{xx}]'(0)\widetilde{w}_0(\widetilde{x}) + J[K_{xxx}](0) \int_{\widetilde{x}}^{\infty} \widetilde{w}_0(\sigma)d\sigma \equiv 0, \tag{4.15$_1$}$$

while $\widetilde{P}_2$ is found to be given as

$$\widetilde{P}_2(\widetilde{x}) = -\widetilde{x}^2\widetilde{w}_0(\widetilde{x}) + 6\widetilde{x} \int_{\widetilde{x}}^{\infty} \widetilde{w}_0(\sigma)d\sigma. \tag{4.15$_2$}$$

The general solution of $(4.14)_0$ is given as a linear combination of the functions

$$e^{+\widetilde{x}},\ e^{-\widetilde{x}/2}\, sin\, \frac{\sqrt{3}\widetilde{x}}{2},\ e^{-\widetilde{x}/2}\, cos\, \frac{\sqrt{3}\widetilde{x}}{2}, \tag{4.16}$$

but the appropriate matching condition from (2.10) eliminates $e^{+\widetilde{x}}$ and so the boundary-layer solution is

$$\widetilde{w}_0(\widetilde{x}) = e^{-\widetilde{x}/2} \left(A_0\, sin\, \frac{\sqrt{3}\widetilde{x}}{2} + B_0\, cos\, \frac{\sqrt{3}\widetilde{x}}{2} \right), \tag{4.17$_0$}$$

and $\widetilde{w}_1$ is obtained similarly from $(4.14)_1$ and $(4.15)_1$ as

$$\widetilde{w}_1(\widetilde{x}) = e^{-\widetilde{x}/2} \left(A_1\, sin\, \frac{\sqrt{3}\widetilde{x}}{2} + B_1\, cos\, \frac{\sqrt{3}\widetilde{x}}{2} \right), \tag{4.17$_1$}$$

for suitable constants of integration A_0, B_0, A_1, B_1. The result $(4.17)_0$ can be used for $\widetilde{w}_0$ in $(4.15)_2$ and then the resulting equation $(4.14)_2$ can be solved for $\widetilde{w}_2$ subject to the usual decay condition at infinity. One can similarly obtain as many of the functions $\widetilde{w}_j$ (for $j = 0, 1, \ldots$) as may be required for accuracy, where each additional $\widetilde{w}_j$ introduces two additional constants A_j, B_j of integration, and where all such layer-correction functions decay exponentially as $\widetilde{x} \to \infty$.

Beginning instead with the integrodifferential equation (4.13), we find for the coefficient functions $\widehat{w}_j$ a collection of differential equations

$$\widehat{w}_j''' + \frac{4}{9}\widehat{w}_j = \begin{cases} 0 & \text{for } j = 0, \\ \widehat{Q}_j(\widehat{x}) & \text{for } j \geq 1, \end{cases} \tag{4.18$_j$}$$

for suitable functions $\widehat{Q}_j$ which are determined recursively in terms of $\widehat{w}_k$ for $k \leq j-1$. For example the function $\widehat{Q}_1$ is given as

$$\widehat{Q}_1(\widehat{x}) = -\frac{16}{27}\widehat{x}\widehat{w}_0(\widehat{x}) + \frac{16}{9} \int_{\widehat{x}}^{\infty} \widehat{w}_0(\sigma)d\sigma. \tag{4.19}$$

The general solution of $(4.18)_0$ is given as a linear combination of the functions

$$e^{-\beta\widehat{x}},\ e^{+\frac{\beta\widehat{x}}{2}}\, sin\,\frac{\sqrt{3}}{2}\beta\widehat{x},\ e^{+\frac{\beta\widehat{x}}{2}}\, cos\,\frac{\sqrt{3}}{2}\beta\widehat{x}, \tag{4.20}$$

with

$$\beta=\left(\frac{4}{9}\right)^{1/3}, \tag{4.21}$$

so that the most general solution with the required decay property is

$$\widehat{w}_0(\widehat{x})=C_0e^{-\beta\widehat{x}} \tag*{$(4.22)_0$}$$

for an arbitrary constant C_0. Inserting $(4.22)_0$ into (4.19) and then solving $(4.18)_1$, we find the general decaying solution for $\widehat{w}_1$ to be

$$\widehat{w}_1(\widehat{x})=\left[C_1+\frac{1}{9}C_0\widehat{x}\,(8-2\beta\widehat{x})\right]e^{-\beta\widehat{x}} \tag*{$(4.22)_1$}$$

for an arbitrary constant C_1. Again one can similarly obtain as many functions $\widehat{w}_j$ as required for accuracy, where all these functions decay exponentially as $\widehat{x}\to\infty$.

The expansions (2.8) for $\widetilde{w}$ and $\widehat{w}$ are inserted now in the right side of the outer equation (4.9), and we find (through the first few orders, using the appropriate Taylor expansions; cf. (3.21))

$$\begin{aligned}
&h(x,\epsilon)+\epsilon^3w^*(x,\epsilon)-\int_0^1K(x,s)w^*(s,\epsilon)ds\\
&\sim\epsilon\widetilde{\phi}(\epsilon)\Bigg[\int_0^{1/\epsilon}\widetilde{w}_0(\sigma)d\sigma+\epsilon\left(\int_0^{1/\epsilon}\widetilde{w}_1(\sigma)d\sigma-x\int_0^{1/\epsilon}\sigma\widetilde{w}_0(\sigma)d\sigma\right)\\
&+\epsilon^2\left(\int_0^{1/\epsilon}\widetilde{w}_2(\sigma)d\sigma-x\int_0^{1/\epsilon}\sigma\widetilde{w}_1(\sigma)d\sigma+\left(\frac{1}{2}+x^2\right)\int_0^{1/\epsilon}\sigma^2\widetilde{w}_0(\sigma)d\sigma\right)+\cdots\Bigg]\\
&+\frac{\epsilon\widehat{\phi}(\epsilon)}{1+\frac{1}{2}x^2}\left[\int_0^{1/\epsilon}\widehat{w}_0(\sigma)d\sigma+\epsilon\int_0^{1/\epsilon}\widehat{w}_1(\sigma)d\sigma+\cdots\right],
\end{aligned} \tag{4.23}$$

or with (4.17), (4.22) and neglecting exponentially small terms,

$$\begin{aligned}
&h(x,\epsilon)+\epsilon^3w^*(x,\epsilon)-\int_0^1K(x,s)w^*(s,\epsilon)ds\\
&\sim\epsilon\widetilde{\phi}(\epsilon)\Bigg[\frac{1}{2}(\sqrt{3}A_0+B_0)+\epsilon\left(\frac{1}{2}(A_1+B_1)+\frac{1}{2}x(-\sqrt{3}A_0+B_0)\right)\\
&+\epsilon^2\left(\left(\frac{1}{2}+x^2\right)\int_0^{\infty}\sigma^2\widetilde{w}_0(\sigma)d\sigma+\int_0^{\infty}\widetilde{w}_2(\sigma)d\sigma+\frac{1}{2}x(-\sqrt{3}A_1+B_1)\right)+\cdots\Bigg]\\
&+\frac{\epsilon\widehat{\phi}(\epsilon)}{1+\frac{1}{2}x^2}\left[\frac{C_0}{\beta}+\epsilon\int_0^{\infty}\left[C_1+\frac{1}{9}C_0\sigma(8-2\beta\sigma)\right]e^{-\beta\sigma}d\sigma+\cdots\right].
\end{aligned} \tag{4.24}$$

A linear combination of 3 independent functions must be produced by the order-unity terms on the right side of (4.24) upon letting $\epsilon \to 0$. Moreover, because there are 2 independent decaying solutions in the left boundary-layer list (4.16) while there is only 1 such decaying solution in the right boundary-layer list (4.20), we anticipate that we should use 2 independent terms with coefficient $\widetilde{\phi}$ associated with the left endpoint on the right side of (4.24), along with 1 function associated with the $\widehat{\phi}$-terms at the right endpoint. It is not possible to satisfy these requirements with the leading terms on the right side. Moreover there must hold $C_0 \neq 0$ (see (2.9)) and so we must set to zero the quantity

$$\sqrt{3}A_0 + B_0 = 0. \tag{4.25}$$

The remaining terms suffice with the choices (cf. (4.5))

$$\widetilde{\phi}(\epsilon) = \frac{1}{\epsilon^2} \quad \text{and} \quad \widehat{\phi}(\epsilon) = \frac{1}{\epsilon}, \tag{4.26}$$

and then (4.24) along with the expansion (2.7) for w^* lead to the equations

$$\int_0^1 K(x,s)w_0^*(s)ds = h_0(x) - \left[c_1 + c_2 x + \frac{c_3}{1+\frac{1}{2}x^2}\right] \tag{4.27}$$

with

$$c_1 = \frac{1}{2}(\sqrt{3}A_1 + B_1), \quad c_2 = \frac{1}{2}(-\sqrt{3}A_0 + B_0), \quad c_3 = \frac{C_0}{\beta}, \tag{4.28}$$

and

$$\begin{aligned} \int_0^1 K(x,s)w_1^*(s)ds = h_1(x) &- \left(\frac{1}{2} + x^2\right)\int_0^\infty \sigma^2 \widetilde{w}_0(\sigma)d\sigma \\ &- \left[d_1 + d_2 x + \frac{d_3}{1+\frac{1}{2}x^2}\right] \end{aligned} \tag{4.29}$$

with

$$\begin{aligned} d_1 &= \int_0^\infty \widetilde{w}_2(\sigma)d\sigma, \quad d_2 = \frac{1}{2}(-\sqrt{3}A_1 + B_1), \\ d_3 &= \int_0^\infty \left[C_1 + \frac{1}{9}C_0\sigma(8 - 2\beta\sigma)\right] e^{-\beta\sigma}d\sigma, \end{aligned} \tag{4.30}$$

and analogous equations for w_j^* for $j \geq 2$. In the following we are content to compute only the first two terms in the expansion ($j = 0, 1$).

The equations (4.27) and (4.29) (and the related equations for higher-order terms) are of the general form

$$\int_0^x \frac{v(s)}{1 + xs - \frac{1}{2}s^2}ds + \frac{1}{1+\frac{1}{2}x^2}\int_x^1 v(s)ds = f(x) - \left[k_1 + k_2 x + \frac{k_3}{1+\frac{1}{2}x^2}\right] \tag{4.31}$$

for a suitable given function $f(x)$ and for suitable constants k_i $(i = 1,2,3)$ that must be chosen in each case so that the first-kind integral equation (4.31) is solvable. The constants can be conveniently determined as follows. The integral equation can be multiplied by $1 + \frac{1}{2}x^2$ and the resulting equation can be differentiated with respect to x to yield

$$\int_0^x \left[\frac{(x-s)\left(1+\frac{1}{2}xs\right)}{\left(1+xs-\frac{1}{2}s^2\right)^2} \right] v(s)ds = \left(\left(1+\frac{1}{2}x^2\right) f(x)\right)' - k_1 x - k_2\left(1+\frac{3}{2}x^2\right). \quad (4.32)$$

Put $x = 0$ in this last equation to find

$$k_2 = f'(0). \quad (4.33)$$

Next differentiate the equation (4.32) and put $x = 0$ in the resulting equation to find

$$k_1 = f(0) + f''(0). \quad (4.34)$$

Finally, to obtain k_3, put $x = 0$ in (4.31) and use (4.34) to find

$$k_3 = -\left[f''(0) + \int_0^1 v(s)ds\right]. \quad (4.35)$$

These results can be used to eliminate the constants in (4.31) to yield

$$\frac{1}{2}\int_0^x \frac{(x-s)^2}{1+xs-\frac{1}{2}s^2} v(s)ds = f''(0) + \left(1+\frac{1}{2}x^2\right)\left[f(x) - f(0) - f'(0)x - f''(0)\right], \quad (4.36)$$

and this latter equation determines $v = v(x)$ uniquely for $0 \le x \le 1$. Indeed three differentiations of (4.36) lead to a Volterra equation of second kind which is uniquely solvable.

