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# Multi-Valued Solutions and Branch Point Singularities for Nonlinear Hyperbolic or Elliptic Systems *

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**Abstract**

Multi-valued solutions are constructed for $2 \times 2$ first order systems using a generalization of the hodograph transformation. The solution is found as a complex analytic function on a complex Riemann surface for which the branch points move as part of the solution. The branch point singularities are envelopes for the characteristics and thus move at the characteristic speeds. We perform an analysis of stability of these singularities with respect to perturbations of the initial data. The generic singularity types are folds, cusps and non-degenerate umbilic points with non-zero 3-jet. An isolated singularity is generically a square root branch

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point corresponding to a fold. Two types of collisions between singularities are generic: At a "tangential" collision between two singularities moving at the same characteristic speed, a cube root branch point is formed, corresponding to a cusp. A "non-tangential" collision, between two square root branch points moving at different characteristic speeds, remains a square root branch point at the collision and corresponds to a non-degenerate umbilic point. These results are also valid for a diagonalizable $n$-th order system for which there are exactly two speeds.

# 1 Introduction

Multi-valued solutions of nonlinear hyperbolic equations occur due to formation of envelopes for the characteristics. Such multi-valued solutions are well-known for Burger's equation (see Section 2). In this paper we construct multi-valued solutions for $2 \times 2$ strictly hyperbolic and elliptic systems, as well as for $n$-th order systems that are diagonalizable and have exactly 2 characteristic speeds. In addition we analyze the generic form of singularities for such solutions.

A multi-valued solution is more naturally described as a complex analytic function of a complex variable on an evolving Riemann surface. The function is singular at branch points, which move on the surface as part of the solution. In general these branch point singularities move along envelopes of the characteristics, but if one of the singularities is sufficiently weak it will move along a characteristic. Interactions, i.e. collisions of singularities, are the most interesting feature of these solutions.

The significance of these multi-valued solutions for applications such as fluid dynamics is that they correctly describe the initial onset of physical singularities. For example analytic solutions of the compressible flow equations will have branch point singularities for complex values of the space variable $x$. This description of the singularities is correct up to and including the time that the branch points hit the real axis, i.e. up to the first time for shock formation. After the time of the first shock, the multi-valued solution is changed: For a scalar equation the change consists of putting in shocks that connect solutions on different branches; for systems, the solution between shocks is not part of the multi-valued solution found here.

A second example is the Birkhoff-Rott equation for evolution of a planar vortex sheet, which can be written as a 4th order system with exactly two characteristic speeds and with lower order non-local terms [8]. Our results indicate that the generic singularity on a vortex sheet is of square root type, as predicted by Moore [23, 24] and numerically verified by Shelley [28].

The multi-valued solution is analyzed by introducing a transformation that unfolds its singularities. For a $2 \times 2$ system the unfolded system may be made linear with non-constant coefficients, using a generalization of the hodograph transformation. For an $n$-th order system with 2 speeds, the unfolded system is still nonlinear, and its solutions are constructed using the Cauchy-Kowalewski Theorem [9]. In either case, our construction of the solution requires extension to complex values of the independent variables $z$ and $t$, so that singularity development on the real line for $t > 0$ can be traced back to a singularity at a complex point in the initial data.

Once a solution is constructed for the unfolded system, its singularities in the original $(z, t)$ plane can be analyzed by geometric methods. The first type of singularity is an isolated branch point, which corresponds to a fold singularity. Second is a collision of two square root branch points traveling at different characteristic speeds, which corresponds to a non-degenerate umbilic singularity with nontrivial 3-jet. Finally a collision between singularities traveling at the same characteristic speed corresponds to a cusp. These types of singularities are stable with respect to perturbations of the initial data; all other types of singularities are unstable. For consistency with the existence result, the singularity analysis will be performed within the class of holomorphic functions. The same results are true, however, for real singularities.

Our main results concern a first order system given by

$$F_t + M(F)F_z = \phi(F) \tag{1.1}$$

$$F(z, t = 0) = F_0(z) \tag{1.2}$$

in which $F_0, F, \phi(F) \in C^n$ (n dimensional complex space) and $M(F)$ is a complex $n \times n$ matrix. Assume that $M$ is diagonalizable and has exactly two characteristic speeds, i.e.

$$M(F) = A(F)^{-1}\Lambda(F)\, A(F)$$

$$\Lambda(F) = diag(\lambda(F), \ldots, \lambda(F), \mu(F), \ldots, \mu(F)) \tag{1.3}$$

$$\tag{1.4}$$

in which $\lambda$ and $\mu$ have multiplicity $k$ and $n - k$, respectively with $0 \leq k \leq n$. We will assume that $\lambda \neq \mu$ and that $M$, $A$, $\lambda$, $\mu$ are all analytic in $F$. We also assume that the system satisfies a condition of nontrivial coupling of the components of $F$. The precise condition is given in Theorem 7.1.

The initial data $F_0$ is assumed to be an analytic function on a Riemann surface. In particular it is allowed to have branch point singularities in $z$. $F_0$ may be locally unfolded by the representation

$$z = \zeta(v)$$

$$F_0(z) = \mathcal{F}_0(v) \tag{1.5}$$

in which $\mathcal{F}_0$ and $\zeta$ are single-valued analytic functions of $v$. The corresponding value of $t$ in (1.5) is $t = \tau(v) \equiv 0$. From the geometric point of view, the natural perturbations of the initial data (1.5) include perturbations of $\tau$, i.e. a perturbation of the initial data is

$$\begin{aligned} z &= \tilde{\zeta}(v) \\ F_0 &= \tilde{\mathcal{F}}_0(v) \\ t &= \tilde{\tau}(v) \end{aligned} \tag{1.6}$$

in which $\tilde{\zeta}$, $\tilde{\mathcal{F}}_0$ and $\tilde{\tau}$ are small perturbations of $\zeta$, $\mathcal{F}_0$ and $\tau \equiv 0$, respectively. The necessity of perturbing $t$ is discussed further in Section 6.

The following two theorems are the main results of this paper:

**Theorem 1.1** *(Existence of Solutions). Assume that $M$ and $F_0$ satisfy the assumptions above. Then for a short time $0 \leq t \leq t_0$, the equation ( 1.1) has a solution $F(z,t)$ which is an analytic function of $z$ on a Riemann surface with a finite number of branch points of finite order for each $t$. Each branch point starts at a branch point in the initial data $F_0$ and moves at speed either $\lambda$ or $\mu$, i.e. each branch point travels along a characteristic or an envelope for one of the two families of characteristics.*

**Theorem 1.2** *(Generic Singularities). Suppose that $M$ and $F_0$ satisfy the assumptions above. Under perturbation of the "initial data" $\zeta$, $\mathcal{F}_0$ and $\tau$ as in (1.6), the stable singularity types for ( 1.1) are the following:*

*1. Fold. This singularity lies on a complex curve $z(t)$ which moves at either speed $\lambda$ or $\mu$. It corresponds to a square root branch point in $z$ for each $t$. In a neighborhood of the branch point, this singularity can be "unfolded" to the normal form*

$$\begin{aligned} z &= u^2 \\ t &= v. \end{aligned}$$

*2. Cusp. This singularity is a single point in $(z,t)$ at which there is a collision of two singularities. The colliding singularities are either both envelope points for the $\lambda$ characteristics or both envelope points for the $\mu$ characteristics. The normal form of this singularity is*

$$\begin{aligned} z &= u^3 + vu \\ t &= v. \end{aligned}$$

*3. Non-degenerate umbilic. This singularity is also a single point at which two singularities collide, one corresponding to $\lambda$ and the other corresponding to $\mu$. Under an additional condition described in Theorem 7.1 below, these are umbilics with non-trivial cubic terms The normal form in this case is*

$$\begin{aligned} z &= u^2 + v^3 \\ t &= v^2 + u^3. \end{aligned}$$

*Given any solution of system ( 1.1) its initial data may always be perturbed so that the resulting solution has only singularities of the above types. In the* *case that there is only one characteristic speed $\lambda$, the non-degenerate umbilic singularity is not generic.*

In Theorem 1.2, the multivalued solution is regarded as a single-valued function on a surface that is a branched covering of the $(z,t)$-plane, with a covering map $z(u,v), t(u,v)$ from $\mathbb{C}^2$ to $\mathbb{C}^2$. The type of a singularity for a map is characterized by equivalence under conjugation, i.e., change of variables in the domain and range. For example, a point $(z,t) = (0,0)$ is a fold, if in a neighborhood of $(0,0)$,

$$\phi \circ (z,t) \circ \psi(u,v) = (u^2, v) \tag{1.7}$$

for some transformations $\phi$ and $\psi$ that are locally smooth and smoothly invertible.

We refer to a map $z(u,v), t(u,v)$ between a surface, locally parameterized by $(u,v)$ on which the multivalued solution becomes single-valued, and the original $(z,t)$-plane as an *unfolding map*. The classical hodograph inversion $z(f,g), t(f,g)$ provides an example of such an unfolding. However, in order to have sufficient flexibility to analyze the initial value problem perturbatively, we work in a much broader class of unfolding maps. Indeed, the principal achievement of this paper is to reduce the classification of generic singularities for systems with two speeds to the classification of singularities of the unfolding map.

*Stability* in this setting means that given a solution whose singularities are only folds, cusps or umbilics with nonzero cubic terms, then under a small perturbation of the initial data the perturbed solution will also have singularities only of these types. In other words the set of initial data producing solutions with only these singularities is an open set. To establish the *genericity* one needs also to show that this set is *dense*; i.e., one can perturb the data so that the perturbed solution has only stable singularities.

For arbitrary mappings $z(u,v), t(u,v)$ from $\mathbb{C}^2$ to $\mathbb{C}^2$, Whitney's Theorem states that the generic singularities are folds and cusps. The additional constraint that $z, t$ is the unfolding for a solution of ( 1.1) restricts the allowable perturbations, so that an additional singularity type, the non-degenerate umbilics, is also stable. Any other type of singularity is either not realizable for a solution of ( 1.1) or is not stable under perturbations of the initial data.

Theorem 1.2 is stated in terms of genericity with respect to perturbations of the initial data ( 1.6). The same result, that folds, cusps and non-degenerate umbilics with non-trivial cubic terms are the generic singularities, is also true with respect to perturbations of the equations themselves. Moreover perturbation within the class of solutions of ( 1.1) on an open set is equivalent to perturbation of the initial data ( 1.6), according to the Cauchy-Kowalewski theorem. Thus Theorem 1.2 is also true with respect to such perturbations.

A related analysis of singularities for nonlinear differential equations was performed by Dubois and Dufour [12], who found that under perturbation of the initial data *and time*, the only stable singularities are folds. Our results show that cusp and umbilic singularities are not removed from the solution by such perturbations, but are only moved to different locations in space-time. Stability results for singularities of a scalar conservation law were earlier obtained in [15, 27]. An analysis of singularities in the complex plane for Burgers equation with viscosity was carried out in [4].

Singular solutions for semi-linear and quasi-linear hyperbolic equations have been constructed, for example, in [1, 3, 6, 17]. These solutions are constructed in Sobolev space using micro-local analysis, and are thus much more general than the analytic solution here. On the other hand their singularities are much weaker than those in the present analysis; in particular, envelopes in the characteristics are not allowed. Analysis of the strong singularities in the present work seems to require the strong restriction to analytic functions. Previous results on the geometry of singularities for nonlinear differential equations are found in [18]. Analysis of singularities in mappings coming from the Riemann problem for a nonlinear hyperbolic equation is found in [19, 20].

This paper is organized as follows. In Section 2 we introduce the basic examples for the unfolding of a tangential and non-tangential collision: respectively, Burgers equation and the wave equation. Section 3 introduces the normal forms for singularities of mappings and defines stability. It also establishes the normal form for umbilic singularities which will be stable for maps that correspond to solutions of ( 1.1). Section 4 introduces the unfolding map which is used to transform the original $2 \times 2$ nonlinear system to a linear system for the unfolding. We use this unfolding transformation to establish existence (Theorem 1.1). Section 5 extends the result to higher order systems with two speeds. In Section 6 we reinterpret the unfolding equation as a condition on the jets of mappings $(u, v) \rightarrow (z, t)$ and use this to express the conditions for a singularity to be a fold, cusp or umbilic just in terms of the 1-jet of $t$. In Section 7 the genericity theorem is proved. All perturbation calculations are carried out in terms of the unfolding variables.

Finally in Section 8 we present a more geometrical view of the unfolding of singularities. The hyperbolic system is encoded in a submanifold $\Gamma$ of jet space. A solution is an integrable surface $S$ for a distribution of planes on $\Gamma$. Generic singularities of the system correspond to generic singularities of the projection of $S$ into the domain $(z, t)$ of the jets. This elegant reformulation of the problem of singularities is based on Elie Cartan's theory of exterior differential systems [10]. The existence theorem reduces to a geometrical version of the Cauchy-Kowalevsky theorem also due to Cartan.

## 2 Examples: Burgers Equation and the Wave Equation

Two simple examples motivate the results of this paper. The first is a cusp singularity and the second is a nondegenerate umbilic. In particular these examples show that such singularities are realizable from solutions of hyperbolic pde's.

## 2.1 Singularities for Burger's Equation

A first example of singularity propagation and interaction is given by Burger's equation

$$f_t + f\, f_z = 0. \tag{2.1}$$

For initial data

$$f(0,z) = f_0(z) \tag{2.2}$$

the solution is given implicitly by the formulas

$$\begin{aligned} f(t,z) &= f_0(z_0) \\ z &= z_0 + t f_0(z_0). \end{aligned} \tag{2.3}$$

The value $z_0$ is the initial position of the straight line characteristic with speed $f_0(z_0)$ on which $f = f_0(z_0)$ is constant.

A singularity occurs when $\partial_z f = \infty$. Since

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z_0}\frac{\partial z_0}{\partial z} = \frac{\partial f}{\partial z_0} / \frac{\partial z}{\partial z_0} \tag{2.4}$$

and if $\partial_{z_0} f_0$ is assumed to be analytic and non-vanishing, then the singularity condition is

$$\partial_{z_0} z = 0 \tag{2.5}$$

i.e.

$$1 + t f_0'(z_0) = 0 \tag{2.6}$$

The singularity is the point at which the solution $f$ becomes multi-valued. For real analytic initial data $f_0$ that is monotone decreasing in $x$ (i.e. for $x = Re z$) as in Figure 1a, the real solution $f(x,t)$ will steepen until a time $t_c$, at which it develops an infinite derivative. At later times the real solution becomes triple valued in an interval of $x$ as shown in Figure 1b. The "turning points" $x_1$ and $x_2$ are the singular points satisfying ( 2.6). The real characteristics for $f$ are drawn in Figure 2a.

If $f$ is analytically continued to complex $z$, the singular points $x_1$ and $x_2$ are seen to be branch points around which $f$ has a complex-valued continuation, as indicated in Figure 1c. The complex solution is triple valued for all $z$. For $x_1 < x < x_2$, the three values come from real characteristics, while for $x < x_1$ or $x > x_2$, the additional 2 complex values of $f$ come from characteristics that start at a complex value of $z_0$. These complex characteristics are sketched in Figure 2b.

Finally the singularity condition ( 2.6) can be interpreted as saying that the singularity curve in $(t,z)$ is an envelope for the straight line characteristics parameterized by their initial value $z_0$. If $f_0$ itself has a branch point singularity, it also will move along a characteristic or an envelope of characteristics.

As an example consider initial data

$$f_0(z_0) = -z_0^{\frac{1}{3}}. \tag{2.7}$$

The solution formula ( 2.3) then becomes

$$\begin{aligned} f(t,z) &= f_0 \\ z &= -f_0^3 + tf_0. \end{aligned} \tag{2.8}$$

The solution $f$ is given by the Cardano formula

$$f(t,z) = (-\frac{1}{2}z + (\frac{z^2}{4} - \frac{t^3}{27})^{1/2})^{1/3} + (-\frac{1}{2}z - (\frac{z^2}{4} - \frac{t^3}{27})^{1/2})^{1/3} \tag{2.9}$$

and the singularities travel on the curves

$$z = \pm\frac{2}{3\sqrt{3}}t^{\frac{3}{2}}. \tag{2.10}$$

For $t < 0$, there are two square root singularities traveling on the imaginary $z$ axis. They collide at time $t = 0$ at the point $z = 0$, forming a cube root. Then they separate as two square root singularities traveling on the real $z$ line. We shall show below that this type of collision of singularities is generic, i.e. it is the only singularity type that is stable with respect to perturbations of the initial data $f_0$.

The formula ( 2.8) is an unfolding of the singularities and their collision. Although $f$ is multi-valued as a function of the variables $(z,t)$, both $f$ and $z$ are single-valued as functions of the unfolding variables $(f_0,t)$.

