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COMPUTATIONAL AND APPLIED MATHEMATICS

Analysis for the Evolution of Vortex Sheets

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November, 1990

CAM Report 91-05

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Abstract

The evolution of a planar vortex sheet in an incompressible, inviscid flow is described by the Birkhoff-Rott equation. These lectures describe mathematical results on Kelvin-Helmholtz instability, singularity formation, roll-up, and convergence of the vortex method for a vortex sheet. The vortex sheet solution is found in the space of analytic functions, and its existence is proved using the abstract Cauchy-Kowalewski theorem. Some conjectures on extensions of this theory are presented at the end.

I. Introduction

A planar vortex sheet is a curve in 2D flow across which the tangential component of the velocity is discontinuous. Vortex sheets occur in many flows, and their instability and roll-up contribute strongly to mixing and production of small-scale variation and turbulence in these flows.

A flat and uniform vortex sheet is a steady but unstable fluid flow, and perturbations of it will rapidly grow due to Kelvin-Helmholtz instability. The growth rate ω for this instability is proportional to the wavenumber k and can thus be arbitrarily large. This leads to ill-posedness of the vortex sheet problem and formation of singularities on the sheet at a finite critical time t_c. The singularities occur as points at which the curvature becomes infinite. Numerical computations indicate that immediately after t_c the sheet rolls up into a spiral with an infinite number of turns and with the diameter of the spiral starting out as 0 at t_c. Subsequent interaction of the rolls of the vortex

[1] AMS Classification 76C05

[2] Research supported in part by the Air Force Office of Scientific Research under grant number AFOSR 90-0003.

[3] This paper is in final form and no version of it will be submitted for publication elsewhere.

sheet may lead to further singularities, such as the energy concentration singularities proposed by DiPerna and Majda [16-18] and further analyzed by Greengard and Thomann [22].

This paper will describe analytic results on well-posedness and singularity formation for vortex sheets up to and including the time t_c. The results are phrased in the space of analytic functions and include existence and uniqueness of solutions, convergence of the numerical vortex method, construction of singular solution and analysis of the generic form of singularities.

All of this analysis is motivated by asymptotic analyses of Moore [28-30] and numerical computations of Krasny [25,26] and Meiron, Baker and Orszag [27], which demonstrate the instability, singularity formation and roll-up. Vortex sheet singularities have an intrinsic mathematical importance and serve as a simple example that may lead to understanding of possible singularities in other flows such as vortex patches [15], the Rayleigh-Taylor problem [2,7,38,39], unsteady Prandtl boundary layers [37,44,45] and 3D inviscid incompressible flows [4,6].

The physical importance of the curvature singularity of a vortex sheet stems from the phenomena that it produces: It is a strong source of sound waves in a slightly compressible flow [11], and it is followed by roll-up that leads to mixing and production of small scales. Although inclusion of viscosity or finite thickness [3] of the vortex sheet would smooth out and eliminate the singularity, these effects of the singularity would remain.

II. Birkhoff-Rott Equation

A planar vortex sheet is a curve in two-dimensional flow across which the tangential velocity is discontinuous. The position of a vortex sheet is described by a complex function $z(\Gamma, t) = x + iy$ in which Γ is a real variable parameterizing the sheet. For a sheet of all positive vorticity, we take Γ to be the circulation variable,

$$\Gamma = \int (u_+ - u_-) \cdot d\tau$$

so that $\gamma = |\partial z / \partial \Gamma|^{-1}$ is the vorticity density along the sheet. Away from the sheet the velocity is incompressible and irrotational, but there is a delta-function of vorticity on the sheet; i.e.

$$\nabla \cdot u = 0$$

$$\nabla \times u = \gamma \delta_{sheet}.$$

The velocity u changes in time according to the momentum equation.

