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C. K. Chu
C. C. Wu

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

MAGNETOHYDRODYNAMIC SHOCK WAVES REVISITED

C. K. Chu*, C. C. Wu**, UCLA

Introduction

The equations of ideal fluid magnetohydrodynamics i.e., the flow of a fluid with no viscosity, heat conductivity, or electrical resistivity, form a set of hyperbolic conservation laws, and they approximate the motion of hot ionized gases or plasmas in stars, solar wind, earth's magnetosphere, or fusion experiments. The discontinuous solutions that form are called magnetohydrodynamic shocks, and they bear the same relations to the basic equations as do ordinary gas dynamic shocks to the Euler equations.

However, in the 1960's, it was discovered that there are too many such discontinuous solutions that all satisfy the (thermodynamic) entropy increase condition. Thus, the solutions to the piston problems, Riemann problems, etc. will all become nonunique, see e.g. [1]. Additional criteria were proposed to select the admissible shocks, based either on stability of the linearized perturbation problem (evolutionary) or on the existence of structure with reasonably arbitrary dissipation included. With these criteria, the extraneous shocks (usually called intermediate shocks) were eliminated, and the solution to initial and mixed initial-boundary value problems again appear unique. In some numerical experiments, Chu and Taussig[2] showed the connection between structure and evolutionarity.

More recently, Wu and Brio [3] noticed the formation of intermediate shocks in many numerical calculations. They suggested that the existence of these shocks and the resulting nonuniqueness of initial value problems are a consequence of the nonconvexity of the hyperbolic system, an important feature hitherto unnoticed. Wu then reconsidered many of the properties of MHD shocks, and reached many new conclusions: the instability of the Alfven wave, the stability of the intermediate shocks, the existence of noncoplanar shocks, and the occurrence of shocks that do not satisfy Hugoniot conditions (e.g. [4]).

This paper presents some results of new numerical experiments, with the aid of which we strengthen several of the earlier conclusions of Wu on the occurrence of the intermediate shocks; in particular, we show further the history-dependent nature of these shocks. We also confirm the existence of noncoplanar shocks. For the Alfven wave and the nonstationary shocks, we offer an alternate interpretation that does not require the abandoning of the Hugoniot conditions. This paper is dedicated to H. O. Kreiss on his 60th birthday as a token of friendship and admiration.

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Equations of Ideal Magnetohydrodynamics

The equations of ideal fluid magnetohydrodynamics, generally known as the Lundquist equations, are expressible in simple form in one space dimension

$$\partial U/\partial t + A\,\partial U/\partial x = 0$$

where $U = (\rho, p, u, v, w, B_y, B_z)$, with ρ and p the density and pressure, u, v, w the x, y, z components of the velocity, and By, Bz the y and z components of the magnetic field. A= A(U) is a 7 x 7 coefficient matrix. The seven equations are, respectively, the conservation of mass, conservation of energy, conservation of momentum (three components), and the two components of the Maxwell's equation $\partial B/\partial t$ - curl E = 0.

The electric field E does not appear, because the Ohm's law for a perfectly conducting moving fluid E + u x B = 0 permits its elimination. The current density j does not appear, because by neglecting the displacement current, the Maxwell's equation j = curl B permits the elimination of j. Also, B_x is not an unknown; because of the Maxwell's equation div B = 0, it only appears in the A matrix as a constant.

The characteristics are the magnetohydrodynamic wave speeds. Because of the presence of the magnetic field B, they are highly anisotropic, and are best presented in a Friedrichs diagram, Fig. 1 [5]. Here a is the usual sound speed, and $A = (B^2/\mu\rho)^{1/2}$ is the Alfven speed. The three MHD wave speeds are represented by the three intersections of the ray in the x-direction, the direction of the wave front normal, and they are called the fast MHD wave, Alfven wave, and slow Alfven wave respectively. (There is also the usual particle path). The eigenspaces of these waves are disjoint: the fast and slow waves have changes in velocities, transverse field magnitudes, and density and pressure, while the Alfven waves have only a rotation of the transverse field and transverse velocity, but no changes in their magnitudes and no changes in the density or pressure.

The system can be written in the usual conservation form

$$\partial U/\partial t + \partial F(U)/\partial x = 0$$

from which one derives Hugoniot conditions for discontinuities. The solution of these nonlinear algebraic equations are the MHD shock waves. Particular cases of these shocks have been studied in the early 1950's, but the first systematic study of MHD waves and shocks from the viewpoint of quasilinear hyperbolic equations was made by Friedrichs in 1954 [5].

