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## COMPUTATIONAL AND APPLIED MATHEMATICS


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# On the motion of a fluid that is incompressible in a generalized sense and its relationship to the Boussinesq Approximation

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**ABSTRACT**
This paper presents a fresh approach to the derivation of the Oberbeck–Boussinesq approximation, an approximation that is widely used in the theory of stratified fluids and thermal convection. Here it is exhibited as a particular case of a new continuum model, the generalized incompressible fluid.

**1.** It has long been appreciated that the theories of fluid dynamics describing incompressible materials are generally more likely to be tractable than their compressible counterparts. In the literature two attitudes to incompressibility are common. The first approach is perhaps the more mathematical and stipulates, by definition, that an incompressible material can only undertake isochoric, that is volume preserving, motions. The velocity field $\mathbf{v}$ is thereby constrained to be solenoidal. Then, from the equation of continuity, it follows that $\dot{\rho} = 0$ where $\rho$ is the density and a superposed dot denotes the motional derivative. The pressure, $p$, that appears in the equation of motion becomes, in effect, a Lagrangian multiplier arising from the constraint $\nabla \cdot \mathbf{v} = 0$. In the second approach the term 'incompressible' is more literally interpreted to mean that the density of the fluid cannot be changed by compression. Thus $(\partial\rho/\partial p)_T = (\partial\rho/\partial p)_S = 0$, where $T$ is the temperature and $S$ is the specific entropy. There remains though the possibility of changing $\rho$ by varying $T$ or $S$ and so violating the central isochoric axiom of the first approach. To avoid confusion we shall describe a material in the less restricted case as being 'incompressible in the generalized sense' and regard 'incompressible' by itself as indicating strictly that no change in volume occurs.

Although the two outlined approaches generally differ, the final results are sometimes identical. To understand why, we must look more closely at the physical reasoning that underpins the second view. There we are concerned with situations in which changes in field quantities take place so slowly compared with the time taken by sound to cross the system that the velocity of sound, $a = \sqrt{(\partial p/\partial\rho)_S}$ can be assumed infinite, i.e. $(\partial\rho/\partial p)_S = 0$, so that $\dot{\rho} = (\partial\rho/\partial S)_p \dot{S}$. Suppose now that the conduction of heat is so inefficient that the entropy of each fluid element changes negligibly. Then $\dot{S} = 0$, so that $\dot{\rho} = 0$ and hence from the continuity equation, $\nabla \cdot \mathbf{v} = 0$. The pressure appearing in the momentum balance equation then represents the deviation in pressure from the initial state rather than a thermodynamic pressure. In both approaches the energy equation is abandoned since it is decoupled from the remaining equations. If these remaining equations are solved, the temperature field can always be determined, if necessary, by returning to the discarded energy equation.

One of the objectives of the present note is to provide a continuum theory for the second approach, and in this way obtain the continuum theory governing a generalized incompressible fluid. We may reasonably expect that such a theory will be a natural environment in which to discuss the celebrated Oberbeck–Boussinesq (OB-) approximation which undoubtedly is the most widely used model in the theory of stratified fluids and thermal convection. Some regard it as merely a simplifying device that recognises the key physical ingredient (buoyancy) but which can have no strict physical justification. Others more ambitiously have sought to deduce the OB–approximation from the full thermo-mechanical equations and have attempted to define strict limits for its validity [see for example, Jeffreys (1930), Spiegel and Veronis (1960), Veronis (1962), Mihaljan (1962), Roberts (1967), Cordon andVelarde (1975), Velarde and Cordon (1976), Roberts and Stewartson (1977), Spiegel and Weiss (1982)]. This has sometimes led to intricate analyses, and occasionally to obscurities. In this paper we attempt a fresh approach which, we feel, has the merit of clarity. We display the OB–approximation as a particular case of a new continuum model, the generalized incompressible fluid. By taking this model as a starting point, we do not have to consider the limit of large sound speeds that has typified earlier derivations, and we avoid reliance not only on physical illumination, as provided for example by Spiegel and Weiss (1982) but also on a systematic mathematical expansion, as for example that of Roberts and Stewartson (1977). Our analysis constitutes a clarification and correction of the approach adopted in chapter 7 of Roberts (1967).

