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Uniform Refinement of a Tetrahedron

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January 1991

CAM Report 91-01

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UNIFORM REFINEMENT OF A TETRAHEDRON

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Abstract. A uniform refinement strategy for a tetrahedron is presented. Most finite element theories are based on the assumption that the tetrahedral elements in the refinement do not have small interior angles. In this paper, we present a strategy that avoids small interior angles. The refinement strategy to be described is non-degenerate and, strictly speaking, quasi-uniform. It can be used to construct nested, multi-level triangulations. At level j of refinement, an arbitrary non-degenerate tetrahedron in the initial triangulation is partitioned into 2^{3j} tetrahedra of equal volume. This refinement strategy can be easily implemented by partitioning block elements instead of the more complicated tetrahedral elements. This feature makes the use of tetrahedral elements attractive in a computer code.

Key Words. finite element, non-degenerate, tetrahedron, triangulation, quasi-uniform, uniform refinement

AMS(MOS) subject classification. 65N30, 65N50

1. Introduction. The finite element method is often used to obtain discretizations of second order elliptic problems. It is based on an underlying triangulation of the domain, which is typically refined to achieve higher accuracy in the discrete solution. When the domain is three-dimensional, the elements are often taken to be tetrahedra. In such cases, it is imperative that the refinement of tetrahedral elements result in non-degenerate tetrahedra. In this paper, we describe a uniform and non-degenerate refinement strategy for an arbitrary tetrahedron.

We first describe a refinement strategy for four model tetrahedra. Each tetrahedron is refined into eight small tetrahedra, each of which is *congruent* to one of the four model tetrahedra. This congruence to the model tetrahedra ensures non-degeneracy in the refinement. We say the refinement is *non-degenerate* if

$$\sigma_T = \frac{h_T}{\rho_T} \leq \sigma \quad \forall\, T \in \mathcal{T}_k,\ k = 0, 1, 2, \ldots \tag{1.1}$$

where $\mathcal{T}_k$ is the triangulation at level k of refinement, h_T is the diameter of tetrahedron T, ρ_T is the diameter of the largest sphere inscribed in T, σ_T is the measure of non-degeneracy of T, and σ is a constant independent of $T \in \mathcal{T}_k$ and the refinement level k. ($k = 0$ is the initial level of refinement.) This implies that the interior angles of the tetrahedra in the refinement do not get smaller (and eventually become "flat" in the worst case) with increasing levels of refinement. When non-degeneracy is satisfied, we have a *regular family of triangulations* according to the definition given in [2]. A single *tetrahedron* is said to be *non-degenerate* if $\sigma_T \neq \infty$. We assume that any tetrahedron that we refine is initially non-degenerate.

The resulting eight small tetrahedra also have equal volumes. We shall call the refinement *uniform* in the sense that every tetrahedron in the domain Ω is refined into eight equi-volume tetrahedra at each level of refinement. A uniform refinement typically implies that the small tetrahedra generated from the refinement of a tetrahedron have the same diameters. The diameters of the tetrahedra generated by the refinement

* Department of Mathematics, University of California, Los Angeles, CA, 90024. This research was supported mainly by AFSOR 86-0154 and partly by NSF ASC85-19353, DOE-627815, NSF ASC90-03002, ONR N00014-86-K-0691, and ARO DAAL03-88-K-0085 .

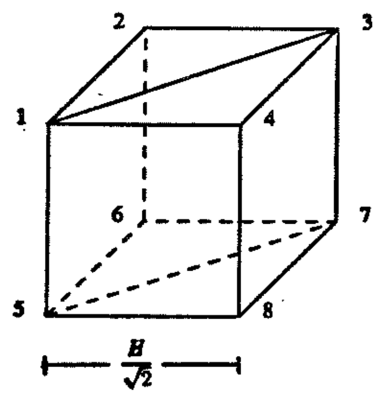

FIG. 1. *Two prisms in a cube.*

strategy that we will describe differ by a factor of $\sqrt{\frac{3}{2}}$. Hence, strictly speaking, the refinement is quasi-uniform. We say the refinement is *quasi-uniform* if

$$\frac{h_{T_1}}{h_{T_2}} \leq c_0 \quad \forall\, T_1, T_2 \in \mathcal{T}_k,\ k = 0, 1, 2, \ldots \tag{1.2}$$

where h_{T_1} and h_{T_2} are the diameters of any two tetrahedra T_1 and T_2, respectively, in the triangulation $\mathcal{T}_k$ and c_o is a constant independent of $T \in \mathcal{T}_k$ and the refinement level k.

The refinement strategy to be described also generates *nested* tetrahedra such that the triangulation $\mathcal{T}_{k+1}$ is obtained by partitioning some or all tetrahedra $T \in \mathcal{T}_k$ for $k = 0, 1, 2, \ldots$.

In Section 2, we describe the uniform refinement strategy for the four model tetrahedra. In Section 3, we use the uniform refinement procedure for any of the four model tetrahedra to show that

1. *Any tetrahedron T can be refined at level j into 2^{3j} tetrahedra that are equi-volume and nested.*
2. *For any tetrahedron T, there exists a refinement that is non-degenerate and quasi-uniform.*

The advantages, usefulness, and limitations of this refinement strategy are discussed in Section 4. A more detailed discussion of this refinement strategy can be found in [8].

2. Refinement of Model Tetrahedra. We make use of the *cube* as a device in the refinement strategy for the four model tetrahedra. Consider a cube with length $H/\sqrt{2}$ and with $(2^0 + 1)^3 = 8$ nodes in the initial refinement, called level 0 refinement and denoted by $\mathcal{T}_0$.

Divide the cube into two prisms, as illustrated in Figure 1, and fit three tetrahedra in each prism as shown in Figures 2 and 3[1]. The tetrahedra in the prism in Figure 2

[1] This particular triangulation of the cube falls under one type of triangulation discussed in [5] and

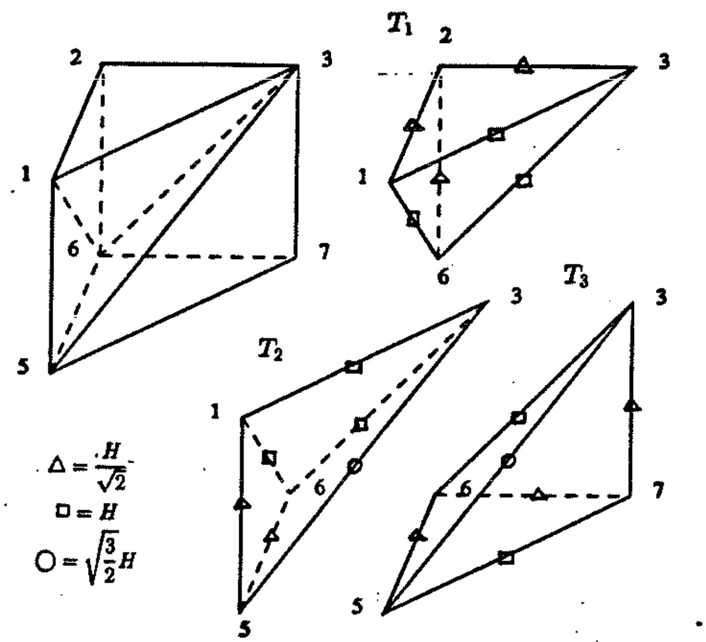

FIG. 2. *Tetrahedra T_1, T_2, T_3 in a prism.*

are denoted by T_1, T_2, and T_3 while the tetrahedra in the prism in Figure 3 are denoted by T_4, T_5, and T_6. Note that T_1 and T_2 are identical to T_5 and T_6, respectively, while T_3 is a reflection of T_4. The tetrahedra $T_1 = T_5$, $T_2 = T_6$, T_3, and T_4 are the four model tetrahedra that we will refine.