## 2.2 The Wave Equation

Consider the linear system

$$\begin{aligned} f_t + f_z &= 0 \\ g_t - g_z &= 0 \end{aligned} \tag{2.11}$$

with initial data

$$\begin{aligned} f(z,t=0) &= f_0(z) \\ g(z,t=0) &= g_0(z). \end{aligned} \tag{2.12}$$

Suppose that $f_0$ and $g_0$ are multi-valued and have branch point singularities. Assume that these initial singularities are unfolded by the transformation $z = \zeta(v)$ so that

$$\begin{aligned} f_0(z) &= F_0(v) \\ g_0(z) &= G_0(v) \end{aligned} \tag{2.13}$$

with $F_0$ and $G_0$ analytic and single-valued in $v$. In addition to the "space-like" unfolding variable $v$, we introduce a "time-like" unfolding variable $u$ and ask that the characteristic lines $z \pm t = constant$ in $(x, t)$ map onto lines $v \pm u = constant$ in $(v, u)$. Then the solution of ( 2.11) and ( 2.12) is given by

$$\begin{aligned} f(z,t) &= f_0(z-t) = F_0(v-u) \\ g(z,t) &= g_0(z+t) = G_0(v+u) \end{aligned} \tag{2.14}$$

in which $z$ and $t$ are given as functions of $u$ and $v$ by

$$\begin{aligned} z+t &= \zeta(v+u) \\ z-t &= \zeta(v-u). \end{aligned} \tag{2.15}$$

For example if $f_0$ and $g_0$ have square root singularities at $z = 1$ and $z = -1$, then the singularities are unfolded by the transformation

$$z = \frac{v^4+4}{4v^2} \tag{2.16}$$

i.e.

$$v = \sqrt{z+1} + \sqrt{z-1}. \tag{2.17}$$

This is the transformation at $t = 0$, corresponding to $u = 0$. From ( 2.15), $z(u, v)$ and $t(u, v)$ are given by

$$\begin{aligned} z+t &= \frac{(v+u)^4+4}{4(v+u)^2} \\ z-t &= \frac{(v-u)^4+4}{4(v-u)^2} \end{aligned} \tag{2.18}$$

or

$$\begin{aligned} v+u &= \sqrt{z+t+1} + \sqrt{z+t-1} \\ v-u &= \sqrt{z-t+1} + \sqrt{z-t-1}. \end{aligned} \tag{2.19}$$

The multi-valuedness and singularities of $f$ and $g$ as function of $z$ and $t$ are given by those of $v$ and $u$ in ( 2.19).

For each $(v, u)$ there is one $(z, t)$; while for each $(z, t)$ there are 16 values of $(v, u)$. For fixed $t$, the surface defined by ( 2.19) is 16-sheeted corresponding to

16 choices of the signs of the 4 square roots. On each of these surfaces there are 4 branch points $z = \pm t \pm 1$, which are joined in pairs to give 32 distinct ramification points on the Riemann surface.

For the initial data ( 2.17) there are only 4 sheets, on all of which $u = 0$. However the simple formulas $f(z,t) = f_0(z-t)$ and $g(z,t) = g_0(z+t)$ are not sufficient if one wants a Riemann surface on which both $f$ and $g$ are defined. In fact an initial branch point splits into two branch points traveling at speeds $+1$ and $-1$. A new cut is opened up between them, connecting the 4 sheets of initial data ( 2.17) to the other 12 sheets of equations ( 2.19). Thus at $t = 0$ each ramification point represent a collision of 4 ramification points. For example the singular point $(t, z, u, v) = (0, 1, 0, \sqrt{2})$ splits into 4 distinct ramification points $(t, z, u, v) = (t, 1+t, \frac{1}{\sqrt{2}}(\sqrt{t+1} \pm \sqrt{t} - 1), \frac{1}{\sqrt{2}}(\sqrt{t+1} \pm \sqrt{t} + 1))$ and $(t, 1-t, -\frac{1}{\sqrt{2}}(\sqrt{1-t} \pm \sqrt{-t} - 1), \frac{1}{\sqrt{2}}(\sqrt{1-t} \pm \sqrt{-t} + 1))$.

Similarly at later collision points, such as $(t, z, u, v) = (1, 0, \frac{1}{\sqrt{2}}(1-i), \frac{1}{\sqrt{2}}(1+i))$, 4 ramification points are colliding. Two of them come from the initial surface ( 2.12), the other 2 come from the other 12 sheets, which we call the "hidden initial data".

# 3 Local Structure of a Singularity

In the present paper we analyze the interaction of propagating singularities for the equation ( 1.1) by replacing the nonlinear evolution with a linear system on a complex surface, locally coordinatized by $(u, v)$. This surface maps onto the complex $(t, z)$-plane by a finite-to-one mapping. The singularities are then encoded in singularities of an analytic map

$$\begin{aligned} G : \mathbb{C}^2 &\rightarrow \mathbb{C}^2 \\ \{u, v\} &\mapsto \{z(u,v), t(u,v)\}. \end{aligned} \tag{3.1}$$

In this section we are going to cite the facts we need about the classification of singularities of maps $\mathbb{C}^2 \rightarrow \mathbb{C}^2$.

Let $\mathcal{D}$=**discriminant variety** = the set of critical points of $G$:

$\{(u, v) \in \mathbb{C}^2 \mid \text{the rank of } DG = rank(\frac{\partial(z,t)}{\partial(u,v)}) < 2\}$.

If $G$ is not constant, then on the complement of a discrete set of points in $\mathbb{C}^2$, $\mathcal{D}$ is a smooth analytic manifold of dimension 1. Collisions correspond to places where *simple* ramification points (in u,v) coalesce. These may also be described as the set of points on $\mathcal{D}$ where either $\mathcal{D}$ has a singularity (i.e. is not a manifold) or where $G\mid_{\mathcal{D}'}$, the restriction of $G$ to the smooth points $\mathcal{D}'$ of $\mathcal{D}$, has a critical point.

Throughout the remainder of this chapter $G$ will denote the map in (3.1) which we will always take to be analytic. At first we will recall some standard definitions.

## 3.1 Equivalence, Stability and Genericity

We now define some terms which will be used frequently in subsequent sections. Two analytic maps, $G$ and $H : \mathcal{V} \to \mathbb{C}^2$, are said to be *equivalent* on $\mathcal{V}$ if there exist a pair of biholomorphic maps $\phi_1 : \mathcal{V} \to \mathcal{V}$, and $\phi_2 : \mathbb{C}^2 \to \mathbb{C}^2$ mapping $G(\mathcal{V})$ on $H(\mathcal{V})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{H} & \mathbb{C}^2 \\ \downarrow \phi_1 & & \downarrow \phi_2 \\ \mathbb{C}^2 & \xrightarrow{G} & \mathbb{C}^2 \end{array}$$

The next two definitions assume that we have introduced a topology in the space of functions mapping $\mathcal{V} \subset \mathbb{C}^2 \to \mathbb{C}^2$. In this paper we will work with the space of solutions of non-linear systems, and the topology will be induced by $C^\infty$-topology on a closed bounded domain $\bar{\mathcal{V}}_1 \subset \mathbb{C}^2$.

A map $G$ is said to be *stable* if any map in a neighborhood of $G$ is equivalent to $G$. A map with some property is said to be *generic* (with respect to that property) if the subset of maps with this property is open and dense.

The goal of our paper is to describe the generic singularities of the unfoldings of solutions of the system (1.1). In the remainder of this section we will describe two classifications of singularities of maps – classification by local rings and the Thom-Boardman classification, which will be useful in our analysis.

## 3.2 Local ring

The classical invariant of a singularity of an analytic map is its *local ring*. $\mathcal{A}_p$ will denote the ring of analytic functions at a point $p \in \mathbb{C}^2$. This ring is naturally identified with the ring of convergent power series in two variables centered at $p$. Furthermore, we let $\mathcal{M}_p$ denote the maximal ideal of analytic functions which vanish at $p$. $\mathcal{A}_p$ is a local ring; $\mathcal{M}_p$ is its unique maximal ideal.

If $G(p) = q$, then $G$ induces a natural mapping of local rings

$$\begin{aligned} G^* : \mathcal{A}_q &\to \mathcal{A}_p \\ f(z,t) &\mapsto f(z(u,v), t(u,v)). \end{aligned} \tag{3.2}$$

Clearly, $G^*\mathcal{M}_q \subset \mathcal{M}_p$. The *local ring of the map $G$ at $p$* is defined to be

$$\mathcal{R}_G(p) := \mathcal{A}_p/\mathcal{A}_pG^*\mathcal{M}_q. \tag{3.3}$$

*Definition.* The $\dim_{\mathbb{C}} \mathcal{R}_G(p)$ as a vector space is called the *multiplicity of $p$* with respect to the map $G$.

The following facts justify this definition:

1. $G$ is locally (near $p$) finite-to-one iff $\dim_{\mathbb{C}} \mathcal{R}_G(p) < \infty$ [16].

2. $\sum_{G(p_i)=q} \dim_{\mathbb{C}} \mathcal{R}_G(p_i)$ is constant for $f$ in an open neighborhood of $q$ [22].

Thus, $\dim_{\mathbb{C}} \mathcal{R}_G(p)$ measures the multiplicity of $p$ as a root of $G(p) = q$.

## 3.3 Classification of Local Rings of Mappings with Multiplicity $\leq 4$

The classification of local rings for singularities of multiplicity $\leq 4$ is well documented in the literature on singularity theory. See for example [14], chapter 7. These results are typically derived for real maps in the $C^\infty$-category. We present a sketch the proof of this classification below which is appropriate for our holomorphic setting.

**Theorem 3.1** *If the local ring of the map $G$ satisfies $2 \leq \dim_{\mathbb{C}} \mathcal{R} \leq 4$ then it is isomorphic, under an analytic change of variables, to one of the following:*

$$\begin{array}{lll} \mathbb{C}[[u,v]]/\{z(u,v),t(u,v)\} \cong & \mathbb{C}[u,v]/\{u^2,v\} & \textit{(fold)} \\ & \mathbb{C}[u,v]/\{u^3,v\} & \textit{(cusp)} \\ & \mathbb{C}[u,v]/\{u^4,v\} & \textit{(swallowtail)} \\ & \mathbb{C}[u,v]/\{u^2,v^2\} & \textit{(nondegenerate umbilic)} \\ & \mathbb{C}[u,v]/\{uv,1-v^2\} & \textit{(parabolic umbilic)} \end{array} \tag{3.4}$$

*Sketch of Proof.* Recall that $\mathcal{M}$ denotes the maximal ideal of $\mathcal{R}$; i.e. the set of functions vanishing at (0,0). As a vector space over $\mathbb{C}$, $\mathcal{R}$ may be decomposed as

$$\mathbb{C} \oplus \mathcal{M}/\mathcal{M}^2 \oplus \mathcal{M}^2/\mathcal{M}^3 \oplus \ldots.$$

Here $\mathbb{C}$ denotes the constants in $\mathcal{R}$. For the first three cases listed in the theorem the dimension over $\mathbb{C}$ of $\mathcal{M}/\mathcal{M}^2$ is 1; while, for the last two cases–the umbilics–it is 2.

We will derive the structure of the local rings for the umbilics, the derivations of the preceding cases being analogous. Since we assume that the mult(0,0) $\leq 4$ and since $\dim(\mathcal{M}/\mathcal{M}^2) = 2$ for umbilics, it follows that dim $\mathcal{M}^2 \leq 1$. So consider the quadratic form induced by multiplication:

$$\mathcal{M}/\mathcal{M}^2 \otimes \mathcal{M}/\mathcal{M}^2 \rightarrow \mathcal{M}^2.$$

If this is a non-degenerate quadratic form, then since all non-degenerate quadratic forms are equivalent over $\mathbb{C}$, we can find a basis $x, y$ of $\mathcal{M}/\mathcal{M}^2$ such that $x \cdot x = y \cdot y = 0$ and $x \cdot y \neq 0$. Setting $u = x, v = y$ we derive the relations for the local ring of the nondegenerate umbilic described above.

In the case where the above quadratic form is degenerate but not identically zero we can find a basis $x, y$ such that $x^2 = 0, x \cdot y = 0$ and $y^2 = 1$. Setting $u = x, v = y$ this gives the relations for the local ring of the parabolic umbilic. ∎

We now list examples of mappings whose local rings realize each of the above types and whose geometric realization gives rise to the name associated with each ring:

$$
\begin{array}{lll}
z = u^2 & t = v & (fold) \\
z = u^3 + vu & t = v & (cusp) \\
z = u^4 + vu^2 + cu & t = v & (swallowtail) \\
 & & \\
z = u^2 & t = v^2 & (nondegenerate\ umbilic) \\
z = u^2 - v^2 & t = uv & \\
z = u^2 - v^2 & t = u^2 + v^2 & \\
 & & \\
z = v^2 & t = uv & (parabolic\ umbilic)
\end{array}
\tag{3.5}
$$

## 3.4 The Jet Stratification of Singularities

We present now a classification of singularities which is due to Thom and Boardman [5]. Though this classification is cruder than that given by the local ring of a singularity, it provides a natural measure of the genericity of a singularity or of a singularity in the presence of a constraint. The latter will be most important when we consider genericity of collisions of singularities in Section 6. Our analysis of singularities for first order systems is divided into two cases: the fold and cusp singularities, which are of of type $\Sigma^1$, and the umbilic singularities, which are of type $\Sigma^2$. Although the details of the Boardman-Thom classification are not used below, a brief discussion of this theory will highlight the geometric significance of these two cases.

The Thom-Boardman classification of mappings between two manifolds M and N (in our case M = N = $\mathbb{C}^2$) is given in terms of a stratification of the space of jets of mappings between M and N which we now describe. Thom partitions points in the domain M of $G$ into strata as follows. For a point $p \in M$, we say that $p \in \Sigma^i$ if

$$ker(dG_p : M_p \rightarrow N_{G(p)})$$

has dimension = i. For example the fold, cusp and swallowtail singularities of (3.5) represent singularities of class $\Sigma^1$. Furthermore in this case rank $DG$ at p will equal 1. The remaining (umbilic) singularities of (3.5) are of class $\Sigma^2$ and rank $DG = 0$. Thom proposed a further refinement of this stratification as follows:

Let I denote a nonincreasing sequence of integers $dimM \geq i_1 \geq i_2 \geq i_3 \geq \ldots \geq 0$. The stratum $\Sigma^I(G) \subset M$ is inductively defined by

$$\Sigma^{i_1,i_2,\ldots,i_k,i_s+1}(G) = \Sigma^{i_s+1}(G \mid \Sigma^{i_1,i_2,\ldots,i_s}(G))$$

For example, the fold is of class $\Sigma^{1,0}$, the cusp is of class $\Sigma^{1,1,0}$, the swallôwtail is of class $\Sigma^{1,1,1,0}$, while the umbilics are all of class $\Sigma^{2,0}$. This definition is flawed in that $\Sigma^I(G)$ is not a manifold in general so that the inductive definition breaks down. Boardman, however, was able to circumvent this difficulty, by defining these strata in terms of the space of jets of maps from M to N, as follows:

Two maps $G, H$ from M to N are said to have $k^{th}$ order contact at p $\in$ M if the Taylor expansions of $G$ and $H$ at p agree up to $k^{th}$ order. We let $J^k(M,N)_{p,q}$ denote the space of equivalence classes of mappings $G$ with $G(p) = q$ under the equivalence relation of $k^{th}$ order contact. The elements of $J^k$ are called k-jets. The k-jet of a map G can be thought of as a map which assigns to each point $p \in M$ the $k^{th}$ order vector Taylor polynomial of G at p. The k-th jet of $G$ at $p$ is denoted $j^k_p G$.

The $\Sigma^I$ are then defined in terms of universal polynomial conditions on the k-jet of a map. See [7] for the explicit form of these conditions. In this setting $\Sigma^I$ is a certain (not necessarily closed) subvariety of $M \times J^k(M,N)$.

For a mapping $G : M \longrightarrow N$ we define the induced mapping $M \longrightarrow J^k(M,N)$ which associates with each point $p \in M$ the jet of G at p:

$$j^k G(p) = j^k_p(G).$$

If for $k$ sufficiently large, $j^k G$ is transverse to all $\Sigma^I$ then the sets $\Sigma^I(G) = (j^k G)^{-1}(\Sigma^I) \subset M$ are smooth subvarieties, and the codimension of $\Sigma^I(G)$ in $M$ is the codimension of $\Sigma^I$ in $J^k(M,N)$.

It is useful to reformulate the definition of folds and cusps. The map $G$ has a fold singularity at the point $p$ if $j^2_p G \in \Sigma^{1,0}$. $G$ has a cusp singularity at $p$ if $j^3_p G \in \Sigma^{1,1,0}$ and $j^1 G$ is transversal to $\Sigma^1$ at $p$.

We are interested in the maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$. When dim M = dim N the only cases of $codim\Sigma^I \leq 2$ are $\Sigma^1$ (codimension 1) and $\Sigma^{1,1}$ (codimension 2) (see [2]). Maps with $j^k G$ transversal to $\Sigma^I$ are generic. Therefore we see that the only generic singularities of maps $\mathbb{C}^2 \longrightarrow \mathbb{C}^2$ are folds and cusps. This result is known as Whitney's theorem.

We are going to show that if we restrict ourselves to the maps which are unfoldings of solutions of the hyperbolic system, then we will find new stable singularities, nondegenerate umbilics.

### 3.5 Normal forms of umbilics

A map, $\mathbb{C}^2 \to \mathbb{C}^2$, with an umbilic singularity is not stable, i.e. the map

$$\begin{aligned} z &= u^2 + \epsilon_1 v \\ t &= v^2 + \epsilon_2 u \end{aligned}$$

has only fold and cusp singularities, if $\epsilon_1$ and $\epsilon_2$ are nonzero. Moreover, two germs of maps $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with umbilic singularities can be non-equivalent. Throughout this subsection, all of the mappings are assumed to map 0 to 0. This is denoted by the notation $(\mathbb{C}^2, 0)$.

**Example** . The maps

$$f_0 = \left[ \begin{array}{ll} z & = u^2 + v^3 \\ t & = v^2 + u^3 \end{array} \right. \quad and \quad f_1 = \left[ \begin{array}{ll} z & = u^2 \\ t & = v^2 \end{array} \right.$$

are not equivalent.

However, among the germs of maps $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with an umbilic singularity the orbit of the map $f_0$ is open and dense.