The vortex sheet then evolves according to the Birkhoff-Rott equation [5]

$$\partial_t \overline{z(\Gamma,t)} = \frac{1}{2\pi i} PV \int_{-\infty}^{\infty} \frac{d\Gamma'}{z(\Gamma,t) - z(\Gamma',t)} \tag{1}$$

in which the bar denotes complex conjugate. Because of the singularity at $\Gamma' = \Gamma$, the integral is understood as a Cauchy principal value integral. The singularity at $\Gamma' = \infty$ is eliminated by taking $\partial_\Gamma z$ to be periodic. A second difficulty of this equation is the conjugation, which is required in order to represent rotation.

The Birkhoff-Rott equation can be extended to complex Γ by modifying it in two ways: First analytically extend the conjugate function $\overline{z(\Gamma)}$ to

$$z^*(\Gamma) = \overline{z(\overline{\Gamma})}.$$

Second, in the principal value integral the contour must be moved along with Γ. The analytic extension of the Birkhoff-Rott equation is then

$$\partial_t z^*(\Gamma,t) = \frac{1}{2\pi i} PV \int_{-\infty}^{\infty} \frac{d\zeta}{z(\Gamma,t) - z(\Gamma+\zeta,t)} \tag{1'}$$

A flat vortex sheet of uniform strength $\gamma = 1$ is given by $z = \Gamma$ which is a steady solution of (1). A perturbation of this solution is written in the form

$$z(\Gamma,t) = \Gamma + s(\Gamma,t),$$

and the linearized equation for s is

$$\partial_t \overline{s(\Gamma,t)} = \frac{1}{2\pi i} PV \int_{-\infty}^{\infty} \frac{s(\Gamma',t) - s(\Gamma,t)}{(\Gamma' - \Gamma)^2} d\Gamma' = \frac{1}{2\pi i} PV \int_{-\infty}^{\infty} \frac{s_\Gamma(\Gamma',t)}{\Gamma' - \Gamma} d\Gamma'$$

$$= \frac{1}{2} H[\partial_\Gamma s](\Gamma,t) \tag{2}$$

in which H is the Hilbert transform

$$H[f](\Gamma) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{f(\Gamma+\zeta)}{\zeta} d\zeta.$$

Since $H[f] = f_+ - f_-$, in which f_+ and f_- are respectively the upper and lower analytic parts of f, then the linearized equation is equivalent to the system

$$\partial_t (s_+)^* = -\frac{1}{2} \partial_\Gamma s_- \tag{3.1}$$

$$\partial_t(s_-)^* = \frac{1}{2}\partial_\Gamma s_+. \tag{3.2}$$

The fundamental modes for this linearized problem are

$$s_{k+} = (1 - i)e^{|k|t/2}\sin(k\Gamma) \tag{4.1}$$

$$s_{k-} = (1 + i)e^{-|k|t/2}\sin(k\Gamma). \tag{4.2}$$

The growth of s_{k+} (for $k > 0$) at exponential rate $k/2$ is the Kelvin-Helmholtz instability of the sheet and shows equation (2) to be elliptic in Γ vs. t.

The analysis below proceeds from a hyperbolic interpretation of the modes in (4). Rewrite $s_{k\pm}$ multiplied by $sgn(k)$ as

$$s_{k+} = \frac{(1 - i)}{2i}(e^{i|k|(\Gamma - it/2)} - e^{-i|k|(\Gamma + it/2)}) \tag{4.1'}$$

$$s_{k-} = \frac{(1 + i)}{2i}(e^{i|k|(\Gamma + it/2)} - e^{-i|k|(\Gamma - it/2)}). \tag{4.2'}$$

This representation shows that equation (2) is hyperbolic in Γ vs t with characteristic speeds $i/2$ and $-i/2$.

Another way to see this is that the system (3) shows that s_+ (and s_-) satisfy Laplace's equation in Γ vs. t, i.e.

$$\partial_{tt}s_+ + \frac{1}{4}\partial_{\Gamma\Gamma}s_+ = 0$$

Replace Γ by iy to get the wave equation

$$\partial_{tt}s_+ - \frac{1}{4}\partial_{yy}s_+ = 0.$$

Such an interpretation of an elliptic problem as hyperbolic in the complex direction has been exploited by Garabedian [2] for a variety of *pde* problems including transonic flow and by Moore [30] for the vortex sheet problem.

