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Russel E. Caflisch

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

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Abstract

Singularities occur in solutions of algebraic equations and partial differential equations and are important for many problems in mathematical physics. A general theory is described here for singularities in the solution of a hyperbolic equations in a complex space variable. The solution is defined on a Riemann surface and the singularities are branch points for this surface. The generic form for a singularity is found to be a square root. There are two types of collisions between singularities. At a tangential collision the singularity remains of square root type generically. At a transverse collision the singularity is generically of cube root type.

1 Introduction

This lecture will describe singularities in systems of partial differential equations (pde's) for which the dependent and independent variables are complex (as opposed to real). The singularities will be branch points on an appropriate Riemann surface.

Examples of fluid dynamic flows for which singularities arise are the Kelvin-Helmholtz problem for vortex sheets [3, 6, 9, 10, 11, 12, 13], the Rayleigh-Taylor problem [15, 16], and unsteady Prandtl boundary layers [14]. The most interesting singularity problem for fluid dynamics is the possibility of singularity formation from smooth initial data for the three-dimensional Euler equations [1, 2].

In each of these problems the singularity would be smoothed out if viscosity, or other smoothing mechanisms, were included. The effects of the

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singularity would remain, however, even if the singularities would not. These effects include roll-up of vortex sheets in the Kelvin-Helmholtz and Rayleigh-Taylor problems and onset of separation in unsteady Prandtl boundary layers. Thus a singularity is an indicator for the onset of complex behavior and serves as a simple localized phenomenon which may be easier to understand that a more realistic, complicated flow.

Singularities are most naturally described through analytic functions and complex variables. The singularity is a branch point at which the analytic function becomes multi-valued. The singularity can be "unfolded" by introducing a "uniformizing variable" in terms of which the function is single valued. The geometric description of this unfolding is the subject of catastrophe theory. We shall use both the analytic and geometric viewpoints in our study of singularities.

2 Singularities for the Laplace Equation

The simplest example of singularities for a pde is in the initial value problem for the Laplace equation in space versus time

$$u_{tt} + u_{xx} = 0 \tag{1}$$

with

$$u(x,0) = u_0(x) \qquad u_t(x,0) = u_1(x) \tag{2}$$

prescribed initially. This problem with both u_0 and u_1 prescribed is well known to be ill-posed [8]. Part of the ill-posedness is that a singularity can form in finite time from initial data u_0, u_1 that is arbitrarily smooth.

Through complex extension of this equation, the singularity formation is easy to understand. Set $x = iy$ and denote $\tilde{u}(y,t) = u(iy,t)$. Then $\partial_x = -i\partial_y$, so that (1) becomes the wave equation

$$\tilde{u}_{tt} - \tilde{u}_{yy} = 0 \tag{3}$$

with initial data

$$\tilde{u}(y,0) = u_0(iy) \qquad \tilde{u}_t(y,0) = u_1(iy). \tag{4}$$

The solution of this equation can be written in the form

$$\tilde{u}(y,t) = \tilde{f}(y+t) + \tilde{g}(y-t) \tag{5}$$

in which $\tilde{f}$ and $\tilde{g}$ and simply related to u_0 and u_1. Denote

$$f(x) = \tilde{f}(-ix) \qquad g(x) = \tilde{g}(-ix) \tag{6}$$

Then

$$u(x,t) = f(x - it) + g(x + it) \tag{7}$$

Now we can easily understand the process of singularity formation for the Laplace equation (1). Suppose that f and g are analytic on the real line, i.e. arbitrarily smooth there, but that g has a singularity off in the complex plane at a point $z_0 = x_0 + iy_0$. Then at time $t = y_0$ the solution u given by equation (7) will have a singularity at the point x_0.

This can be stated in a way that is more global and geometric. The formula (7) says that the Laplace equation has complex characteristics, which are the lines $z = z_0 \pm it$. Singularities, as well as other features of the solution, will move along those lines. This also shows the advantage of describing singularities in terms of complex variables. For complex space variables there is no difference between elliptic and hyperbolic equations since they both have characteristics. In a later section this result will be generalized to nonlinear hyperbolic systems.

Finally note that for Laplace's equation the singularities on the real line at later times are of the same type as the initial singularities in the complex plane. In order to understand the generic form of these singularities, we must find the generic form of singularities for analytic functions. This is discussed in the next section.