MHD Shock Waves -- Classical Theory

We summarize very briefly the state of knowledge of MHD shocks through the middle 1960's. The discontinuous solutions from the Hugoniot conditions must first satisfy the thermodynamic entropy rise (or at least, nondecrease) condition. This condition eliminates all the expansion shocks, exactly as in ordinary gas dynamics, but no more. The remaining admissible solutions can be classified into many families, which is most easily achieved by refering to Fig. 1, where we define the four regions **1** - **4** of fluid velocity as super-fast, sub-fast but super-Alfven, sub-Alfven but super-slow, or sub-slow respectively. Each MHD shock represents a transition of the fluid velocity from one of these four regions into another; the Friedrichs diagrams on the two sides of the shock are different, but each has these four regions uniquely defined.

For all subsequent discussion, it is most convenient to make a Galilean transformation to align the field and velocity; the continuity of tangential E field across the front insures that this can be done for both sides simultaneously. MHD shock waves can be classified thus: (1) Fast shocks: the velocity changes from region **1** to region **2**, (Fig. 1) and (together with the magnetic field) bends away from the normal (Fig. 2). (2) Slow shocks: the velocity changes from region **3** to region **4**, and bends toward the normal. (3) Intermediate or trans-Alfvenic shocks: the velocity changes from **1** or **2** to **3** or **4**, thereby crossing the Alfven wave circle, and bends beyond the normal. These are all compressive and entropy increasing solutions. In addition, there are: (4) the usual contact discontinuities, in which normal stress (pressure plus magnetic) and normal velocities must be continuous, but the density, temperature, etc. can have arbitrary jumps, and (5) Alfven discontinuities, or finite amplitude Alfven waves, with transverse field and velocity rotation but no changes in thermodynamic state. The former three classes of shocks are nonlinear, but the latter two classes of discontinuities are linear and do not steepen. This fact for the Alfven discontinuity will have important implications later on.

With these discontinuities, the solutions to piston problems, Riemann problems, etc. are nonunique. There are too many shocks. To overcome this nonuniqueness, additional criteria must be invoked.

The more common and simple criterion was introduced by many Soviet workers (e.g. Akhiezer, Polovin,[6] etc.) and detailed in [1]. It is usually called evolutionarity, and requires that a linearized perturbation of the shock possess a unique solution. This immediately translates into counting characteristics, that the number emanating from the shock path in the x-t plane should be one less than the number of Hugoniot conditions (the unknowns are the linearized Riemann invariants outgoing from the shock path and the perturbed shock speed). Under this criterion, the intermediate shocks were rejected, and all the others remain; the solutions to initial value problems then became unique.

The other criterion, introduced by Germain [7], is that with reasonably arbitrary dissipation coefficients, the discontinuities should have a structure. The intermediate shocks again turn out to have structure only for special values of dissipation coefficients, or to have nonunique structure, and again were rejected.

The only exception not answered by either criterion was the switch-on or switch-off shock, where the field (and velocity) is normal to the shock on one side and oblique on the other side. This was a limiting case for the evolutionarity criterion, and could not be answered by the structure criterion. In 1966, Chu and Taussig[2] resolved this problem by numerical experimentation. They showed that in the case of intermediate shocks, the nonunique shock structure depends on the history of the perturbations that the shock has experienced: since there are not enough characteristics to carry away the perturbing field, the field builds up inside the structure to different levels, then saturates and splits the shock. In the case of the switch-on shock, the perturbations repolarize the plane of the shock, but does not break up the shock; in fact the thermodynamic structure of the shock is stable. Thus, by the late 1960's, the nature of MHD shocks has been completely understood.

Recent Conclusions

In 1985, Brio and Wu [3] found the existence of intermediate shocks in numerical calculations of the MHD Riemann problem, using many different schemes. They also noticed the important fact that the MHD equations are not genuinely nonlinear, since the coefficient matrix lacks convexity. They suggested that the nonevolutionarity of the intermediate shocks and the nonconvexity of the equations are related. In a long series of papers [4], Wu made detailed studies of MHD shocks, and came to many new surprising conclusions, which we summarize very briefly:

1. The MHD equations are not genuinely nonlinear, because of the nonconvexity.

2. Intermediate shocks should not be rejected, as recommended in the classical theory. All of the transitions, from regions **1→3, 2→3, 1→4, 2→4,** can be formed from steepening. Their structure is dissipation dependent and history dependent.

3. The Alfven discontinuity, or rotational discontinuity, is unstable in the presence of dissipation, and breaks up into all kinds of waves.

4. There exist non-coplanar shocks: these are shocks where the upstream and downstream field and velocity are in the same plane, but the field and velocity in the structure have components out of the plane.

5. There exist time-dependent shocks, not satisfying Rankine-Hugoniot conditions, and for these shocks, even the upstream and downstream fields and velocities are not coplanar.