From this hyperbolic interpretation it follows that features of the solution s, such as singularities, will approximately move along the linear characteristics with speeds $\pm i/2$.

III. Localized Approximation

As shown above, the linearization of the Birkhoff-Rott equation around the flat sheet results in a system of first order linear partial differential equations that are elliptic in real Γ vs. t or hyperbolic in imaginary Γ vs. t. Now we make a nonlinear generalization of this approximation of the nonlocal equations (3) by a system of local equations. To do this a certain amount of nonlocality is included by introducing some new dependent variables that must be initially defined by a nonlocal operation. Define

$$f = (\partial_\Gamma z)^{-1} \qquad f^*(\Gamma) = \overline{f(\overline{\Gamma})}$$

$$g = Hf \qquad g^*(\Gamma) = \overline{g(\overline{\Gamma})}$$

in which H is the Hilbert transform. Note that f^*, g and g^* are all defined by nonlocal operators applied to f. The function f may be interpreted as complexified vortex sheet strength, since the real strength is $\gamma = |\partial_\Gamma z|^{-1}$, and g is the approximate velocity of the sheet.

The remarkable fact is that these nonlocal quantities suffice to get a good nonlinear approximation of the Birkhoff-Rott equation in terms of a system of first order pde's. Equation (1) is equivalent to the system

$$\partial_t F = M(F)\,\partial_\Gamma F + E[F] \tag{5}$$

in which

$$M(F) = \frac{1}{2}\begin{pmatrix} 0 & f^2 & 0 & 0 \\ -f^{*2} & 0 & 0 & 0 \\ 0 & 0 & 0 & f^{*2} \\ 0 & 0 & -f^2 & 0 \end{pmatrix}.$$

and the vector F is $(f,\ g^*,\ f^*,\ g)$. It is important to note that the matrix $M(F)$ has two double eigenvalues

$$\lambda = \frac{i}{2} f f^* \qquad \mu = -\frac{i}{2} f f^*$$

and a full set of eigenvectors.

This system was derived in [12] as a generalization of the approximate equations of Moore [29,30]. It is a nonlinear hyperbolic system with 2 double characteristics for which the speeds are λ and μ.

The error terms E are small compared with the derivative terms g_Γ and f_Γ, because E consists of terms like the commutator $[f, H[\partial_\Gamma f]] = fH[\partial_\Gamma f] - H[f\partial_\Gamma f]$. This is precisely stated in the following theorem.

<u>Theorem 1</u> (Error Estimates) [12]. *Suppose that* $|f| > c_1 > 0$. *Then for* $i = 1,2$

$$||E_i||_r \le c||\partial_\Gamma^{1/2} f||_p \ ||\partial_\Gamma^{1/2} g||_p$$

in which $r = p/2$.

The physical interpretation of this approximation is not clear, although Nicholas Rott has suggested that it may be something like a slender airfoil approximation. Such an interpretation could be quite helpful because it might show how to extend this approximation to other problems, such as three dimensional vortex sheets.

If the error terms $E[F]$ are omitted, the resulting system

$$\partial_t F \ + \ M(F)\, \partial_\Gamma F \ = \ 0 \tag{6}$$

is a complex hyperbolic system of pde's with two double characteristics. Since the error terms are small, even in the presence of singularities, Birkhoff-Rott solutions can be found by first solving the hyperbolic equations (6), then adding the E_i as perturbations using the abstract Cauchy-Kowalewski Theorem.

This localized approximation for the vortex sheet problem is the main tool in all of the following theory. It is a generalization of Moore's approximation [29, 30].