**Theorem 3.2** *(Normal form theorem). Suppose that a germ of a map $f$: $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ has 2-jet $f_1$. If $\partial^3 z/\partial v^3$ and $\partial^3 t/\partial u^3$ are non-zero, then $f$ is equivalent to $f_0$.*

**Remark 3.1** *By a linear changes of variables $(u, v)$ and $(z, t)$, the 2-jet of a map with an umbilic singularity can be taken to a form $f_1$.*

We will say that a map has an *umbilic singularity with nontrivial cubic terms* if it is equivalent to $f_0$.

By completing a square it is easy to show the following:

**Lemma 3.1** *If $f$ satisfies the conditions of Theorem 3.2, then it is equivalent to a map with 3-jet $f_0$.*

Thus the proof of the theorem is reduced to showing that such a map is equivalent to its own 3-jet. We will outline the proof of this proposition. Denote by $V_a$ a ball $|z|^2 + |t|^2 < a^2$ and $U_a$ the domain $f_0^{-1}(V_a)$.

**Proposition 3.1** *For any $a > 0$ there exists a positive constant $C_a$ such that if $h$ is analytic in a domain $U_a$, $j^2 h = 0$ at 0, and $|h| < C_a$, then there exist maps*

$$k : V_{a/2} \to V_a \quad and \quad g : U_{a/2} \to U_a$$

*such that $k \circ (f_0 + h) \circ g = f_0$ .*

The proof is based on an infinitesimal normal form theorem. Consider a family of maps $f_\tau = f_0 + \tau h$, $0 \le \tau \le 1$. We want to find families $k_\tau$ and $g_\tau$ such that

$$k_\tau \circ f_\tau \circ g_\tau = f_0 \tag{3.6}$$

for $0 \le \tau \le 1$ with $k_0 = g_0 = identity$ at $\tau = 0$. Differentiating ( 3.6) with respect to $\tau$ at $\tau = 0$ we obtain an equation

$$h = (df_0)\alpha(u,v) + \beta(f_0), \tag{3.7}$$

where $\alpha = -\partial_\tau g$ and $\beta = -\partial_\tau k$ are vector fields on $C^2_{u,v}$ and $C^2_{z,t}$ respectively.

**Theorem 3.3** *(Infinitesimal normal form theorem) For any map $h$ with zero 2-jet at the origin the equation ( 3.7) can be solved for $\alpha$ and $\beta$. If $h$ is analytic in a domain $U_a$, then $\alpha$ is analytic in $U_a$, $\beta$ is analytic in $V_a$, and*

$$\max_{U_a} |\alpha(u,v)| \le C_1(a)|h|, \quad \max_{V_a} |\beta(z,t)| \le C_2(a)|h|.$$

**Proof of the existence of vector fields $\alpha$ and $\beta$.** To find $\alpha$ and $\beta$ one has to solve equations

$$\begin{aligned} h_1(u,v) &= 2u\alpha_1(u,v) + 3v^2\alpha_2(u,v) + \beta_1(u^2+v^3, v^2+u^3) \\ h_2(u,v) &= 3u^2\alpha_1(u,v) + 2v\alpha_2(u,v) + \beta_2(u^2+v^3, v^2+u^3). \end{aligned}$$

Solve this system for $\alpha_1$ and $\alpha_2$:

$$\begin{aligned} \alpha_1(u,v) = \ & \{2[h_1(u,v) - \beta_1(u^2+v^3, v^2+u^3)] \\ & -3v[h_2(u,v) - \beta_2(u^2+v^3, v^2+u^3)]\}/(u(4-9uv)) \\ \alpha_2(u,v) = \ & \{2[h_2(u,v) - \beta_2(u^2+v^3, v^2+u^3)] \\ & -3u[h_1(u,v) - \beta_1(u^2+v^3, v^2+u^3)]\}/(v(4-9uv)). \end{aligned}$$

Functions $\alpha_1$ and $\alpha_2$ exist if and only if the system

$$\begin{aligned} 2[h_1(0,v) - \beta_1(v^3,v^2)] - 3v[h_2(0,v) - \beta_2(v^3,v^2)] &= 0 \\ 2[h_2(u,0) - \beta_2(u^2,u^3)] - 3u[h_1(u,0) - \beta_1(u^2,u^3)] &= 0 \end{aligned}$$

can be solved for $\beta_1$ and $\beta_2$.

We are looking now for functions $\beta_1(z,t)$ and $\beta_2(z,t)$, such that

$$\begin{aligned} \beta_1(x^3,x^2) - (3/2)x\beta_2(x^3,x^2) &= h_1(0,x) - (3/2)xh_2(0,x) &= \tilde{h}_1(x) \\ \beta_2(x^2,x^3) - (3/2)x\beta_1(x^2,x^3) &= h_2(x,0) - (3/2)xh_1(x,0) &= \tilde{h}_2(x). \end{aligned}$$

Note that $\tilde{h}_1(0) = \tilde{h}_1'(0) = 0$ since $\beta_i(0,0) = 0$. Hence we can express $\tilde{h}_1$ as a sum

$$\tilde{h}_1(x) = \varphi(x^2) + x^3\psi(x^2)$$

with $\varphi(0) = 0$. Let $\beta_{1,0}(z,t) = \varphi(t) + z\psi(t)$, $\beta_{2,0}(z,t) = 0$. We will look for $\beta_1$ and $\beta_2$ in a form $\beta_i = \beta_{i,0} + \beta_{i,1}$, $i = 1,2$.

$$\begin{aligned}\beta_{1,1}(x^3,x^2)-(3/2)x\beta_{2,1}(x^3,x^2) &= 0\\ \beta_{2,1}(x^2,x^3)-(3/2)x\beta_{1,1}(x^2,x^3) &= \tilde{h}_2(x)+(3/2)x\beta_{1,0}=\hat{h}(x).\end{aligned} \tag{3.8}$$

Again let $\hat{h}(x)=\hat{\varphi}(x^2)+x^3\hat{\psi}(x^2)$, and look for $\beta_{2,1}$ in the form $\beta_{2,1}(z,t)=\mu(z)+t\nu(z)$. From the first equation in ( 3.8) we obtain

$$\beta_{1,1}(x^3,x^2)=(3/2)x(\mu(x^3)+x^2\nu(x^3))=(3/2)\{(x^2)^2(\mu(x^3)/x^3)+x^3\nu(x^3)\}.$$

A solution is

$$\beta_{1,1}(z,t)=(3/2)\{t^2\mu(z)/z+z\nu(z)\}.$$

The second equation becomes

$$\{\mu(x^2)+x^3\nu(x^2)\}-(3/2)x(3/2)\{x^6\mu(x^2)/x^2+x^2\nu(x^2)\}=\hat{\varphi}(x^2)+x^3\hat{\psi}(x^2),$$

which has solution

$$\mu(z)=\hat{\varphi}(z)$$

$$\nu(z)=-(4/5)\hat{\psi}(z)-(9/5)z\hat{\varphi}(z).$$

Note that $\hat{\varphi}(0)=0$ , hence $\hat{\varphi}(z)/z$ and $\beta_{1,1}$ are well defined.

The estimates in the Theorem 3.3 follow from the formulas for $\alpha$ and $\beta$. ∎

The transition from the infinitesimal normal form theorem to the proof of Proposition 3.1 can be done by standard methods (see [2, 14]). This finishes the proof of Theorem 3.2

# 4 The Unfolding Transformation

Consider the $2\times 2$ nonlinear hyperbolic system

$$\begin{aligned} f_t+\lambda f_z &= 0\\ g_t+\mu g_z &= 0\\ f(t=0,z) &= f_0(z)\\ g(t=0,z) &= g_0(z)\end{aligned} \tag{4.1}$$

in which $\lambda=\lambda(f,g)$, $\mu=\mu(f,g)$. We shall assume throughout that $\lambda\neq\mu$ in the region of interest. In Section 5, larger systems with more equations and forcing terms will also be considered. Introduce new independent variables $u,v$

and look for $(z, t, f, g)$ to be functions of $(u, v)$. The equations ( 4.1) for $f$ and $g$ then become

$$f_u - \frac{z_u - \lambda t_u}{z_v - \lambda t_v} f_v = 0$$
$$g_u - \frac{z_u - \mu t_u}{z_v - \mu t_v} g_v = 0. \tag{4.2}$$

At singular points the denominator $(z_v - \lambda t_v)$ (or $(z_v - \mu t_v)$) will vanish (this is shown directly below). The transformation $(z(u,v),\ t(u,v))$ must be chosen so that the numerator $(z_u - \lambda t_u)$ (or $(z_u - \mu t_u)$) will vanish at the same point, to make equations ( 4.2) non-singular. A convenient choice of $(z, t)$ to insure this non-singularity is to require that

$$\frac{z_u - \lambda t_u}{z_v - \lambda t_v} = a \qquad \frac{z_u - \mu t_u}{z_v - \mu t_v} = b \tag{4.3}$$

in which $a$ and $b$ are given constants (this could be generalized to choosing $a$ and $b$ as given functions of $(u, v)$). The resulting equation for $f, g, z, t$ are

$$\begin{aligned} f_u - a f_v &= 0 \\ g_u - b g_v &= 0 \end{aligned} \tag{4.4}$$

$$\begin{pmatrix} z \\ t \end{pmatrix}_u - \begin{pmatrix} 1 & -\lambda \\ 1 & -\mu \end{pmatrix}^{-1} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -\lambda \\ 1 & -\mu \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}_v = 0 \tag{4.5}$$

The solution of ( 4.4) (for constant $a, b$) is

$$\begin{aligned} f(u,v) &= f_0(v + au) \\ g(u,v) &= g_0(v + bu) \end{aligned} \tag{4.6}$$

in which $f, g$ are given at $u = 0$ by $f_0$ and $g_0$.

The system ( 4.5), with $\lambda = \lambda(f, g) = \lambda_0(v + au,\ v + bu)$, is equivalent to the original system (4.1) and is the system that will be used for most of the remainder of the paper. It is a linear system with nonconstant coefficients and with straight characteristics $du/dv = -a$ and $du/dv = -b$.

Now we are ready to prove Theorem 1.1 for systems of the form (4.1).

*Proof of Theorem 1.1* If "initial data"

$$\begin{aligned} z(u = 0, v) &= \zeta_0(v) \\ t(u = 0, v) &= \tau_0(v) \end{aligned} \tag{4.7}$$

is provided for $v \in \Omega$, then ( 4.5) has a solution for all $(u, v) \in \{(u, v) :\ v + au \in \Omega,\ v + bu \in \Omega\}$. If $f_0, g_0, \lambda, \mu, \zeta_0$ and $\tau_0$ are analytic, then the resulting solution $(f, g, z, t)$ is analytic in $(u, v)$.

All that is now needed is to relate the "initial data" ($u = 0$) for $f_0, g_0, \zeta_0, \tau_0$ to the original equation ( 4.1). Assume that there is an unfolding variable $v$ such that

$$\begin{aligned} z &= \zeta_0(v) \\ t &= \tau_0(v) \\ f_0(z) &= f_0(v) \\ g_0(z) &= g_0(v) \end{aligned} \tag{4.8}$$

with $f_0, g_0, \tau_0$ and $\zeta_0$ analytic for $v \in \Omega$. If this data for $u = 0$ corresponds to initial data with $t = 0$, choose $\tau_0(v) = 0$.

Now we examine the singularities of the mapping $(z(u,v), t(u,v))$ coming from this transformation. The discriminant $D = z_u t_v - z_v t_u$ has a simple form

$$\begin{aligned} D &= \frac{(z_u - \lambda t_u)(z_v - \mu t_v) - (z_v - \lambda t_v)(z_u - \mu t_u)}{\lambda - \mu} \\ &= \frac{a-b}{\lambda - \mu}(z_v - \lambda t_v)(z_v - \mu t_v) \end{aligned} \tag{4.9}$$

Since $a \neq b$ and $\lambda \neq \mu$ then singularities in the mapping $(z, t)$ occur precisely at zeroes of $z_v - \lambda t_v$ and $z_v - \mu t_v$. Thus for the system ( 4.1) singularities are found by first solving the regular system ( 4.4), ( 4.5), then locating zeroes of $z_v - \lambda t_v$ or $z_v - \mu t_v$. Since the singularity positions do not have to be kept as an unknown for the solution, we call this a "singularity capturing" transformation.

The velocity $dz_0/dt$ of a singularity $z_0(t)$ at a zero of $z_v - \lambda t_v$ is calculated as follows: Let $(u, v_0(u))$ be the path of the singularity in $(u, v)$. On this curve

$$\begin{aligned} dz(u, v_0(u))/du = z_u + z_v v_{0u} \\ dt(u, v_0(u))/du = t_u + t_v v_{0u} \end{aligned} \tag{4.10}$$

so that

$$dz_0/dt = \frac{z_u + z_v v_{0u}}{t_u + t_v v_{0u}} = \lambda \tag{4.11}$$

since $z_u = \lambda t_u$ and $z_v = \lambda t_v$ on $v = v_0(u)$. Thus the singularities move at the characteristic speed, i.e. they move either on a characteristic or on an envelope of the characteristics. Similarly a singularity at a zero of $z_v - \mu t_v$ moves with speed $dz_0/dt = \mu$.

This concludes the proof of Theorem 1.1 for systems of the form ( 4.1). ∎

Before proceeding several remarks on this transformation are made:

(1) For $f_0(z) = g_0(z) = \sqrt{z}$, we may choose $\zeta_0(v) = v^2$. This is a simple example of unfolding of initial singularities.

(2) The transformation ( 4.3) may be compared to the hodograph transformation $z = z(f, g)$, $t = t(f, g)$. It is similar to the hodograph transformation in

that the resulting system is linear with non-constant coefficients and straight-line characteristics. This transformation overcomes two defects of the hodograph transformation: For the hodograph transformation the "initial surface" $t = 0$ is complicated. Moreover it may be parallel to the characteristics in some places, which leads to $z$ and $t$ that are multi-valued functions of $f$ and $g$. Both these defects are absent from the transformation ( 4.3). The initial surface is $u = 0$, which is never characteristic if $a \neq 0,\ b \neq 0$.

(3) Under a non-degeneracy condition, the singularity does not move on a characteristic. Differentiate the equations

$$\begin{aligned} z_u(u, v_0(u)) - \lambda(u, v_0(u))t_u(u, v_0(u)) &= 0 \\ z_u - \lambda t_u &= a(z_v - \lambda t_v) \end{aligned} \tag{4.12}$$

and recombine to get

$$dv_0(u)/du = -a(1 + \frac{\lambda_v t_u - \lambda_u t_v}{z_{uv} - \lambda t_{uv} - \lambda_v t_u}) \tag{4.13}$$

which is in general not equal to the characteristic speeds $-a$. In this case the path of the singularity is not a characteristic and must be an envelope for the $\lambda$ characteristics.

(4) Finally, there are two types of collisions between singularities: If two $\alpha = 0$ singularities meet, they are both moving at speed $\lambda$. Thus they collide tangentially in $(z, t)$, so that we call this a *tangential collision.* On the other hand if a zero of $z_v - \lambda t_v$ collides with a zero of $z_v - \mu t_v$, they are moving at different speeds $\lambda \neq \mu$, so that we call this a *non-tangential collision.*

# 5 High Order Systems

In this section the results concerning unfolding are generalized to $n$th order systems that are diagonalizable and have exactly two speeds. The main difference is that the unfolding transformation results in nonlinear equations, which are solved using the Cauchy-Kovalevski theorem.

First we prove the existence Theorem 1.1 for systems. Consider

$$F_t + M(F)F_z = \phi(F). \tag{5.1}$$

The matrix $M$ is diagonalizable, and has eigenvalues $\lambda_1 = \ldots = \lambda_k = \lambda(F)$, $\lambda_{k+1} = \ldots = \lambda_n = \mu(F)$. Choose the left eigenvectors $l_1, \ldots, l_n$ and the right eigenvectors $r_1, \ldots, r_n$ satisfying

$$\begin{aligned} l_i M &= \lambda_i l_i = \lambda l_i, \quad i = 1, \ldots, k, \\ l_j M &= \lambda_j l_j = \mu l_j, \quad j = k+1, \ldots, n, \end{aligned} \tag{5.2}$$

and

$$l_s r_m = \delta_{sm},\ s, m = 1, \ldots, n,$$

in which $0 < k < n$.

Take the inner product of the left eigenvectors with (5.1) to obtain

$$\begin{aligned} l_i \cdot (\partial_t + \lambda \partial_z) F &= l_i \cdot \phi,\ i = 1, \ldots, k \\ l_j \cdot (\partial_t + \mu \partial_z) F &= l_j \cdot \phi,\ j = k+1, \ldots, n \end{aligned} \tag{5.3}$$

Use the unfolding transformation (4.5) to convert (5.3) to

$$l_i \cdot (\partial_u - a\partial_v) F = \frac{a-b}{\lambda - \mu}(z_v - \mu t_v) l_i \cdot \phi \tag{5.4}$$

for $i = 1, \ldots, k$ and

$$l_j \cdot (\partial_u - b\partial_v) F = \frac{a-b}{\lambda - \mu}(z_v - \lambda t_v) l_j \cdot \phi \tag{5.5}$$

for $j = k+1, \ldots n$. The equations for $z$ and $t$ are

$$\begin{pmatrix} z \\ t \end{pmatrix}_u - \begin{pmatrix} 1 & -\lambda \\ 1 & -\mu \end{pmatrix}^{-1} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -\lambda \\ 1 & -\mu \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}_v = 0. \tag{5.6}$$

The initial data is

$$\begin{aligned} t &= \tau_0(v) \\ z &= \zeta_0(v) \\ F &= F_0(v) \end{aligned} \tag{5.7}$$

for $u = 0$. The system (5.4), (5.5), (5.6) with data (5.7) has analytic coefficients and analytic initial data in $v$. The Cauchy-Kovalevski Theorem provides a solution $F, z, t$ for this system, with $F, z, t$ analytic in $u, v$.