In the present paper, we perform two additional numerical experiments, which permit us to view these features in a different light. Essentially, we confirm and reenforce the conclusions on the nature of intermediate shocks and non-coplanar shocks, and offer an alternate interpretation on the instability of the Alfven wave and the existence of non-Hugoniot shocks.

Intermediate Shocks

The numerical experiment is a repetition of that of Chu and Taussig [2]. The intermediate shock chosen is a normal shock for simplicity. The shock is no different from an ordinary gas dynamic normal shock. except for the presence of a B_x field. The transverse conponents B_y, B_z, v, w all vanish. The upstream and downstream velocities are governed by the usual Hugoniot relations of gas dynamics, as the B_x field is parallel to u and contributes nothing to the steady state. The B_x field is chosen so that the Alfven velocities upstream and downstream are such that the shock represents a transition from region **1** to region **4** in fig. 1.

A steady transverse perturbation B_y of 10% of Bx is applied upstream. As seen in [2]. the transverse field is trapped in the shock structure, and builds up in time. When the transverse field corresponding to a switch on shock is reached, which we denote by B_y*, the shock splits into a switch-on switch-off pair. (Fig.3) If the perturbing field is withdrawn any time before the split, the shock remains a normal shock externally, but contains an amount of B_y and v, corresponding to the total transverse fluxes trapped up to that time. Thus, before splitting, the structure of the shock is nonunique, even though the external Hugoniot conditions remain valid and yield a unique gas dynamic normal shock. This nonuniquneness in the shock structure was discovered as early as 1958 by Ludford [8], though no connection is made to the history of the perturbations that the shock has received.

It was concluded in [2] that such a shock will always split up into a switch-on switch-off pair, because sooner or later, there will be enough transverse flux built up in the shock structure. Thus, the structural property and the stability (evolutionarity) property are shown to be linked. This conclusion, however, is faulty, because in actual physical situations, particularly in space plasmas, the perturbing fields are rarely steady-state, and are nearly always stochastic. Thus, it is necessary to redo the numerical experiment with a perturbing field which is alternating in sign. This was done, and the results are in Fig.3. The transverse field first builds up as in the steady state case, but when the field reverses, the trapped field is annihilated, and then builds up in the reverse direction. In other words, most of the time, such shocks remain unsplit, but has trapped fields in arbitrary directions depending on the history of the perturbation.

A simple analysis clarifies this situation. Fig.5(a) is th u-B_y phase plane for such shocks, taken from Ludford [8]. The singular points 1 and 2 refer respectively to the upstream and downstream states, while 3 refers to the switch-on (and switch-off) state. The straight line 1-2 represents a normal shock with no trapped fields, the curved lines 1-2 represent the many nonunique transitions with trapped fields, while the curve 1-3-2 represents the switch-on switch-off pair. Now splitting occurs when the maximum B_y reaches $B_y{}^*$, which is a function of the upstream conditions only, i.e.

$$B_y{}^* = f(u_o, a_o, A_o)$$

On the other hand, the thickness of the shock δ before splitting is a function of the upstream velocity and sound speed, and is directly proportional to the dissipation ν, i.e.

$$\delta = \nu\ g(u_o, a_o)$$

Hence the total transverse flux at splitting is determined:

$$\Phi_{y^*} = \text{const}\cdot\nu\cdot f(u_o, a_o, A_o)\cdot g(u_o, a_o)$$

Now when the shock is subject to perturbations as shown in Fig.5(b), splitting will occur only when any one of the fluxes in the perturbing field exceeds this value of $\Phi_y{}^*$. Otherwise, no splitting occurs, and the intermediate shock remains stable and thus occurs in nature.

There are claims that these shocks have been seen in the earth's magnetosphere regions. The present calculation not only illustrates but reenforces the conclusions of Brio and Wu [3], that intermediate shocks are real and should not be rejected. The nonuniqueness of the structure has to be resolved from a knowledge of the history of the perturbations with dissipation.

These experiments also confirm and reenforce the conclusions of Wu on the noncoplanar shocks. As stated earlier, noncoplanar shocks are those with upstream and downstream fields and velocities spanning the same plane, e.g. x-y plane, but transverse fields and velocities (B_z and w) appear in the structure. If we had taken an oblique MHD shock, instead of a normal shock, with $w = B_z = 0$ both upstream and downstream, then perturbed it with B_z and w but withdrawing the perturbations before splitting occurs, we would have generated precisely such a non-coplanar shock. Hence, the noncoplanar shocks bear the same relation to the oblique shock as do the shocks with trapped transverse field structures to the normal shock. Their occurrence is reasonable in the same sense as the intermediate normal shocks.