IV. Well-Posedness Results

The vortex sheet problem (1) has been shown to be well-posed in an analytic function space for a short time. Ill-posedness in Sobolev space will be shown in the next section. The first existence theorem for analytic solutions of (1) was derived by Sulem, Sulem, Bardos and Frisch [36]. A slightly different result, which exploits proximity to the flat equilibrium solution, is presented here:

Define

$$B_\rho \ = \ \{s : s(\Gamma \ + \ 2\pi) \ = \ s(\Gamma), \ \ \text{s is analytic in } |Im\Gamma| \ < \rho, ||s||_\rho \ < \infty\}$$

$$||s||_\rho \ = \ \ sup_{|Im\Gamma| \ < \rho} |s(\Gamma)|.$$

Theorem 2 (Existence and Uniqueness) [8] *Suppose that* $s_0(\Gamma) = s(\Gamma, t = 0) \epsilon B_{\rho_0}$ *and* $||s_0||_{\rho_0} \ < \varepsilon$. *Then for* ε *sufficiently small, the Birkhoff-Rott equation (1) has a unique analytic solution* $z \ = \ \Gamma \ + \ s$ *for* $0 \ \le \ t \ \le \ (2 \ - \ \delta)\rho_0$.

This theorem, as well as the earlier result [36], is proved using the abstract Cauchy-Kowalewski theorem [1,31,32,42]. Since the initial data is analytic in a strip of width ρ_0 and the linear characteristics move at speed $\pm i/2$, the solution at time t should be analytic in the strip of width $\rho(t) \ = \ \rho_0 \ - \ t/2$. A correction term δ is needed since the linear characteristics are only approximately correct. These nonlinear error terms are handled using the abstract Cauchy-Kowalewski theorem. Construction of exact solutions with singularities, which is described in the next two sections, shows this result to be nearly optimal.

The necessity of the analytic function setting for this theorem is seen by considering the linear Kelvin-Helmholtz instability. For initial data $s_0(\Gamma)$, the k-th Fourier component grows like $e^{|k|t/2}$, i.e.

$$\hat{s}(k,t) \approx \hat{s}_0(k) e^{|k|t/2}.$$

This is bounded for $0 \le t \le T$ only if

$$|\hat{s}_0(k)| \le c e^{-|k|T/2}$$

which is roughly equivalent to the requirement that s_0 be analytic in a strip $|Im\Gamma| < T/2$.

Next consider a numerical method for the Birkhoff-Rott equation (1). Desingularize the integral operator on the right of (1) as

$$B_\delta[s] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma - \Gamma' + \overline{s(\Gamma)} - \overline{s(\Gamma')}}{|\Gamma - \Gamma' + s(\Gamma) - s(\Gamma')|^2 + \delta^2} d\Gamma'$$

with a desingularization parameter δ. Rewrite B_δ using periodicity and discretize it as

$$B_{\delta h}[s](m) = \frac{h}{4\pi i} \sum_{l=1}^{N} \frac{\cos\frac{1}{2}(s - s' - lh) \; \sin\frac{1}{2}(\overline{s} - \overline{s'} - lh)}{\sin\frac{1}{2}(s - s' - lh) \; \sin\frac{1}{2}(\overline{s} - \overline{s'} - lh) + \delta^2} \tag{7}$$

in which $s = s(m), s' = s(l)$ and h is the spatial grid size. The vortex blob method for evolution of a vortex sheet consists of the system of ode's

$$\dot{s}_D(m) = B_{\delta h}[s_D](m) + e_r \tag{8}$$

in which e_r is a forcing function that simulates roundoff. If δ is set to zero, then (8) is the equations of the point vortex method.

Because of the ill-posedness of the Kelvin-Helmholtz instability, this numerical method can only be analyzed in the class of analytic functions. This is interpreted in terms of the Fourier transform or its discrete analogue. Analyticity of s in the strip $|Im\Gamma| < \rho$ is roughly equivalent to the inequality

$$|\hat{s}(k)| < c e^{-\rho|k|}. \tag{9}$$

If s is discretized with grid size h, then the maximal wavenumber k is

$$k_{max} = h^{-1}. \tag{10}$$

Therefore the discrete analogue of analyticity is the inequality (9) for $|k| \leq k_{max}$. In particular this requires that roundoff errors satisfy

$$|e_r| < \varepsilon_r e^{-1/h} \tag{11}$$

for some constant ε_r independent of h. This is a severe restriction on the allowable range of h for a given level of precision, but seems to be optimal for this ill-posed problem. By contrast for well-posed problems most numerical schemes only require $|e_r| < \varepsilon_r h^n$ for some n.