Analysis of the singularities proceeds exactly as for 2×2 case. That is, $F(z,t)$ being the solution of a system (5.1) is equivalent to $(F, z, t)$ satisfying (5.4), (5.5), (5.6). Under this constraint all the stable singularities are exactly those for 2×2 system. As before the singularities move at speed $\lambda$ or $\mu$. This finished the proof of Theorem 1.1 for systems.

# 6 The Differential Equation as a Constraint

## 6.1 Characteristic Variables

In the rest of this paper it will frequently be convenient to write the unfolded system in terms of the characteristic variables $U = u - v$ and $V = u + v$. The system (4.5) or (5.6) then takes the form

$$\begin{aligned} z_U &= \lambda(f, g) t_U, \\ z_V &= \mu(f, g) t_V, \end{aligned} \tag{6.1}$$

We wish to describe the singularities for a solution $G : \mathbb{C}^2_{UV} \to \mathbb{C}^2$ of the system (6.1).

The system (6.1) acts as a constraint on the mapping $(z(U,V), t(U,V))$. Set $p = t_U$, $q = t_V$. The Jacobi matrix of the map $G$ is

$$\begin{pmatrix} z_U & t_U \\ z_V & t_V \end{pmatrix} = \begin{pmatrix} \lambda p & p \\ \mu q & q \end{pmatrix}. \tag{6.2}$$

The discriminant is $D = (\lambda - \mu)pq$ and thus the map $G$ is singular if either $p$ or $q$ or both of them vanish. The constraint (6.1) forces the discriminant $D$ to be a product of terms that can each vanish separately, which makes the umbilic singularity stable, as we will show.

We will also occasionally use the following change of variables on the target of the unfolding map. Set $\lambda_0 = \lambda(u_0, v_0)$, $\mu_0 = \mu(u_0, v_0)$, $z_0 = z(u_0, v_0)$, $t_0 = t(u_0, v_0)$ and consider the new coordinates $(Z, T)$ in $\mathbb{C}^2_{z,t}$:

$$Z = (z - z_0) - \lambda_0(t - t_0), \qquad T = (z - z_0) - \mu_0(t - t_0).$$

Using (6.1) we find the first derivatives of the functions $Z(U,V)$ and $T(U,V)$ to be

$$\begin{aligned} Z_U &= z_U - \lambda_0 t_U = (\lambda - \lambda_0)p, \\ Z_V &= z_V - \lambda_0 t_V = (\mu - \lambda_0)q, \\ T_U &= z_U - \mu_0 t_U = (\lambda - \mu_0)p, \\ T_V &= z_V - \mu_0 t_V = (\mu - \mu_0)q \end{aligned}$$

in which $\lambda = \lambda(f(U,V), g(U,V))$ and $\mu = \mu(f(U,V), g(U,V))$.

Differentiate (6.1) and use the relations $t_{UV} = t_{VU}$, $z_{UV} = z_{VU}$ to obtain

$$\begin{aligned} p_V &= q_U, \\ (\lambda p)_V &= (\mu q)_U, \end{aligned}$$

or

$$\begin{aligned} p_V &= q_U, \\ p_V &= \tfrac{1}{\lambda - \mu}\left(\tfrac{\partial \mu}{\partial U} q - \tfrac{\partial \lambda}{\partial V} p\right). \end{aligned} \tag{6.3}$$

Let $(u_0, v_0)$ be an arbitrary point in the domain $V_1$. Given the values $z_0 = z(u_0, v_0)$ and $t_0 = t(u_0, v_0)$, equations (6.3) determine the Taylor series at $(u_0, v_0)$ for the functions $z$ and $t$ uniquely. Therefore the system (6.3) suffices to describe the singularities of the map $G$..

## 6.2 Classification of Singularity Types in Terms of $p, q$

We will now use the system ( 6.3) to express the conditions for a fold, cusp or umbilic singularity directly in terms of $p, q$ and their derivatives. The following lemma provides a characterization of umbilic singularities (which are in $\Sigma^2$).

**Lemma 6.1** *The map $G : (U,V) \to (z,t)$ has an umbilic singularity at the point $(u_0, v_0)$ if and only if $p(u_0,v_0) = q(u_0,v_0) = 0$, but $p_U(u_0,v_0)$ and $q_V(u_0,v_0)$ do not vanish. If in addition $\frac{\partial\lambda(f,g)}{\partial U}|_{(u_0,v_0)}$ and $\frac{\partial\mu(f,g)}{\partial V}|_{(u_0,v_0)}$ are non-zero, this umbilic has nontrivial cubic terms.*

**Proof.** The Jacobi matrix (6.2) is zero if and only if both $p$ and $q$ are zero. At a point $(u_0,v_0)$, where $p = q = 0$ we have

$$\begin{array}{llll} Z_{UU}|_{(u_0,v_0)} & = Z_{UV}|_{(u_0,v_0)} = 0 & T_{UV}|_{(u_0,v_0)} & = T_{VV}|_{(u_0,v_0)} = 0 \\ Z_{VV}|_{(u_0,v_0)} & = (\mu_0 - \lambda_0) q_V(u_0,v_0) & T_{UU}|_{(u_0,v_0)} & = (\lambda_0 - \mu_0) p_U(u_0,v_0) \\ Z_{UUU}|_{(u_0,v_0)} & = 2\frac{\partial\lambda}{\partial U} p_U(u_0,v_0) & T_{VVV}|_{(u_0,v_0)} & = 2\frac{\partial\mu}{\partial V} q_V(u_0,v_0). \end{array}$$

From these formulas and Theorem 3.2, one can see that the map $G : (U,V) \to (Z,T)$ has an umbilic singularity if $p_U(u_0,v_0) \neq 0$ and $q_V(u_0,v_0) \neq 0$. Furthermore, this umbilic has nontrivial cubic terms if, in addition, $\frac{\partial\lambda}{\partial U}$ and $\frac{\partial\mu}{\partial V}$ are non-zero. ∎

According to (6.1) the characteristic $\frac{dz}{dt} = \lambda$ corresponds to $V =$ constant and the characteristic $\frac{dz}{dt} = \mu$ corresponds to $U =$ constant. Thus the conditions $\partial\lambda/\partial U \neq 0$ and $\partial\mu/\partial V \neq 0$ guarantee nontrivial interactions between the characteristic quantities. Genuine nonlinearity, on the other hand, would be the conditions $\partial\lambda/\partial V \neq 0$ and $\partial\mu/\partial U \neq 0$.

We consider now singularities of rank 1, i.e. those in the Boardman-Thom class $\Sigma^1$. The Jacobi matrix (6.2) has rank one if $p = 0$, but $q \neq 0$ , or vice versa if $q = 0$, $p \neq 0$. These two cases are symmetric and we will consider only the first.

If the map $G : \mathbb{C}^2_{U,V} \to \mathbb{C}^2_{z,t}$ has a fold or cusp singularity then the map $j^1G : \mathbb{C}^2_{U,V} \to J^1(\mathbb{C}^2,\mathbb{C}^2)$ is transversal to the manifold $\Delta$: $z_U t_V - z_V t_U = 0$. We say that the map $G : \mathbb{C}^2 \to \mathbb{C}^2$ *satisfies the transversality condition* if $j^1G$ is transversal to $\Delta$.

**Lemma 6.2** *Let $q = t_V \neq 0$ in the domain of consideration. The map $G$ satisfies the transversality condition if either $p_U \neq 0$ or $p_V \neq 0$ (or both of them are non-zero) at any point where $p = 0$.*

**Proof.** In the domain $t_V \neq 0$ the manifold $\Delta \subset J^1(\mathbb{C}^2,\mathbb{C}^2)$ is a smooth hypersurface. Hence, the map $j^1(G)$ is transversal to $\Delta$ if $\nabla\delta \neq 0$, at any point where $\delta(U,V) = z_U t_V - z_V t_U$ is zero. Using the system (6.1) one can express $\delta$ as $\delta = p(U,V)q(U,V)(\lambda(F(U,V)) - \mu(F(U,V)))$. In the domain of consideration $q$ and $\lambda - \mu$ are non-zero. Therefore $\delta$ and $\nabla\delta$ are not simultaneously zero if $p$ and $\nabla p$ do not vanish simultaneously. ∎

The following lemma follows from Lemma 6.2 and the last equation in (6.3):

**Lemma 6.3** *Let $q(u_0, v_0) = t_V(u_0, v_0) \neq 0$. The map $j^1(G)$ is transversal to $\Delta$ at the point $(u_0, v_0)$ where $p = 0$ if the three functions*

$$p, \ p_U, \ \text{and} \ \frac{\partial \mu}{\partial U}$$

*are not simultaneously zero.*

In fact in the 2x2 case $\frac{\partial \mu}{\partial U} = \frac{\partial \mu}{\partial f} f_U + \frac{\partial \mu}{\partial g} g_U = \frac{\partial \mu}{\partial g} g_U$, because $f_U = 0$.

Finally a characterization of the fold and cusp singularities can be given.

**Lemma 6.4** *Let $G : (U, V) \to (z, t)$ be a solution of the system (6.3) that satisfies the transversality condition and is singular at the point $(u_0, v_0)$. Let $q(u_0, v_0) = t_V(u_0, v_0) \neq 0$. Then the map $G$ has at $(u_0, v_0)$*

*(i) a fold, if $p(u_0, v_0) = 0$, but $p_U(u_0, v_0) \neq 0$, and*

*(ii) a cusp, if $p(u_0, v_0) = p_U(u_0, v_0) = 0$, but $p_{UU}(u_0, v_0) \neq 0$.*

**Proof.** Using the notation above, at $(u_0, v_0)$ one has

$$\begin{aligned} &Z_U(u_0, v_0) = 0, && Z_V(u_0, v_0) = (\mu - \lambda_0) q(u_0, v_0) \neq 0; \\ &T_U(u_0, v_0) = (\lambda - \mu_0) p(u_0, v_0), && T_V(u_0, v_0) = 0; \end{aligned}$$

The value $p(u_0, v_0)$ is zero because $G$ is singular at $(u_0, v_0)$ and $q(u_0, v_0) \neq 0$. Then

$$\begin{aligned} T_{UU}(u_0, v_0) &= (\lambda_0 - \mu_0) p_U(u_0, v_0), \\ T_{UV}(u_0, v_0) &= \tfrac{\partial \mu}{\partial U}(u_0, v_0) q(u_0, v_0). \end{aligned}$$

If $p_U(u_0, v_0) \neq 0$, then the map $(U, V) \to (Z, T)$ has a fold . If $p_U(u_0, v_0) = 0$, then

$$T_{UUU} = (\lambda - \mu_0) p_{UU}(u_0, v_0).$$

If $p(u_0, v_0) = p_U(u_0, v_0) = 0$ then $\frac{\partial \mu}{\partial U}(u_0, v_0) \neq 0$ by Lemma 6.3, and therefore to have cusp we only need $p_{UU} \neq 0$. ∎

The definition of transversality has not been given here, since the only attributes of transversality needed are those of the preceeding Lemmas. In fact Lemma 6.2 could be used as a working definition of transversality here.

At this point the significance of the perturbations of initial data, including perturbation of $t$ as in (1.6), can be understood. If $t \equiv 0$ for $u = 0$ then also $t_v \equiv 0$. Then at a branch point in the initial data where also $z_v = 0$, the Jacobi matrix (6.2) must vanish because of the equations (6.1), and so the singularity must then be an umbilic point, unless there is some degeneracy. This implies that an initial singularity will split into two folds moving at the two characteristic speeds. The branch cut opened up between them is the source of the "hidden initial data" mentioned in Section 2. On the other hand under perturbations of the form (1.6), including a perturbation of $t$, such a degeneracy does not occur.

We will now illustrate the effect the constraint has in the situation where the speeds are constant: $\lambda = \lambda_0, \mu = \mu_0$. For appropriate values of these constants, the solution of the unfolding system is simply

$$Z = V^2$$
$$T = -U^2.$$

Now suppose we consider a linear perturbation of this unfolding:

$$\tilde{Z} = \epsilon(aU + bV) + V^2$$
$$\tilde{T} = \epsilon(cU + dV) + U^2.$$

The jacobian of this transformation becomes degenerate along the locus where its determinant vanishes:

$$ad\epsilon^2 - (2U + \epsilon c)(2V + \epsilon b) = 0.$$

As long as $a, d \neq 0$ this locus is smooth, along it the rank of the jacobian is one and singularities of the map are folds.

However, in the limit as $a \to 0$ or $d \to 0$, the determinant locus becomes singular at $U = -\frac{\epsilon c}{2}, V = -\frac{\epsilon b}{2}$ where the rank of the jacobian drops to zero and the folds limit to an umbilic point.

Figure 3 shows, for real values of the parameters $a, b, c, d$, the real part of the discriminant loci which consist of a family of hyperbolas. For $a, d \neq 0$ these are smooth corresponding to folds while if either of these parameters vanishes the hyperbola degenerates to a pair of intersecting lines corresponding to a hyperbolic umbilic.

Now in our example the differential constraint reduces to $\tilde{Z}_U = 0, \tilde{T}_V = 0$ or equivalently $a = d = 0$. Thus, the constraint forces the hyperbolic umbilic to persist under perturbation.

# 7 Genericity Theorem.

## 7.1 Statement of the theorem.

In this section we will consider singularities of the system (5.1) with zero right-hand side:

$$F_t + M(F)F_z = 0. \tag{7.1}$$

The analysis for systems with non-zero right hand side is similar. We are interested in the singularities of the solutions of (7.1) in the domain $D \subset \mathbb{C}^{n+2}_{z,t,F}$ where the system has two distinct characteristic speeds. That is, in the domain $D$ where $\lambda(F) \neq \mu(F)$.

Take again left and right eigenvectors $l_m$ and $r_s$ of the matrix $M$, satisfying (5.2). Let $F = (f_1, \ldots, f_n)$. We can rewrite the system (7.1) as follows:

$$\begin{array}{lll} \sum_{s=1}^{n} l_{is}((f_s)_t + \lambda(f_s)_z) & = 0 & i = 1, \ldots, k, \\ \sum_{s=1}^{n} l_{js}((f_s)_t + \mu(f_s)_z) & = 0 & j = k+1, \ldots, n. \end{array} \tag{7.2}$$

Suppose that the solution $F = F(z,t)$ of the system (7.2) defines the surface $S$ in $D \subset \mathbb{C}^{n+2}$. Let $z(u,v), t(u,v), F(u,v)$ be a parameterization of a solution such that $u - v = const$ and $u + v = const$ are characteristics on $S$. Then $z(u,v), t(u,v), F(u,v)$ satisfy the system:

$$\begin{array}{lll} \sum_{s=1}^{n} l_{is}((f_s)_u - (f_s)_v) & = 0, & i = 1, \ldots, k, \\ \sum_{s=1}^{n} l_{js}((f_s)_u + (f_s)_v) & = 0, & j = k+1, \ldots, n, \\ z_u - \lambda t_u & = z_v - \lambda t_v, & \\ z_u - \mu t_u & = -(z_v - \mu t_v). & \end{array} \tag{7.3}$$

**Remarks.** (1) The parameterization $z(u,v), t(u,v)$ is the unfolding of Section 4 with $a = 1$, $b = -1$. The lines $\{z = z(u, u+c),\ t = t(u, u+c)\}$ and $\{z = z(u, -u+c),\ t = t(u, -u+c)\}$ are the characteristics of the initial system (7.1).

(2) The initial value problem with data on the lines $u = const$, $F = F_0(v)$, $z = z_0(v)$, $t = t_0(v)$ is well posed for the system (7.3).

(3) The parameterization of the solution is determined uniquely up to the change of variables:$u_1 + v_1 = \phi(u+v)$, $u_1 - v_1 = \psi(u - v)$, with $\phi'$ and $\psi'$ nonzero.

As in Section 3, we will say that a generic solution satisfies a given property **A** if

(1) the set of solutions with property **A** is open, and

(2) the set of solutions, satisfying property **A** is dense in the following sense: For any solution $s : \mathcal{V} \subset \mathbb{C}^2_{uv} \to D \subset \mathbb{C}^{n+2}_{ztF}$ and for any bounded subdomain $\mathcal{V}_1 \subset \mathcal{V}$, there is a solution $s_1 : \mathcal{V}_1 \to D$ arbitrarily close to $s|_{\mathcal{V}_1}$ satisfying property **A**.

In this section we are going to prove Theorem 1.2, which is restated more explicitly as the following:

**Theorem 7.1** *(Genericity Theorem). For a generic solution s of (7.1) the only singularities of the unfolding $(u,v) \to (z,t)$ are folds, cusps and nondegenerate umbilics.*

*If, in addition, at least one of the functions*

$$\sum_{m=1}^{n} \frac{\partial \lambda}{\partial f_m} \Big( \sum_{j=k+1}^{n} r_{mj} l_{js} \Big), \qquad s = 1, \ldots, n,$$

*and at least one of the functions*

$$\sum_{m=1}^{n} \frac{\partial \mu}{\partial f_m} (\sum_{i=1}^{k} r_{mi} l_{is}), \qquad s = 1, \ldots, n,$$

*are not identically zero, then a generic solution has nondegenerate umbilics with nontrivial cubic terms.*

## 7.2 Overview of the proof for the Genericity Theorem

Consider a solution $s : \mathcal{V} \to D$ and a bounded domain $\mathcal{V}_1 \subset \mathcal{V}$. At first we will construct $s_1$ such that the only singularities of $s_1$ with

$$rank \begin{pmatrix} z_u & z_v \\ t_u & t_v \end{pmatrix} = 0 \qquad (7.4)$$

are umbilics (Section 7.3). Then we will perturb $s_1$ to obtain a solution which satisfies the transversality condition (cf. Section 6.2) at its singularities. After perhaps another perturbation, this solution will have only folds and cusps as singularities at points where the rank of the Jacobian is equal to 1 (Section 7.5).