Alfven Discontinuities and Non-Hugoniot Shocks

The other numerical experiment we ran is also a repetition of the previous calculation in [2], namely, the perturbation of a switch-on shock polarized in the x-y plane by a B_z field. More specifically, the switch-on shock has the upstream field and velocity B_x and u only, and the downstream field and velocity in the x-y plane, i.e., B_y and $v \neq 0$; the perturbation then is a constant B_z and w upstream.

Fig. 6 shows the density, pressure, and normal velocity u profiles, and the B_y, B_z and $B_t = (B_y^2 + B_z^2)^{1/2}$ profiles. Clearly, the front part is the switch-on shock repolarized, and the long rear section is an Alfven rotation wave. Note that the pressure profile remains unchanged, while the Alfven wave is a region of nearly constant pressure and nearly constant transverse B field.

This very simple experiment permits us to give an alternate interpretation to the instability of the Alfven discontinuity and to the occurrence of the non-Hugoniot shocks. If we apply the ideal theory naively, we might consider the solution as a repolarized shock followed by an Alfven discontinuity; then according to Fig. 7, taken from [4(b)], the Alfven discontinuity would not be stable. Moreover, in the presence of dissipation, the region of the Alfven wave tail is not exactly uniform, hence we are tempted to consider the entire solution as a single shock, which satisfies no Hugoniot condition.

We first address the instability of the Alfven discontinuity. It is true that an Alfven discontinuity will break up, as shown in Fig. 7 ; physically it is quite clear, since an Alfven discontinuity is a discontinuity in B_y and B_z, thus is a current sheet of infinite current density, which in the presence of dissipation will produce enormous heating, hence resulting in enormous pressure built up and all kinds of waves and shocks.

The proper reinterpretation is that indeed there are no Alfven discontinuities in a dissipative plasma, and giving a Alfven discontinuity as an initial condition is highly artificial. In realistic cases, such as in the present experiment, the rotation is achieved by a finite width Alfven wave. Thus, instead of Alfven discontinuities, we should use finite width Alfven waves, the exact thickness of which depends on its production mechanism and the thickness is time-dependent.

On the other hand, the Alfven wave is not a strictly uniform region, and the density and pressure, for example, can either increase or decrease slightly depending on the speed regime and the dissipation. Thus, it does not satisfy a Hugoniot condition strictly. Yet, this is not surprising, since the Alfven wave is a linear wave and is non-steepening. The situation is similar to, but a bit more complicated than, the uniform flow of a gas. The uniform state of constant pressure and velocity, under the presence of dissipation, has to be changed to one in which the pressure either rises or drops, depending on the velocity regime.

Thus, instead of considering the entire system as a single non-Hugoniot shock, an alternate, more appealing interpretation is to consider it as a switch-on shock, which satisfies Hugoniot conditions, and an Alfven wave, which satisfies dissipative modifications of Hugoniot conditions. Actually, Wu had previously suggested a similar interpretation earlier [4(c)], previously suggested a similar interpretation earlier [4(c)], but the emphasis shifted to non-Hugoniot shocks. While these different interpretations are a matter of personal taste, the present characterisation does permit a more orderly classification of MHD shocks and waves.

On the other hand, it must be admitted that the present case of a switch-on shock being perturbed by an out-of-plane field component is a simple case. If we had taken an oblique intermediate shock, say a **1 - 3** or a **2 - 4** shock, with B_y and B_z components present both upstream and downstream, then it has been shown in [4(c)] that instead of an Alfven wave, there now is a **2 - 3** intermediate shock (with fields noncoplanar upstream and downstream), in addition to a fast **1 - 2** or slow **3 - 4** shock. The **2 - 3** intermediate shock slowly weakens into an Alfven wave, requiring infinite time to achieve this. Thus, this **2 - 3** shock is in one sense a non-Hugoniot shock, but in another sense a dissipative modification of the Alfven wave. Which interpretation one chooses is, perhaps even more than in the previous simpler case, a matter of taste.