This analytic approach leads to the following convergence theorem for vortex sheets.

Theorem 3 (Convergence of the Vortex Method) [10,24]. *Suppose that the initial data s_0 satisfies the assumptions of Theorem 1 and that the roundoff error e_r satisfies*

$$e_r < \varepsilon_r \Delta t\, e^{-1/h}.$$

Let s solve the Birkhoff-Rott equation (1) and s_D solve the discrete system (8) with this initial data. Then for a short time and for h, δ and ε_r sufficiently small,

$$sup_m |s(t = mh) - s_D(m)| < c(\delta + \varepsilon_r + h). \tag{12}$$

Note the following:

(1) This shows convergence of both the vortex blob ($\delta > 0$) and point vortex ($\delta = 0$) methods for the vortex sheet.

(2) Small roundoff error is needed because the roundoff error can grow under Kelvin-Helmholtz instability.

(3) With the δ desingularization of Krasny, stability is increased and the condition on roundoff error level can be relaxed to

$$|e_r| < c\varepsilon_r max(e^{-1/h}, e^{-1/\delta^2}).$$

We have not however been able to fully incorporate the advantages of Krasny's filtering method into this theorem.

V. Construction of Singular Solutions

Examples of vortex sheet solutions that develop singularities in finite time from smooth initial data can be constructed using the approach of the previous section. One way to do this is to first consider initial data with a singularity at $\Gamma = 0$. The singularity will split up and move approximately along the linear characteristics in the imaginary Γ direction. Thus for $t > 0$ the solution will be analytic on the real Γ line. This solution may be run backwards to obtain singularities from initially smooth data. The result is summarized in the next theorem.

Theorem 4 (Construction of Singular Solutions) [9,19]. *Let $\kappa > 1, t_0 > 0$ and let $\varepsilon > 0$ be sufficiently small. Then there is a solution $z = \Gamma + s(\Gamma, t)$ for $0 \leq t \leq t_0$ of the Birkhoff-Rott equation with s analytic for $t = 0$ and*

$$s(\Gamma, t_0) \approx \varepsilon \; sgn\,(\Gamma)|\Gamma|^{\kappa}$$

for $\Gamma \approx 0$.

This shows that singularities of any order larger than one can be produced. However, as described in the next section the generic form of the singularity is $\kappa = 3/2$. The singular solutions may also be used to show ill-posedness for the vortex sheet problem in any Sobolev space H_n with $n > 3/2$ [9,20]; i.e. for arbitrarily small initial perturbations s_0 in H_n, there can be a singularity in an arbitrarily short time.

The importance of these singularities is both physical and mathematical. The singularity leads to roll-up of the sheet and the resulting production of small scale variation in the flow. It also is a strong source of noise in a slightly compressible flow. Although smoothing effects such as viscosity, surface tension and finite thickness would smooth out the singularity, these effects of the singularity would remain. Thus we may consider the singularity as an idealized phenomenon that may simplify the description of more realistic and complicated phenomena.

Mathematically the singularity is important for any understanding of the well-posedness of the vortex sheet problem. The vortex sheet singularity may also serve as a model problem for other, more difficult singularity problems, such as possible singularities for the three dimensional Euler equations.

VI. Generic Form of Vortex Sheet Singularities

To understand the generic form of singularities for the Birkhoff-Rott equation, it is necessary to investigate motion of singularities in the complex Γ plane. We present several results which indicate that the generic form of singularities is $s \approx \Gamma^{3/2}$. The square root behavior in s_Γ is associated with an envelope of characteristics, and near such an envelope the generic behavior is a square root. This is most easily understood by considering several examples:

Example 1. Algebraic equations. Square root singularities are generic for algebraic equations. Suppose that a function $f(z)$ is defined through an algebraic equation, such as

$$f^n + \varepsilon f = z.$$

For $\varepsilon = 0$ this functions has an n-th root; i.e. $f(z) = z^{1/n}$. For $\varepsilon \neq 0$ all of the singularities are of square root type. This is seen by noting that a singularity is a point at which $\partial_z f = \infty$, i.e. $\partial_f z = 0$. If $\partial_{ff} z \neq 0$ then the singular point is of square root type; while an n-th root is characterized by the conditions

$$\partial_f z = \ldots = \partial_f^{n-1} z = 0 \qquad \partial_f^n z \neq 0.$$

Since $\partial_f z = n f^{n-1} + \varepsilon$ and $\partial_{ff} z = n(n-1) f^{n-2}$ in this example, the only multiple-order singularities occur for $\varepsilon = 0$.