Applying these results to (4.27) for $v = w_0^*$ with $f = h_0$, we have

$$c_1 = h_0(0) + h_0''(0), \quad c_2 = h_0'(0), \quad c_3 = -\left[h_0''(0) + \int_0^1 w_0^*(s)ds\right], \quad (4.37)$$

with w_0^* determined by the equation

$$\int_0^x \frac{(x-s)^2}{(2+x^2)\left(1+xs-\frac{1}{2}s^2\right)} w_0^*(s)ds = h_0(x) - h_0(0) - h_0'(0)x - \frac{h_0''(0)x^2}{2+x^2}. \quad (4.38)$$

Similarly, applying the results to (4.29) with $v = w_1^*$ and with (cf. $(4.17)_0$)

$$\begin{aligned} f(x) &= h_1(x) - \left(\frac{1}{2} + x^2\right)\int_0^\infty \sigma^2 \widetilde{w}_0(\sigma)d\sigma \\ &= h_1(x) + \left(1+2x^2\right)B_0, \end{aligned} \quad (4.39)$$

we find

$$d_1 = h_1(0) + h_1''(0) + 5B_0, \quad d_2 = h_1'(0), \quad d_3 = -\left[h_1''(0) + 4B_0 + \int_0^1 w_1^*(s)ds\right], \tag{4.40}$$

with w_1^* determined by the equation

$$\int_0^x \frac{(x-s)^2 w_1^*(s)}{(2+x^2)\left(1+xs-\frac{1}{2}s^2\right)}ds = h_1(x) - h_1(0) - h_1'(0)x - \frac{h_1''(0)x^2}{2+x^2} + \frac{2x^4 B_0}{2+x^2}. \tag{4.41}$$

These equations can be used to obtain the constants A_j, B_j, C_j appearing in the first two layer-corrections $(4.17)_j$ and $(4.22)_j$ for $j = 0, 1$. From (4.28) and (4.37),

$$-\sqrt{3}A_0 + B_0 = 2h_0'(0), \quad C_0 = -\beta\left[h_0''(0) + \int_0^1 w_0^*(s)ds\right], \tag{4.42}$$

which with (4.25) yield

$$A_0 = -\frac{h_0'(0)}{\sqrt{3}}, \quad B_0 = h_0'(0), \quad C_0 = -\beta\left[h_0''(0) + \int_0^1 w_0^*(s)ds\right]. \tag{4.43}$$

Turning to the case $j = 1$, there follows from (4.28) and (4.37)

$$\sqrt{3}A_1 + B_1 = 2\left[h_0(0) + h_0''(0)\right], \tag{4.44}$$

while (4.30) and (4.40) yield

$$\begin{aligned} -\sqrt{3}A_1 + B_1 &= 2h_1'(0), \\ -\left[h_1''(0) + 4B_0 + \int_0^1 w_1^*(s)ds\right] &= \frac{C_1}{\beta} + \frac{4C_0}{9\beta^2}. \end{aligned} \tag{4.45}$$

From (4.43)-(4.45) we have

$$\begin{aligned} A_1 &= \frac{1}{\sqrt{3}}\left[h_0(0) + h_0''(0) - h_1'(0)\right] \\ B_1 &= h_0(0) + h_0''(0) + h_1'(0) \\ C_1 &= \frac{4}{9}\left[h_0''(0) + \int_0^1 w_0^*(s)ds\right] - \beta\left[h_1''(0) + 4h_0'(0) + \int_0^1 w_1^*(s)ds\right]. \end{aligned} \tag{4.46}$$

The procedure can be continued to determine as many terms as may be required for accuracy.

In particular we find for the solution of the given integral equation of jump-order 2, to first-order,

$$\begin{aligned} w(x,\epsilon) = \frac{1}{\epsilon^2}\left[A_0\, sin\,\frac{\sqrt{3}x}{2\epsilon} + B_0\, cos\,\frac{\sqrt{3}x}{2\epsilon} + O(\epsilon)\right] e^{-x/(2\epsilon)} \\ + w_0^*(x) + O(\epsilon) + \frac{1}{\epsilon}\left[C_0 + O(\epsilon)\right] e^{-\beta(1-x)/\epsilon} \end{aligned} \tag{4.47}$$

uniformly for $0 \le x \le 1$ as $\epsilon \to 0^+$, with β given by (4.21), with the constants A_0, B_0, C_0 given by (4.43), and where the outer function $w_0^*(x)$ is determined uniquely by the (regular) equation (4.38). Similarly, to second-order we have

$$\begin{aligned} w(x,\epsilon) = &\frac{1}{\epsilon^2}\left[(A_0+\epsilon A_1)\, sin\,\frac{\sqrt{3}x}{2\epsilon} + (B_0+\epsilon B_1)\, cos\,\frac{\sqrt{3}x}{2\epsilon} + O(\epsilon^2)\right] e^{-x/(2\epsilon)} \\ &+ w_0^*(x) + \epsilon w_1^*(x) + O(\epsilon^2) \\ &+ \frac{1}{\epsilon}\left[C_0 + \epsilon\left[C_1 + \frac{8}{9}C_0\frac{1-x}{\epsilon} - \frac{2\beta}{9}C_0\left(\frac{1-x}{\epsilon}\right)^2\right] + O(\epsilon^2)\right] e^{-\beta(1-x)/\epsilon} \end{aligned} \tag{4.48}$$

where A_1, B_1, C_1 are given by (4.46) and the function $w_1^*(x)$ is determined by (4.41).

— *Figure 2 Here* —

Fig. 2. Solution of (4.6) with $h(x,\epsilon) = e^{-x}$ and $\epsilon = 0.01$.

For the special case

$$h(x,\epsilon) = e^{-x} \equiv h_0(x), \tag{4.49}$$

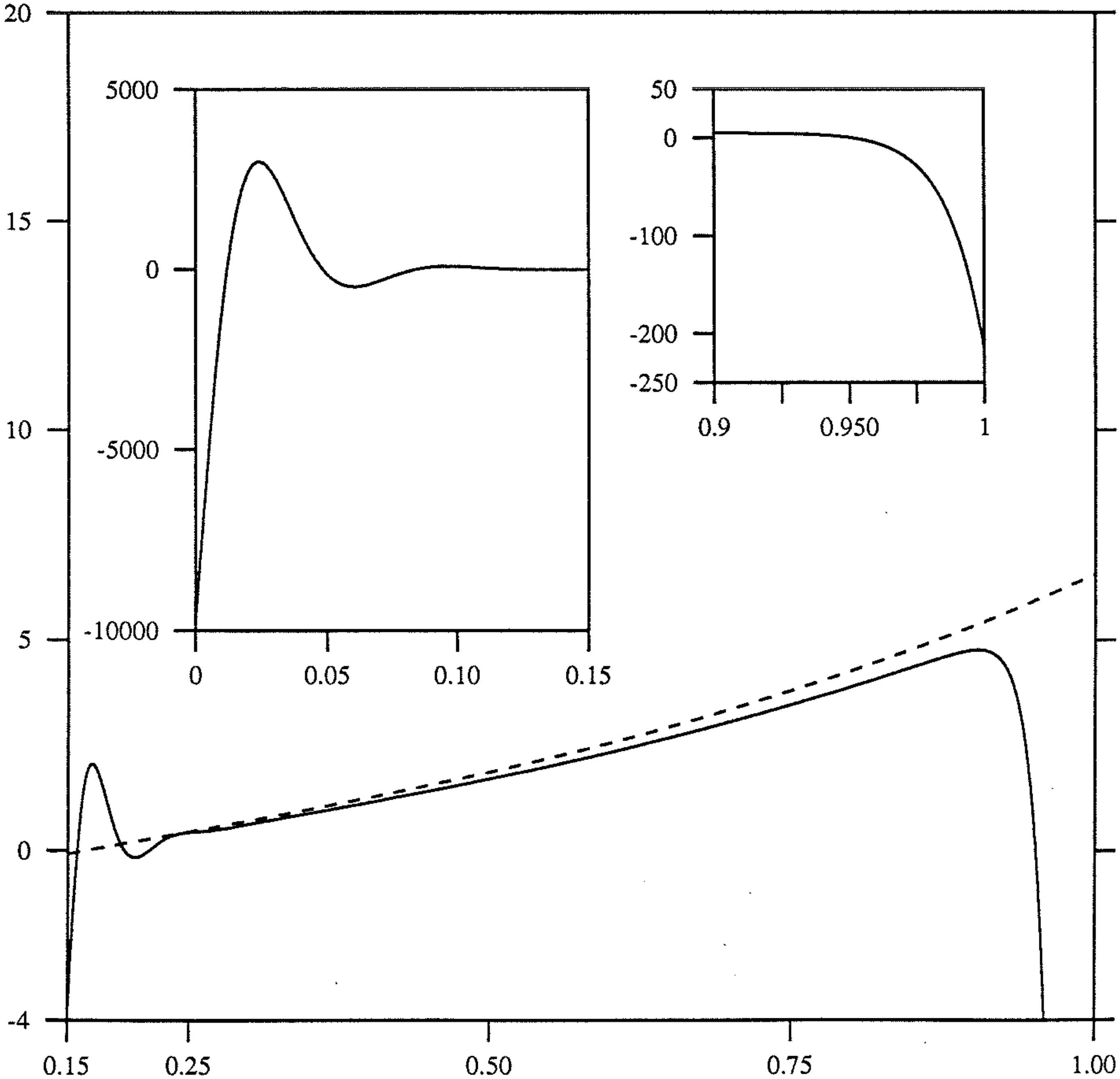
the numerical solutions of (4.38) and (4.41) imply

$$\int_0^1 w_0^*(s)ds \doteq 2.196688 \quad \text{and} \quad \int_0^1 w_1^*(s)ds \doteq -21.58155, \tag{4.50}$$

and we have

$$\begin{aligned} A_0 &= \frac{1}{\sqrt{3}}, & A_1 &= \frac{2}{\sqrt{3}} \\ B_0 &= -1, & B_1 &= 2, \\ C_0 &\doteq -2.43953, & C_1 &\doteq 20.9431. \end{aligned} \tag{4.51}$$

Figure 2


20
15
10
5
0
-4
0.15
0.25
0.50
0.75
1.00
5000
0
-5000
-10000
0
0.05
0.10
0.15
50
0
-100
-200
-250
0.9
0.950
1

The graph of the resulting first-order outer solution w_0^* is indicated in *Figure 2* by the dashed curve for the case $\epsilon = 0.01$. The solid curve in *Figure 2* indicates the second-order asymptotic approximation (4.48) which coincides (within the precision of the graph) with the numerical solution obtained by a direct numerical solution of (4.6). The solution values at the endpoints are found to be $w(0) \doteq -9.8 \times 10^3$ and $w(1) \doteq -2.17 \times 10^2$. The direct numerical solution was obtained with a finite-element scheme using a variable mesh with 1000 grid points, on an *Alliant FX-80* machine. The direct numerical solution proved to be expensive to obtain for small ϵ, in contrast with the asymptotic solution which was cheaply obtained. Note that the first-order asymptotic approximation is reasonably satisfactory here, but the second-order approximation is nevertheless useful due to the relatively larger values of A_1, B_1, C_1, w_1^* as compared to A_0, B_0, C_0, w_0^*. The second-order asymptotic approximation gives excellent accuracy.

We shall now consider several examples for which at least one of the three functions of (4.2) vanishes or for which those functions are otherwise linearly dependent. In such cases at least one of the layer-amplitudes is of larger order than indicated by (4.26) for the standard case.