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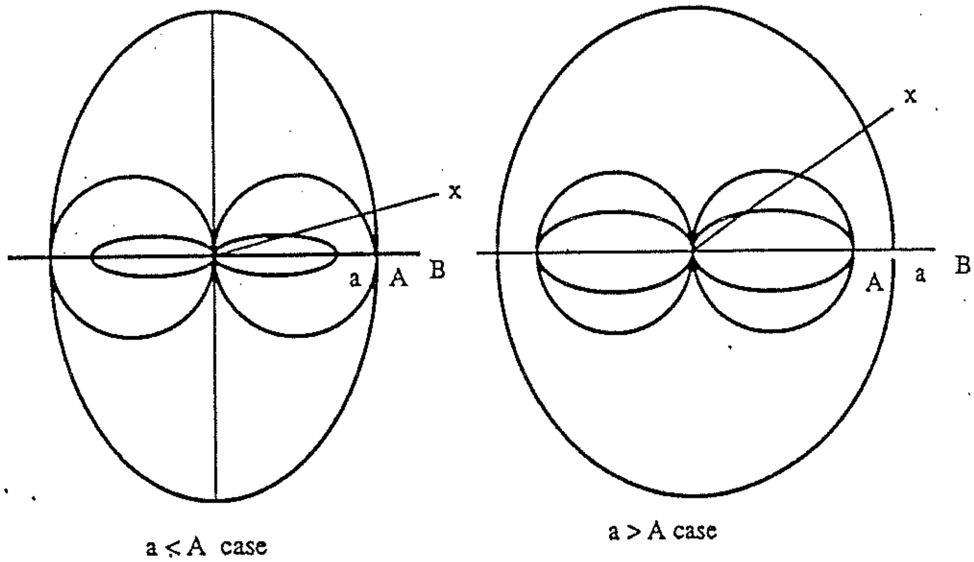