This is easily seen to be generic, since zeroes of $\partial_f z$ and $\partial_{ff} z$ will only rarely coincide. We next claim that the same is true for solutions of pde's and for vortex sheets.

Example 2. Burgers' Equation. Burger's equation is

$$\partial_t f + f \partial_z f = 0 \qquad f(z,0) = f_0(z).$$

The solution is given implicitly by the formula

$$f(z,t) = f_0(z_0)$$

$$z = z_0 + t f_0(z_0).$$

This says that f is constant and equal to its initial value $f_0(z_0)$ on the characteristic line of points z with initial point z_0 and slope $f_0(z_0)$.

Singularities are points at which

$$\infty = \partial_z f = (\partial_{z_0} f_0)(\partial_{z_0} z)^{-1}.$$

If the initial data f_0 is analytic then this is possible only if

$$\partial_{z_0} z = 0$$

which is exactly the condition that the characteristic lines $z(t; z_0)$ have an envelope as the parameter z_0 is varied. At such an envelope the generic behavior is again of square root type, since the next derivative $\partial_{z_0}^2 z$ will almost always be nonzero.

Example 3. Hyperbolic systems. Consider a system of two equations in Riemann invariant form

$$\partial_t f + \lambda(f,g)\partial_z f = 0$$

$$\partial_t g + \mu(f,g)\partial_z g = 0.$$

For this system singularities will again move along envelopes for the characteristics. Thus there are two types of singularities, those for the λ-characteristics and those for the μ-characteristics. The generic form of singularities is again of square root type, but now we are also interested in interactions, i.e. collisions, of the singularities.

There are two types of collisions: When two λ singularities (or two μ singularities) collide, they are traveling at the same speed λ (or μ), so that they meet tangentially. We call this a tangential collision. On the other hand when a λ singularity hits a μ singularity, they will meet transversely (since $\lambda \neq \mu$), and we call it a transverse collision. The generic behavior is different at the two types of collisions, as stated in the following theorem:

Theorem 5 (Singularities for Complex Hyperbolic PDE's) [13]. *Consider the system*

$$\partial_t F + M(F)\partial_z F = 0$$

satisfying the following conditions:
(i) F is a nearly constant.
(ii) M(F) is analytic.
(iii) M(F) has exactly two (or one) distinct eigenvalues λ and μ and a full set of eigenvectors.
Then
(1) The generic form of singularities for F is of square root type.
(2) Singularities for F move on envelopes of characteristics; in particular a singular point z_s moves at velocity either λ or μ.
(3) The generic form of F at a collision of singularities is square root type for a transverse collision and cube root type for a tangential collision.

The precise meaning of the term "generic" is made clear in [13]. The proof of this theorem is based on a modification of the hodograph transformation, which causes the restriction to two speeds. The behavior of systems with more than two speeds is an open question. Also the analysis of this theorem is based on assumption of singularities in the initial data. Another outstanding problem is how singularities would form from entire initial data. For example a singularity might form at points z where $\lambda = \mu$

Finally we consider singularities for the Birkhoff-Rott equation. The first analysis of such singularities was through a very clever asymptotic analysis by Moore [29,30]. This has been improved by Cowley [43] and by Caflisch, Ercolani and Hou [14]. Numerical observations of singularities have been performed by Meiron, Baker and Orszag [27], Krasny [25] and Shelley [35]. This presentation follows the approach of [14].