Example 4.2: *A Nonstandard Equation* ($K^-(x,0), K_s^-(x,0)$ and $K^+(x,1)$ linearly dependent). The equation

$$\int_0^x \left(x^2 - \frac{xs}{2} + \frac{s^2}{2}\right) w(s)ds + \int_x^1 \left(\frac{x^2 + xs}{2}\right) w(s)ds = h(x,\epsilon) + \epsilon^3 w(x) \tag{4.52}$$

corresponds to (2.1) with kernel

$$K(x,s) = \begin{cases} K^-(x,s) = (2x^2 - xs + s^2)/2 & \text{for } s < x \\ K^+(x,s) = (x^2 + xs)/2 & \text{for } s > x, \end{cases} \tag{4.53}$$

with $\nu = 2$ and with n and p given by (4.1). This is a nonstandard case because of the linear dependence of the three functions of (4.2) ($K^-(x,0) = x^2$, $K_s^-(x,0) = -x$, $K^+(x,1) = (x^2 + x)/2$), resulting in boundary-layer amplitudes of larger order than occur in the standard case. Here the asymptotic splitting produces the outer equation

$$\begin{aligned} h(x,\epsilon) + \epsilon^3 w^*(x,\epsilon) - \int_0^1 K(x,s)w^*(s,\epsilon)ds \\ = \epsilon\widetilde{\phi}(\epsilon) \int_0^{1/\epsilon} x\left(x^2 - \frac{\epsilon}{2}x\sigma + \frac{\epsilon^2}{2}\sigma^2\right)\widetilde{w}(\sigma,\epsilon)d\sigma \\ + \frac{\epsilon}{2}\widehat{\phi}(\epsilon)\int_0^{1/\epsilon} \left(x^2 + x - \epsilon x\sigma\right)\widehat{w}(\sigma,\epsilon)d\sigma, \end{aligned} \tag{4.54}$$

along with the boundary-layer equations

$$\widetilde{w}(\widetilde{x},\epsilon) = -\frac{1}{2}\int_{\widetilde{x}}^{\infty} (\widetilde{x} - \sigma)^2 \widetilde{w}(\sigma,\epsilon)d\sigma \quad \text{and} \quad \widehat{w}(\widehat{x},\epsilon) = \frac{1}{2}\int_{\widehat{x}}^{\infty} (\widehat{x} - \sigma)^2 \widehat{w}(\sigma,\epsilon)d\sigma. \tag{4.55}$$

From (4.55) we have the differential equations

$$\widetilde{w}'''(\widetilde{x},\epsilon) - \widetilde{w}(\widetilde{x},\epsilon) = 0 \quad \text{for} \quad \widetilde{x} > 0, \qquad \widehat{w}'''(\widehat{x},\epsilon) + \widehat{w}(\widehat{x},\epsilon) = 0 \quad \text{for} \quad \widehat{x} > 0, \tag{4.56}$$

with solutions given respectively as linear combinations of the functions

$$e^{+\widetilde{x}},\ e^{-\widetilde{x}/2}\, sin\, \frac{\sqrt{3}\widetilde{x}}{2},\ e^{-\widetilde{x}/2}\, cos\, \frac{\sqrt{3}\widetilde{x}}{2} \qquad (\text{for } \widetilde{w}) \tag{4.57}$$

and

$$e^{-\widehat{x}},\ e^{+\widehat{x}/2}\, sin\, \frac{\sqrt{3}\widehat{x}}{2},\ e^{+\widehat{x}/2}\, cos\, \frac{\sqrt{3}\widehat{x}}{2} \qquad (\text{for } \widehat{w}). \tag{4.58}$$

There are 2 decaying solutions in the list (4.57) associated with the left endpoint $x = 0$ because of the result $n = 2$ (cf. (4.1)), while there is just 1 decaying solution in the list (4.58) because $p = 1$. The general solutions of the boundary-layer equations (4.55) coincide with the general decaying solutions of the differential equations (4.56) and are

$$\begin{aligned} \widetilde{w}(\widetilde{x},\epsilon) &= e^{-\widetilde{x}/2}\left(A(\epsilon)\, sin\, \frac{\sqrt{3}\widetilde{x}}{2} + B(\epsilon)\, cos\, \frac{\sqrt{3}\widetilde{x}}{2}\right) \\ \widehat{w}(\widehat{x},\epsilon) &= C(\epsilon)e^{-\widehat{x}} \end{aligned} \tag{4.59}$$

for constants $A(\epsilon), B(\epsilon), C(\epsilon)$ that are to have asymptotic expansions in ϵ resulting in expansions for $\widetilde{w}, \widehat{w}$ of the form (cf. (2.8))

$$\begin{aligned} \widetilde{w}(\widetilde{x},\epsilon) &\sim \sum_{j=0}^{\infty} e^{-\widetilde{x}/2}\left(A_j\, sin\, \frac{\sqrt{3}\widetilde{x}}{2} + B_j\, cos\, \frac{\sqrt{3}\widetilde{x}}{2}\right)\epsilon^j \\ \widehat{w}(\widehat{x},\epsilon) &\sim \sum_{j=0}^{\infty} e^{-\widehat{x}}\, C_j \epsilon^j. \end{aligned} \tag{4.60}$$

The expansions (2.7) and (4.60) are inserted into (4.54) and a now-routine calculation gives

$$\begin{aligned} &\sum_{j=0}^{\infty}\Big[h_j(x) + w_{j-3}^*(x) - \int_0^1 K(x,s)w_j^*(s)ds\Big]\epsilon^j \\ &\quad \sim \sum_{j=0}^{\infty}\left[\widetilde{\phi}(\epsilon)\left[\left(\sqrt{3}A_j + B_j\right)\frac{x^2}{2} + \left(-\sqrt{3}A_{j-1} + B_{j-1}\right)\frac{x}{4} - B_{j-2}\right]\right. \\ &\qquad\qquad \left. + \widehat{\phi}(\epsilon)\left[\left(\frac{x^2+x}{2}\right)C_j - \left(\frac{x}{2}\right)C_{j-1}\right]\right]\epsilon^{j+1} \end{aligned} \tag{4.61}$$

with $w_j^* \equiv 0$ and $A_j = B_j = C_j = 0$ for $j < 0$. Exhibiting the dominant terms we have

$$
\begin{aligned}
h_0(x) - \int_0^1 K(x,s) w_0^*(s) ds + \cdots \sim\ & \epsilon\widetilde{\phi}(\epsilon)\left(\sqrt{3}A_0 + B_0\right)\frac{x^2}{2} + \epsilon\widehat{\phi}(\epsilon) C_0 \left(\frac{x^2+x}{2}\right) \\
& + \epsilon^2\widetilde{\phi}(\epsilon)\left[\left(\sqrt{3}A_1 + B_1\right)\frac{x^2}{2} + \left(-\sqrt{3}A_0 + B_0\right)\frac{x}{4}\right] \\
& + \epsilon^2\widehat{\phi}(\epsilon)\left[C_1\left(\frac{x^2+x}{2}\right) - C_0\left(\frac{x}{2}\right)\right] \\
& + \epsilon^3\widetilde{\phi}(\epsilon)\left[\left(\sqrt{3}A_2 + B_2\right)\frac{x^2}{2} + \left(-\sqrt{3}A_1 + B_1\right)\frac{x}{4} - B_0\right] \\
& + \epsilon^3\widehat{\phi}(\epsilon)\left[C_2\left(\frac{x^2+x}{2}\right) - C_1\left(\frac{x}{2}\right)\right] + \cdots,
\end{aligned} \tag{4.62}
$$

and upon letting $\epsilon \to 0$ it is not possible to obtain a linear combination of 3 independent functions from the leading terms on the right side. Moreover the leading $\widetilde{\phi}$-term involves only 1 function. As in the earlier example with (4.24) here also we must set to zero the quantity $\sqrt{3}A_0 + B_0$, giving again the condition (4.25). There will now be two independent functions on the right side of (4.62) associated with the dominant $\widetilde{\phi}$-term, but the leading terms there still cannot produce 3 independent functions. Since C_0 must not vanish, we impose without loss the condition

$$
\epsilon\widehat{\phi}(\epsilon) = \epsilon^2\widetilde{\phi}(\epsilon) \tag{4.63}
$$

and combine the remaining terms of (4.62) as

$$
\begin{aligned}
& h_0(x) - \int_0^1 K(x,s) w_0^*(s) ds + \cdots \\
& \sim \epsilon^2\widetilde{\phi}(\epsilon)\left[\left(\sqrt{3}A_1 + B_1 + C_0\right)\frac{x^2}{2} + \left(-\sqrt{3}A_0 + B_0 + 2C_0\right)\frac{x}{4}\right] \\
& + \epsilon^3\widetilde{\phi}(\epsilon)\left[\left(\sqrt{3}A_2 + B_2 + C_1\right)\frac{x^2}{2} + \left(-\sqrt{3}A_1 + B_1 + 2C_1 - 2C_0\right)\frac{x}{4} - B_0\right] \\
& \quad + \cdots.
\end{aligned} \tag{4.64}
$$

Now we obtain 3 independent functions with the conditions

$$
\begin{aligned}
\sqrt{3}A_1 + B_1 + C_0 &= 0 \\
-\sqrt{3}A_0 + B_0 + 2C_0 &= 0,
\end{aligned} \tag{4.65}
$$

and with $\epsilon^3\widetilde{\phi}(\epsilon) = 1$, so that (4.63) gives

$$
\widetilde{\phi}(\epsilon) = \frac{1}{\epsilon^3} \quad \text{and} \quad \widehat{\phi}(\epsilon) = \frac{1}{\epsilon^2}. \tag{4.66}
$$

With these choices we obtain from (4.61) the conditions

$$\int_0^1 K(x,s)w_j^*(s)ds = h_j^*(x) - \left[c_{j,1}x^2 + c_{j,2}x + c_{j,3}\right] \tag{4.67}$$

for $j = 0, 1, \cdots$, with K given by (4.53), $h_j^*(x)$ defined by

$$h_j^*(x) := h_j(x) + w_{j-3}^*(x), \tag{4.68}$$

and with

$$\begin{aligned} c_{j,1} &= \frac{1}{2}\left(\sqrt{3}A_{j+2} + B_{j+2} + C_{j+1}\right) \\ c_{j,2} &= \frac{1}{4}\left(-\sqrt{3}A_{j+1} + B_{j+1} + 2C_{j+1} - 2C_j\right) \\ c_{j,3} &= -B_j. \end{aligned} \tag{4.69}$$

The constants $c_{j,i}$ $(i = 1, 2, 3)$ can be evaluated from (4.67) to give

$$\begin{aligned} c_{j,1} &= \frac{1}{2}\left[h_j^{*''}(0) - \int_0^1 w_j^*(s)ds\right] \\ c_{j,2} &= h_j^{*'}(0) - \int_0^1 \frac{s}{2} w_j^*(s)ds \\ c_{j,3} &= h_j^*(0), \end{aligned} \tag{4.70}$$

which can be used back in (4.67) to yield

$$\frac{1}{2}\int_0^x (x-s)^2 w_j^*(s)ds = h_j^*(x) - h_j^*(0) - h_j^{*'}(0)x - \frac{1}{2}h_j^{*''}(0)x^2. \tag{4.71}$$

This equation determines w_j^* uniquely as

$$w_j^*(x) = h_j^{*'''}(x) \tag{4.72}$$

for $j = 0, 1, \ldots$ which with (4.70) gives

$$\begin{aligned} c_{j,1} &= h_j^{*''}(0) - \frac{1}{2}h_j^{*''}(1) \\ c_{j,2} &= \frac{1}{2}\left[h_j^{*'}(1) - h_j^{*''}(1)\right] \\ c_{j,3} &= h_j^*(0). \end{aligned} \tag{4.73}$$