The cube has volume $V = (H/\sqrt{2})^3$. The volume of a tetrahedron is given by:

$$\frac{1}{3} \times (area\ of\ base) \times height. \tag{2.3}$$

It can be easily shown that each of the six tetrahedra T_1 to T_6 has volume $\frac{1}{6}V$.

We now describe the procedure to refine a tetrahedron uniformly. We use a standard tetrahedron $\hat{T}$ shown in Figure 4 to illustrate the refinement procedure since any non-degenerate tetrahedron T can be mapped to the standard tetrahedron $\hat{T}$; that is, for any non-degenerate tetrahedron T, there exists a *unique invertible affine mapping* [1, 2, 10]

$$F : \boldsymbol{x} \in \Re^3 \mapsto F(\boldsymbol{x}) = B\boldsymbol{x} + \boldsymbol{b} \tag{2.4}$$

where B is an invertible 3×3 matrix and $\boldsymbol{b}$ is a vector in $\Re^3$ such that

$$F(\hat{\imath}) = i, \quad 1 \leq i \leq 4. \tag{2.5}$$

Here, $\hat{\imath}$ and i are the vertices of $\hat{T}$ and corresponding vertices of T, respectively. Since midpoints are preserved by the affine mapping, it follows that

$$F(\hat{m}) = m, \tag{2.6}$$

appears in [11]. Other triangulations of the cube that are amenable to the refinement strategy are discussed in Section 4.

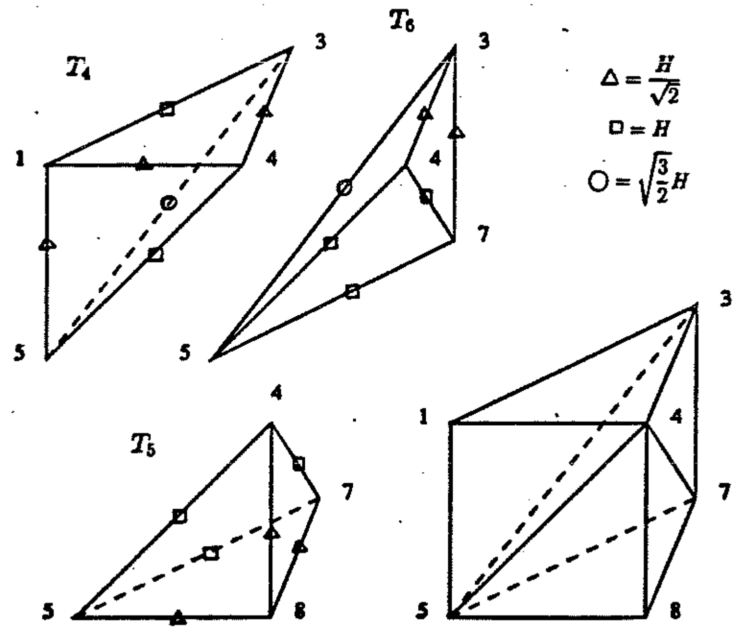

FIG. 3. *Tetrahedra T_4, T_5, T_6 in a prism.*

where $\hat{m}$ and m are the midpoints of $\hat{T}$ and T, respectively.

The first step is to connect the midpoints of the edges of the tetrahedron as shown in Figure 5. This gives four tetrahedra – $\hat{t}_1, \hat{t}_2, \hat{t}_3$, and $\hat{t}_4$ – formed by chopping off the four corners of the tetrahedron $\hat{T}$, and an octahedron $\hat{p}$ in the middle section of $\hat{T}$. See Figure 6. In three different orientations, the octahedron $\hat{p}$ can be viewed as two skewed pyramids patched together on a common base as shown in Figure 7.

The choice of a diagonal that connects two of the four base nodes in Figure 7 determines the four other tetrahedra – $\hat{t}_5, \hat{t}_6, \hat{t}_7$, and $\hat{t}_8$ – obtained from the octahedron $\hat{p}$. There are three possible diagonals and hence three possible sets of four tetrahedra that can be generated from octahedron $\hat{p}$. The three diagonals are formed by connecting the following pairs of nodes: $\hat{5}$ and $\hat{9}$ (shown in Figure 8), $\hat{7}$ and $\hat{8}$, and $\hat{6}$ and $\hat{10}$. Regardless of which diagonal is chosen, we are able to refine the tetrahedron $\hat{T}$ into eight small tetrahedra. The choice of a particular diagonal, and hence of a particular orientation, will be determined when we go back to the original model tetrahedra in the cube. One orientation will be preferred over the other two.

Let us proceed to show that the eight tetrahedra for any choice of diagonal have the same volume. If we denote by H the diameter of $\hat{T}$ as shown in Figure 5, then $\hat{T}$ has volume $V_{\hat{T}} = \frac{1}{6}(H/\sqrt{2})^3$. Now the four tetrahedra $\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_4$ shown in Figure 6, are similar to the standard tetrahedron $\hat{T}$ but with diameter $H/2$. Hence each of these four tetrahedra has volume $\frac{1}{6}(\frac{H}{2}/\sqrt{2})^3 = \frac{1}{8}V_{\hat{T}}$. Subtracting the volumes of these four tetrahedra from $V_{\hat{T}}$, we obtain the volume of the octahedron $\hat{p}$ which is $\frac{1}{2}V_{\hat{T}}$. What remains to be shown is that each of the four tetrahedra, $\hat{t}_5$, $\hat{t}_6$, $\hat{t}_7$, and $\hat{t}_8$, in $\hat{p}$ has volume $\frac{1}{8}V_{\hat{T}}$.