Again, as in Section 6, introduce new coordinates $U = u - v$ and $V = u + v$ and denote $p = t_U, q = t_V$. Let $\Pi$ denote the standard projection $\Pi : D \to \mathbb{C}^2_{zt}$. The system (7.3) then takes the form:

$$\begin{array}{ll} \sum_{s=1}^{n} l_{is}(F)(f_s)_U = 0, & i = 1, \ldots, k, \\ \sum_{s=1}^{n} l_{js}(F)(f_s)_V = 0, & j = k+1, \ldots, n, \\ p_V = q_U & \\ p_V = \frac{1}{\lambda - \mu}(\frac{\partial \mu}{\partial U} q - \frac{\partial \lambda}{\partial V} p) & \end{array} \qquad (7.5)$$

The singularities of the map $\Pi \circ s$ are described in terms of functions $p$, $q$ and their derivatives by Lemmas 6.1 – 6.4. In the cases of umbilics and cusps Lemmas 6.1 and 6.4 can be summarized in the following way: The singularity in each of these cases is described in terms of a certain pair of analytic functions $\alpha$ and $\beta$ in $\mathcal{V}$, which depend on the 1-jet of $s$. For the cusp $\alpha = p$ and $\beta = p_U$ (or $q$ and $q_V$); while for the umbilic $\alpha = p$ and $\beta = q$. The map $\Pi \circ s$ will have a singularity of the prescribed normal form (either an umbilic or a cusp) if the corresponding map $(\alpha, \beta) : \mathcal{V} \to \mathbb{C}^2$ has a zero of multiplicity 1.

The openness of the set of functions with fold, cusp or umbilic singularities is an immediate corollary of this representation: Suppose that the map $(\alpha, \beta) = (\alpha(\cdot; F, p, q), \beta(\cdot; F, p, q))$ has zeros only of multiplicity 1, and let $(F_1, p_1, q_1) : \mathcal{V} \to \mathbb{C}^{n+2}$ be a small perturbation of $(F, p, q)$. Then the map $(\alpha_1, \beta_1) = (\alpha(\cdot; F_1, p_1, q_1), \beta(\cdot; F_1, p_1, q_1))$ will have zeros only of multiplicity 1, close to those of $(\alpha, \beta)$.

We now turn to a main principle in our analysis of density of maps with folds, cusps and umbilics. We want to show that given any solution of system (7.5) in the bounded region $\mathcal{V}$ we may, by perturbing initial data, arrive at a solution whose branch point singularities are all equivalent to one of our normal forms. First note that because our solutions are all analytic in the unfolding variables $(u, v)$, with existence of solutions given by the Cauchy-Kowalevsky theorem, perturbations with respect to initial data are entirely equivalent to perturbations within the class of solutions on $\mathcal{V}$. In particular perturbation of the data along $u = 0$ is equivalent to perturbation along the line $u = u_0$ for any $u_0$.

The following Reduction Principle exploits this observation to provide a method for establishing the density of solutions having singularities of a prescribed type (namely fold, cusp or umbilic). This Principle will be used repeatedly throughout the remainder of this section, and it will be directly verified in each application. The proof of the Principle always follows the same pattern, however, which is sketched below.

**Reduction Principle.** *Let $(\alpha, \beta) : \mathcal{V} \times J^1 \to \mathbb{C}^2$ be a map which is analytic in $\mathcal{V}$ and depends continuously on $J^1 = \{\jmath^1(p), \jmath^1(q), \jmath^1(F)\}$ the 1-jets of solutions to system (7.5). Suppose that, for the 1-jet of a particular solution $p, q, F$, the zeroes of $(\alpha, \beta)$ are isolated points in $\mathcal{V}$. Then for generic perturbations of the 1-jet of $p, q, F$, the perturbed map $(\tilde{\alpha}, \tilde{\beta})$ has only isolated zeroes of multiplicity 1, i.e. at each of which the local ring for $(\tilde{\alpha}, \tilde{\beta})$ has dimension 1.*

*Scheme of the Proof:* Since $(\alpha, \beta)$ is analytic and $\mathcal{V}$ is bounded, the zeroes of $(\alpha, \beta)$ are a finite set, $\{x_1, \ldots, x_n\}$ at each of which the local ring of $(\alpha, \beta)$ has finite multiplicity $m_i$. Let $N = \sum_{i=1}^n m_i$. The principle is established by (decreasing) induction on $\sum_{i=1}^n (m_i - 1)$.

The induction step is effected through a succession of perturbations. At the i-th point $x_i = (u_i, v_i)$, perturb $(p, q, F)$ along the line $u = u_i$, so that this point remains a zero of $(\alpha, \beta)$ but with multiplicity one. One then uses the Cauchy-Kowalevsky theorem to construct a solution initialized along this line. In this way the multiplicity at $x_i$ is reduced and the perturbation remains within the class of solutions to system (7.5).

We must still insure that the perturbation does not introduce new singularities other than by splitting the old ones. On each induction step choose a small ball $B_i$ around each $x_i$ such that no two of these balls intersect. We will always take perturbations sufficiently small so that, under the perturbation, the total multiplicity of zeroes within a given $B_i$ remains constant as does the total multiplicity within $\mathcal{V}$ . This can be established by a covering argument since the number of zeroes and their multiplicities is finite and because we have continuous dependence of the solution on the data, which is prescribed on the lines $u = u_i$. ∎

In order to verify this principle in a specific application it will suffice to show that one can perturb $(p, q, F)$ along the line $u = u_i$, so that $x_i$ remains a zero of $(\alpha, \beta)$ but with multiplicity decreased by at least one. In each of the following subsections we will describe corresponding functions $\alpha$ and $\beta$, then show that for a generic solution the map $(\alpha, \beta)$ has only isolated zeros, and finally verify the Reduction Principle.

Throughout the remainder of this section denote $F_0(v) = (f_1^0(v), \ldots, f_n^0(v))$, $p_0(v)$, and $q_0(v)$ to be the restrictions of a solution $s = (F, p, q)$ of the system (7.5) on a line $u = u_0$.

## 7.3 Genericity of umbilics.

By Lemma 6.1 the map $\Pi \circ s$ has an umbilic singularity at a point $(u_0, v_0)$ if the map $(p, q) : \mathcal{V} \to \mathbb{C}^2$ has a zero of multiplicity 1; i.e., $p = q = 0$ but $p_U \neq 0, q_V \neq 0$ at $(u_0, v_0)$.

In this section we will show that generically solutions of the system (7.5) have only umbilic singularities with nontrivial cubic terms, at points where $p = q = 0$. The argument is developed in three steps. First we show that the common zeroes $p = q = 0$ in $\mathcal{V}$ are a set of isolated points. Then we show that the solution may be perturbed so that when $p = q = 0$, $p_U \neq 0$ and $q_V \neq 0$. Finally we show that after perturbation $\frac{\partial \lambda}{\partial U}(u_0, v_0) \neq 0$ and $\frac{\partial \mu}{\partial V}(u_0, v_0) \neq 0$ at the umbilic points. Thus by Lemma 6.1 these umbilic points have nontrivial cubic terms, so that such singularities are generic.

We start by characterizing the structure of the set $p = q = 0$.

**Lemma 7.1** *Given a solution $s$ of the system (7.5), consider the set $p = q = 0$. Its connected components are either all of $\mathbb{C}^2_{uv}$, or isolated points, or characteristic lines $V = const$, or $U = const$.*

**Proof.** Let functions $p(U, V)$ and $q(U, V)$ give a solution of the system (7.5). Suppose that a point $(u_0, v_0)$ is a non-isolated point of the set $p = q = 0$. Suppose also that in a neighborhood of $(u_0, v_0)$ this set is not one of the characteristics $U = U_0$, or $V = V_0$. If the point $(u_0, v_0)$ belongs to a smooth arc of the set $p = q = 0$, then there exists a neighborhood $\mathcal{U}$ of $(u_0, v_0)$, where this arc is a zero set of a function $U - \gamma(V)$, and $\gamma'$ in $\mathcal{U}$ is not identically zero. Then

$$p|_{\mathcal{U}} = (U - \gamma(V))^k p_1, \qquad q|_{\mathcal{U}} = (U - \gamma(V))^l q_1, \tag{7.6}$$

where $p_1(\gamma(V), V)$, $q_1(\gamma(V), V)$ are not identically zero. From equations (7.5) one has:

$$\begin{aligned} k(U - \gamma(V))^{k-1}\gamma'(V)p_1 = \; & l(U - \gamma(V))^{l-1}q_1 + (U - \gamma(V))^l \tfrac{\partial q_1}{\partial U} \\ & -(U - \gamma(V))^k \tfrac{\partial p_1}{\partial V}, \end{aligned} \tag{7.7}$$

$$\begin{aligned} k(U - \gamma(V))^{k-1}\gamma'(V)p_1 = \; & (U - \gamma(V))^l q_1 \tfrac{\partial \mu}{\partial U}/(\lambda - \mu) \\ & -(U - \gamma(V))^k (\tfrac{\partial \lambda}{\partial V} p_1/(\lambda - \mu) + \tfrac{\partial p_1}{\partial V}). \end{aligned} \tag{7.8}$$

From the equation (7.7) one has $k = l$, and from the (7.8) $k - 1 = l$. But this is a contradiction, which proves the lemma. ∎

Now we are ready to establish the first step of our argument.

**Lemma 7.2** *A generic solution of (7.5) in a bounded domain $\mathcal{V}$ has only isolated zeros of the map $(p, q)$.*

**Proof.** Note that if $p = q = 0$ at the point $(u_0, v_0)$, then $p_V(u_0, v_0) = q_U(u_0, v_0) = (\frac{\partial \mu}{\partial U} q - \frac{\partial \lambda}{\partial V} p)/(\lambda - \mu) = 0$. If in addition $p_U = 0$ at this point, then the derivative of $p$ in any direction is zero, and in particular $p_0'(v_0) = 0$. Similarly, if $q(u_0, v_0) = q_V(u_0, v_0) = 0$ then $q_0'(v_0) = 0$. If $p|_{V=const} \equiv 0$ then $p_U|_{V=const} \equiv 0$, and, similarly, if $q|_{U=const} \equiv 0$ then $q_V|_{U=const} \equiv 0$. Perturb now the restriction of the solution on the line $u = u_0$ in such a way that $p_0(v)$ and $(p_0)'(v)$ are not simultaneously zero, and also $q_0(v)$ and $(q_0)'(v)$ are not simultaneously zero. We can retain the restriction $F_0(v)$ unchanged. Let $\tilde{s} : \mathcal{V}' \to D$ be the solution of the system (7.5) with the functions $p_0$, $q_0$, and $F_0$ on the line $u = 0$ as initial data. For the solution $\tilde{s}$ none of the characteristic lines $U = const$ or $V = const$ can belong to the set $p = q = 0$. Otherwise at the point of intersection of this characteristic and the line $u = u_0$ either $p_0 = p_0' = 0$ or $q_0 = q_0' = 0$. Therefore the set $p = q = 0$ for $\tilde{s}$ will consist only of isolated points. ∎

We now turn the second step: perturbation of the common zeroes of $p, q$ to umbilic points. This requires verification of the Reduction Principle for the map $(\alpha, \beta) = (p, q)$.

Consider a solution $s$ which has a finite number of points where $p = q = 0$ in $\bar{\mathcal{V}}_1$. Let $(u_0, v_0)$ be a point where $p = q = 0$ with multiplicity $k \geq 1$. Let $p_0, q_0$ be the restrictions of $p$, $q$ to the line $u = u_0$. To justify the usage of the Reduction Principle, we have to verify that we can perturb the restrictions $p_0$ and $q_0$ so that the point $(u_0, v_0)$ will be zero of $(p, q)$ of multiplicity 1 for the corresponding solution of (7.5). At $(u_0, v_0)$, $p_U(u_0, v_0) = p_V(u_0, v_0) - p_0'(v_0) = -p_0'(v_0)$, and $q_V(u_0, v_0) = q_0'(v_0) + q_U(u_0, v_0) = q_0'(v_0)$. Therefore it suffice to take $p_0$ and $q_0$ which have zero values and non-zero derivatives at $v_0$. This establishes the Reduction Principle in this case, and shows that, at points with $p = q = 0$, the singularity is generically a nondegenerate umbilic.

Lastly we show that generic umbilic singularities have nontrivial cubic terms under the conditions of Theorem 7.1. Along the line $u = u_0$, $(f_s)_V = (f_s^0)'(v) + (f_s)_U$. Therefore, on $u = u_0$ the first two equations of (7.5) are

$$\begin{aligned} \sum_{s=1}^{n} l_{is}(f_s)_U &= 0, \qquad i = 1, \ldots, k, \\ \sum_{s=1}^{n} l_{js}(f_s)_U &= -\textstyle\sum_{s=1}^{n} l_{js}(f_s^0)'(v), \qquad j = k+1, \ldots, n. \end{aligned}$$

Also on $u = u_0$ we can express

$$(f_m)_U = -\sum_{j=k+1}^{n}\sum_{s=1}^{n} r_{mj} l_{js} (f_s^0)'(v),$$

and

$$\frac{\partial\lambda}{\partial U} = \sum_{s=1}^{n} \frac{\partial\lambda}{\partial f_s}(f_s)_U$$

$$= -\sum_{s=1}^{n}\left(\sum_{m=1}^{n} \frac{\partial\lambda}{\partial f_m} \sum_{j=k+1}^{n} r_{mj} l_{js}\right)(f_s^0)'(v).$$

If $(\sum_{m=1}^{n} \frac{\partial\lambda}{\partial f_m} \sum_{j=k+1}^{n} r_{mj} l_{js})$ are not all identically zero, then at each umbilic point $(u_0, v_0)$ we can perturb the initial functions $f_s^0$ (retaining the generic properties of $p_0$ and $q_0$) so that $(u_0, v_0)$ will become an umbilic point with $\frac{\partial\lambda}{\partial U}(u_0, v_0) \neq 0$.

In a similar way, if not all $(\sum_{m=1}^{n} \frac{\partial\mu}{\partial f_m} \sum_{i=1}^{k} r_{mi} l_{is})$ are identically zero then we can also find a perturbation of the initial functions such that $\frac{\partial\mu}{\partial V}(u_0, v_0) \neq 0$.

## 7.4 Generic maps with singularities $\Sigma^1$.

Since the Jacobi matrix has the form (6.2), it has rank 0 only at points where $p = q = 0$, which are generically umbilic points according to the previous subsection. Now we consider singular points where the Jacobi matrix has rank 1, which are the singularities in $\Sigma^1$. In a neighborhood of an umbilic point all other singular points are folds. Therefore we need only consider $\Sigma^1$ singularities that are away from umbilics, i.e. we can now assume that in the domain $\mathcal{V}$ the rank of the Jacobi matrix of a map $\Pi \circ s : (U, V) \rightarrow (z, t)$ is at least 1.

In this section we are going to prove the following:

**Lemma 7.3** *Let $s$ be a solution for which the rank of the Jacobi matrix of $\Pi \circ s : \mathcal{V} \rightarrow D$ is at least 1. For any bounded subdomain $\mathcal{V}_1$, with $\bar{\mathcal{V}}_1 \subset \mathcal{V}$, there exists a solution $\tilde{s}$, arbitrary close to $s|_{\mathcal{V}_1}$, such that $\Pi \circ \tilde{s}$ satisfies the transversality condition (Lemmas 6.2, 6.3).*

The proof of Lemma 7.3 will follow the scheme stated in Section 7.2. We will exploit the assumption that $\lambda$, $\mu$, and the matrix $(l_{rs})$ do not depend on $z$ and $t$. Therefore by fixing a solution $F$ of the first $n$ equations in (7.5) we fix the functions $\frac{\partial\lambda}{\partial V}$ and $\frac{\partial\mu}{\partial U}$. We will show that transversality can be satisfied by perturbing only $p$ and $q$. We continue to assume that $q \neq 0$ in the domain under consideration.

We must first treat separately the extreme situation where $\frac{\partial\mu}{\partial U}$ is identically zero.