These results along with (4.69) and (4.25) suffice to determine the values of the constants A_j, B_j, C_j appearing in the layer-corrections $\widetilde{w}, \widehat{w}$. To lowest order we find

$$\begin{aligned} w(x,\epsilon) = {} & \frac{1}{\epsilon^3}\left[A_0 \, sin\, \frac{\sqrt{3}x}{2\epsilon} + B_0 \, cos\, \frac{\sqrt{3}x}{2\epsilon} + O(\epsilon)\right] e^{-x/(2\epsilon)} \\ & + \frac{1}{\epsilon^2}\left[C_0 + O(\epsilon)\right] e^{-(1-x)/\epsilon} + h_0'''(x) + O(\epsilon) \end{aligned} \tag{4.74}$$

uniformly for $0 \leq x \leq 1$ as $\epsilon \to 0^+$, with

$$A_0 = \frac{1}{\sqrt{3}} h_0(0), \quad B_0 = -h_0(0), \quad C_0 = h_0(0). \tag{4.75}$$

In this example the basic three functions $K^-(x,0)$, $K_s^-(x,0)$, $K^+(x,1)$ of (4.2) are linearly dependent, but the following extended list of functions

$$\begin{aligned} &K^-(x,0) = x^2, \qquad K_s^-(x,0) = -\tfrac{x}{2}, \qquad K_{ss}^-(x,0) = 1, \\ &K^+(x,1) = \frac{x^2+x}{2} \end{aligned} \tag{4.76}$$

suffices to provide three linearly independent functions with two of these functions associated with K^- at $s = 0$ ($n = 2$ in (4.1)). Due to the nonstandard nature of this example, each of the amplitudes $\widetilde{\phi}(\epsilon) = 1/\epsilon^3$ and $\widehat{\phi}(\epsilon) = 1/\epsilon^2$ of the solution at the endpoints is one order larger than indicated by (4.26) in the earlier standard case (4.6). The single condition $h_0(0) = 0$ would suffice to reduce the layer-amplitudes in (4.74) back to the standard amplitudes.

Example 4.3: *A Nonstandard Equation* ($K^+(x,1) = 0$). As another example consider the equation

$$\int_0^x (1-x)\left[s - \frac{1+x}{2}\right] w(s)ds - \int_x^1 \frac{(1-s)^2}{2} w(s)ds = h(x,\epsilon) + \epsilon^3 w(x) \tag{4.77}$$

which is (2.1) with kernel

$$K(x,s) = \begin{cases} (1-x)\left[s - \frac{1+x}{2}\right] & \text{for } s < x \\ -\frac{(1-s)^2}{2} & \text{for } s > x, \end{cases} \tag{4.78}$$

with $\nu = 2$ and with n and p given again by (4.1). Here again the basic three functions of (4.2) are linearly dependent, but the following extended list of functions

$$\begin{aligned} &K^-(x,0) = \tfrac{x^2-1}{2}, \qquad K_s^-(x,0) = 1-x, \\ &K^+(x,1) = 0, \qquad K_s^+(x,1) = 0, \qquad K_{ss}^+(x,1) = -1 \end{aligned} \tag{4.79}$$

contains three linearly independent functions, 2 associated with the left endpoint and 1 with the right endpoint. In this case our approach reveals a larger boundary-amplitude at $x = 1$ than at $x = 0$, with (cf. (4.5))

$$\widetilde{\phi}(\epsilon) = \frac{1}{\epsilon^2} \quad \text{and} \quad \widehat{\phi}(\epsilon) = \frac{1}{\epsilon^3}, \tag{4.80}$$

where the amplitude follows that of the standard case (4.26) here at the left endpoint $x = 0$ but the boundary amplitude $\widehat{\phi}$ is two orders larger than the corresponding standard case at the right endpoint $x = 1$.

In this example the asymptotic splitting produces the outer equation

$$\begin{aligned} h(x,\epsilon) + \epsilon^3 w^*(x,\epsilon) - \int_0^1 K(x,s)w^*(s,\epsilon)ds \\ = \epsilon\widetilde{\phi}(\epsilon)\int_0^{1/\epsilon}(1-x)\left[-\frac{1+x}{2}+\epsilon\sigma\right]\widetilde{w}(\sigma,\epsilon)d\sigma \\ -\frac{1}{2}\epsilon^3\widehat{\phi}(\epsilon)\int_0^{1/\epsilon}\sigma^2\widehat{w}(\sigma,\epsilon)d\sigma, \end{aligned} \tag{4.81}$$

along with the same boundary-layer equations of (4.55) with solutions $\widetilde{w}$ and $\widehat{w}$ given by (4.59)-(4.60). A routine calculation yields (4.80) along with the following leading-order expansion

$$\begin{aligned} w(x,\epsilon) = \frac{1}{\epsilon^2}\left[A_0\, sin\,\frac{\sqrt{3}x}{2\epsilon} + B_0\, cos\,\frac{\sqrt{3}x}{2\epsilon} + O(\epsilon)\right]e^{-x/(2\epsilon)} \\ + \frac{1}{\epsilon^3}\left[C_0 + O(\epsilon)\right]e^{-(1-x)/\epsilon} + h_0'''(x) + O(\epsilon) \end{aligned} \tag{4.82}$$

with

$$A_0 = -\frac{1}{\sqrt{3}}h_0'(0), \quad B_0 = h_0'(0), \quad C_0 = -h_0(1). \tag{4.83}$$

Example 4.4: *A Nonstandard Equation* (exponentially large solution). As a final example in this section consider the equation

$$\int_0^x \frac{1}{2}s^2 w(s)ds + \int_x^1 x\left(s-\frac{1}{2}x\right)w(s)ds = h(x,\epsilon) + \epsilon^3 w(x) \tag{4.84}$$

which is (2.1) with kernel

$$K(x,s) = \begin{cases} K^-(x,s) = \frac{s^2}{2} & \text{for } s < x \\ K^+(x,s) = x\left(s-\frac{x}{2}\right) & \text{for } s > x, \end{cases} \tag{4.85}$$

with $\nu = 2$, $J\left[K_{xx}\right] = 1$, and with n and p given by (4.1). Again one checks that the three functions of (4.2) are linearly dependent. Moreover there hold

$$\left.\frac{\partial^j K^-(x,s)}{\partial s^j}\right|_{s=0} = \begin{cases} 1 & \text{for } j = 2 \\ 0 & \text{for } j \neq 2, \end{cases} \tag*{$(4.86)_j$}$$

and

$$\left.\frac{\partial^j K^+(x,s)}{\partial s^j}\right|_{s=1} = \begin{cases} x - \frac{x^2}{2} & \text{for } j = 0 \\ x & \text{for } j = 1 \\ 0 & \text{for } j \geq 2, \end{cases} \tag*{$(4.87)_j$}$$

so that it is ***not possible*** to produce a collection of functions analogous to (4.76) or (4.79) containing three linearly independent functions of the type $(4.86)_j$ and $(4.87)_j$ with 2 ($= n$) such functions associated with the left endpoint. Hence our construction of a boundary-layer solution must generally fail for this equation (4.84), and it is then not surprising that the solution of (4.84) in fact generally fails to exhibit boundary-layer behavior.

Indeed upon differentiation one sees that any solution of the integral equation (4.84) must also satisfy the differential equation

$$\epsilon^3 w'''(x) - w(x) = -h'''(x, \epsilon), \tag{4.88}$$

and the most general solution of (4.88) can be easily represented by variation of parameters in terms of three integration constants that can be determined by inserting the solution back into (4.84). For example in the case

$$h(x, \epsilon) \equiv e^{\beta x} \tag{4.89}$$

the resulting solution is given as

$$\begin{aligned} w(x,\epsilon) = {} & \frac{\beta^3 e^{\beta x}}{1-\epsilon^3\beta^3} - \frac{2e^{-x/(2\epsilon)}\, cos\, \frac{\sqrt{3}(1-x)}{2\epsilon}}{\epsilon^3\left(1-\epsilon^3\beta^3\right)\left(2\, cos\, \frac{\sqrt{3}}{2\epsilon} + e^{-3/(2\epsilon)}\right)} \\ & - e^{(1-x)/(2\epsilon)} \left[\frac{2\beta e^{\beta}(1-\epsilon\beta)\, sin\, \frac{\sqrt{3}x}{2\epsilon}}{\sqrt{3}\epsilon^2\left(1-\epsilon^3\beta^3\right)\left(2\, cos\, \frac{\sqrt{3}}{2\epsilon} + e^{-3/(2\epsilon)}\right)} \right] \\ & - e^{-(1-x)/\epsilon} \left[\frac{\beta e^{\beta}\left[\sqrt{3}(1+\epsilon\beta)\, cos\, \frac{\sqrt{3}}{2\epsilon} + (1-\epsilon\beta)\, sin\, \frac{\sqrt{3}}{2\epsilon}\right] + \sqrt{3}\epsilon^{-1}e^{-1/(2\epsilon)}}{\sqrt{3}\epsilon^2\left(1-\epsilon^3\beta^3\right)\left(2\, cos\, \frac{\sqrt{3}}{2\epsilon} + e^{-3/(2\epsilon)}\right)} \right] \\ & + \beta e^{\beta} e^{-(2-x)/(2\epsilon)} \left[\frac{(1-\epsilon\beta)\, sin\, \frac{\sqrt{3}(1-x)}{2\epsilon} + (1+\epsilon\beta)\, cos\, \frac{\sqrt{3}(1-x)}{2\epsilon}}{\sqrt{3}\epsilon^2\left(1-\epsilon^3\beta^3\right)\left(2\, cos\, \frac{\sqrt{3}}{2\epsilon} + e^{-3/(2\epsilon)}\right)} \right]. \end{aligned} \tag{4.90}$$

This solution is generally *exponentially large* for $0 < x < 1$ except in the case $\beta = 0$. In this latter case there is a layer at $x = 0$ and the solution is exponentially small for $x > 0$ if $cos\,\sqrt{3}/(2\epsilon)$ is bounded away from zero. The solution of the integral equation (4.84) is generally not of the boundary-layer type (2.2)-(2.4) employed here.

Section 5. General Case

In this section we discuss briefly the general equation (cf. (1.33)-(1.34))

$$\int_0^1 K(x,s)w(s)ds = h(x,\epsilon) + \epsilon^{\nu+1}w(x) \tag{5.1}$$

of jump-order ν subject to the earlier assumption (cf. (1.36))

$$|\mathrm{Re}\ \mu(x)| \geq \kappa > 0 \tag{5.2}$$

for the roots $\mu = \mu(x)$ of the equation

$$\mu^{\nu+1} = J[\partial^\nu K/\partial x^\nu](x). \tag{5.3}$$

For solutions of boundary-layer type with bounded outer solution we employ the decomposition (cf. (2.2))

$$w(x,\epsilon) \sim \widetilde{\phi}(\epsilon)\widetilde{w}(\widetilde{x},\epsilon) + w^*(x,\epsilon) + \widehat{\phi}(\epsilon)\widehat{w}(\widehat{x},\epsilon) \tag{5.4}$$

with layer-variables $\widetilde{x} = x/\epsilon$ and $\widehat{x} = (1-x)/\epsilon$ as in (2.3). The layer-corrections $\widetilde{w}$ and $\widehat{w}$ are expected to decay for large values of their variables, resulting in an asymptotic splitting of (5.1) into the outer equation

$$\begin{aligned} h(x,\epsilon) + \epsilon^{\nu+1}w^*(x,\epsilon) &= \int_0^1 K(x,s)w^*(s,\epsilon)ds \\ &+ \widetilde{\phi}(\epsilon)\int_0^1 K^-(x,s)\widetilde{w}(s/\epsilon,\epsilon)ds + \widehat{\phi}(\epsilon)\int_0^1 K^+(x,s)\widehat{w}((1-s)/\epsilon,\epsilon)ds \end{aligned} \tag{5.5}$$

and the two boundary-layer equations

$$\epsilon^\nu \widetilde{w}(\widetilde{x},\epsilon) = \int_{\widetilde{x}}^\infty [K^+ - K^-](\epsilon\widetilde{x},\epsilon\sigma)\,\widetilde{w}(\sigma,\epsilon)d\sigma \tag{5.6}$$

and

$$\epsilon^\nu \widehat{w}(\widehat{x},\epsilon) = -\int_{\widehat{x}}^\infty [K^+ - K^-](1-\epsilon\widehat{x},1-\epsilon\sigma)\,\widehat{w}(\sigma,\epsilon)d\sigma, \tag{5.7}$$

where the functions $K^-(x,s)$ $(s < x)$ and $K^+(x,s)$ $(s > x)$ of (1.3) are assumed to be extended smoothly here for all relevant values of x and s. If such extensions of K^- and K^+ are not convenient then the related splitting procedure described in Part I can be used in terms of suitable Taylor expansions.