3 Singularities for Algebraic Equations.

Square root singularities are generic for algebraic equations. Suppose that a function $f(z)$ is defined through an algebraic equation, such as

$$f^n + \varepsilon f = z. \tag{8}$$

For $\varepsilon = 0$ this functions has an n-th root; i.e. $f(z) = z^{1/n}$. For $\varepsilon \neq 0$ all of the singularities are of square root type. This is seen by noting that a singularity is a point at which $\partial_z f = \infty$, i.e. $\partial_f z = 0$. If $\partial_{ff} z \neq 0$ then the singular point is of square root type; while an n-th root is characterized by the conditions

$$\partial_f z = ... = \partial_f^{n-1} z = 0 \qquad \partial_f^n z \neq 0. \tag{9}$$

Since $\partial_f z = nf^{n-1} + \varepsilon$ and $\partial_{ff} z = n(n-1)f^{n-2}$ in this example, the only multiple-order singularities occur for $\varepsilon = 0$.

In general the solution f of an algebraic equation will generically have square root singularities (i.e. branch points), since zeroes of $\partial_f z$ and $\partial_{ff} z$ will only rarely coincide.

Next consider a collision between two square root branch points. Although a collision is a non-generic event, it might occur due to some symmetry requirments or other constraints. Collision of two roots of $\partial_f z = 0$ implies that $\partial_{ff} z = 0$, but generically $\partial_{fff} z \neq 0$. Therefore a generic collision of square root branch points will form a cube root branch point.

We next claim that the same is true for solutions of complex hyperbolic pde's.

4 Singularities for Burger's Equation.

The simplest example of a nonlinear hyperbolic equation is Burger's equation, which is used as a test problem for all nonlinear hyperbolic theories. Burger's equation is

$$\partial_t f + f \partial_z f = 0 \qquad f(z, 0) = f_0(z). \tag{10}$$

The solution is given implicitly by the formula

$$f(z, t) = f_0(z_0) \tag{11}$$

$$z = z_0 + t f_0(z_0). \tag{12}$$

This says that f is constant and equal to its initial value $f_0(z_0)$ on the characteristic line of points z with initial point z_0 and slope $f_0(z_0)$.

Singularities are points at which

$$\infty = \partial_z f = (\partial_{z_0} f_0)(\partial_{z_0} z)^{-1}. \tag{13}$$

If the initial data f_0 is analytic then this is possible only if

$$\partial_{z_0} z = 0 \tag{14}$$

which is exactly the condition that the characteristic lines $z(t;z_0)$ have an envelope as the parameter z_0 is varied. At such an envelope the generic behavior is again of square root type, since the next derivative $\partial^2_{z_0} z$ will almost always be nonzero.

From equation 12 the condition for a singularity is

$$1 + t f_0'(z_0) = 0 \tag{15}$$

If f_0 is real analytic and entire entire in z_0, then for small t the solutions of 15 must be near ∞. Moreover since $\overline{f_0(z)} = f_0(\overline{z})$ then solutions of 15 must come inconjugate pairs. Thus singularities for Burger's equation start come from ∞ in the complex plane at $t = 0$. When a singularity appears on the real line it must correspond to a collision of two singularities, one coming from above the real line and the second, its complex conjugate, coming from below. As described in the previous section, at such a collision the generic form of the singularity is a cube root.

The nature of the singularities for Burger's equation can be understood also from the implicit solution 11, 12. Consider initial data f_0 that is decreasing as in figure 1a. Since points on the solution move at speed f_0, then points on the upper part of the initial curve will overtake points on the lower part. At some time $t = t_c$ the solution $f(x, t_c)$ will have a vertical tangent at some point x_c as in figure 1b. At later times $t > t_c$ the curve will turn over on itself as in figure 1c, so that $f(x,t)$ will be multi-valued. These turning points x_1 and x_2 are singular points as described above; so that they are actually branch points for the solution. There are branches of the solution that extend beyond these branch points, but on these branches the solution f is complex valued. This is indicated by the dashed lines in figure 1d.

The multi-valuedness of the solution and its branch points can also be understood from the description of the solution through characteristics. In figure 2a starting at time t_c, the real characteristics are shown with their envelopes. Outside the characteristics, the solution f is single valued; while inside it has three real values. When f is analytically extended it must have three values everywhere, but two of the values will be complex outside the envelopes. The real values of f come from characteristics that start on the real line and stay real. The complex values of f at time t come from characteristics that start off in the complex plane and pierce through the real line at that time. This is indicated in figure 2b.

The effect of viscosity on complex singularities for Burger's equation is analyzed in [17].

5 Singularities for a 2×2 System with Riemann Invariants

The results above for Burger's equation can now be generalized to systems of pde's for which there are exactly two characteristic speeds. First this will be done for a system of two equations in diagonal form, i.e. in Riemann invariant form. Later it will be stated for more general systems.

Consider a system of two equations in Riemann invariant form

$$\partial_t f + \lambda(f, g)\partial_x f = 0 \tag{16}$$

$$\partial_t g + \mu(f, g)\partial_x g = 0. \tag{17}$$

For this system singularities will again move along envelopes for the characteristics. Thus there are two types of singularities, those for the λ-characteristics and those for the μ-characteristics. The generic form of singularities is again of square root type, but now we are also interested in interactions, i.e. collisions, of the singularities.