First note that the localized approximation (6) for the Birkhoff-Rott equation satisfies the assumptions of Theorem ; i.e. for a small perturbation of a flat sheet, it has two distinct characteristic speeds $\lambda \approx i/2$ and $\mu \approx -i/2$ and a full set of eigenvectors.

Singularities may occur in the complex Γ plane, but a singularity is physical only when it hits the real Γ line. Since the localized system (6) contains both f and f^*, then the presence of a singularity at a point Γ_s implies one also at $\overline{\Gamma_s}$. Moreover these two singularities will move at the two different speeds λ and μ. Thus when a physical singularity appears on the real Γ line, it must be a transverse collision of two singularities, one coming form the upper half plane and one from the lower half plane. Therefore the generic form of singularities for the Birkhoff-Rott equation is of square root type in $f = (\partial_\Gamma z)^{-1}$, so that generically at the singularity time

$$f = f_0 + f_1\sqrt{\Gamma - \Gamma_0} + \ldots$$

and

$$z = z_0 + f_0\Gamma + \frac{2}{3}f_1(\Gamma - \Gamma_0)^{3/2}.$$

VII. Roll-Up

The analytic results described above are valid up to and including the singularity time t_c but not beyond it. Experimental observations and numerical computations [26] show that after the singularity time the vortex sheet rolls up into a spiral with an infinite number of turns. The spiral diameter starts at zero at the singularity time and then grows, with the spiral rolling up the vortex sheet. Thus the only length scale in the solution comes from a time scale, and so the solution can be expected to be of similarity form.

There are no rigorous mathematical results on the roll-up after the singularity, but there are several asymptotic and numerical results. A solution may be sought in similarity form

$$z(\Gamma, t) \;=\; t^{\frac{1}{2-p}} z(\xi \;=\; \Gamma/t^{\frac{p}{2-p}}) \tag{13}$$

for any p in the range $0 < p < 2$. The equation for z is

$$p\xi\overline{z_{\xi}(\xi)} - \overline{z(\xi)} = \frac{(2-p)}{2\pi i} PV \int_{-\infty}^{\infty} \frac{d\xi'}{z(\xi) - z(\xi')}. \tag{14}$$

Moore [28], Guiraud and Zeytounian [23], and Pullin [33] have analyzed (14) for $|\xi| \ll 1$ and $|\xi| \gg 1$ and shown that

$$z(\xi) \cong \begin{cases} z_0 |\xi|^{1/p}\, sgn(\xi) & \text{for } |\xi| \gg 1 \\ z_1 |\xi|^{1/p'} e^{i|\xi|^{1-2/p'}}\, sgn(\xi) & \text{for } 0 \le \xi \ll 1 \end{cases} \tag{15}$$

which shows the roll-up to be an algebraic spiral in its center. On physical grounds it has been supposed that $p' = p$, but the mathematical basis of this assumption is unclear.

Pullin and Phillips [34] have performed numerical solutions of equation (14). They found rolled up solutions which agree with (15) for $0 < p < 1$. Pullin [33] also found a branch of smooth solutions and a branch of solutions consisting of two half-infinite vortex sheets that separately roll-up. Most strikingly he found multiple similarity solutions with the same initial data. For example for $p = 1$ one solution is $z(\xi) = \xi$ corresponding to the steady solution $z(\Gamma, t) \equiv \Gamma$. A second numerical solution consists of two separate rolled-up sheets, which start off as a flat sheet with $z = \Gamma$ at $t = 0$. The sheet is effectively cut at $\Gamma = 0$ initially, and the two halves roll-up. The resulting solution is not analytic, so that the well-posedness theorems of section III are not violated. Resolution of this non-uniqueness may require some selection principle, such as the Kutta condition for flow around a boundary with a corner.

A Model Equation for Roll-Up

The asymptotic form (15) in the center of the spiral $|\xi| \ll 1$ is based on the following approximate equation

$$\partial_t \overline{z(\Gamma, t)} = \frac{1}{2\pi i} \frac{\Gamma}{z}. \tag{16}$$

The origin of this approximation is that the spiral is algebraic and tightly wound-up, so that the vorticity density in the center is nearly radially symmetric. As a result, points on the vortex sheet move on circular paths, with a velocity due to the total circulation inside the circle. This circulation is just the value Γ at the point, and the velocity is then given by (16).