The layer equation (5.6) can be differentiated $\nu + 1$ times with respect to $\widetilde{x}$ to give

$$\frac{d^{\nu+1}}{d\widetilde{x}^{\nu+1}}\widetilde{w}(\widetilde{x},\epsilon) - J\left[\frac{\partial^\nu K}{\partial x^\nu}\right](\epsilon\widetilde{x})\widetilde{w}(\widetilde{x},\epsilon) = \epsilon\int_{\widetilde{x}}^\infty \left[\frac{\partial^{\nu+1}}{\partial x^{\nu+1}}[K^+ - K^-](\epsilon\widetilde{x},\epsilon\sigma)\right]\widetilde{w}(\sigma,\epsilon)d\sigma. \tag{5.8}$$

The expansion (cf. (2.8))

$$\widetilde{w}(\widetilde{x},\epsilon) \sim \sum_{j=0}^\infty \widetilde{w}_j(\widetilde{x})\epsilon^j \tag{5.9}$$

is inserted into (5.8) along with appropriate Taylor expansions for the data, and we find a sequence of differential equations of the form

$$\widetilde{w}_j^{(\nu+1)}(\widetilde{x}) - J\left[\frac{\partial^\nu K}{\partial x^\nu}\right](0)\widetilde{w}_j(\widetilde{x}) = \begin{cases} 0 & \text{for } j=0, \\ \widetilde{P}_j(\widetilde{x}) & \text{for } j \geq 1, \end{cases} \tag{5.10$_j$}$$

for suitable functions $\widetilde{P}_j$ which are determined by the data recursively in terms of $\widetilde{w}_k$ for $k \leq j-1$. For example in the case $j=1$ the function $\widetilde{P}_1$ is given as

$$\widetilde{P}_1(\widetilde{x}) = \widetilde{x}J\left[\frac{\partial^\nu K}{\partial x^\nu}\right]'(0)\widetilde{w}_0(\widetilde{x}) + J\left[\frac{\partial^{\nu+1} K}{\partial x^{\nu+1}}\right](0)\int_{\widetilde{x}}^\infty \widetilde{w}_0(\sigma)d\sigma \tag{5.11}$$

which depends only on $\widetilde{w}_0$ (and on the kernel K). The assumption (5.2) implies that the equations $(5.10)_j$ can be solved recursively subject to the decay conditions of (2.10), and we find

$$\widetilde{w}_j(\widetilde{x}) = \sum_{i=1}^n a_{j,i}e^{\mu_i(0)\widetilde{x}} + \widetilde{W}_j(\widetilde{x}) \tag{5.12$_j$}$$

for arbitrary constants $a_{j,i}$ $(i=1,\cdots,n)$, where the $n+p$ roots $\mu_i(x)$ of (5.3) are ordered here so that (cf. (2.11)-(2.12))

$$\begin{aligned} &\operatorname{Re}\mu_i < 0 \quad \text{for} \quad i = 1,2,\cdots,n, \quad \text{and} \\ &\operatorname{Re}\mu_i > 0 \quad \text{for} \quad i = n+1, n+2, \cdots, n+p = \nu+1, \end{aligned} \tag{5.13}$$

and where the function $\widetilde{W}_j$ in $(5.12)_j$ can be given uniquely in terms of earlier $\widetilde{w}_k$ for $k \leq j-1$ $(j \geq 1)$, with

$$\widetilde{W}_0 \equiv 0. \tag{5.14}$$

The construction of the function $\widetilde{w}_j$ of (5.12) from (5.10) and (2.10) yields directly the exponential decay property

$$|\widetilde{w}_j(\widetilde{x})| \leq C_j e^{-\tilde{\kappa}\tilde{x}} \quad \text{for} \quad \widetilde{x} \geq 0 \tag{5.15}$$

for a fixed constant $\tilde{\kappa}$ that can be taken to be any positive number less than all the numbers $\operatorname{Re}[-\mu_i(0)]$ for $i=1,\cdots,n$.

For $\widehat{w}$ we find similarly upon differentiation of (5.7) with respect to $\widehat{x}$,

$$\begin{aligned} \frac{d^{\nu+1}}{d\widehat{x}^{\nu+1}}\widehat{w}(\widehat{x},\epsilon) - (-1)^{\nu+1}J\left[\frac{\partial^\nu K}{\partial x^\nu}\right](1-\epsilon\widehat{x})\widehat{w}(\widehat{x},\epsilon) \\ = (-1)^\nu \epsilon \int_{\widehat{x}}^\infty \left[\frac{\partial^{\nu+1}}{\partial x^{\nu+1}}\left[K^+ - K^-\right](1-\epsilon\widehat{x}, 1-\epsilon\sigma)\right]\widehat{w}(\sigma,\epsilon)d\sigma. \end{aligned} \tag{5.16}$$

The expansion

$$\widehat{w}(\widehat{x},\epsilon) \sim \sum_{j=0}^{\infty} \widehat{w}_j(\widehat{x})\epsilon^j \tag{5.17}$$

is inserted into this equation and we are led similarly to expressions of the form

$$\widehat{w}_j(\widehat{x}) = \sum_{i=n+1}^{n+p} a_{j,i} e^{-\mu_i(1)\widehat{x}} + \widehat{W}_j(\widehat{x}) \qquad (5.18)_j$$

where $\widehat{W}_0 \equiv 0$ and where $\widehat{W}_j$ is determined uniquely in terms of earlier $\widehat{w}_k$ for $k \leq j-1$, for $j \geq 1$. The functions $\widehat{w}_j(\widehat{x})$ satisfy exponential decay properties analogous to (5.15).

The expansion (2.7) for w^* is now inserted into the outer equation (5.5) along with the expansions for h, $\widetilde{w}$ and $\widehat{w}$, and we find

$$\begin{aligned}
&\sum_{j=0}^{\infty} \epsilon^j \left[h_j(x) + w^*_{j-\nu-1}(x) - \int_0^1 K(x,s) w^*_j(s)ds\right] \\
&\qquad \sim \epsilon\widetilde{\phi}(\epsilon) \sum_{j=0}^{\infty} \epsilon^j \sum_{k=0}^{j} \frac{1}{k!}\left[\frac{\partial^k K^-(x,s)}{\partial s^k}\right]_{s=0} \int_0^\infty \sigma^k \widetilde{w}_{j-k}(\sigma)d\sigma \\
&\qquad + \epsilon\widehat{\phi}(\epsilon) \sum_{j=0}^{\infty} \epsilon^j \sum_{k=0}^{j} \frac{1}{k!}\left[\frac{\partial^k K^+(x,s)}{\partial s^k}\right]_{s=1} \int_0^\infty (-\sigma)^k \widehat{w}_{j-k}(\sigma)d\sigma,
\end{aligned} \tag{5.19}$$

where appropriate Taylor expansions in ϵ have been inserted on the right side here for $K^-(x,\epsilon\sigma)$ and $K^+(x,1-\epsilon\sigma)$, and where $w^*_j := 0$ for negative j. As noted in Section 2, upon letting $\epsilon \to 0^+$, we insist that the right side of (5.19) should produce a linear combination of $N = \nu+1 = n+p$ linearly independent functions so that the resulting first-kind equation (2.19) for w^*_0 can be reduced to a more tractable equation of the second kind analogous to (2.24).

The standard case occurs when the following collection of functions is linearly independent (cf. (2.34)),

$$g_k(x) = \begin{cases} \left.\dfrac{\partial^{k-1}K^-(x,s)}{\partial s^{k-1}}\right|_{s=0} & \text{for } k = 1,\cdots,n \\ \left.\dfrac{\partial^{k-n-1}K^+(x,s)}{\partial s^{k-n-1}}\right|_{s=1} & \text{for } k = n+1,\cdots,n+p, \end{cases} \tag{5.20}$$

with $\widetilde{\phi}$ and $\widehat{\phi}$ given as (cf. (2.32))

$$\widetilde{\phi}(\epsilon) = \frac{1}{\epsilon^n} \quad \text{and} \quad \widehat{\phi}(\epsilon) = \frac{1}{\epsilon^p}. \tag{5.21}$$

Indeed with (5.21) we find, upon letting $\epsilon \to 0$ in (5.19),

$$h_0(x) - \int_0^1 K(x,s) w_0^*(s) ds = \sum_{k=1}^{n+p} c_k g_k(x) \tag{5.22}$$

with the functions g_k given by (5.20) and with the constants c_k given as

$$c_k = \begin{cases} \frac{1}{(k-1)!} \int_0^\infty \sigma^{k-1} \widetilde{w}_{n-k}(\sigma) d\sigma & \text{for } k = 1, 2, \cdots, n, \\ \frac{1}{(k-n-1)!} \int_0^\infty (-\sigma)^{k-n-1} \widehat{w}_{n+p-k}(\sigma) d\sigma & \text{for } k = n+1, n+2, \cdots, n+p, \end{cases} \tag{5.23}$$

provided that we impose the conditions

$$\int_0^\infty \sigma^k \widetilde{w}_{j-k}(\sigma) d\sigma = 0 \quad \text{for} \quad k = 0, 1, \cdots, j \quad \text{and for all} \quad j = 0, 1, \cdots, n-2, \tag{5.24}$$

and

$$\int_0^\infty (-\sigma)^k \widehat{w}_{j-k}(\sigma) d\sigma = 0 \quad \text{for} \quad k = 0, 1, \cdots, j \quad \text{and for all} \quad j = 0, 1, \cdots, p-2, \tag{5.25}$$

where these conditions (5.24) and (5.25) are required in this case to eliminate unbounded terms on the right side of (5.19).

The equation (5.22) of first kind can now be replaced by an associated equation analogous to (2.24) of second kind, and *we assume that this latter equation is uniquely solvable* with solution w_0^*. This function w_0^* will also solve (5.22) for suitable constants c_k given as (cf. (2.29))

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \\ c_{n+1} \\ \vdots \\ c_{n+p} \end{pmatrix} = W^{-1}(x_1) \Big[[h_0(x_1), h_0'(x_1), \cdots, h_0^{(\nu)}(x_1)]^T - \int_0^1 [K(x_1,s), K_x(x_1,s), \cdots, \partial^\nu K(x_1,s)/\partial x^\nu]^T w_0^*(s) ds \Big] \tag{5.26}$$

for any fixed $0 \leq x_1 \leq 1$, where the matrix function W is given by (2.31) and is invertible because of the linear independence of the functions (5.20). One can evaluate the constants c_k by solving (numerically) the linear system obtained by pre-multiplying (5.26) by the matrix $W(x_1)$. Note that the right side of (5.26) is independent of the particular x_1 used (see the discussion following (2.28)), and in fact a different $x_1 = x_k$ may be used for each component c_k ($k = 1, \cdots, n+p$) if this is convenient. Moreover if the outer equation is uniquely solvable as assumed here, then the c_k's can often be eliminated from (5.22) by several different approaches based on using different $x_k's$ in various differentiated versions of (5.22), as illustrated by (3.28)-(3.31) and by (4.31)-(4.36). In this way in practice one can sometimes obtain an alternative (though equivalent) integral equation for w_0^* that is simpler to handle than the general equation analogous to (2.24).