There are two types of collisions: When two λ singularities (or two μ singularities) collide, they are traveling at the same speed λ (or μ), so that they meet tangentially. We call this a tangential collision. On the other hand when a λ singularity hits a μ singularity, they will meet transversely (since $\lambda \neq \mu$), and we call it a transverse collision. This is indicated in figure 3.

The generic behavior of f and g at these collisions can be analyzed using a generalization of the hodograph transformation [7] in which the independent variables x, t are replaced by new independent variables marking the characteristics. The result from this is that at a tangential collision the generic form of the singularity is a cube root, as for a collision of singularities in Burger's equation. This is to be expected since for singularities on the same characteristic family, the local variation of the other characteristic family could be ignored.

On the other hand at a transverse collision, the generic singularity form is a square root. This is rather surprising since the square root form is not generic for singularities of algebraic equations, as described in section 3. The reason for the square root behavior for the transverse collision is that the differential equation entails some constraints that force the singularity to be special at a transverse collision.

Although this result can be proved analytically, it is most easily seen geometrically by looking at the relevant catastrophe surface. For the collision of two square roots, the catastrophe surface is the swallowtail, which is drawn in figure 4. The meaning of this picture takes some discussion. The coordinates of the diagram are $(a,\ b,\ c)$ which represent the coefficients of the fourth order polynomial

$$q(w) = w^4 + aw^2 + bw + c. \tag{18}$$

This polynomial represents the local expansion for the Riemann surface at a fixed time, and the variable $w = z - z_1$ is a shifted spatial variable, in which the shift is chosed to make the cubic term in 18 vanish. This means that near a point z^0 the relationship between z and f (or between z and g) can be expressed as $q(w) = 0$ in which $(a,\ b,\ c)$ are single valued functions of f (or g). Actually the surface in figure 4 represents only the swallowtail surface for real values of $(a,\ b,\ c)$. The surface has an extension for complex values of $(a,\ b,\ c)$.

The swallowtail surface is the set of values $(a,\ b,\ c)$ for which $q(w)$ has a double root; i.e. at which $q(w) = q'(w) = 0$ for some w. As discussed in section 3, this implies that the Riemann surface has a square root branch point. Collisions between two square root branch points occur along the two curves S_1 and S_2 at which the swallowtail intersects itself and at the cusp point S_3. S_1 corresponds to phony collisions, for which the two branch points have the same value of w (or z) but do not meet on the Riemann surface because they lie on different sheets. Geometrically this is because there is no degeneracy of the surface at this self-intersection.

On the other hand on the curve S_2 the two branches of the swallowtail surface are tangential to each other, which is a degeneracy of the surface. This corresponds to tru collisions of the square root branch points. Since the two branches of the surface are tangential, the singularities must meet tangentially. Thus the curve S_2 corresponds to tangential collisions. At points on S_2 the polynomial $q(w)$ satisfies $q(w) = q'(w) = q''(w) = 0$ so that the Riemann surface has a cube root branch point.

Finally the transverse collisions must correspond to the cusp point S_3, at which $q(w) = w^4$. This point corresponds to a square root branch point when time variation is also included.

6 General Systems with Two Characteristic Speeds

Finally we describe a generalization of these results for systems of equations for which there are exactly two characteristic speeds. Consider the system

$$\partial_t F + M(F)\partial_z F = 0 \tag{19}$$

in which F is a vector and $M(F)$ is a matrix. The characteristic speeds for 19 are the eigenvalues of the matrix M. The generic behavior of singularities for this system is stated in the following theorem:

Theorem 5 (Singularities for Complex Hyperbolic PDE's) [13].

Suppose that the system 19 satisfies the following conditions:
(i) F is a nearly constant.
(ii) M(F) is analytic.
(iii) M(F) has exactly two (or one) distinct eigenvalues λ and μ and a full set of eigenvectors.
Then
(1) The generic form of singularities for F is of square root type.
(2) Singularities for F move on envelopes of characteristics; in particular a singular point z_s moves at velocity either λ or μ.
(3) The generic form of F at a collision of singularities is square root type for a transverse collision and cube root type for a tangential collision.

The precise meaning of the term "generic" is made clear in [5]. The proof of this theorem is based on the abstract Cauchy-Kowalewski theorem [4] and on a modification of the hodograph transformation, which causes the restriction to two speeds. The behavior of systems with more than two speeds is an open question. Also the analysis of this theorem is based on assumption of singularities in the initial data. Another outstanding problem is how singularities would form from entire initial data. For example a singularity might form at points z where $\lambda = \mu$.

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Mathematics Department, UCLA, Los Angeles, CA 90024
E-mail address: caflisch@math.ucla.edu