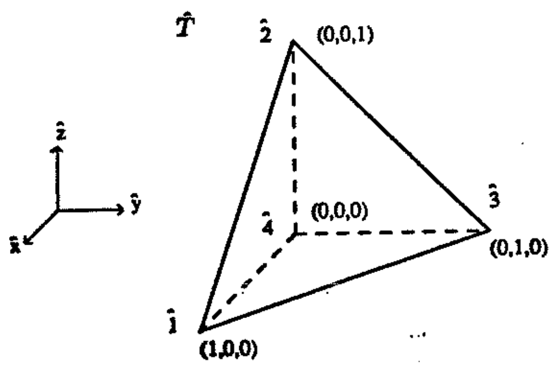

FIG. 4. *Standard tetrahedron.*

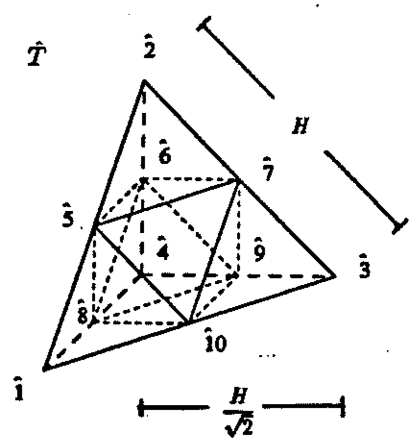

FIG. 5. *Connect midpoints of the tetrahedron.*

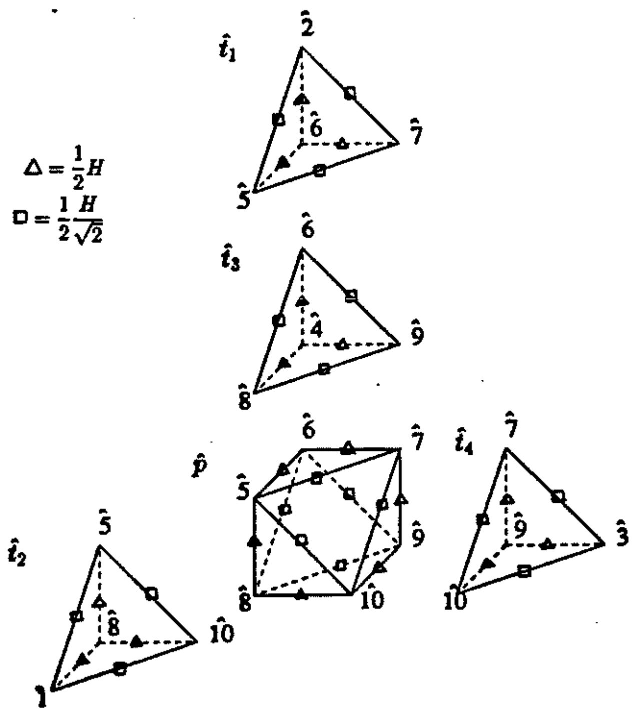

FIG. 6. *Four tetrahedra $\hat{t}_1, \hat{t}_2, \hat{t}_3, \hat{t}_4$ and octahedron $\hat{p}$.*

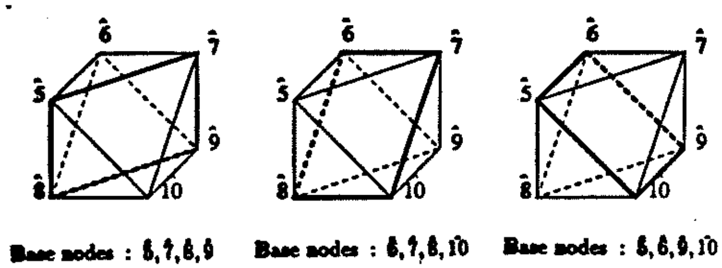

FIG. 7. *Octahedron as two patched up pyramids in three orientations.*

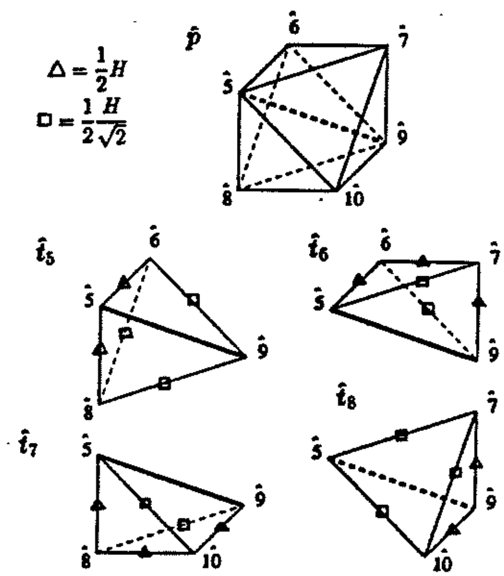

FIG. 8. *Four tetrahedra $\hat{t}_5, \hat{t}_6, \hat{t}_7, \hat{t}_8$ obtained from octahedron $\hat{p}$ when nodes $\hat{5}$ and $\hat{9}$ are connected to form a diagonal.*

In Figure 8, where nodes $\hat{5}$ and $\hat{9}$ are connected to form the diagonal, $\hat{t}_5 = \hat{t}_8$ and $\hat{t}_6$ is a reflection of $\hat{t}_7$. Hence, $\hat{t}_5$ and $\hat{t}_8$ have the same volume and so do $\hat{t}_6$ and $\hat{t}_7$ for any choice of diagonal. It is sufficient to show that the volume of *any* of the four tetrahedra is $\frac{1}{8}V_{\hat{T}}$. Once any of the tetrahedra is shown to have volume $\frac{1}{8}V_{\hat{T}}$, it follows that each of the four tetrahedra in $\hat{p}$ has volume $\frac{1}{8}V_{\hat{T}}$.

Looking at tetrahedron $\hat{t}_6$ in Figure 8, the right triangles $\triangle_{\hat{5}\hat{6}\hat{7}}$ and $\triangle_{\hat{6}\hat{7}\hat{9}}$ are perpendicular to each other. The legs of the two right triangles have length $\frac{H}{2}/\sqrt{2}$. The volume of $\hat{t}_6$ is easily calculated to be $\frac{1}{6}(\frac{H}{2}/\sqrt{2})^2(\frac{H}{2}/\sqrt{2}) = \frac{1}{8}V_{\hat{T}}$.

By the same procedure, it can be shown that each of the four tetrahedra in $\hat{p}$ obtained by connecting nodes $\hat{7}$ and $\hat{8}$ or nodes $\hat{6}$ and $\hat{10}$ has volume equal to $\frac{1}{8}V_{\hat{T}}$. Hence all the eight tetrahedra making up the standard tetrahedron $\hat{T}$, regardless of the diagonal chosen, have the same volume, satisfying the rule for the *uniform refinement strategy.*

By property (2.6) of the affine mapping, midpoints of the standard tetrahedron $\hat{T}$ will map to midpoints of any non-degenerate tetrahedron T. Moreover, we have the following differential volume relations:

$$dx\,dy\,dz = |det(B)|\,d\hat{x}\,d\hat{y}\,d\hat{z} \tag{2.7}$$

where $|det(B)|$ is the absolute value of the determinant of the affine mapping matrix B and is constant by the property of the affine mapping. Hence *any tetrahedron can be refined into eight tetrahedra of equal volume by connecting the midpoints of its edges.* In particular, *each of the six tetrahedra T_1 through T_6 in the cube in Figures 2 and 3 can be refined into eight tetrahedra of equal volume.*

Recall that we have three choices of diagonals when refining any tetrahedron in the cube. We will choose a particular diagonal so that we maintain the tetrahedral structure in the refinement. In other words, if we refine the cube in Figure 1 uniformly into eight small cubes, each of the small cubes will have a tetrahedral structure identical to that of the big cube shown in Figures 2 and 3. We illustrate this in Figure 9. Such a restriction forces the small tetrahedra to be *congruent* to one of the four parent model tetrahedra $T_1 = T_5$, $T_2 = T_6$, T_3, T_4. The diagonals chosen and the refinement of each of the model tetrahedra $T_1 = T_5$, $T_2 = T_6$, T_3, and T_4 are shown in Figures 10 to 13, respectively, where we denote by $T_i^{(1)}$ the small tetrahedra congruent to T_i, $i = 1, \ldots, 6$. (The superscript (1) indicates one level of refinement.) Notice the *inclusion* of the small tetrahedra in their parent tetrahedron. Thus, the *nesting condition is satisfied.*

This procedure is repeated at each level of refinement. A tetrahedron is refined into 2^{3j} tetrahedra after j levels of uniform refinement. *By the uniform refinement strategy, any tetrahedron can be refined at level j into 2^{3j} tetrahedra that are nested and have equal volumes. In particular, each of the six tetrahedra T_1 through T_6 in the cube in Figures 2 and 3 can be refined at level j into 2^{3j} tetrahedra that are equi-volume and nested.* Notice that after one level of refinement, the cube has $(2^1+1)^3 = 27$ nodes. In general, we have $(2^j + 1)^3 = N$ nodes after j levels of uniform refinement.