**Lemma 7.4** *If $\frac{\partial \mu}{\partial U}$ is identically zero, then after small perturbation the solution has only fold singularities in the domain where $q \neq 0$.*

**Proof.** If $\frac{\partial \mu}{\partial U} \equiv 0$ then the last equation in (7.5) takes the form

$$p_V = -p\frac{\partial \lambda}{\partial V}/(\lambda - \mu). \tag{7.9}$$

We solve (7.9) with respect to $V$ starting from the line $u = 0$, i.e. $V = -U = v$. Denote $p_0(v)$ to be the restriction of a solution $p(U, V)$ on the line $u = 0$. The solution $p(U, V)$ of this equation can be written as follows:

$$\begin{aligned} p(U,V) &= p(U,-U)\exp(\int_{-U}^{V}(\frac{\partial \lambda}{\partial V}/(\lambda-\mu))dV \\ &= p_0(-U)\exp(\int_{-U}^{V}(\frac{\partial \lambda}{\partial V}/(\lambda-\mu))dV. \end{aligned} \tag{7.10}$$

In particular $p(U, V) = 0$ on the whole line $U = U_0$ if $p_0(-U_0) = 0$. If also $p(U_0, V) = 0$ then by differentiating (7.10) with respect to $U$ and substituting $p_0(-U_0) = 0$ one has

$$p_U(U_0, V) = -p_0'(-U_0)\exp(\int_{-U_0}^{V}(\frac{\partial \lambda}{\partial V}/(\lambda-\mu))dV.$$

Therefore, the set $p = p_U = 0$ also consists of a collection of whole lines $U = const$. One can now perturb the values of $p_0$ on the line $u = 0$ in such a way that $p_0$ and $p_0'$ do not vanish simultaneously. Then for the solution $\tilde{s}$ with this initial data, $p$ and $p_u$ do not vanish simultaneously. By Lemma 6.4 all the singularities of $\Pi \circ \tilde{s}$ at which the Jacobi matrix has rank one are then folds. ∎

We assume now that $\frac{\partial \mu}{\partial U}$ is not identically 0, and proceed according to the scheme of subsection 7.2, with $\alpha = p$ and $\beta = \frac{\partial \mu}{\partial U}$. We will first show that the common zeroes of $\alpha$ and $\beta$ are isolated.

**Lemma 7.5** *After a small perturbation of the solution $z, t$ the points where $p = \frac{\partial \mu}{\partial U} = 0$ in a bounded domain $\mathcal{V}$ are all isolated.*

**Proof.** The set $\{\frac{\partial \mu}{\partial U} = 0\}$ in $\bar{\mathcal{V}}$ consists of a finite number of smooth components and a finite number of critical points. Let $Y_1, \ldots, Y_r$ be the smooth components of $\{\frac{\partial \mu}{\partial U} = 0\}$ belonging to the set $\{p = 0\}$. Consider perturbations of the solution so small that no components of $\{\frac{\partial \mu}{\partial U} = 0\}$ other than $Y_i$ $(i = 1, \ldots, r)$ (and isolated points) can completely belong to the set $\{p = 0\}$. Note that we keep the $F(U, V)$ and, therefore, the set $\{\frac{\partial \mu}{\partial U} = 0\}$ fixed, because the first $n$ equations in (7.5) depend only on $F$, and we are perturbing only the solution of the last two equations.

Let $(u_0, v_0) \in Y_1$. We can always perturb the restriction of the solution on the line $u = u_0$ so that $p(u_0, v_0) \neq 0$, and hence the set $\{p = 0\}$ intersects with $Y_1$ only in isolated points for the perturbed solution. Continuing this procedure (and restricting to smaller and smaller perturbations) in no more than $r$ steps we can make all the solutions of $p = \frac{\partial \mu}{\partial U} = 0$ in $\bar{\mathcal{V}}$ isolated. ∎

To establish transversality we use the characterization given in Lemma 6.3. We will show that generically $p_U \neq 0$ at the points where $p = \frac{\partial \mu}{\partial U} = 0$, which are isolated in $\mathcal{V}$ by the previous lemma. We use a slightly modified version of the Reduction Principle to establish this.

Consider all such zeros $x_1, \ldots x_K$ of $(p, \frac{\partial \mu}{\partial U})$ in $\bar{\mathcal{V}}_1$ where $p_U$ also vanishes. Each point $x_i = (u_i, v_i)$ has a finite multiplicity $m_i$ as a zero of $(p, \frac{\partial \mu}{\partial U})$. Proceed by induction on $N = \sum_{i=1}^{K} m_i$. At each step we perturb restrictions of $p$ and $q$ on the line $u = u_i$ in such a way that the point $(u_i, v_i)$ remains a zero of $p = \frac{\partial \mu}{\partial U} = 0$ but $p_U$ becomes non-zero. As in the case of the Reduction Principle, in each induction step cover points $x_i$ by nonintersecting balls, and consider sufficiently small perturbations of $s$ so that the sum of multiplicities of zeros of $(p, \frac{\partial \mu}{\partial U})$ inside these balls remains constant. Then after each step the sum of multiplicities of zeros of $(p, \frac{\partial \mu}{\partial U})$ where $p_U$ is also zero is reduced by at least 1.

Take a zero $(u_0, v_0)$ of $(p, \frac{\partial \mu}{\partial U})$, where also $p_U = 0$ and let $p_0, q_0$ denote the restriction of $p, q$ to the line $u = u_0$. To establish the Reduction Principle it suffices to to find a perturbation such that $(u_0, v_0)$ remains a zero of $(p, \frac{\partial \mu}{\partial U})$, but $p_U(u_0, v_0) \neq 0$. Perturb $p_0$ and $q_0$ such that $p_0 = 0$ but $p_0' \neq q_0 \frac{\partial \mu}{\partial U} / (\lambda - \mu)$ at $(u_0, v_0)$. Then according to (7.5) on $u = u_0$, $p_U = p_V - p_0' = q_0 \frac{\partial \mu}{\partial U} / (\lambda - \mu) - p_0' \neq 0$ at $(u_0, v_0)$. This establishes the Reduction Principle and finishes the proof of Lemma 7.3.

## 7.5 Genericity of folds and cusps.

We have proven that a generic solution $s$ of the system (7.5) satisfies the transversality condition. For $\Sigma^1$ singularities this was established in the previous subsection; while for umbilic points the map $j^1(\Pi \circ s)$ is transversal to $\Delta = \{z_U t_V - z_V t_U = 0\}$. In the rest of this section, we will consider only solutions with $\Sigma^1$ singularities that satisfy the transversality condition. As in the previous subsection we can assume that in the domain of consideration the rank of the Jacobi matrix of the map $(U, V) \to (z, t)$ is not less than one, and that $q = t_V \neq 0$. The singular points of the map $\Pi \circ s$ are the points where $p = 0$.

By Lemma 6.4 at a point $(u_0, v_0)$ the map $\Pi \circ s$, satisfying the transversality condition, has a fold if $p = 0$, but $p_U \neq 0$, and a cusp if $p = p_U = 0$, but $p_{UU} \neq 0$. At a point where $p = p_U = 0$, the derivative $p_V = -q \frac{\partial \mu}{\partial U} \neq 0$ for a solution satisfying the transversality condition (see Lemma 6.3). Hence, the condition $p_{UU} \neq 0$ at a point means that this point is a zero of $(p, p_U)$ with multiplicity 1. In this final subsection we will show that we can perturb $s$ in such a way that every zero of $(p, p_U)$ has multiplicity 1.

Once more we isolate the points where $p = p_U = 0$. As shown above, we may assume that $\partial\mu/\partial U \neq 0$ at such points.

**Lemma 7.6** *For generic solutions, the set $p = p_U = 0$ consists of isolated points.*

**Proof.** If $p = 0$, then $p_V = q\frac{\partial\mu}{\partial U}/(\lambda - \mu) \neq 0$ outside $\{\frac{\partial\mu}{\partial U} = 0\}$. If $p_U = 0$ along the smooth curve $p = 0$ then, by the implicit function theorem, it is a line $V = const$ (note, that $p_V \neq 0$). Therefore the only non-isolated components of the set $p = p_U = 0$ are characteristics $V = const$. But we always can choose initial data on the line $u = 0$ in such a way that $p|_{u=0} = p_0(v)$ and $p_U|_{u=0} = -p_0'(v) + q_0(v)\frac{\partial\mu}{\partial U}/(\lambda - \mu)$ are not simultaneously zero. ∎

We can assume now that the points where $p = p_U = 0$ (which are the points where $\Pi \circ s$ does not have a fold) are isolated. Suppose that $p = p_u = 0$ at $(u_0, v_0)$. To establish the Reduction Principle with $\alpha = p, \beta = p_U$, it suffices to find a perturbation of $F_0, p_0, q_0$ on $u = u_0$ such that $p = p_u = 0$ but $p_{UU} \neq 0$ at $(u_0, v_0)$.

From (7.5) calculate

$$\begin{aligned} p_{VU} &= \tfrac{\partial}{\partial U}(\tfrac{\partial\mu}{\partial U}/(\lambda - \mu))q - \tfrac{\partial}{\partial U}(\tfrac{\partial\lambda}{\partial V}/(\lambda - \mu))p \\ &\quad +(\tfrac{\partial\mu}{\partial U}/(\lambda - \mu))q_U - (\tfrac{\partial\lambda}{\partial V}/(\lambda - \mu))p_U, \\ p_{VV} &= \tfrac{\partial}{\partial V}(\tfrac{\partial\mu}{\partial U}/(\lambda - \mu))q - \tfrac{\partial}{\partial V}(\tfrac{\partial\lambda}{\partial V}/(\lambda - \mu))p \\ &\quad +(\tfrac{\partial\mu}{\partial U}/(\lambda - \mu))q_V - (\tfrac{\partial\lambda}{\partial V}/(\lambda - \mu))p_V. \end{aligned}$$

Note that $q_U = p_V = (\frac{\partial\mu}{\partial U}q - \frac{\partial\lambda}{\partial V}p)/(\lambda - \mu)$, and that $q_0'(v) = -q_U + q_V$ on $u = u_0$. Therefore if $p = p_U = 0$ at $(u_0, v_0)$, then

$$0 = p_U = p_0'(v_0) - p_V(u_0, v_0) = p_0'(v_0) - (\frac{\partial\mu}{\partial U}/(\lambda - \mu))q_0(v_0),$$

and

$$\begin{aligned} p_{UU}(u_0, v_0) &= p_0''(v_0) - 2p_{UV}(u_0, v_0) - p_{VV}(u_0, v_0) \\ &= p_0''(v_0) - 2(\tfrac{\partial}{\partial U}(\tfrac{\partial\mu}{\partial U}/(\lambda - \mu))q_0(v_0) + (\tfrac{\partial\mu}{\partial U}/(\lambda - \mu))^2 q_0(v_0)) \\ &\quad -\tfrac{\partial}{\partial V}(\tfrac{\partial\mu}{\partial U}/(\lambda - \mu))q_0 - (\tfrac{\partial\mu}{\partial U}/(\lambda - \mu))q_0'(v_0)) \\ &\quad -(\tfrac{\partial\mu}{\partial U} - \tfrac{\partial\lambda}{\partial v})p_0'(v_0)/(\lambda - \mu). \end{aligned}$$

Therefore by retaining the values of $p_0(v_0)$, $q_0(v_0)$, $p_0'(v_0)$, $q_0'(v_0)$, and changing the value of $p_0''(v_0)$) we can perturb $p_0$ and $q_0$ in such a way that $p(u_0, v_0) = p_U(u_0, v_0) = 0$, but $p_{UU}(u_0, v_0) \neq 0$ . The point $(u_0, v_0)$ remains a solution of $p = p_U = 0$, but has now multiplicity 1. This establishes the Reduction Principle and finishes the proof of the Theorem 7.1.

# 8 Geometrical Constructions.

The theory of first-order nonlinear systems can be geometrically reformulated in terms of jet spaces. Though it is not directly used in our analysis, the geometrical picture gives another point of view of the unfolding construction, and we hope it will help to extend our results to more complicated systems. The geometrical theory of differential equations is highly developed, and only a few facts are cited here. A more detailed review and references can be found in [29]

For simplicity consider the diagonalized system

$$F_t + \Lambda F_z = 0, \tag{8.1}$$

where $F$ maps $\mathbb{C}^2$ with coordinates $z, t$ to $\mathbb{C}^n$ with coordinates $f_1, \ldots, f_n$ and $\Lambda$ is a diagonal matrix

$$\Lambda = diag\{\lambda_1, \ldots, \lambda_n\},$$

with $\lambda_1 = \ldots = \lambda_k = \lambda(F)$, $\lambda_{k+1} = \ldots = \lambda_n = \mu(F)$. The functions $\lambda$, $\mu$, $l_{ij}$ are supposed to be analytic, and $\lambda(F) \neq \mu(F)$.

## 8.1 Solutions as Submanifolds in the Jet Space.

Consider the space $J^1(\mathcal{V},\mathbb{C}^n)$ of 1-jets of maps $\mathcal{V}$ to $\mathbb{C}^n$ with coordinates $z$, $t$, $F = (f_1, \ldots, f_n)$, and denote $\xi_i = \partial_z f_i, \eta_i = \partial_t f_i$. Any mapping $\mathcal{V} \to \mathbb{C}^n$ naturally defines a submanifold in the jet space. A 2-dimensional submanifold $S$ of $J^1(\mathcal{V},\mathbb{C}^n)$ corresponds to some function, if its projection to $\mathbb{C}^2_{zt}$ is nondegenerate (i.e., it is a local diffeomorphism), and for all $s$ $df_s = \xi_s dz + \eta_s dt$ on $S$. The last condition just says that $\xi_s$ and $\eta_s$ are partial derivatives of the function $f_s = f_s(z,t)$.

**Definition**. Given a manifold M and $r$ 1-forms $\alpha_1, \ldots, \alpha_r$, a submanifold $S \subset M$ is called an *integrable manifold* of a distribution of planes $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$, if for all $s = 1, \ldots, r$ the forms $\alpha_s|_S$ vanish.

We can define now a *multi-valued function* as an integrable manifold in $J^1(\mathcal{V},\mathbb{C}^n)$ of a distribution

$$\alpha_s = df_s - \xi_s dz - \eta_s dt = 0, \qquad s = 1, \ldots, n. \tag{8.2}$$

The equations of (8.1) can be considered as equations on $J^1(\mathcal{V},\mathbb{C}^n)$. They define a submanifold $\Gamma \subset J^1(\mathcal{V},\mathbb{C}^n)$ on which $\eta_s = -\lambda_s \xi_s$. Hence, we can say that a *multi-valued solution* of the system is a submanifold of $\Gamma$ which is an integrable manifold of the distribution (8.2).

## 8.2 Integrable planes, Characteristics and Cartan's Theorem.

If an integrable manifold $S$ of a distribution (8.2) is the graph of a function, $F = F(z,t)$, with $\xi_s = (f_s)_z$, $\eta_s = (f_s)_t$, then $\frac{\partial \xi_s}{\partial t} = \frac{\partial \eta_s}{\partial z}$, i.e.

$$d\alpha_1|_{T_xS} = \ldots = d\alpha_n|_{T_xS} = 0.$$

Consider a manifold $M$ and a distribution of planes $K_x = \{\alpha_1 = \ldots = \alpha_r = 0\}$. If $S$ is an integrable submanifold of the distribution of planes $\alpha_1 = \ldots = \alpha_r = 0$, then for any point $x \in S$

$$d\alpha_1|_{T_xS} = \ldots = d\alpha_r|_{T_xS} = 0.$$

**Definition** ([29]) A plane $\pi \subset K_x \subset T_xM$ is called *integrable* if

$$d\alpha_1|_\pi = \ldots = d\alpha_r|_\pi = 0.$$

**Definition** ([29]) A non-zero vector $l \in K_x$ is called a *non-characteristic vector* if 1-forms

$$d\alpha_1|_{K_X}(l,\cdot), \ldots, d\alpha_r|_{K_x}(l,\cdot)$$

are linearly independent. Otherwise $l$ is called a *characteristic* vector.

The generalization of the Cauchy-Kovalevsky theorem for the systems $\alpha_1 = \ldots = \alpha_r = 0$ is due to Cartan [10]; see also [29]. We will not formulate it here in full generality, and give only the variant we need.

**Definition.** The distribution of planes has *genre 2* if at any point and for any noncharacteristic direction, the maximal dimension of an integrable plane, containing this direction is 2.

**Remark.** If the distribution of planes has genre 2, then the integrable plane containing the given noncharacteristic direction is determined uniquely.

**Theorem 8.1** *Let an analytic distribution of planes $\{K_x\}$ have genre 2. Then for any noncharacteristic curve $C$, tangent to the distribution $\{K_x\}$ and any point $x \in C$ there exists (locally) a unique 2-dimensional analytic integrable manifold containing $C$.*

We conclude this subsection with the description of the characteristic directions and integrable planes in $\Gamma$ for the distribution (8.2). On $\Gamma$, $\eta_i = -\lambda\xi_i$, $i = 1,\ldots,k$, and $\eta_j = -\mu\xi_i$, $j = k+1,\ldots,n$. The functions $z, t, F$ and $\xi_s$, $s = 1,\ldots,n$ can be taken as coordinates on $\Gamma$, and instead of the distribution (8.2) we can work with the distribution

$$\begin{aligned} \hat{\alpha}_i &= df_i - \xi_i(dz - \lambda(F)dt) = 0, \quad i = 1,\ldots,k, \\ \hat{\alpha}_j &= df_j - \xi_j(dz - \mu(F)dt) = 0, \quad j = k+1,\ldots,n. \end{aligned} \tag{8.3}$$

We will use the notation $\hat{\alpha} = 0$ for the distribution (8.3). A direct computation gives

$$\begin{aligned} d\hat{\alpha}_i|_{\hat{\alpha}=0} &= (-d\xi_i \wedge (dz - \lambda(F)dt) + \xi_i \textstyle\sum_{s=1}^{n} \frac{\partial\lambda(F)}{\partial f_s} df_s \wedge dt)|_{\hat{\alpha}=0} \\ &= -d\xi_i \wedge (dz - \lambda dt) + \xi_i (\textstyle\sum_{s=1}^{n} \frac{\partial\lambda}{\partial f_s}\xi_s) dz \wedge dt \\ &= (-d\xi_i + \frac{\xi_i}{\mu-\lambda}(\textstyle\sum_{s=1}^{n} \frac{\partial\lambda}{\partial f_s}\xi_s)(dz - \mu dt)) \wedge (dz - \lambda dt). \end{aligned} \tag{8.4}$$

In a similar way,

$$d\hat{\alpha}_j|_{\hat{\alpha}=0} = (-d\xi_j + \frac{\xi_j}{\lambda-\mu}(\sum_{s=1}^{n} \frac{\partial\mu}{\partial f_s}\xi_s)(dz - \lambda dt)) \wedge (dz - \mu dt). \tag{8.5}$$

From equations (8.4) and (8.5) we see that

$$\begin{aligned} d\hat{\alpha}_i|_{\hat{\alpha}=0} &= \gamma_i \wedge \beta_\lambda, \quad i = 1, \ldots, k, \\ d\hat{\alpha}_j|_{\hat{\alpha}=0} &= \gamma_j \wedge \beta_\mu, \quad j = k+1, \ldots, n \end{aligned} \tag{8.6}$$

where $\beta_\lambda = dz - \lambda dt$, $\beta_\mu = dz - \mu dt$, and the 1-forms $\gamma_1, \ldots, \gamma_n$, $\beta_\lambda, \beta_\mu$ are linearly independent.