**2.** In suffix notation, and with the summation convention, the conservation laws for mass, linear momentum and energy are

$$\dot{\rho} + \rho v_{i,i} = 0, \tag{1}$$

$$\rho \dot{v}_i = \rho F_i + \sigma_{ji,j}, \tag{2}$$

$$\rho Q - \rho \dot{U} - q_{k,k} + \sigma_{ki} d_{ik} = 0. \tag{3}$$

Here $F_i$ is the applied body force per unit mass, $\sigma_{ij}$ the (symmetric) stress tensor, $Q$ the heat supply per unit mass per unit time, $U$ the specific internal energy, $q_k$ the heat flux vector and $d_{ik} = \frac{1}{2}(v_{i,k} + v_{k,i})$ the rate of deformation tensor. The entropy production inequality takes the form

$$\rho(T\dot{S} - \dot{U}) + \sigma_{ki} d_{ik} - q_k T_{,k}/T \geq 0, \tag{4}$$

where $S$ is the specific entropy and $T$ the temperature.

We are interested in materials whose density can be changed by variations in the temperature, $T$, but not in the pressure $P$. This suggests that we formulate the constitutive theory in terms of $P$ and $T$ and then the natural thermodynamic potential is the Gibbs energy

$$G = U - ST + P/\rho. \tag{5}$$

In terms of $G$ the inequality (4) becomes

$$-\rho\left[\dot{G} + S\dot{T}\right] + \dot{P} + \left[\sigma_{ki} + P\delta_{ki}\right] d_{ik} - q_k T_{,k}/T \geq 0. \tag{6}$$

The simplest constitutive fluid model that embraces viscosity and heat conduction, and which eliminates the pressure dependence of $\rho$, is

$$\begin{gathered} G = G(T,P), \qquad S = S(T,P), \qquad \rho = \rho(T), \\ \sigma_{ij} = -p\delta_{ij} + \lambda d_{uu}\delta_{ij} + 2\mu d_{ij}, \qquad q_k = -\kappa T_{,k}, \end{gathered} \tag{7}$$

where the mechanical pressure, $p$, and the material coefficients $\lambda$, $\mu$, $\kappa$ are general functions of $P$ and $T$. The inequality (6) becomes

$$\begin{aligned} -\rho\left(\frac{\partial G}{\partial T} + S\right)\dot{T} - \rho\left(\frac{\partial G}{\partial P} - \frac{1}{\rho}\right)\dot{P} - (p - P)d_{ii} \\ + \lambda d_{uu} d_{ss} + 2\mu d_{ij} d_{ij} + \kappa T_{,i} T_{,i}/T \geq 0, \end{aligned} \tag{8}$$

and this must hold for all motions and thermal states that also satisfy the reduced form of (1), *viz.*

$$\alpha\dot{T} = v_{i,i}, \tag{9}$$

where $\alpha$ $\left(= -\rho^{-1}d\rho/dT\right)$ is the thermal expansion coefficient of the material. To account for the constraint (9) we introduce the Lagrangian multiplier $\Gamma$ and then in the usual way we deduce

$$S = -\left(\frac{\partial G}{\partial T} + \frac{\Gamma\alpha}{\rho}\right), \quad \frac{\partial G}{\partial P} = \frac{1}{\rho}, \quad p = P + \Gamma, \quad \Gamma = \Gamma(T, P), \tag{10}$$

and

$$\lambda + \tfrac{2}{3}\mu \geq 0, \qquad \mu \geq 0, \qquad \kappa \geq 0. \tag{11}$$