Fig. 1. Friedrichs Diagram or Wave Diagram for MHD

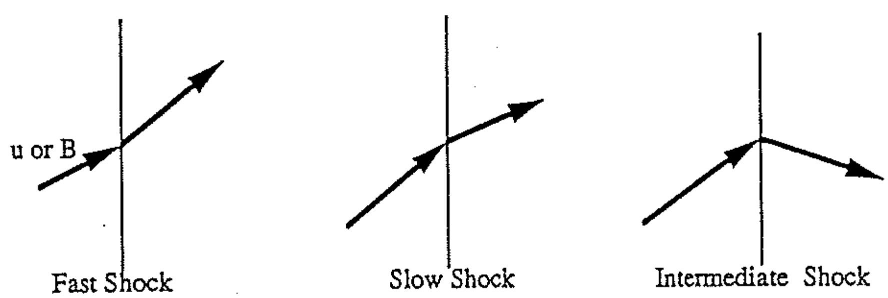

Fig. 2. Fields and Velocities in MHD Shocks

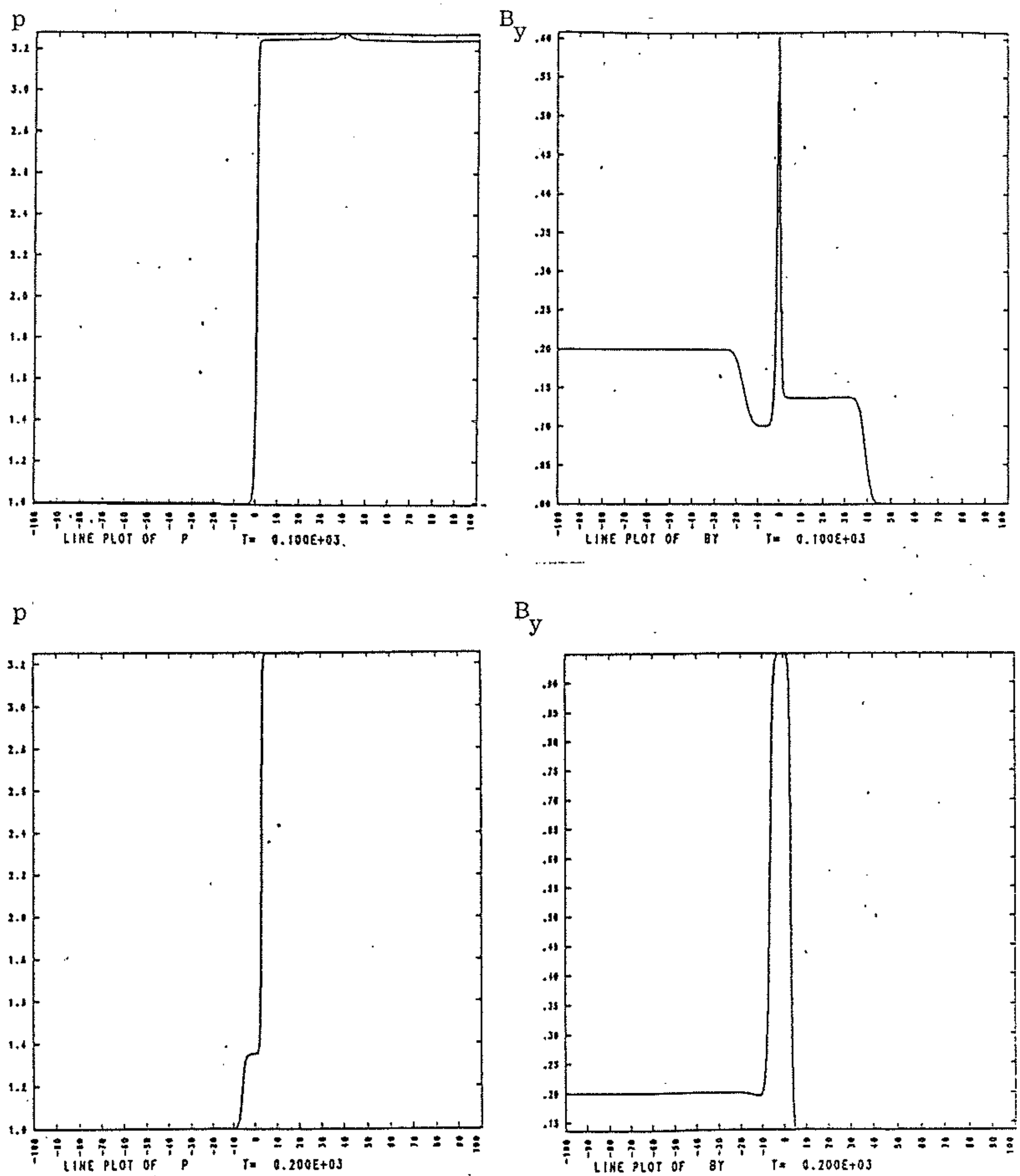

Fig. 3. Intermediate Shock Perturbed by Steady B_y Field. Upper Figures: Early Time, Before Saturation. Lower Figures, After Saturation and Splitting.