Now that the lowest order outer solution $w_0^*(x)$ is uniquely determined for $0 \leq x \leq 1$, we can complete the determination of the lowest order layer-corrections $\widetilde{w}_0$ and $\widehat{w}_0$ given respectively by $(5.12)_0$ and $(5.18)_0$ as

$$\widetilde{w}_0(\widetilde{x}) = \sum_{i=1}^{n} a_i e^{\mu_i(0)\widetilde{x}} \quad \text{and} \quad \widehat{w}_0(\widehat{x}) = \sum_{i=n+1}^{n+p} a_i e^{-\mu_i(1)\widehat{x}} \tag{5.27}$$

for suitable constants $a_i \equiv a_{0,i}$ which must now be determined. Note that any complex-valued constants will always appear in conjugate pairs (if the data h and K are real-valued) so there are exactly n real coefficients to be determined for $\widetilde{w}_0$ and similarly there are p coefficients to be determined for $\widehat{w}$. Turning first to $\widetilde{w}$, the choice $k = n$ in (5.23) along with the choices $k = j$ (for $j = 0, \cdots, n-2$) in (5.24) yield

$$\int_0^\infty \sigma^j \widetilde{w}_0(\sigma) d\sigma = \begin{cases} 0 & \text{for } j = 0, 1, \cdots, n-2, \\ (n-1)! c_n & \text{for } j = n-1, \end{cases} \tag{5.28}$$

and a similar argument for $\widehat{w}$ yields

$$\int_0^\infty (-\sigma)^j \widehat{w}_0(\sigma) d\sigma = \begin{cases} 0 & \text{for } j = 0, 1, \cdots, p-2, \\ (p-1)! c_{n+p} & \text{for } j = p-1. \end{cases} \tag{5.29}$$

The constant c_n is already determined by (5.26), and now (5.28) provides n independent equations which uniquely fix the values of the n coefficients for $\widetilde{w}_0$ in (5.27). Similarly the p equations of (5.29) serve to fix the values of the required coefficients for $\widehat{w}_0$, and the leading terms w_0^*, $\widetilde{w}_0$, $\widehat{w}_0$ are therefore uniquely determined. The procedure can be continued recursively to provide as many terms in the expansions (2.7) and (2.8) as may be required for accuracy, as illustrated in Example 3.1 and Example 4.1 of the previous sections in the standard case.

In a nonstandard case for which the functions of (5.20) are linearly dependent, it may still be possible to obtain a linear combination of $\nu+1 = n+p$ independent functions from the order-unity terms on the right side of (5.19) upon letting $\epsilon \to 0$ with suitable choices for the layer-amplitudes $\widetilde{\phi}$ and $\widehat{\phi}$, with n of the resulting independent functions associated with the left endpoint and p associated with the right endpoint, as illustrated in Example 3.2, Example 3.3, Example 4.2, and Example 4.3; see also the following Example 5.1. In such cases the original integral equation can be expected to have a solution of boundary-layer type as considered here and our technique produces a uniform approximation to the solution for small ϵ.

Finally, if as in Example 4.4 it is not possible simultaneously to obtain n independent functions of the form

$$\left.\frac{\partial^j K^-(x,s)}{\partial s^j}\right|_{s=0} \quad \text{for} \quad j = 0,1,2,\cdots \tag{5.30}$$

and p independent functions of the form

$$\left.\frac{\partial^j K^+(x,s)}{\partial s^j}\right|_{s=1} \quad \text{for} \quad j = 0,1,2,\cdots \tag{5.31}$$

from the order-unity limiting terms on the right side of (5.19), then one can expect that the equation has no boundary-layer solution of the type sought here.

Example 5.1: *A Beam Equation.* The integral equation

$$\begin{aligned}\frac{1}{6}(1-x)^2 \int_0^x s^2[-3x+(1+2x)s]w(s)ds + \frac{1}{6}x^2 \int_x^1 (1-s)^2[-3s+(1+2s)x]w(s)ds \\ = h(x,\epsilon) + \epsilon^4 w(x)\end{aligned} \tag{5.32}$$

is equivalent to the earlier boundary value problem (1.39) from linearized beam theory if the function h is taken as in (1.40) and if there holds $b(x) \equiv 1$ in (1.39). The jump-order is 3 with $J[K_{xxx}] = -1$ so that (cf. (2.12))

$$n = p = 2. \tag{5.33}$$

The equation is not of standard type because

$$\left.\frac{\partial^j K^-(x,s)}{\partial s^j}\right|_{s=0} = \begin{cases} -x(1-x)^2 & \text{for } j=2 \\ (1-x)^2(1+2x) & \text{for } j=3 \\ 0 & \text{for all other } j, \end{cases} \tag{5.34}$$

and

$$\left.\frac{\partial^j K^+(x,s)}{\partial s^j}\right|_{s=1} = \begin{cases} -(1-x)x^2 & \text{for } j=2 \\ x^2(-3+2x) & \text{for } j=3 \\ 0 & \text{for all other } j, \end{cases} \tag{5.35}$$

so that the $n+p=4$ functions g_k of (5.20) all vanish. However the following functions

$$g_1(x) = K^-_{ss}(x,s)|_{s=0} = -x(1-x)^2, \quad g_2(x) = K^-_{sss}(x,s)|_{s=0} = (1+2x)(1-x)^2,$$
$$g_3(x) = K^+_{ss}(x,s)|_{s=1} = (x-1)x^2, \quad g_4(x) = K^+_{sss}(x,s)|_{s=1} = (2x-3)x^2 \tag{5.36}$$

are linearly independent with 2 functions associated with each endpoint, and so our technique suffices to handle (5.32).

There holds $K^+(x,s) - K^-(x,s) = \frac{1}{6}(x-s)^3$ so that (5.8) and (5.16) lead to the layer equations

$$\widetilde{w}^{(4)}(\widetilde{x},\epsilon) + \widetilde{w}(\widetilde{x},\epsilon) = 0, \qquad \widehat{w}^{(4)}(\widehat{x},\epsilon) + \widehat{w}(\widehat{x},\epsilon) = 0 \tag{5.37}$$

with general, decaying solutions given by the expansions (5.9) and (5.17) with coefficients

$$\widetilde{w}_j(\widetilde{x}) = e^{-\widetilde{x}/\sqrt{2}}\left[A_j\, cos\, \frac{\widetilde{x}}{\sqrt{2}} + B_j\, sin\, \frac{\widetilde{x}}{\sqrt{2}}\right] \tag{5.38$_j$}$$

and

$$\widehat{w}_j(\widehat{x}) = e^{-\widehat{x}/\sqrt{2}}\left[C_j\, cos\, \frac{\widehat{x}}{\sqrt{2}} + D_j\, sin\, \frac{\widehat{x}}{\sqrt{2}}\right], \tag{5.39$_j$}$$

for suitable constants A_j, B_j, C_j, D_j. The outer equation (5.19) becomes with (5.34) and (5.35),

$$\begin{aligned}
&\sum_{j=0}^{\infty} \epsilon^j \left[h_j(x) + w^*_{j-4}(x) - \int_0^1 K(x,s) w^*_j(s)ds\right] \\
&\quad \sim -\frac{1}{2}\epsilon^3\left[\widetilde{\phi}(\epsilon)x(1-x)^2 \int_0^\infty \sigma^2 \widetilde{w}_0(\sigma)d\sigma + \widehat{\phi}(\epsilon)x^2(1-x)\int_0^\infty \sigma^2\widehat{w}_0(\sigma)d\sigma\right] \\
&\quad + \epsilon\widetilde{\phi}(\epsilon)\sum_{j=3}^{\infty}\epsilon^j\left[-\frac{x}{2}(1-x)^2\int_0^\infty \sigma^2\widetilde{w}_{j-2}(\sigma)d\sigma + \frac{1+2x}{6}(1-x)^2\int_0^\infty \sigma^3\widetilde{w}_{j-3}(\sigma)d\sigma\right] \\
&\quad - \epsilon\widehat{\phi}(\epsilon)\sum_{j=3}^{\infty}\epsilon^j\left[\frac{1-x}{2}x^2\int_0^\infty \sigma^2\widehat{w}_{j-2}(\sigma)d\sigma + \frac{2x-3}{6}x^2\int_0^\infty \sigma^3\widehat{w}_{j-3}(\sigma)d\sigma\right].
\end{aligned} \tag{5.40}$$

It is not possible to obtain a linear combination of $\nu+1=4$ functions from the leading terms on the right side of (5.40) and so we must set those terms to zero by imposing the conditions

$$\int_0^\infty \sigma^2\widetilde{w}_0(\sigma)d\sigma = 0, \qquad \int_0^\infty \sigma^2\widehat{w}_0(\sigma)d\sigma = 0. \tag{5.41}$$

Now the leading terms of (5.40) produce a linear combination of the 4 independent functions of (5.36) with the choices

$$\widetilde{\phi}(\epsilon) = \widehat{\phi}(\epsilon) = \frac{1}{\epsilon^4}. \tag{5.42}$$

Indeed (5.40), (5.41) and (5.42) lead directly to the outer equations

$$\int_0^1 K(x,s)w_j^*(s)ds = h_j(x) + w_{j-4}^*(x) - \sum_{k=1}^{4} c_{j,k}g_k(x) \tag{5.43$_j$}$$

for $j = 0, 1, \cdots$ with the functions g_k given by (5.36) and with the constants $c_{j,k}$ ($k = 1, 2, 3, 4$) given as

$$\begin{aligned} c_{j,1} &= \frac{1}{2}\int_0^\infty \sigma^2 \widetilde{w}_{j+1}(\sigma)d\sigma, \quad c_{j,2} = \frac{1}{6}\int_0^\infty \sigma^3 \widetilde{w}_j(\sigma)d\sigma \\ c_{j,3} &= \frac{1}{2}\int_0^\infty \sigma^2 \widehat{w}_{j+1}(\sigma)d\sigma, \quad c_{j,4} = -\frac{1}{6}\int_0^\infty \sigma^3 \widehat{w}_j(\sigma)d\sigma. \end{aligned} \tag{5.44$_j$}$$

Putting $x = 0$ and then $x = 1$ in both $(5.43)_j$ and in the equation obtained by differentiating $(5.43)_j$, we find that these constants can be represented as

$$\begin{aligned} c_{j,1} &= -\left[h_j'(0) + w_{j-4}^{*\prime}(0)\right], \quad c_{j,2} = h_j(0) + w_{j-4}^*(0) \\ c_{j,3} &= h_j'(1) + w_{j-4}^{*\prime}(1), \quad c_{j,4} = -\left[h_j(1) + w_{j-4}^*(1)\right], \end{aligned} \tag{5.45$_j$}$$

and then $(5.43)_j$ and these results determine the outer function $w_j^*(x)$ as

$$w_j^*(x) = -\frac{d^4}{dx^4}\left[h_j(x) + w_{j-4}^*(x)\right]. \tag{5.46$_j$}$$

The equations (5.41), $(5.44)_j$ and $(5.45)_j$ can be used recursively to determine the constants A_j, B_j, C_j, D_j in $(5.38)_j$ and $(5.39)_j$. Indeed, the first equation of (5.41) along with the equations involving $c_{0,2}$ of $(5.44)_0$ and $(5.45)_0$ provide the system

$$\int_0^\infty \sigma^2 \widetilde{w}_0(\sigma)d\sigma = 0 \quad \text{and} \quad \int_0^\infty \sigma^3 \widetilde{w}_0(\sigma)d\sigma = 6h_0(0). \tag{5.47}$$

The expression $(5.38)_0$ for $\widetilde{w}_0$ can be inserted into (5.47) and the resulting two equations serve to determine the values

$$A_0 = B_0 = -h_0(0). \tag{5.48}$$

Similarly, from the second equation of (5.41) along with the equations involving $c_{0,4}$ of $(5.44)_0$ and $(5.45)_0$, we find for the constants in $(5.39)_0$,

$$C_0 = D_0 = -h_0(1). \tag{5.49}$$

Similarly at the next level we find for the constants in $(5.38)_1$ and $(5.39)_1$,

$$\begin{aligned} A_1 &= -h_1(0), \quad B_1 = -h_1(0) - \sqrt{2}h_0'(0) \\ C_1 &= -h_1(1), \quad D_1 = -h_1(1) + \sqrt{2}h_0'(1). \end{aligned} \tag{5.50}$$

For example the results for A_1 and B_1 are obtained by solving the two equations (cf. (5.44)-(5.45))

$$\int_0^\infty \sigma^2 \widetilde{w}_1(\sigma)d\sigma = -2h_0'(0) \quad \text{and} \quad \int_0^\infty \sigma^3 \widetilde{w}_1(\sigma)d\sigma = 6h_1(0). \tag{5.51}$$

The process can be continued to obtain the constants in $(5.38)_j$ and $(5.39)_j$ for as many values of j as are required for accuracy in the resulting expansion for the solution of the integral equation.