If $k$ and $n-k$ are not less than 2, then any vector $l$ such that $\beta_\lambda(l) = 0$ or $\beta_\mu(l) = 0$ is a characteristic vector. If $l$ is non-characteristic, then the system

$$\begin{aligned} d\hat{\alpha}_i|_{\hat{\alpha}=0}(l, \cdot) &= \gamma_i(l)\beta_\lambda - \beta_\lambda(l)\gamma_i = 0, \quad i = 1, \ldots, k \\ d\hat{\alpha}_j|_{\hat{\alpha}=0}(l, \cdot) &= \gamma_j(l)\beta_\mu - \beta_\mu(l)\gamma_j = 0, \quad j = k+1, \ldots, n \end{aligned} \tag{8.7}$$

defines a 2-dimensional plane. Therefore, the distribution $\hat{\alpha} = 0$ is a distribution of genre 2.

**Remark.** If $k = 1, n-k \geq 2$ then for $l$ to be characteristic one needs either $\beta_\lambda(l) = \gamma_1(l) = 0$, or $\beta_\mu(l) = 0$. Similarly, if $k \geq 2, n = k+1$, either $\beta_\lambda(l) = 0$, or $\beta_\mu(l) = \gamma_{k+1}(l) = 0$. At last, in the case $k = n-k = 1$ the characteristic vector satisfies either $\beta_\lambda(l) = \gamma_1(l) = 0$, or $\beta_\mu(l) = \gamma_2(l) = 0$.

Though all the vectors $l$, such that $(dz - \lambda dt)(l)$ or $(dz - \mu dt)(l)$ vanish, are characteristic, the characteristic vectors which lie in the integrable planes satisfy additional equations.

**Lemma 8.1** *An integrable plane of the distribution $\hat{\alpha} = 0$ in $\Gamma$, such that $\beta_\lambda|_\pi$ and $\beta_\mu|_\pi$ are linearly independent, is spanned by two characteristic vectors $l_\lambda$ and $l_\mu$, satisfying the equations*

$$\begin{aligned} \beta_\lambda(l_\lambda) &= \gamma_i(l_\lambda) = 0, \quad 1 \leq i \leq k, \\ \beta_\mu(l_\mu) &= \gamma_j(l_\mu) = 0, \quad k+1 \leq j \leq n. \end{aligned}$$

**Proof.** Take an integrable plane $\pi$. If $\beta_\lambda|_\pi$ and $\beta_\mu|_\pi$ are independent, then $\pi$ is spanned by two characteristic vectors $l_\lambda$ and $l_\mu$, such that $\beta_\lambda(l_\lambda) = \beta_\mu(l_\mu) = 0$, but $\beta_\lambda(l_\mu)$ and $\beta_\mu(l_\lambda)$ do not vanish. The lemma follows now from the formulas (8.7). ∎

The independence of $\beta_\lambda|_\pi$ and $\beta_\mu|_\pi$ means that $\pi$ has nondegenerate projection on $\mathbb{C}^2_{zt}$. Using (8.4), (8.5) and Lemma 8.1 we can write the general form of an integrable plane $\pi$ in $\Gamma$ which has non-degenerate projection on $\mathbb{C}^2_{zt}$ as $\pi =< l_\lambda, l_\mu >$, where

$$\begin{aligned} l_\lambda &= \tfrac{\partial}{\partial t} + \lambda\tfrac{\partial}{\partial z} + \textstyle\sum_{j=k+1}^n \xi_j(\lambda-\mu)\tfrac{\partial}{\partial f_j} + \\ &\textstyle\sum_{i=1}^k \xi_i(\sum_{s=1}^n \tfrac{\partial\lambda}{\partial f_s}\xi_s)\tfrac{\partial}{\partial \xi_i} + \sum_{j=k+1}^n b_j \tfrac{\partial}{\partial \xi_j}, \\ l_\mu &= \tfrac{\partial}{\partial t} + \mu\tfrac{\partial}{\partial z} + \textstyle\sum_{i=1}^k \xi_i(\mu-\lambda)\tfrac{\partial}{\partial f_i} + \\ &\textstyle\sum_{j=k+1}^n \xi_j(\sum_{s=1}^n \tfrac{\partial\mu}{\partial f_s}\xi_s)\tfrac{\partial}{\partial \xi_j} + \sum_{i=1}^k a_i \tfrac{\partial}{\partial \xi_i}. \end{aligned}$$

Here $a_i$, $1 \le i \le k$ and $b_j$, $k+1 \le j \le n$ are arbitrary numbers.

**Remark** If the 2-dimensional plane $\pi \subset \{\hat\alpha = 0\}$ has a non-degenerate projection on $\mathbb{C}^{n+2}_{F,z,t}$, then the restrictions $(dz - \lambda dt)|\pi$ and $(dz - \mu dt)|_\pi$ are linearly independent.

## 8.3 Infinite Derivatives.

We present now a construction that will allow us to take into consideration solutions that have infinite derivatives. The manifold $\Gamma$ has the structure of a product of $J^0(\mathbb{C}^2, \mathbb{C}^n)$ with coordinates $z, t, f_1, \ldots, f_n$ and $\mathbb{C}^n_\xi = \mathbb{C}^k_{\xi_1,\ldots,\xi_k} \times \mathbb{C}^{n-k}_{\xi_{k+1},\ldots,\xi_n}$. Consider now a product $\tilde\Gamma = J^0(\mathbb{C}^2, \mathbb{C}^n) \times \mathbb{P}^k \times \mathbb{P}^{n-k}$ with homogeneous coordinates $(q_0 : q_1 : \ldots : q_k) \times (p_{k+1} : \ldots : p_n : p_{n+1})$ in $\mathbb{P}^k \times \mathbb{P}^{n-k}$. Let $0 \le \kappa \le k$, $k+1 \le \nu \le n+1$. Denote $\mathcal{A}_{\kappa,\nu}$ the affine chart $q_\kappa \ne 0, p_\nu \ne 0$ in $\tilde\Gamma$. The manifold $\Gamma$ is just the chart $\mathcal{A}_{0,n+1}$:

$$\xi_i = \frac{q_i}{q_0}, \qquad \xi_j = \frac{p_j}{p_{n+1}}, \qquad 1 \le i \le k, \quad k+1 \le j \le n.$$

In the remainder of this section, the indices $i$ and $i_0$ will always take values between 1 and $k$, and the indices $j$ and $j_0$ between $k+1$ and $n$.

**Lemma 8.2** . *A distribution of planes (8.3) on $\Gamma$ can be extended to a distribution of $(n+2)$-dimensional planes $\{K_x\} \subset T_x\tilde\Gamma$ on $\tilde\Gamma$ of genre 2.*

**Proof.** On $\Gamma$ we have $\eta_i = -\lambda\xi_i$ and $\eta_j = -\mu\xi_j$. Therefore on $\Gamma = \mathcal{A}_{0,n+1}$, the zeroes of the $n$ 1-forms

$$\alpha'_i = q_0 df_i - q_i(dz - \lambda dt) \quad i = 1, \ldots, k$$

and

$$\alpha'_j = p_{n+1} - p_j(dz - \mu dt) \quad j = k+1, \ldots, n$$

define the same distribution of planes as (8.3). Consider now a chart $\mathcal{A}_{i_0,n+1}$ with affine coordinates $\tilde{q} = q_0/q_{i_0}, \tilde{\xi}_i = q_i/q_{i_0}$, where $i \neq i_0$, and $\xi_j = p_j/p_{n+1}$. Define the forms

$$\begin{aligned} \tilde{\alpha}_{i_0} &= \alpha'_{i_0}/q_{i_0} = \tilde{q}df_{i_0} - (dz - \lambda dt), \\ \tilde{\alpha}_i &= (q_{i_0}\alpha'_i - q_i\alpha'_{i_0})/(\tilde{q}q_{i_0}^2) = (df_i - \tilde{\xi}_i df_{i_0}), \quad 1 \leq i \leq k,\ i \neq i_0. \end{aligned} \tag{8.8}$$

The $n$ linearly independent forms $\tilde{\alpha}_{i_0}, \tilde{\alpha}_i$, $i \neq i_0$ and $\hat{\alpha}_j$ define a distribution of $(n+2)$-dimensional planes in $\mathcal{A}_{i_0,n+1}$ which coincides with (8.3) on $\mathcal{A}_{0,n+1} \cap \mathcal{A}_{i_0,n+1}$.

In a similar way, in a chart $\mathcal{A}_{0,j_0}$ with coordinates $\xi_i$, $\tilde{p} = p_{n+1}/p_{j_0}$ and $\tilde{\xi}_j = p_j/p_{j_0}$ for $j \neq j_0$ the distribution is defined as zeros of forms $\hat{\alpha}_i$ and

$$\begin{aligned} \tilde{\alpha}_{j_0} &= \alpha'_{j_0}/p_{j_0} = \tilde{p}df_{j_0} - (dz - \mu dt), \\ \tilde{\alpha}_j &= (p_{j_0}\alpha'_j - p_j\alpha'_{j_0})/(\tilde{p}p_{j_0}^2) = df_j - \tilde{\xi}_j df_{i_0},\ k+1 \leq j \leq n,\ j \neq j_0. \end{aligned} \tag{8.9}$$

Lastly, in a chart $\mathcal{A}_{i_0,j_0}$ we take zeros of the forms $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n$, defined by (8.8) and (8.9).

We can calculate now the differentials in each of the charts. For convenience we will put $\tilde{\xi}_{i_0} = \tilde{\xi}_{j_0} = 1$. In $\mathcal{A}_{i_0,n+1}$ we have:

$$\begin{aligned} d\tilde{\alpha}_{i_0}|_{K_x} &= (d\tilde{q} - (\textstyle\sum_{i=1}^k \frac{\partial\lambda}{\partial f_i}\tilde{\xi}_i + \sum_{j=k+1}^n \frac{\partial\lambda}{\partial f_j}\tilde{q}\xi_j)dt) \wedge df_{i_0}, \\ d\tilde{\alpha}_i|_{K_x} &= d\tilde{\xi}_i \wedge df_{i_0}, \qquad i \neq i_0, \\ d\hat{\alpha}_j|_{K_x} &= (d\xi_j + \tfrac{\xi_j}{\lambda-\mu}(\textstyle\sum_{i=1}^k \frac{\partial\mu}{\partial f_i}\tilde{\xi}_i + \frac{\partial\mu}{\partial f_j}\tilde{q}\xi_j)dt) \wedge (\tilde{q}df_{i_0} + (\lambda-\mu)dt), \end{aligned}$$

where $1 \leq i \leq k$, $k+1 \leq j \leq n$.

We see that these differentials have the same structure as $d\hat{\alpha}_s|_{\hat{\alpha}=0}$. By the same reasoning as in the previous subsection, we conclude, that the distribution has genre 2 in the chart $\mathcal{A}_{i_0,n+1}$, and that any integrable plane $\pi$ with linearly independent $df_{i_0}|_\pi$ and $dt|_\pi$ is spanned by

$$\begin{aligned} l_\lambda &= \tfrac{\partial}{\partial t} + \lambda\tfrac{\partial}{\partial z} + \textstyle\sum_{j=k+1}^n \xi_j(\lambda-\mu)\frac{\partial}{\partial f_j} + \\ &\quad (\textstyle\sum_{i=1}^k \frac{\partial\lambda}{\partial f_i}\tilde{\xi}_i + \sum_{j=k+1}^n \frac{\partial\lambda}{\partial f_j}\tilde{q}\xi_j)\frac{\partial}{\partial\tilde{q}} + \sum_{j=k+1}^n b_j\frac{\partial}{\partial\xi_j}, \\ l_\mu &= \tfrac{\tilde{q}}{\mu-\lambda}\tfrac{\partial}{\partial t} + \mu\tfrac{\tilde{q}}{\mu-\lambda}\tfrac{\partial}{\partial z} + \textstyle\sum_{i=1}^k \tilde{\xi}_i\frac{\partial}{\partial f_i} + \\ &\quad \textstyle\sum_{j=k+1}^n \frac{\xi_j}{\mu-\lambda}(\sum_{i=1}^k \frac{\partial\mu}{\partial f_i}\tilde{\xi}_i + \sum_{j=k+1}^n \frac{\partial\mu}{\partial f_j}\tilde{q}\xi_j)\frac{\partial}{\partial\xi_j} + \sum_{i\neq i_0} a_i\frac{\partial}{\partial\tilde{\xi}_i} + a_{i_0}\frac{\partial}{\partial\tilde{q}}. \end{aligned}$$

Here again $a_i$, $1 \leq i \leq k$, and $b_j$, $k+1 \leq j \leq n$, are arbitrary numbers. As before, if a 2-dimensional plane $\pi \subset K_x$ is projected non-degenerately on $\mathbb{C}^{n+2}_{Fzt}$, then $df_{i_0}|\pi$ and $dt|\pi$ are independent.

The case $\mathcal{A}_{0,j_0}$ is symmetric to $\mathcal{A}_{i_0,n+1}$. Integrable planes that project non-degenerately on $\mathbb{C}^{n+2}_{Fzt}$ are spanned by pairs of vectors

$$\begin{aligned} l_\lambda &= \tfrac{\tilde{p}}{\lambda-\mu}\tfrac{\partial}{\partial t} + \lambda\tfrac{\tilde{p}}{\lambda-\mu}\tfrac{\partial}{\partial z} + \textstyle\sum_{i=1}^k \tilde{\xi}_j\frac{\partial}{\partial f_j} + \\ &\quad \textstyle\sum_{i=1}^k \frac{\xi_i}{\lambda-\mu}(\sum_{i=1}^k \frac{\partial\lambda}{\partial f_i}\tilde{p}\xi_i + \sum_{j=k+1}^n \frac{\partial\lambda}{\partial\tilde{\xi}_j})\frac{\partial}{\partial\xi_i} + \sum_{j\neq j_0} b_j\frac{\partial}{\partial\tilde{\xi}_j} + b_{j_0}\frac{\partial}{\partial\tilde{p}}, \\ l_\mu &= \tfrac{\partial}{\partial t} + \mu\tfrac{\partial}{\partial z} + \textstyle\sum_{i=1}^k \xi_i(\lambda-\mu)\frac{\partial}{\partial f_i} + \\ &\quad (\textstyle\sum_{i=1}^k \frac{\partial\mu}{\partial f_i}\tilde{p}\xi_i + \sum_{j=k+1}^n \frac{\partial\mu}{\partial f_j}\tilde{\xi}_j)\frac{\partial}{\partial\tilde{p}} + \sum_{i=1}^k a_i\frac{\partial}{\partial\xi_i}. \end{aligned}$$

To finish the proof of lemma 8.2 we have to look at the chart $\mathcal{A}_{i_0,j_0}$.

$$\begin{aligned} d\tilde{\alpha}_{i_0} &= (d\tilde{q} - \tfrac{1}{\lambda-\mu}(\textstyle\sum_{i=1}^k \frac{\partial\lambda}{\partial f_i}\tilde{\xi}_i\tilde{p} + \frac{\partial\lambda}{\partial f_j}\tilde{\xi}_j\tilde{q})df_{j_0}) \wedge df_{i_0}, \\ d\tilde{\alpha}_i &= d\tilde{\xi}_i \wedge df_{i_0}, \quad ,1 \leq i \leq k, i \neq i_0, \\ d\tilde{\alpha}_{j_0} &= (d\tilde{p} - \tfrac{1}{\mu-\lambda}(\textstyle\sum_{i=1}^k \frac{\partial\mu}{\partial f_i}\tilde{\xi}_i\tilde{p} + \frac{\partial\mu}{\partial f_j}\tilde{\xi}_j\tilde{q})df_{i_0}) \wedge df_{j_0}, \\ d\tilde{\alpha}_j &= d\tilde{\xi}_j \wedge df_{j_0}, \quad ,k+1 \leq j \leq n, j \neq j_0. \end{aligned} \tag{8.10}$$

From (8.10) we derive that the distribution $\tilde{\alpha}_1 = \ldots = \tilde{\alpha}_n = 0$ has genre 2, and also find the characteristic vectors, spanning integrable planes that project non-degenerately on $\mathbb{C}^{n+2}_{Fzt}$:

$$\begin{aligned} l_\lambda &= \tfrac{\tilde{p}}{\lambda-\mu}\tfrac{\partial}{\partial t} + \lambda\tfrac{\tilde{p}}{\lambda-\mu}\tfrac{\partial}{\partial z} + \textstyle\sum_{j=k+1}^n \tilde{\xi}_j \frac{\partial}{\partial f_j} + \\ &\tfrac{1}{\lambda-\mu}(\textstyle\sum_{i=1}^k \frac{\partial\lambda}{\partial f_i}\tilde{\xi}_i\tilde{p} + \sum_{j=k+1}^n \frac{\partial\lambda}{\partial f_j}\tilde{\xi}_j\tilde{q})\frac{\partial}{\partial\tilde{q}} + \sum_{j\neq j_0} b_j \frac{\partial}{\partial\tilde{\xi}_j} + b_{j_0}\frac{\partial}{\partial\tilde{p}}, \\ l_\mu &= \tfrac{\tilde{q}}{\mu-\lambda}\tfrac{\partial}{\partial t} + \mu\tfrac{\tilde{p}}{\mu-\lambda}\tfrac{\partial}{\partial z} + \textstyle\sum_{i=1}^k \tilde{\xi}_i \frac{\partial}{\partial f_i} + \\ &\tfrac{1}{\mu-\lambda}(\textstyle\sum_{i=1}^k \frac{\partial\mu}{\partial f_i}\tilde{\xi}_i\tilde{p} + \sum_{j=k+1}^n \frac{\partial\mu}{\partial f_j}\tilde{\xi}_j\tilde{q})\frac{\partial}{\partial\tilde{p}} + \sum_{i\neq i_0} a_i \frac{\partial}{\partial\tilde{\xi}_i} + a_{i_0}\frac{\partial}{\partial\tilde{q}}. \end{aligned} \tag{8.11}$$

Let $\tilde{\Pi}$ denote the projection $\tilde{\Gamma}$ onto $J^0(\mathbb{C}^2,\mathbb{C}^n)$. As in previous sections we will denote by $\Pi$ the standard projection $J^0(\mathbb{C}^n,\mathbb{C}^2) = \mathbb{C}^{n+2}_{Fzt} \rightarrow C^2_{zt}$. We can define now *a multi-valued solution of the system (1.1)* as an integrable submanifold $S$ of a distribution of planes $\{K_x\}$ in $\tilde{\Gamma}$, such that the projection $\tilde{\Pi}_*|_S$ is non-degenerate.