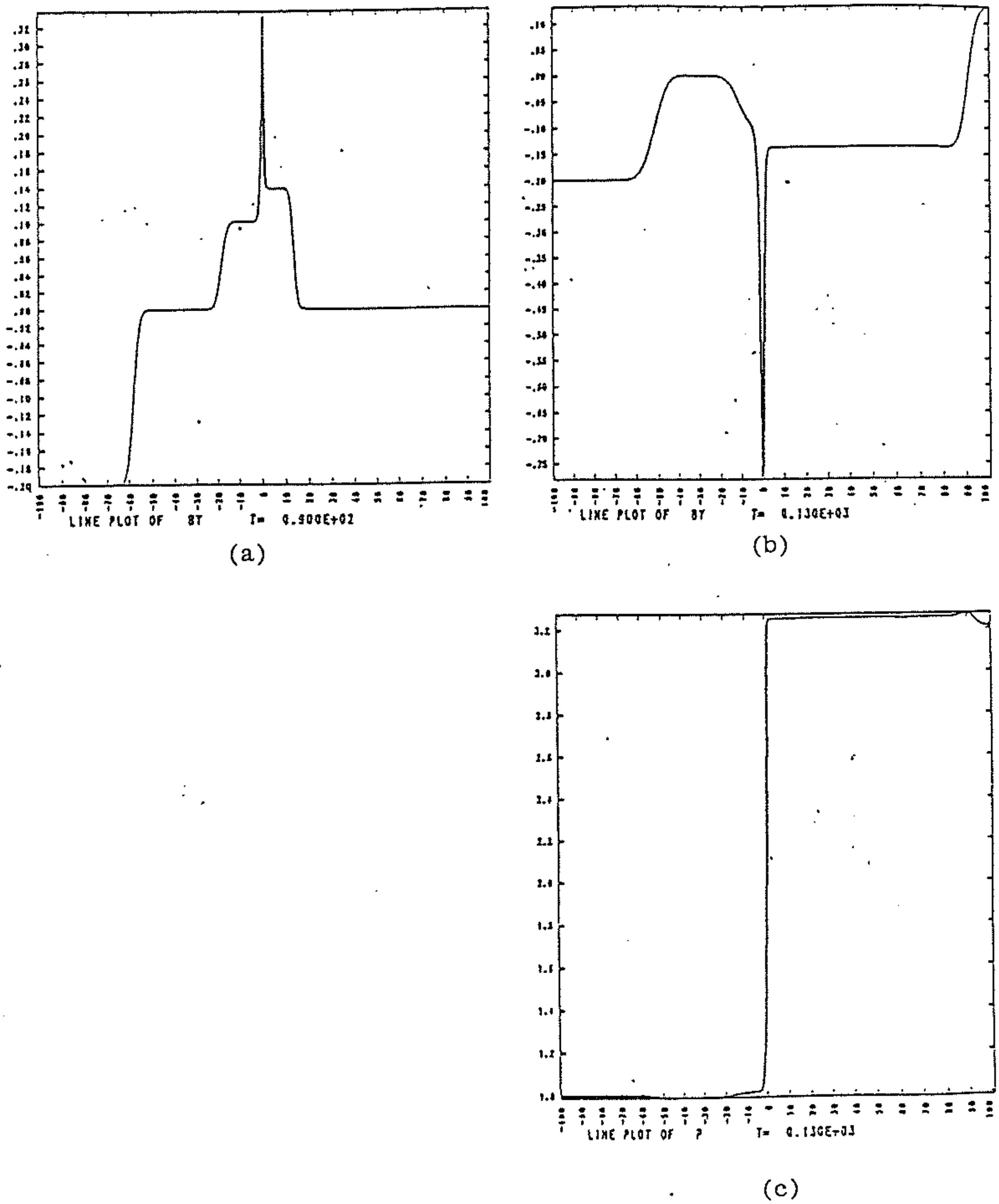

Fig. 4. Intermediate Shock Perturbed by Alternating B_y Field.
(a) B_y in Shock Before Reversal. (b) B_y After Reversal.
(c) Pressure Profile After Reversal --- Not Split.

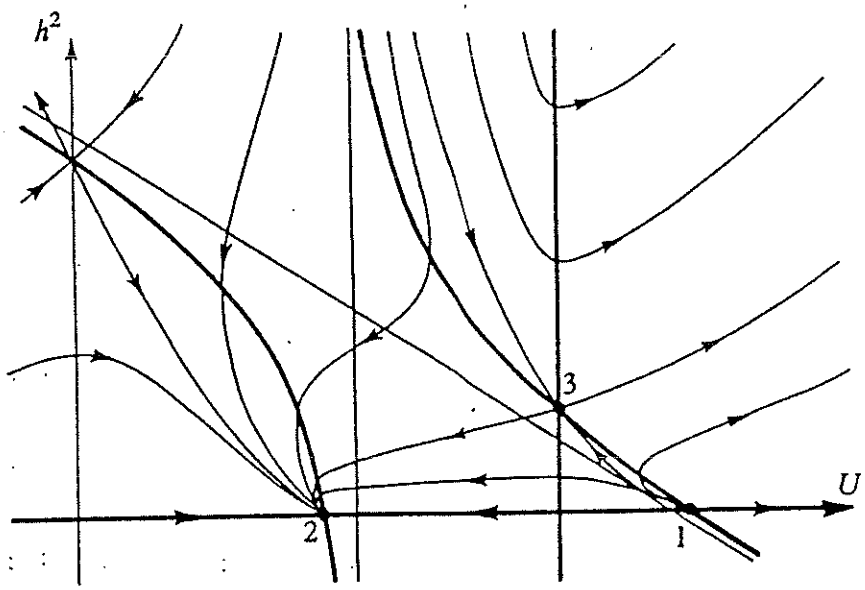

Fig. 5(a) u-B_y Phase Plane for Intermediate Normal Shocks and Switch-On Shock. (From Ludford , Ref. 8.