To measure *non-degeneracy of the model tetrahedra*, we use Zhang's formula [10] to compute the diameter ρ_T of the largest sphere inscribed in a tetrahedron T. This is given by

$$\rho_T = 6\frac{V_T}{S_T}, \tag{2.8}$$

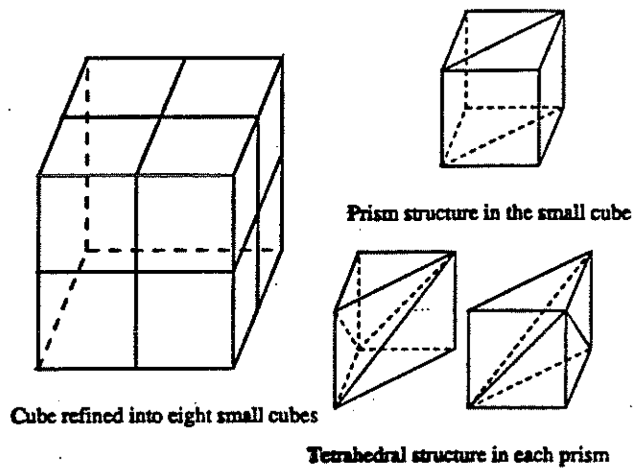

FIG. 9. *Tetrahedral structure preserved in small cubes.*

where V_T is the volume and S_T is the surface area of tetrahedron T. The surface area S_T is the sum of the area of the triangles that make up the faces of tetrahedron T. The following surface areas of the model tetrahedra can be easily calculated:

$$S_{T_1} = S_{T_5} = \frac{3+\sqrt{3}}{4}H^2 \tag{2.9}$$

$$S_{T_2} = S_{T_6} = \frac{1+2\sqrt{2}+\sqrt{3}}{4}H^2 \tag{2.10}$$

$$S_{T_3} = S_{T_4} = \frac{1+\sqrt{2}}{2}H^2. \tag{2.11}$$

Since the volume of each of the six tetrahedra T_1 through T_6 is $\frac{1}{6}V = \frac{1}{6}(H/\sqrt{2})^3$, we have by (2.8) the following diameters of the largest spheres inscribed in the six tetrahedra:

$$\rho_{T_5} = \rho_{T_1} = 6\frac{V_{T_1}}{S_{T_1}} = \frac{\sqrt{2}}{3+\sqrt{3}}H \tag{2.12}$$

$$\rho_{T_6} = \rho_{T_2} = 6\frac{V_{T_2}}{S_{T_2}} = \frac{2}{4+\sqrt{2}(1+\sqrt{3})}H \tag{2.13}$$

$$\rho_{T_4} = \rho_{T_3} = 6\frac{V_{T_3}}{S_{T_3}} = \frac{1}{2+\sqrt{2}}H. \tag{2.14}$$

Since the diameter of $T_1 = T_5$ is H and the diameter of $T_2 = T_6$, T_3, or T_4 is $\sqrt{\frac{3}{2}}H$, then (1.1) and the values in (2.12)–(2.14) give the following measures of non-degeneracy of

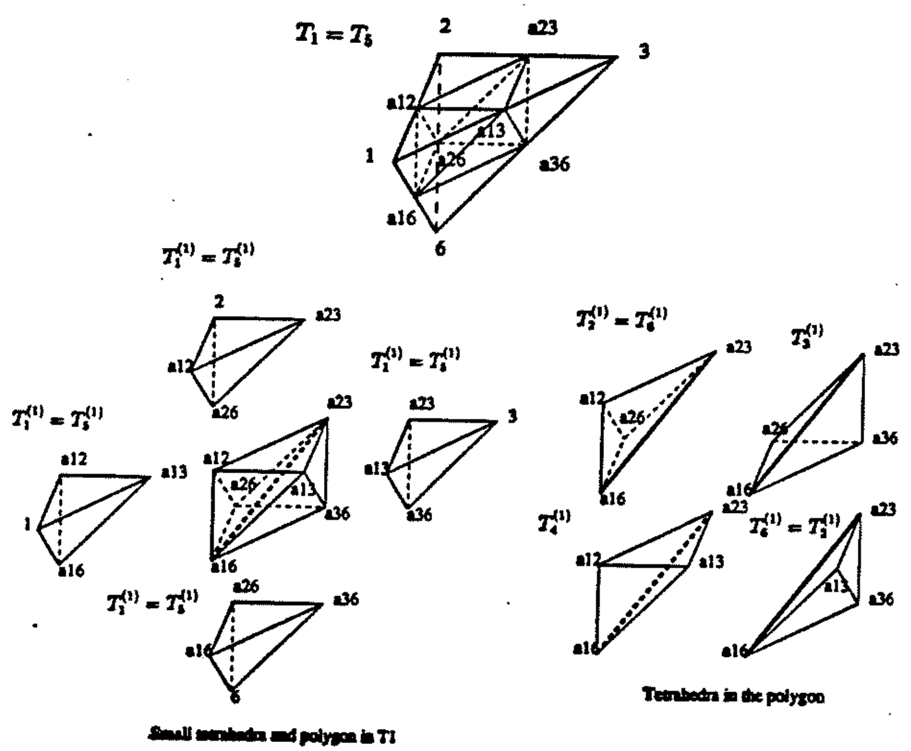

FIG. 10. *Refinement of tetrahedron* T_1.