If the integral equation (5.32) originates with the boundary value problem (1.39) (with $b = 1$ there), then the expansion (1.34) for the function h satisfies (cf. (1.40))

$$h_j(x) = \begin{cases} \int_0^1 K(x,s)f(s)ds & \text{for } j = 0, \\ -[\alpha_0 + (2\alpha_0 + \alpha_1)x](1-x)^2 - [\beta_0 + (2\beta_0 - \beta_1)(1-x)]x^2 & \text{for } j = 4 \\ 0 & \text{for all other } j. \end{cases} \tag{5.52}$$

It follows with (5.48) and (5.49) that $A_0 = B_0 = C_0 = D_0 = 0$. In fact the following constants vanish by (5.45),

$$c_{j,1} = c_{j,2} = c_{j,3} = c_{j,4} = 0 \quad \text{for} \quad j = 0,1,2,3, \tag{5.53}$$

and then (5.41) and (5.44) yield

$$A_j = B_j = C_j = D_j = 0 \quad \text{for} \quad j = 0,1,2,3. \tag{5.54}$$

The outer functions $w_j^*(x)$ of (5.46) are given as

$$w_j^*(x) = \begin{cases} f(x) & \text{for } j = 0 \\ 0 & \text{for } j = 1,2,3 \\ -f^{(4)}(x) & \text{for } j = 4 \\ 0 & \text{for } j = 5,6,7, \end{cases} \tag{5.55}$$

and so forth, and then (5.44) and (5.45) yield directly

$$A_4 = B_4 = \alpha_0 - f(0), \qquad C_4 = D_4 = \beta_0 - f(1), \tag{5.56}$$

and

$$A_5 = 0, \quad B_5 = \sqrt{2}\left[\alpha_1 - f'(0)\right], \quad C_0 = 0, \quad D_5 = \sqrt{2}\left[-\beta_1 + f'(1)\right]. \tag{5.57}$$

The solution w satisfies

$$\begin{aligned} w(x,\epsilon) = {} & e^{-\frac{x}{\sqrt{2\epsilon}}}\left[(\alpha_0 - f(0))\left(cos\,\frac{x}{\sqrt{2}\epsilon} + sin\,\frac{x}{\sqrt{2}\epsilon}\right) + \sqrt{2}\epsilon(\alpha_1 - f'(0))\,sin\,\frac{x}{\sqrt{2}\epsilon}\right] \\ & + e^{-\frac{1-x}{\sqrt{2\epsilon}}}\left[(\beta_0 - f(1))\left(cos\,\frac{1-x}{\sqrt{2}\epsilon} + sin\,\frac{1-x}{\sqrt{2}\epsilon}\right) + \sqrt{2}\epsilon(f'(1) - \beta_1)\,sin\,\frac{1-x}{\sqrt{2}\epsilon}\right] \\ & + f(x) + O\left(\epsilon^2\right), \end{aligned} \tag{5.58}$$

so that the layer-amplitudes and the solution are bounded in this case. The present analysis also handles the more general problem with any smooth positive-valued coefficient function $b(x)$ in (1.39); details are omitted.

By way of comparison, if the forcing function in equation (5.32) is taken instead as

$$h(x,\epsilon) \equiv h_0(x) := e^{\beta x} \tag{5.59}$$

for some fixed constant β, then the above procedure leads directly to the results

$$\begin{array}{llll} A_0 = -1, & B_0 = -1, & C_0 = -e^{\beta}, & D_0 = -e^{\beta}, \\ A_1 = 0, & B_1 = -\sqrt{2}\beta, & C_1 = 0, & D_1 = \sqrt{2}\beta e^{\beta}, \\ A_2 = 0, & B_2 = 0, & C_2 = 0, & D_2 = 0, \\ A_3 = 0 & B_3 = 0, & C_3 = 0, & D_4 = 0, \end{array} \tag{5.60}$$

along with

$$w_j^*(x) = \begin{cases} -\beta^4 e^{\beta x} & \text{for } j = 0, \\ 0 & \text{for } j = 1, 2, 3, \end{cases} \tag{5.61}$$

so that the solution satisfies

$$\begin{aligned} w(x,\epsilon) &= -\beta^4 e^{\beta x} + O\left(\epsilon^4\right) \\ &+ \frac{1}{\epsilon^4} e^{-x/(\sqrt{2}\epsilon)} \left[-\cos\frac{x}{\sqrt{2}\epsilon} - (1 + \sqrt{2}\beta\epsilon)\sin\frac{x}{\sqrt{2}\epsilon} + O\left(\epsilon^4\right) \right] \\ &+ \frac{e^{\beta}}{\epsilon^4} e^{-(1-x)/(\sqrt{2}\epsilon)} \left[-\cos\frac{1-x}{\sqrt{2}\epsilon} - (1 - \sqrt{2}\beta\epsilon)\sin\frac{1-x}{\sqrt{2}\epsilon} + O\left(\epsilon^4\right) \right]. \end{aligned} \tag{5.62}$$

In this case the layer-amplitudes are $O\left(1/\epsilon^4\right)$ as suggested by (5.42). The result (5.62) is in agreement with the explicitly known exact solution which can be readily obtained in this case by solving the appropriate 4^{th}-order boundary value problem obtained from (5.32) by repeated differentiation.

The solution (5.58) of the integral equation (5.32) is of order unity when the forcing function h arises from the conversion (via the Green function (1.42)) of the boundary value problem (1.39) into the integral equation, resulting thereby in the special forcing function (1.40). However, small errors in evaluating this forcing function during a numerical computation of the solution of the integral equation will generally be amplified substantially in the layer regions where such errors can be magnified by a factor of ϵ^{-4} (cf. (5.62)). Special care must therefore be taken when using such an integral equation formulation in the numerical computation of the solution of a stiff boundary value problem.

Section 6. Modification Due to Nontrivial Null Space

It is possible for the kernel K in (1.1) to have a nontrivial null space. Olmstead and Angell [7] discuss the nice example

$$\int_0^x 2(1-x)(s-1)w(s)ds + \int_x^1 2x(2-s)w(s)ds = h(x) + \epsilon w(x). \tag{6.1}$$

In this case the kernel

$$K(x,s) = \begin{cases} 2(1-x)(s-1) & \text{for } s < x \\ 2x(2-s) & \text{for } s > x \end{cases} \tag{6.2}$$

has a 1-dimensional null space spanned by the function

$$v(x) = e^{-x} \qquad (Kv = 0 \quad \text{for} \quad v \neq 0). \tag{6.3}$$

Olmstead and Angell show how to solve (6.1) and related problems having a 1-dimensional null space using their perturbation method.

The integral equation (6.1) is equivalent to the boundary value problem

$$\begin{gathered} \epsilon w'' + 2w' + 2w = -h''(x) \quad \text{for} \quad 0 < x < 1 \\ w(0) = -\frac{h(0)}{\epsilon}, \quad w(1) = -\frac{h(1)}{\epsilon}, \end{gathered} \tag{6.4}$$

and one sees that the function $v = e^{-x}$ is a solution of the reduced ($\epsilon = 0$) homogeneous ($h = 0$) differential equation

$$2w' + 2w = 0. \tag{6.5}$$

The dominant differential expression $\epsilon w'' + 2w'$ in (6.4) determines the balance in the layer and governs the scaling of the layer width. The numerical difference between the order of the full differential equation and the order of the reduced differential equation is 1 ($= 2 - 1$), which will imply that the kernel (6.2) has jump-order 0.

A kernel K with a nontrivial null space need not be associated with a boundary value problem for a differential equation. But it is interesting to note that such kernels as are typically obtained by repackaging singularly perturbed boundary value problems as integral equations will generally have this property, so that boundary value problems provide an important source of this phenomenon for integral equations. For example, the third-order boundary value problem

$$\begin{gathered} \epsilon w''' + w'' - w = f(x) \quad \text{for} \quad 0 < x < 1, \\ w(0) = a, \quad w(1) = b, \quad w'(1) = c \end{gathered} \tag{6.6}$$

is equivalent to the integral equation (1.1) with forcing function $h(x,\epsilon) := x(1-x)b - \epsilon\left[(1-x)^2 a + x(2-x)b - x(1-x)c\right] - \int_0^x (1/2)(1-x)^2 s^2 f(s)ds + \int_x^1 (1/2)x[s^2(x-2) - x + 2s]f(s)ds$ and with kernel

$$K(x,s) = \begin{cases} (1-x)^2\left(-2+s^2\right)/2 & \text{for } s < x \\ x\left(4-3x+2s+xs^2-2s^2\right)/2 & \text{for } s > x. \end{cases} \tag{6.7}$$

The reduced homogeneous differential equation is

$$w'' - w = 0 \tag{6.8}$$

with a 2-dimensional solution space spanned by

$$v_1(x) = e^x, \quad v_2(x) = e^{-x}, \tag{6.9}$$

where these same two functions v_1, v_2 also span the null space of the kernel (6.7) with $Kv_j = 0$ for $j = 1, 2$. The difference between the orders of the full and reduced differential equations is 1 $(= 3-2)$, and the kernel (6.7) has jump-order 0.

As another example, the boundary value problem

$$\begin{aligned} \epsilon w''' - w' + w &= f(x) \quad \text{for} \quad 0 < x < 1, \\ w(0) = a, \quad w(1) &= b, \quad w'(1) = c \end{aligned} \tag{6.10}$$

is equivalent to the integral equation (1.1) with forcing function $h(x,\epsilon) := -\epsilon[(1-x)^2 a + x(2-x)b - x(1-x)c] - \int_0^x (1/2)(1-x)^2 s^2 f(s)ds + \int_x^1 (1/2)x[s^2(x-2) - x + 2s]f(s)ds$ and with kernel

$$K(x,s) = \begin{cases} -(1-x)^2 s(2+s)/2 & \text{for } s < x \\ x(-2+2s+x+2s^2-xs^2-2sx)/2 & \text{for } s > x. \end{cases} \tag{6.11}$$

The reduced homogeneous differential equation here is

$$w' - w = 0 \tag{6.12}$$

with 1-dimensional solution space spanned by

$$v_1(x) = e^x, \tag{6.13}$$

where v_1 also spans the null space of the kernel (6.11). The difference between the orders of the full and reduced differential equations is 2 $(= 3-1)$ which will imply that the kernel (6.2) has jump-order 1.