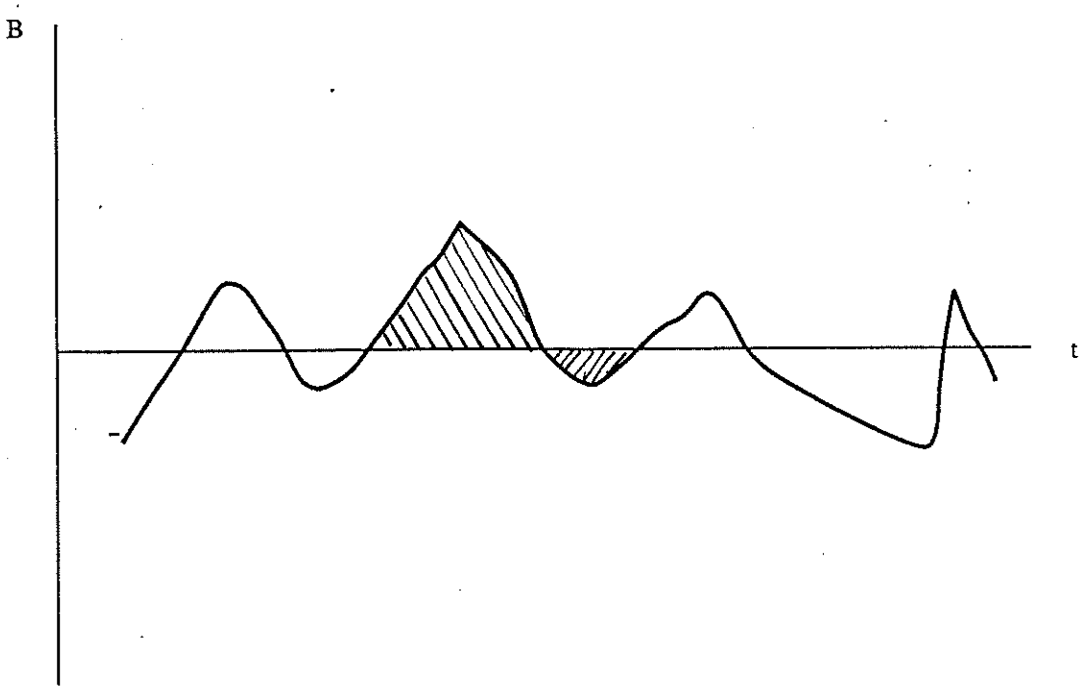

Fig. 5(b) Stochastic Fields

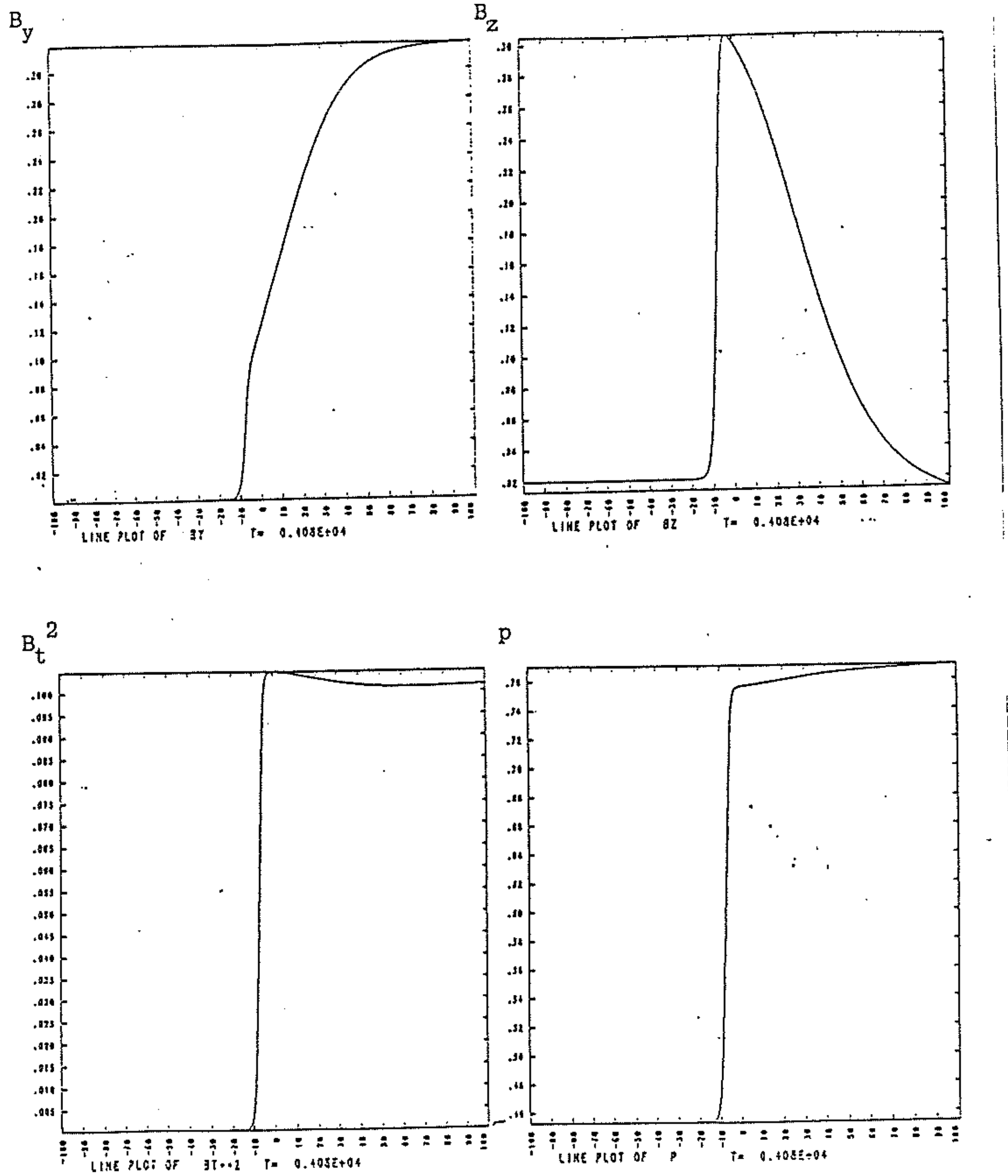

Fig. 6. Perturbation of Switch-on Shock in x-y Plane by B_z Field. Splitting Does Not Occur.