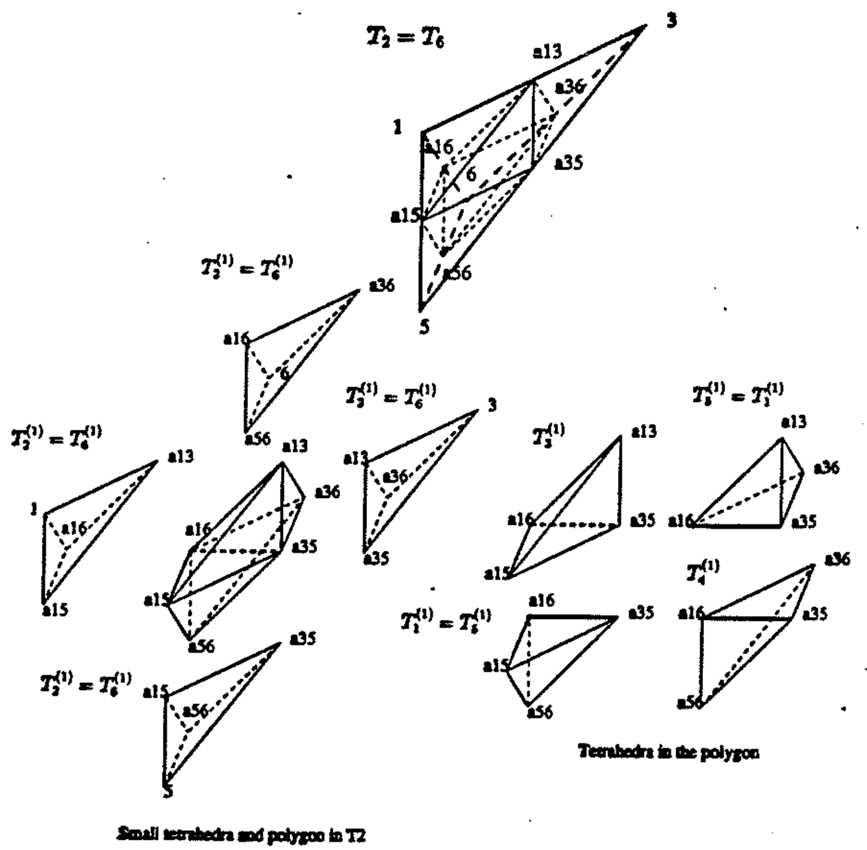

FIG. 11. *Refinement of tetrahedron T_2.*

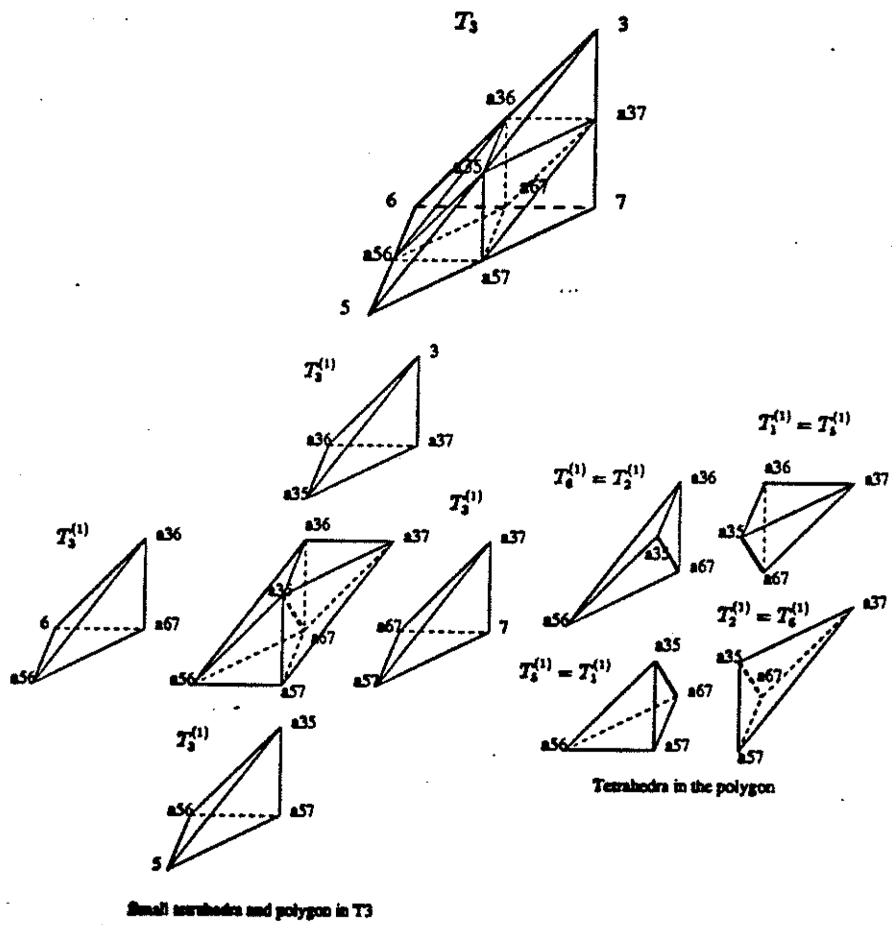

FIG. 12. *Refinement of tetrahedron* T_3.

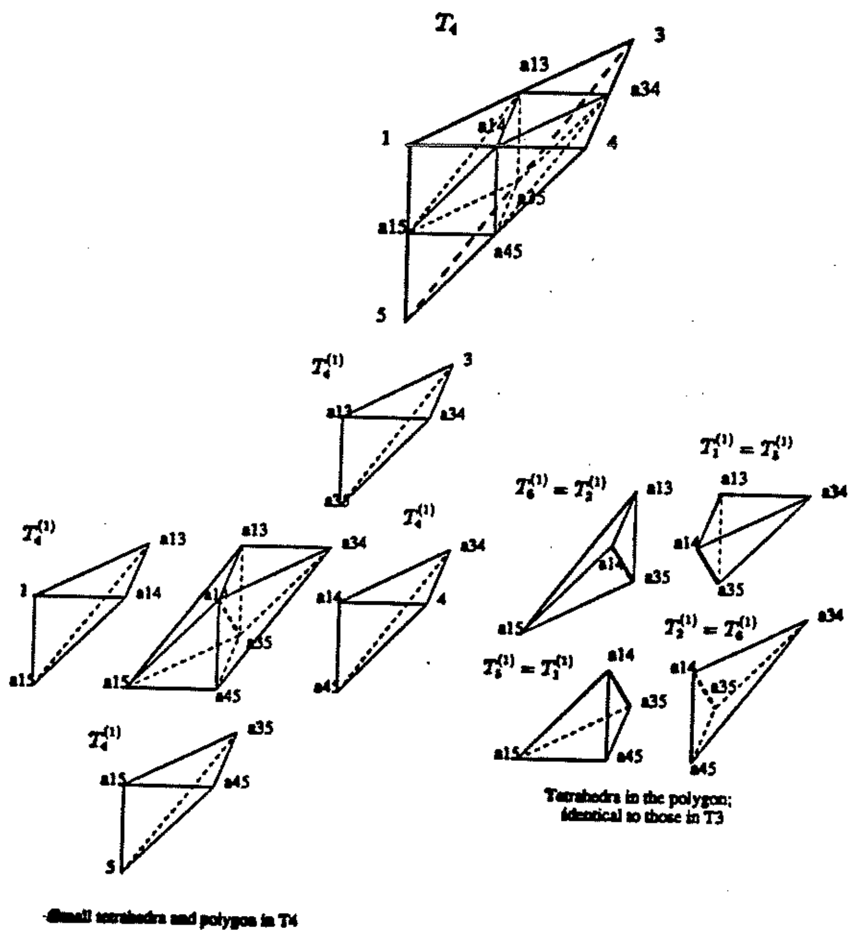

FIG. 13. *Refinement of tetrahedron T_4.*

the model tetrahedra:

$$(2.15) \qquad \sigma_{T_1} = \frac{h_{T_1}}{\rho_{T_1}} = \sigma_{T_5} = \frac{h_{T_5}}{\rho_{T_5}} = \frac{3+\sqrt{3}}{\sqrt{2}} \approx 3.35$$

$$(2.16) \qquad \sigma_{T_2} = \frac{h_{T_2}}{\rho_{T_2}} = \sigma_{T_6} = \frac{h_{T_6}}{\rho_{T_6}} = \sqrt{\frac{3}{2}} \left(\frac{4+\sqrt{2}(1+\sqrt{3})}{2}\right) \approx 4.82$$

$$(2.17) \qquad \sigma_{T_3} = \frac{h_{T_3}}{\rho_{T_3}} = \sigma_{T_4} = \frac{h_{T_4}}{\rho_{T_4}} = \sqrt{\frac{3}{2}} \left(2+\sqrt{2}\right) \approx 4.18.$$

We proceed to measure the *quasi-uniformity of the model tetrahedra.* Using (1.2), we have

$$(2.18) \qquad \frac{h_{T_i}}{h_{T_l}} \leq \sqrt{\frac{3}{2}} \quad 1 \leq i, l \leq 6.$$