Equation (16) is valid only near the center of the spiral, since for example it is not translation invariant in Γ. A translation invariant model equation similar to (16) is

$$\partial_t \overline{z(\Gamma, t)} = \frac{1}{2\pi i} \frac{1}{z_\Gamma}. \tag{17}$$

Look for similarity solutions for (17) of the form (13) with

$$z(\xi) = \xi^{1/2} \, s \, e^{i\theta} \tag{18}$$

and denote $\eta = log \xi$. The equations for s and θ are then

$$\frac{ds}{d\eta} = \frac{s}{2p}(1 - p \pm \sqrt{1 - s^{-4}}) \tag{19}$$

$$\frac{d\theta}{d\eta} = \frac{2 - p}{2\pi} s^{-2}. \tag{20}$$

These equations are easily solved, and for $1 < p < 2$, there are spiral solutions of the form

$$z(\xi) \cong \begin{cases} z_0 |\xi|^{1/p} \, sgn(\xi) & \text{for } |\xi| \gg 1 \\ z_1 \, |\xi|^{1/2+i} \, sgn(\xi) & \text{for } 0 \leq |\xi| \ll 1 \end{cases} \tag{21}$$

This is a logarithmic spiral at the center.

We hope that some new understanding of vortex sheet roll-up can come from this model. In particular, one may look for solutions of the model (16) with smooth initial data, which some time later roll-up. One defect of this model is that the linearized equation is hyperbolic rather than elliptic. It is probably possible to find better model equations that are elliptic in Γ vs. t and have spiral similarity solutions.

VIII. Extensions

In this last section, we present several speculations or conjectures about additional properties for vortex sheets. This presentation is primarily intended to stimulate the reader to think about these questions. There is no claim that these conjectures are original.

Regularization of Vortex Sheets. Consider the addition of viscosity ν, finite thickness d, numerical desingularization δ or surface tension σ.

Conjecture 1. In the limit $(\nu, d, \delta) \to 0$ with $\sigma = 0$, the Birkhoff-Rott equation is obtained even after the singularity time t_c.

Conjecture 2. In the complex Γ plane, branch cuts in the solution for $\nu = 0$ are replaced by sequences of interlacing poles and zeroes in the solution for $\nu > 0$.

Such a transition from branch cuts to poles and zeroes is seen in Pade approximation of analytic functions by rational functions and for Burger's equation [41,46,47].

Conjecture 3. In the zero surface tension limit, there are some solutions which develop increasingly rapid oscillations, so that the vortex sheet equations are not obtained.

There are several pieces of evidence for this: Numerically the limit $\sigma \rightarrow 0$ has been difficult to compute. One explanation is that an order 1 amount of energy may go into surface waves. Since the surface energy is σ times the length of the interface, this would require the interface to become increasingly long, i.e. to develop many oscillations. A mechanism for transferring energy into surface modes is found in the solitary wave solutions of Beale [40] for $\sigma \neq 0$. He shows (at least formally) that these solitary waves have an oscillatory part at infinity of very small amplitude. Since the oscillatory part goes all the way to infinity, it contains an infinite amount of energy. Thus an attempt to form such a wave could drive all of the energy into the oscillatory part. Barry Merriman has pointed out a weakness in this result; namely that the propagation speed for this energy goes to zero as $\sigma \rightarrow 0$, so that the energy may be unable to get out to the oscillatory waves in any finite time.

The Birkhoff-Rott Equation after Singularities.

Conjecture 4. The solution of the Birkhoff-Rott equation has a branch point in time at the singularity time $t = t_c$.

The motivation for this conjecture is that at a transverse collision of branch points, the Hilbert transform may have a branch point in time at which it is infinitely multi-valued. If this conjecture is correct, it would seem to say that there is no unique way to continue the Birkhoff-Rott solution past the singularity time. Determination of the correct solution would then require regularization, for example taking the limit $\nu \rightarrow 0$.

IX. References

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