For a more general boundary value problem of boundary-layer type for the linear scalar m^{th}-order differential equation (for integers $m > q > 0$)

$$\epsilon w^{(m)} + \sum_{j=0}^{q} a_j(x) w^{(j)}(x) = f(x) \quad \text{for} \quad 0 < x < 1 \tag{6.14}$$

(with $a_q(x) \neq 0$ for $0 \leq x \leq 1$, to avoid turning points), the reduced q^{th}-order homogeneous differential equation

$$\sum_{j=0}^{q} a_j(x) w^{(j)}(x) = 0 \tag{6.15}$$

will have a q-dimensional solution space spanned by q independent solutions

$$v_1(x), v_2(x), \ldots, v_q(x). \tag{6.16}$$

The difference between the orders of the full and reduced differential equations is $m - q$, and the corresponding integral equation will have kernel K with jump-order $\nu = m - 1 - q$. To see this recast (6.14) as an integral equation of the form (1.1) using the Green function for the operator d^m/dx^m.

When the kernel K of jump-order ν in (2.1) has a q-dimensional null space, we must modify our *ansatz* (2.2) as

$$w(x, \epsilon) \sim \widetilde{\phi}(\epsilon) \widetilde{w}(\widetilde{x}, \epsilon) + \phi^*(\epsilon) \sum_{j=1}^{q} d_j v_j(x) + w^*(x, \epsilon) + \widehat{\phi}(\epsilon) \widehat{w}(\widehat{x}, \epsilon), \tag{6.17}$$

where the outer term $w^*(x)$ in (2.2) is supplemented here with the additional outer terms

$$\phi^*(\epsilon) \sum_{j=1}^{q} d_j v_j(x), \tag{6.18}$$

for a suitable amplitude $\phi^*(\epsilon)$ and for suitable coefficients $d_j = d_j(\epsilon)$ that are expected to have asymptotic expansions in powers of ϵ, with

$$d_j = O(1). \tag{6.19}$$

In the asymptotic splitting, the modified ansatz (6.17) adds the supplementary term

$$\epsilon^{\nu+1} \phi^*(\epsilon) \sum_{j=1}^{q} d_j v_j(x) \tag{6.20}$$

to the left side of (5.5). We anticipate that these additional terms will generally be needed to obtain a solvable equation for the leading term in w^*, in which case (6.19) leads to the amplitude choice

$$\phi^*(\epsilon) = \frac{1}{\epsilon^{\nu+1}}. \tag{6.21}$$

The earlier outer equation (5.22) for the leading term in w^* is now modified as

$$h_0(x) - \int_0^1 K(x,s) w_0^*(s) ds = \sum_{k=1}^{n+p} c_k g_k(x) + \sum_{k=1}^{q} d_k(0) v_k(x), \tag{6.22}$$

where the right side can be interpreted as the summation

$$\sum_{k=1}^{N} c_k g_k(x) \tag{6.23}$$

in (2.13) with

$$N = \nu + 1 + q = n + p + q, \tag{6.24}$$

with the last q of the c_k's and g_k's given in (6.23) by the appropriate d_j's and v_j's. Uniqueness will fail for (6.22), but the resulting asymptotic approximation (6.17) will be unique as a consequence of the choice of the d_j's.

The highest order term $w^{(m)}$ multiplying ϵ in (6.14) can be replaced by a suitable, more general m^{th}-order differential operator applied to w, in which case the present approach based on the integral equation (1.1) generalizes known results for singularly perturbed boundary value problems for scalar differential equations and equivalent first order systems (cf. [4], [8], [9], and other references given in [6]).

Finally we mention that the possibility exists for an integral equation to have a kernel with an infinite dimensional null space. As an example consider the simple 2×2 (Volterra) system

$$\int_0^x \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} w_1(s) \\ w_2(s) \end{pmatrix} ds = \begin{pmatrix} h_1(x) \\ h_2(x) \end{pmatrix} + \epsilon \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix} \tag{6.25}$$

with matrix kernel

$$K(x,s) = \begin{cases} K^- = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} & \text{for } 0 \le s < x \le 1 \\ K^+ = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } 1 \ge s > x \ge 0 \end{cases} \tag{6.26}$$

with nonzero, though singular, jump matrix $J[K] = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$. The null space of the operator with kernel (6.26) consists of vector functions of the form

$$v(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.27}$$

where $v = v(x)$ can be an arbitrary scalar function. Our asymptotic splitting can be carried through based on the solution ansatz (cf. (6.17))

$$\begin{pmatrix} w_1(x,\epsilon) \\ w_2(x,\epsilon) \end{pmatrix} = \widetilde{\phi}(\epsilon) \begin{pmatrix} \widetilde{w}_1(\widetilde{x},\epsilon) \\ \widetilde{w}_2(\widetilde{x},\epsilon) \end{pmatrix} + \phi^*(\epsilon) v(x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} w_1^*(x,\epsilon) \\ w_2^*(x,\epsilon) \end{pmatrix}, \tag{6.28}$$

and we find

$$\begin{pmatrix} w_1(x,\epsilon) \\ w_2(x,\epsilon) \end{pmatrix} = \frac{1}{\epsilon}\left(-f_1(0) + \epsilon f_1'(0)\right) e^{-x/\epsilon} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\epsilon}(f_1(x) - f_2(x)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} - f_1'(x) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + O(\epsilon), \tag{6.29}$$

where the asymptotic splitting leads to $\phi^*(\epsilon) = 1/\epsilon$ along with $v(x) = f_1(x) - f_2(x)$ as a solvability condition in the outer equation. The result (6.29) is in agreement with the exact (unique) solution of (6.25).

Section 7. Existence, Uniqueness, and Error estimates

We show that the scalar equation (2.1) of jump-order $\nu > 0$ can be reduced to a vector system of jump-order 0 to which the methods of Part I apply, leading to an existence and uniqueness result for (2.1) along with error estimates on the difference between the exact solution and an approximate solution such as produced by our asymptotic splitting. For brevity we include here only the case $\nu = 1$, for the equation

$$\epsilon^2 w(x,\epsilon) + h(x,\epsilon) = \int_0^1 K(x,s) w(s) ds \tag{7.1}$$

and its differentiated equation

$$\epsilon^2 w'(x,\epsilon) + h'(x,\epsilon) = \int_0^1 K_x(x,s) w(s) ds \tag{7.2}$$

with $J[K](x) \equiv 0$ and $J[K_x](x) \neq 0$ for $0 \leq x \leq 1$.

We introduce the vector function (cf. (7.15))

$$u(x,\epsilon) = \begin{pmatrix} u_1(x,\epsilon) \\ u_2(x,\epsilon) \end{pmatrix} := \epsilon \begin{pmatrix} w(x,\epsilon) \\ \epsilon w'(x,\epsilon) \end{pmatrix} \tag{7.3}$$

for any solution w of (7.1), and find upon integration of $\epsilon^2 w' = u_2$,

$$\epsilon^2 [w(s,\epsilon) - w(0,\epsilon)] = \int_0^s u_2(\sigma,\epsilon) d\sigma. \tag{7.4}$$

We multiply (7.4) by $K^+(0,s)$ and integrate to find

$$\epsilon^2 \int_0^1 K^+(0,s)w(s,\epsilon)ds = \epsilon^2\kappa(0)w(0,\epsilon) + \int_0^1 \kappa(s)u_2(s,\epsilon)ds \tag{7.5}$$

with

$$\kappa(s) := \int_s^1 K^+(0,\sigma)d\sigma, \tag{7.6}$$

and then $\int_0^1 K^+(0,s)w(s,\epsilon)ds$ can be eliminated between (7.6) and (7.1) evaluated at $x = 0$, to yield

$$[\kappa(0) - \epsilon^2]w(0,\epsilon) = h(0,\epsilon) - \epsilon^{-2}\int_0^1 \kappa(s)u_2(s)ds. \tag{7.7}$$

The quantity $\kappa(0) - \epsilon^2$ is nonzero for all small enough ϵ so that (7.7) can be used to eliminate $w(0,\epsilon)$ in (7.4). The resulting equation gives with (7.3),

$$\epsilon u_1(x,\epsilon) = \left[\kappa(0) - \epsilon^2\right]^{-1}\left[\epsilon^2 h(0,\epsilon) - \int_0^1 \kappa(s)u_2(s)ds\right] + \int_0^x u_2(s,\epsilon)ds, \tag{7.8}$$

while (7.2) and (7.3) yield

$$\epsilon u_2(x,\epsilon) + \epsilon h'(x,\epsilon) = \int_0^1 K_x(x,s)u_1(s,\epsilon)ds. \tag{7.9}$$

It follows that the vector function u of (7.3) satisfies the system

$$\epsilon\begin{pmatrix} u_1(x,\epsilon) \\ u_2(x,\epsilon)\end{pmatrix} + \begin{pmatrix}(\epsilon^2 - \kappa(0))^{-1}\epsilon^2 h(0,\epsilon) \\ \epsilon h'(x,\epsilon)\end{pmatrix} = \int_0^1 \mathcal{K}(x,s,\epsilon)\begin{pmatrix} u_1(s,\epsilon) \\ u_2(s,\epsilon)\end{pmatrix} ds \tag{7.10}$$

with matrix kernel

$$\mathcal{K}(x,s,\epsilon) = \mathcal{K}_0(x,s) + \mathcal{K}_1(s,\epsilon) \tag{7.11}$$

given with

$$\mathcal{K}_0(x,s) = \begin{cases} \begin{pmatrix} 0 & 1 \\ K_x^-(x,s) & 0\end{pmatrix} & \text{for } s < x \\ \begin{pmatrix} 0 & 0 \\ K_x^+(x,s) & 0\end{pmatrix} & \text{for } s > x,\end{cases} \tag{7.12}$$

and

$$\mathcal{K}_1(s,\epsilon) = \begin{pmatrix} 0 & -\left(\kappa(0) - \epsilon^2\right)^{-1}\kappa(s) \\ 0 & 0\end{pmatrix}. \tag{7.13}$$

Note that $\mathcal{K}$ is unrelated here to the earlier $\mathcal{K}$ of (2.24)-(2.25). The matrix function $\mathcal{K}_1$ is independent of x and smooth with no jump, $J[\mathcal{K}_1](x) \equiv 0$, so that

$$J[\mathcal{K}](x) = J[\mathcal{K}_0](x) = \begin{pmatrix} 0 & 1 \\ J[K_x](x) & 0\end{pmatrix}, \tag{7.14}$$

where this latter jump matrix is smooth and invertible, uniformly for $0 \le x \le 1$. The scalar equation (7.1) of jump-order 1 is thus equivalent, via (7.3), to the vector equation (7.10)-(7.13) of jump-order 0. For the more general equation (2.1) of jump-order ν, the relation (7.3) is replaced by

$$u(x,\epsilon) = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{\nu+1} \end{pmatrix} := \epsilon^{\nu} \begin{pmatrix} w \\ \epsilon w' \\ \vdots \\ \epsilon^{\nu} w^{(\nu)} \end{pmatrix}. \tag{7.15}$$

Our asymptotic procedure can be extended so as to handle such kernels as (7.11) depending on ϵ, and then a modified version of the theorem of Section 7 of Part I holds for the vector problem (7.10). In this way we obtain an existence and uniqueness result including error estimates for (7.1) subject to the assumptions of Part I. Alternatively the method of proof used in Part I can be applied directly to (7.1) without repackaging the latter equation as the system (7.10). Details are omitted. A related theorem is given in [1], based on the work of Èskin [2, 3].

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