The small tetrahedra into which T_1 through T_6 are refined are congruent to these parent tetrahedra. After k levels of refinement, $k = 1, 2, \ldots$, the small tetrahedra congruent to T_i, which we denote by $T_i^{(k)}$, $i = 1, \ldots, 6$, have diameter $h_{T_i^{(k)}} = h_{T_i}/2^k$, where $h_{T_1} = h_{T_5} = H$ and $h_{T_2} = h_{T_3} = h_{T_4} = h_{T_6} = \sqrt{\frac{3}{2}}H$. The diameter of the largest inscribed sphere in these tetrahedra is $\rho_{T_i^{(k)}} = \rho_{T_i}/2^k$, where $\rho_{T_i}, i = 1, \ldots, 6$, are given in (2.12)–(2.14). Hence, the measures of non-degeneracy

$$(2.19) \qquad \sigma_{T_i^{(k)}} = \sigma_{T_i}, \quad i = 1, \ldots, 6,$$

where $\sigma_{T_i}, i = 1, \ldots, 6$ are given in (2.15) to (2.17), remain the same for all k. Likewise, the measure of quasi-uniformity

$$(2.20) \qquad \frac{h_{T_i^{(k)}}}{h_{T_l^{(k)}}} = \frac{h_{T_i}}{h_{T_l}} \leq c_0 = \sqrt{\frac{3}{2}} \quad 1 \leq i, l \leq 6$$

remains the same for all k. Because of the properties given in (2.19) and (2.20), it follows that the uniform refinement strategy for the model tetrahedra is non-degenerate and quasi-uniform.

In the next section we show that non-degeneracy and quasi-uniformity are preserved when the uniform refinement strategy is applied to any non-degenerate tetrahedron.

3. Extensions to Arbitrary Tetrahedra. In the previous section, we have shown that with uniform refinement, any tetrahedron can be refined at level j into 2^{3j} tetrahedra that are equi-volume and nested. We now show that *for any non-degenerate tetrahedron T, there exists a refinement that is non-degenerate and quasi-uniform.*

We denote by F_i the affine mapping that maps the model tetrahedron T_i, $i = 1, 2, 3$, ($i = 5, 6, 4$, respectively, yield the same results) to an arbitrary non-degenerate tetrahedron T:

$$(3.21) \qquad T = F_i(T_i) : \boldsymbol{x} \in T_i \mapsto F_i(\boldsymbol{x}) = B_i\boldsymbol{x} + \boldsymbol{b}_i \in T.$$

We proceed to prove the following theorem, which shows non-degeneracy of the refinement of T into 2^{3k} small tetrahedra at level k, where T is the image of any of the model tetrahedra T_i, $i = 1,2,3$. This theorem is similar to Theorem 2.2.8 in Zhang's thesis [10].

THEOREM 3.1. *For any non-degenerate tetrahedron T with*

$$\sigma_T \le c_0,$$

where σ_T is given by (1.1), there exists a refinement of T into 2^{3k} small tetrahedra $\{T_{s,\ell}^{(k)}\}_{\ell=1}^{2^{3k}}$ for levels $k = 1,2,\dots$ such that[2]

1. *Case 1. If T is the image of the model tetrahedron T_1, then*

$$\sigma_{T_{s,\ell}^{(k)}} \le c_0 \sigma_{T_1} \sigma_{T_2} \quad \ell = 1,\dots,2^{3k} \quad \textit{for all } k.$$

2. *Case 2. If T is the image of the model tetrahedron T_2, then*

$$\sigma_{T_{s,\ell}^{(k)}} \le c_0 \sigma_{T_2}^2 \quad \ell = 1,\dots,2^{3k} \quad \textit{for all } k.$$

3. *Case 3. Is T is the image of the model tetrahedron T_3, then*

$$\sigma_{T_{s,\ell}^{(k)}} \le c_0 \sigma_{T_3} \sigma_{T_2} \quad \ell = 1,\dots,2^{3k} \quad \textit{for all } k.$$

Proof.

1. *Case 1.* Let T be the image of the model tetrahedron T_1 shown in Figure 10 where we have the following affine mapping from (3.21):

$$T = F_1(T_1) : \boldsymbol{x} \in T_1 \mapsto F_1(\boldsymbol{x}) = B_1\boldsymbol{x} + \boldsymbol{b}_1 \in T. \tag{3.22}$$

Let T and T_1 be refined into 2^{3k} tetrahedra $\{\, T_{s,\ell}^{(k)}\}_{\ell=1}^{2^{3k}}$ and $\{\, T_{1:s,\ell}^{(k)}\}_{\ell=1}^{2^{3k}}$, respectively, at level k. In all four subcases below, we use the relations found in Theorem 3.1.3 in Ciarlet [2], namely,

$$\|B_1\| \le \frac{h_T}{\rho_{T_1}}, \quad \|B_1^{-1}\| \le \frac{h_{T_1}}{\rho_T} \tag{3.23}$$

where $\|\cdot\|$ denotes the Euclidean or 2-norm. We also have

$$\begin{aligned} h_{T_{1:s,l}^{(k)}} &= \frac{h_{T_i}}{2^k} \\ \rho_{T_{1:s,l}^{(k)}} &= \frac{\rho_{T_i}}{2^k} \end{aligned} \tag{3.24}$$

if $T_{1:s,l}^{(k)}$ is congruent to $T_i, i = 1,2,3,4$.

Subcase 1.1. If $T_{s,\ell}^{(k)}$ is the image of $T_{1:s,\ell}^{(k)}$, which is congruent to T_1, we have

$$\begin{aligned} \sigma_{T_{s,\ell}^{(k)}} = \frac{h_{T_{s,\ell}^{(k)}}}{\rho_{T_{s,\ell}^{(k)}}} &\le \frac{\|B_1\| h_{T_{1:s,\ell}^{(k)}}}{\|B_1^{-1}\|^{-1}\rho_{T_{1:s,\ell}^{(k)}}} \\ &\le \frac{h_T}{\rho_{T_1}}\frac{h_{T_1}}{\rho_T}\frac{h_{T_1}}{\rho_{T_1}} \\ &= \sigma_T \sigma_{T_1}^2. \end{aligned} \tag{3.25}$$

[2] *The subscript s is used to denote* small.

Subcase 1.2. If $T^{(k)}_{s,\ell}$ is the image of $T^{(k)}_{1:s,\ell}$, which is congruent to T_2, we have

$$\begin{aligned}
\sigma_{T^{(k)}_{s,\ell}} = \frac{h_{T^{(k)}_{s,\ell}}}{\rho_{T^{(k)}_{s,\ell}}} &\leq \frac{\|B_1\| h_{T^{(k)}_{1:s,\ell}}}{\|B_1^{-1}\|^{-1}\rho_{T^{(k)}_{1:s,\ell}}} \\
&\leq \frac{h_T}{\rho_{T_1}} \frac{h_{T_1}}{\rho_T} \frac{h_{T_2}}{\rho_{T_2}} \\
&= \sigma_T \sigma_{T_1} \sigma_{T_2}. \qquad (3.26)
\end{aligned}$$

Subcase 1.3. If $T^{(k)}_{s,\ell}$ is the image of $T^{(k)}_{1:s,\ell}$, which is congruent to T_3, we have

$$\begin{aligned}
\sigma_{T^{(k)}_{s,\ell}} = \frac{h_{T^{(k)}_{s,\ell}}}{\rho_{T^{(k)}_{s,\ell}}} &\leq \frac{\|B_1\| h_{T^{(k)}_{1:s,\ell}}}{\|B_1^{-1}\|^{-1}\rho_{T^{(k)}_{1:s,\ell}}} \\
&\leq \frac{h_T}{\rho_{T_1}} \frac{h_{T_1}}{\rho_T} \frac{h_{T_3}}{\rho_{T_3}} \\
&= \sigma_T \sigma_{T_1} \sigma_{T_3}. \qquad (3.27)
\end{aligned}$$