**Remark** The last condition guarantees the projection $\tilde{pr}(S)$ to be a smooth submanifold of $J^0(\mathbb{C}^2,\mathbb{C}^n)$.

The problem of describing singularities of the solutions of the system (8.1) can be reformulated as a problem of describing the singularities of a projection of an integrable manifolds of the distribution $\{K_x\}$ in $\tilde{\Gamma}$ onto $\mathbb{C}^2_{zt}$. The difference between the geometrical construction and the unfolding transformation, the main theme of our paper, is that in this section we realize the unfolding as the projection of an integrable manifold $S$ in $\tilde{\Gamma}$ onto $\mathbb{C}^2_{zt}$. This choice of the unfolding transformation amounts to parameterization of the surface $S$ by characteristic coordinates $U, V$, viewed as a proper choice of the local coordinates on $S$.

In $\tilde{\Gamma}$ we have a subvariety at "infinity": $I = \tilde{\Gamma} - \Gamma = \tilde{\Gamma} - \mathcal{A}_{0,n+1}$. The projection $\Pi \circ \tilde{\Pi}$ is singular at the points of intersection of a multi-valued solution $S$ and the subvariety $I$. $I$ is a union of two sets $I_\lambda = \{p_{n+1} = 0\}$ and $I_\mu = \{q_0 = 0\}$ (recall that $(q_0 : \ldots : q_k)$ and $(p_{k+1} : \ldots : p_{n+1})$ are homogeneous coordinates in $\mathbb{P}^k \times \mathbb{P}^{n-k}$.

**Lemma 8.3** *At a point $x$ where $S$ meets $I$ transversely, the projection $S \rightarrow \mathbb{C}^2_{zt}$ has*

*(i) a fold, if either $x \in I_\lambda - I_\mu$, and the characteristic direction $l_\lambda$ is transversal to $I_\lambda$, or $x \in I_\mu - I_\lambda$, and $l_\mu$ is transversal to $I_\mu$; or*

*(ii) a cusp, if either $x \in I_\lambda - I_\mu$ and the field $l_\lambda$ is tangent to $I_\lambda$ with first*

*order of tangency, or $x \in I_\mu - I_\lambda$ and the direction field $l_\mu$ has first order tangency with $I_\mu$; or*

*(iii) an umbilic singularity, if $x \in I_\lambda \cap I_\mu$.*

Lemma 8.3 is a reformulation of lemmas 6.1, 6.3, and 6.4 in geometrical terms. We show here how this reformulation goes in the umbilic case.

Consider a chart $\mathcal{A}_{i_0,j_0}$. At the point $x$, the tangent plane $T_x S$ is spanned by characteristic vectors (8.11). We can take $U = f_{j_0}$ and $V = f_{i_0}$ as the coordinates on $S$. We have now

$$\begin{aligned} t_U &= \tilde{p}/(\lambda-\mu), & t_V &= \tilde{q}/(\mu-\lambda), \\ z_U &= \lambda\tilde{p}/(\lambda-\mu), & z_V &= \mu\tilde{q}/(\mu-\lambda), \\ (f_i)_U &= 0, & (f_i)_V &= \tilde{\xi}_i, \\ (f_j)_U &= \tilde{\xi}_j, & (f_j)_V &= 0. \end{aligned} \tag{8.12}$$

$$\begin{aligned} \tilde{p}_V &= \tfrac{1}{\mu-\lambda}\left(\textstyle\sum_{i=1}^{k} \frac{\partial\mu}{\partial f_i}\tilde{\xi}_i\tilde{p} + \sum_{j=k+1}^{n} \frac{\partial\mu}{\partial f_j}\tilde{\xi}_j\tilde{q}\right), \\ \tilde{q}_U &= \tfrac{1}{\lambda-\mu}\left(\textstyle\sum_{i=1}^{k} \frac{\partial\lambda}{\partial f_i}\tilde{\xi}_i\tilde{p} + \sum_{j=k+1}^{n} \frac{\partial\lambda}{\partial f_j}\tilde{\xi}_j\tilde{q}\right) \end{aligned} \tag{8.13}$$

The projection $S$ to $\mathbb{C}^2_{zt}$ has zero differential if $\tilde{p} = \tilde{q} = 0$. The manifold $S$ is transversal to $I_\lambda \cap I_\mu = \{\tilde{p} = \tilde{q} = 0\}$, if $a_{i_0} = \tilde{p}_U \neq 0$ and $b_{j_0} = \tilde{q}_V \neq 0$. Take now $p = \tilde{p}/(\lambda-\mu)$ and $q = \tilde{q}/(\mu-\lambda)$. The statement of Lemma 8.3 in the umbilic case is exactly the same as that of Lemma 6.1.

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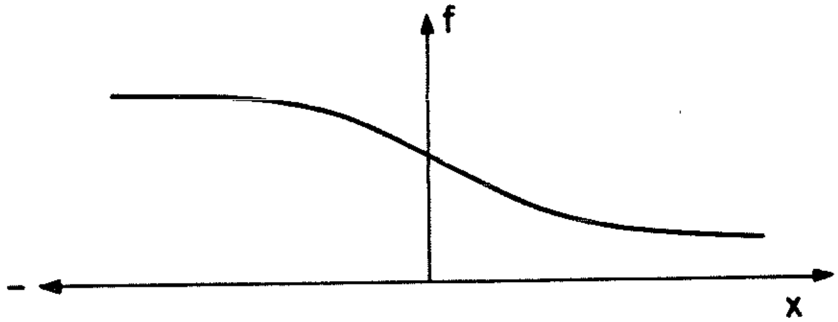
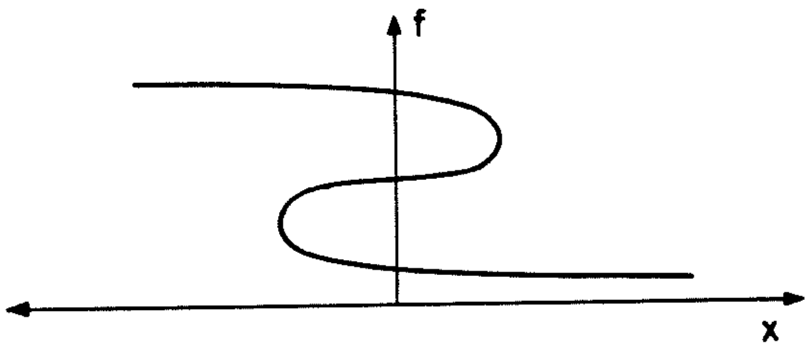
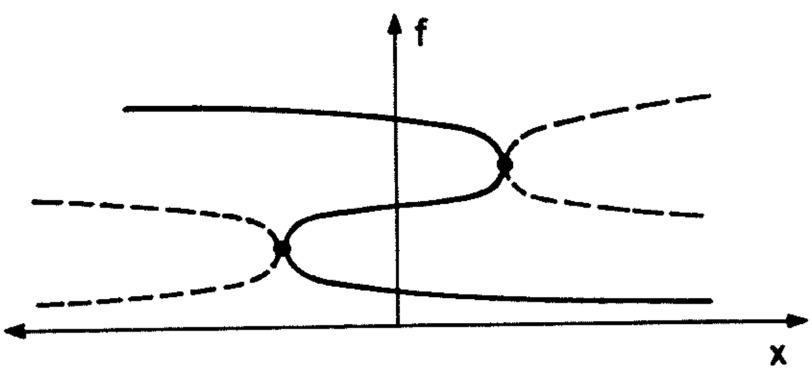


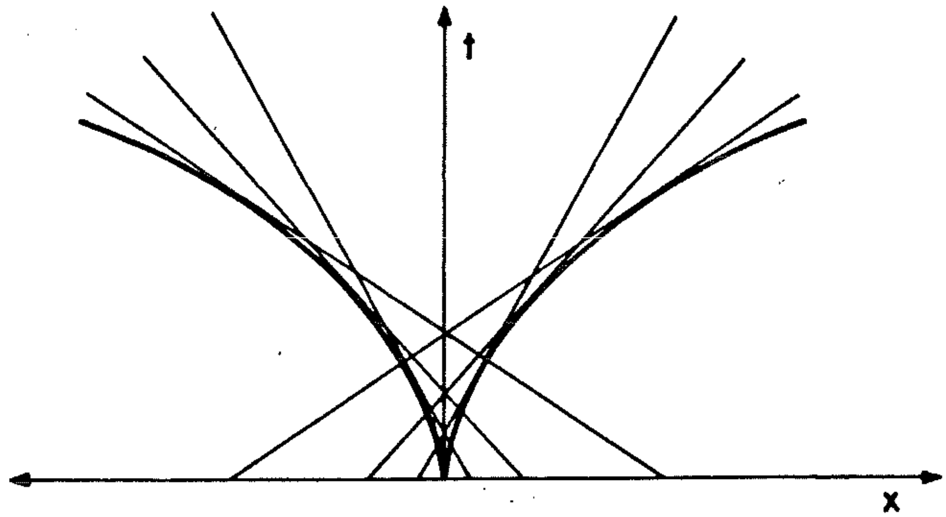

(1a) $t = 0$



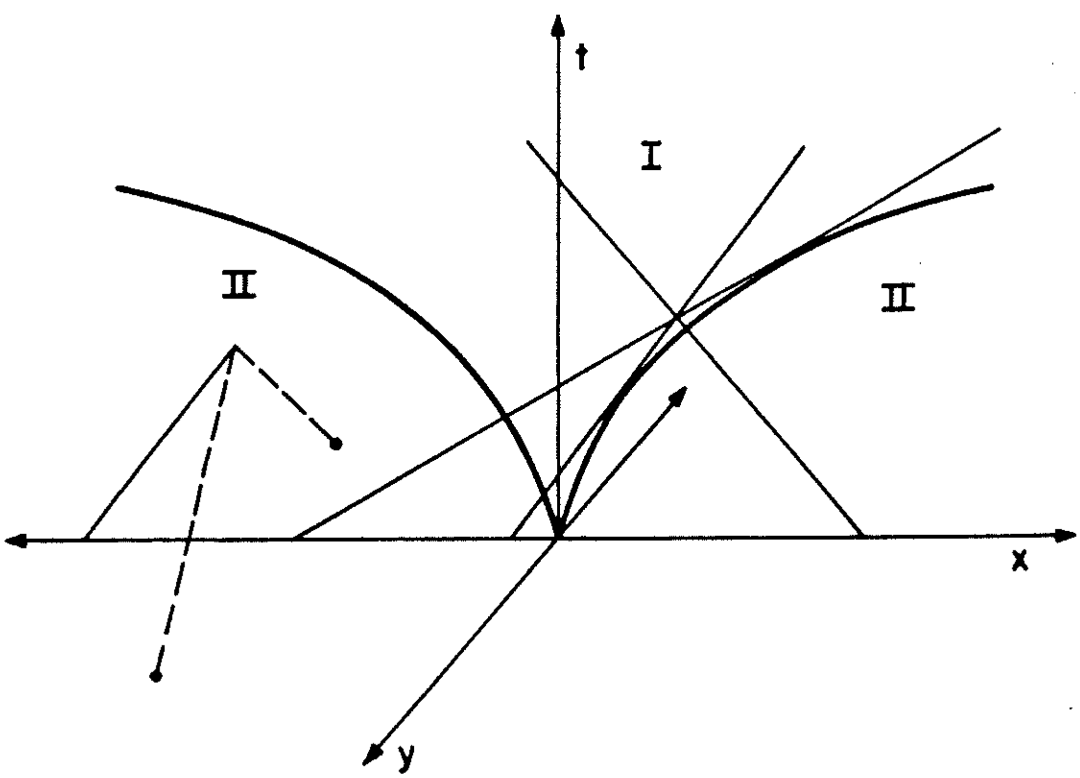

(1b) $t = t_1 > 0$




(1c) $t = t_1$ (analytic extension)




(2a)




(2b)

Figure 2. Characteristics in $x$ vs. $t$ for multi-valued solutions: solid lines are real characteristics, dashed lines are complex characteristics, dark curves are envelopes of characteristics. (2a) shows real characteristics and their envelopes. (2b) shows real and complex characteristics coming to points in two different regions.

Figure 3a. Folds $(a \neq 0, \ d \neq 0)$

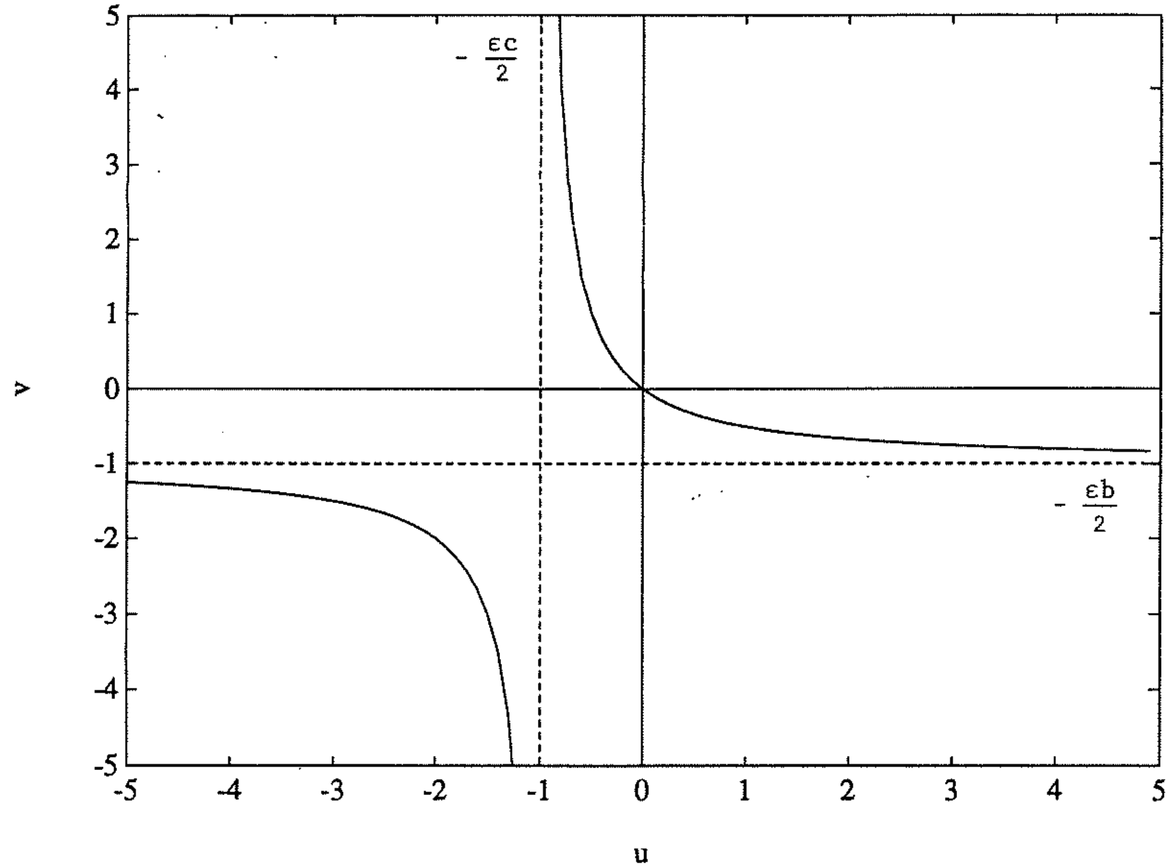



Figure 3b. Umbilics $(a = 0 \text{ or } d = 0)$