Subcase 1.4. If $T^{(k)}_{s,\ell}$ is the image of $T^{(k)}_{1:s,\ell}$, which is congruent to T_4, we have

$$\begin{aligned}
\sigma_{T^{(k)}_{s,\ell}} = \frac{h_{T^{(k)}_{s,\ell}}}{\rho_{T^{(k)}_{s,\ell}}} &\leq \frac{\|B_1\| h_{T^{(k)}_{1:s,\ell}}}{\|B_1^{-1}\|^{-1}\rho_{T^{(k)}_{1:s,\ell}}} \\
&\leq \frac{h_T}{\rho_{T_1}} \frac{h_{T_1}}{\rho_T} \frac{h_{T_4}}{\rho_{T_4}} \\
&= \sigma_T \sigma_{T_1} \sigma_{T_4}. \qquad (3.28)
\end{aligned}$$

Comparing σ_{T_i} in (2.15)–(2.17), we take the maximum upper bound of the four cases to obtain

$$\sigma_{T^{(k)}_{s,\ell}} \leq \sigma_T \sigma_{T_1} \sigma_{T_2} \leq c_0 \sigma_{T_1} \sigma_{T_2}.$$

2. *Case 2.* The proof is similar to that for Case 1. T is now the image of the model tetrahedron T_2 shown in Figure 11 and the affine mapping is now given by

$$T = F_2(T_2) : \boldsymbol{x} \in T_2 \mapsto F_2(\boldsymbol{x}) = B_2\boldsymbol{x} + \boldsymbol{b}_2 \in T. \qquad (3.29)$$

The relations in Ciarlet are

$$\|B_2\| \leq \frac{h_T}{\rho_{T_2}}, \quad \|B_2^{-1}\| \leq \frac{h_{T_2}}{\rho_T}. \qquad (3.30)$$

3. *Case 3.* The proof is again similar to that for Case 1. T is now the image of the model tetrahedron T_3 shown in Figure 12 and the affine mapping is given by

$$T = F_3(T_3) : \boldsymbol{x} \in T_3 \mapsto F_3(\boldsymbol{x}) = B_3\boldsymbol{x} + \boldsymbol{b}_3 \in T. \qquad (3.31)$$

The relations in Ciarlet are

$$\|B_3\| \leq \frac{h_T}{\rho_{T_3}}, \quad \|B_3^{-1}\| \leq \frac{h_{T_3}}{\rho_T}. \quad \square \qquad (3.32)$$

In the following theorem, we show that *for any non-degenerate tetrahedron T, there exists a quasi-uniform refinement of T into 2^{3k} tetrahedra at level k for $k = 1, 2, \ldots$.*

THEOREM 3.2. *For any non-degenerate tetrahedron T with*

$$\sigma_T \le c_0$$

there exists a refinement of T into 2^{3k} small tetrahedra $\{T_{s,\ell}^{(k)}\}_{\ell=1}^{2^{3k}}$ for levels $k = 1, 2, \ldots$ such that

1. *Case 1. If T is the image of the model tetrahedron T_1, then*

$$\frac{h_{T_{s,m}^{(k)}}}{h_{T_{s,n}^{(k)}}} \le \sqrt{\frac{3}{2}}\, c_0 \sigma_{T_1} \quad m, n = 1, \ldots, 2^{3k} \quad \textit{for all } k.$$

2. *Case 2. If T is the image of the model tetrahedron T_2, then*

$$\frac{h_{T_{s,m}^{(k)}}}{h_{T_{s,n}^{(k)}}} \le \sqrt{\frac{3}{2}}\, c_0 \sigma_{T_2} \quad m, n = 1, \ldots, 2^{3k} \quad \textit{for all } k.$$

3. *Case 3. If T is the image of the model tetrahedron T_3, then*

$$\frac{h_{T_{s,m}^{(k)}}}{h_{T_{s,n}^{(k)}}} \le \sqrt{\frac{3}{2}}\, c_0 \sigma_{T_3} \quad m, n = 1, \ldots, 2^{3k} \quad \textit{for all } k.$$

Proof.

1. *Case 1.* Let T be the image of the model tetrahedron T_1 where the affine mapping is given in (3.22) and the relations in (3.23) hold. Let T be refined into 2^{3k} tetrahedra $\{ T_{s,\ell}^{(k)} \}_{\ell=1}^{2^{3k}}$ and let T_1 be refined into 2^{3k} tetrahedra $\{ T_{1:s,\ell}^{(k)} \}_{\ell=1}^{2^{3k}}$ at level k. Let $T_{s,m}^{(k)}$ and $T_{s,n}^{(k)}$ be the image of $T_{1:s,m}^{(k)}$ and $T_{1:s,n}^{(k)}$, respectively, where $m, n \in \{1, \ldots, 2^{3k}\}$. Let $T_{1:s,m}^{(k)}$ and $T_{1:s,n}^{(k)}$ be congruent to the model tetrahedra T_i and T_l, respectively, where $i, l = 1, \ldots, 4$, so that $T_{1:s,m}^{(k)} = T_i^{(k)}$ and $T_{1:s,n}^{(k)} = T_l^{(k)}$. Using (2.20), (3.23) and (3.24), we have for all k

$$\begin{aligned}
\frac{h_{T_{s,m}^{(k)}}}{h_{T_{s,n}^{(k)}}} \le \frac{\|B_1\| h_{T_{1:s,m}^{(k)}}}{\|B_1^{-1}\|^{-1} h_{T_{1:s,n}^{(k)}}} &= \frac{\|B_1\| h_{T_i^{(k)}}}{\|B_1^{-1}\|^{-1} h_{T_l^{(k)}}} \\
&\le \frac{h_T}{\rho_{T_1}} \frac{h_{T_1}}{\rho_T} \frac{h_{T_i^{(k)}}}{h_{T_l^{(k)}}}. \\
&\le \sqrt{\frac{3}{2}}\, \sigma_T \sigma_{T_1} \\
&\le \sqrt{\frac{3}{2}}\, c_0 \sigma_{T_1} \quad m, n = 1, \ldots, 2^{3k}.
\end{aligned}$$

2. *Case 2.* The proof is similar to that for Case 1. T is now the image of the model tetrahedron T_2, the affine mapping is given by (3.29), and the relations in (3.30) are used.
3. *Case 3.* The proof is similar to that for Case 1. T is now the image of the model tetrahedron T_3, the affine mapping is given by (3.31), and the relations in (3.32) are used. □

COROLLARY 3.1. *At the initial refinement $\mathcal{T}_0$, let the polygonal domain Ω be partitioned into blocks, each containing six tetrahedra, where each block is the image of the cube containing the model tetrahedra T_1 to T_6 shown in Figures 1, 2, and 3. If*

$$\sigma_T \le c_0 \quad \forall T \in \mathcal{T}_0$$

then there exists a refinement wherein each $T \in \mathcal{T}_0$ is refined into 2^{3k} tetrahedra $\{T_{s,l}^{(k)}\}_{l=1}^{2^{3k}}$ at level $k = 1, 2, \ldots$ such that

$$\begin{aligned} \sigma_{T_{s,l}^{(k)}} &\le c_0\sigma_{T_2}^2 \quad l = 1, \ldots, 2^{3k} \\ \frac{h_{T_{s,m}^{(k)}}}{h_{T_{s,n}^{(k)}}} &\le \sqrt{\frac{3}{2}}c_0\sigma_{T_2} \quad m, n = 1, \ldots, 2^{3k} \end{aligned} \tag{3.33}$$

for all $T \in \mathcal{T}_0$ and all levels k. Here, $T_{s,1}^{(0)} = T \in \mathcal{T}_0$ which is not refined.
Proof. This follows from Theorems 3.1 and 3.2 and comparing the values $\sigma_{T_1}, \sigma_{T_2}$ and σ_{T_3} given in (2.15), (2.16) and (2.17), respectively. □

4. Applications and Limitations. We have described a uniform refinement strategy for any non-degenerate tetrahedron T that results in a triangulation that is *nested* and generates, at level k, 2^{3k} tetrahedra that are *equi-volume.* The refinement strategy, which employs the model cube containing six tetrahedra, is *quasi-uniform* and *non-degenerate.* (Though refinement of the six tetrahedra in the cube generates small tetrahedra congruent to the initial six tetrahedra, this is not true of the refinement of arbitrary tetrahedra.) This strategy applied to the four model tetrahedra happens to satisfy Zhang's rule of choosing the shortest edge, that is, the diagonal with the shortest length, even though the rule is not invoked in choosing the diagonal. Recall that the strategy chooses the diagonal so that the tetrahedral structure in the cubes is preserved.

The refinement strategy is *easy to implement and can be automated.* A polygonal domain Ω can be partitioned into blocks, each block partitioned into eight blocks at each level of refinement. At the final level of refinement, say level j, six tetrahedra will be fitted into each of the 8^j blocks in the same manner as six model tetrahedra were fitted into the model cube. This achieves the same triangulation as when each of the initial blocks in the domain Ω are fitted with six tetrahedra and the tetrahedra refined according to the uniform refinement strategy. This can be easily seen by assembling the refined tetrahedra in Figures 10 through 13 into a cube.

The refinement strategy has *applications to finite element methods* in general. A typical concern in using tetrahedral elements is the generation of small interior angles in the refinement which leads to poorly conditioned systems of equations [3, 9]. The non-degeneracy of the refinement strategy avoids this problem.

The refinement yields a *nested triangulation* that can be used for multigrid and other nested multi-level methods [4, 7]. The optimal operation count of $O(N)$, where N is the number of unknowns, is likewise obtained. This is easily shown by summing over all levels the number of unknowns or nodes in the cube which is the same number of nodes in the blocks making up the domain Ω. That is,

$$\sum_{k=0}^{j}(2^k + 1)^3 = O(N)$$

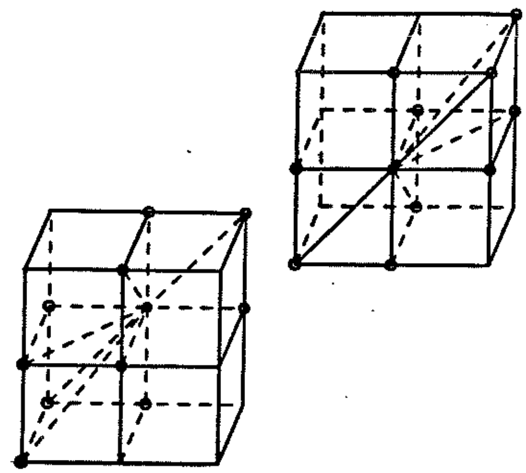

FIG. 14. *Connectivity of a center node.*

since $(2^j+1)^3 = N$.

There are some *limitations* to this refinement strategy. First, the *non-degeneracy* and *quasi-uniform bounds* obtained using the model tetrahedra in the cube are *not optimal.* A refinement using the model tetrahedra in [10] produces smaller bounds. However, the tetrahedra in [10] cannot be used to triangulate a cube.

Second, the *coefficient matrix* associated with the tetrahedral refinement is *not relatively sparse.* There can be as many as 15 nonzeros in each row of the matrix as opposed to 7 nonzeros in the usual 7-point discretization of a second order differential operator in three dimensions. Figure 14 shows the connectivity of 14 nodes to a center node in a cube partitioned into six model tetrahedra and refined according to the uniform refinement strategy.

An interesting question is how many ways can we triangulate a cube into tetrahedra and which of these triangulations is amenable to the uniform refinement strategy. The refinement strategy applied to a cube generates tetrahedra that are nested, equivolume, congruent to the model tetrahedra, and conforming so that the tetrahedral elements match at block interfaces. Moreover, small cubes into which a parent cube is partitioned have the same tetrahedral structure as the parent cube.

A theorem in [5] states that there exist exactly ten essentially different vertex-true triangulations of a cube into tetrahedra. A vertex-true triangulation requires that each vertex of a tetrahedron in the cube is a vertex of the cube. These triangulations are classified into six cases depending on the number k_m of tetrahedra with $m = 0, 1, 2, 3$ external faces on the cube. The triangulation of the cube presented here is the case with $k_0 = 0, k_1 = 2, k_2 = 2$, and $k_3 = 2$ since there are two T_2 tetrahedra with one face on the cube, two tetrahedra (T_3 and T_4) with two faces on the cube, and two T_1 tetrahedra with three faces on the cube. Another realization of this case which is amenable to the refinement strategy is shown in Figure 15 and is basically a different

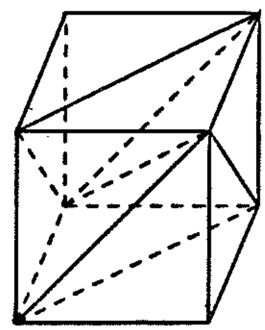

FIG. 15. *Triangulation of a Cube,* $k_0 = 0, k_1 = 2, k_2 = 2, k_3 = 2$.

choice of diagonal. The only other case amenable to the refinement strategy has $k_0 = 0, k_1 = 0, k_2 = 6, k_3 = 0$. This is shown in Figure 16 and appears in [6]. It has only two types of tetrahedra, namely, T_3 and T_4. In the case of the cube containing the four model tetrahedra in Figures 2 and 3 and the cases represented by Figures 15 and 16, a key thing to note is that the diagonals on parallel faces of the cube align. This guarantees that the tetrahedra match at block interfaces. Again, uniform refinement strategies that have the properties of the uniform refinement strategy discussed in the previous sections can be defined for the cases represented in Figures 15 and 16. These strategies will be non-degenerate and quasi-uniform. The tetrahedra generated from the refinement will be equi-volume and nested. The small cubes generated from the refinement will have the same tetrahedral structure as the parent cube. This refinement strategy can also be automated by working with the cubes. Since the cube in Figure 16 has only types T_3 and T_4 tetrahedra, it is interesting to note that the tetrahedra will have the same diameter $\sqrt{\frac{3}{2}}H/2^k$ at level k of refinement (the conventional uniform refinement strategy) and the measures of non-degeneracy will be better (compare $\sigma_{T_3} = \sigma_{T_4}$ with σ_{T_2} in (2.16) and (2.17), respectively).

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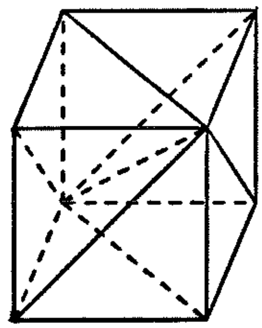

FIG. 16. *Triangulation of a Cube,* $k_0 = 0, k_1 = 0, k_2 = 6, k_3 = 0$.

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