If we introduce the Gibbs energy $\widehat{G} = G + \Gamma/\rho$ we find from (10) that we may eliminate $G$, $P$, $\Gamma$ in favour of $\widehat{G}$, $p$ and

$$\widehat{G} = \widehat{G}(T,p) = G_0(T) + \frac{p}{\rho}, \qquad \frac{\partial \widehat{G}}{\partial T} = -S, \qquad \frac{\partial \widehat{G}}{\partial p} = \frac{1}{\rho}. \tag{12}$$

In addition, the material parameters are now assumed functions of $p$ and $T$. Using these results, we find that the governing equations for a generalized incompressible linear viscous fluid are

$$\alpha\dot{T} = v_{i,i}, \tag{13}$$

$$\rho\dot{v}_i = \rho F_i - p_{,i} + \left[\lambda v_{k,k}\delta_{ij} + 2\mu d_{ij}\right]_{,j}, \tag{14}$$

$$\rho c_p \dot{T} - \alpha T\dot{p} = \rho Q + (\kappa T_{,i})_{,i} + \lambda d_{uu}d_{vv} + 2\mu d_{ij}d_{ij}, \tag{15}$$

where $c_p = T(\partial S/\partial T)_p$ is the specific heat at constant pressure.

At a boundary of the fluid the usual thermal conditions apply, that is continuity of the temperature and the normal component of the heat flux. At a stationary, non-slip surface we have $\mathbf{v} = 0$ while at a free surface, $\mathcal{S}$, at an ambient pressure $\pi$ the stress vector is continuous. Using $(7)_4$, we may express this condition by

$$-\pi = -p + \lambda\,\nabla\cdot\mathbf{v} + 2\mu(\mathbf{n}\cdot\nabla)\mathbf{v}, \qquad \mathbf{n}\times\boldsymbol{\omega} = 0, \qquad \text{on } \mathcal{S}, \tag{16}$$

where $\mathbf{n}$ is the unit normal to $\mathcal{S}$ and $\boldsymbol{\omega}$ the vorticity vector. Condition (16) must be supplemented by the kinematic condition that

$$\mathbf{n}\cdot\mathbf{v} = U, \qquad \text{on } \mathcal{S}, \tag{17}$$

where $U$ is the velocity of $\mathcal{S}$ along its normal $\mathbf{n}$.

**3.** To discuss the derivation of the OB-approximation we select a simple system: a horizontal layer of fluid contained between a fixed lower boundary $x_3 = L$ and a free top surface $x_3 = f(x_1, x_2, t)$: condition (17) may then be written as

$$v_3 = \dot{f} \qquad \text{on } x_3 = f. \tag{18}$$

We also have $\mathbf{F} = \mathbf{g}$ where $\mathbf{g}$, the uniform acceleration due to gravity, is parallel to the $x_3$–axis. We choose a convenient reference state ($T_r$, $p_r$) and will shortly expand the thermodynamic variables and material parameters about that state using the notation $\alpha_r = \alpha(T_r)$, $\kappa_r = \kappa(p_r, T_r)$, etc. The expansion for the density about $T_r$ has the form

$$\rho(T) = \rho_r\big[1 - \alpha_r(T - T_r) + \ldots\big]. \tag{19}$$

The central physical idea is that typical accelerations promoted in the fluid by variations in the density are always much less than the acceleration of gravity. As observed by Roberts (1967), the OB–equations result from taking the simultaneous limits, $g \to \infty$, $\alpha_r \to 0$ but with the restriction that $g\alpha_r$ remains finite. As we shall see this last requirement is essential because otherwise, buoyancy forces are lost. Since $g\alpha_r$ is dimensional, it is more precise to stipulate that the Rayleigh number

$$R = g\alpha_r\beta L^4/\nu_r k_r, \tag{20}$$

is $O(1)$ as the double limit is taken. Here $k = \kappa/\rho c_p$ is the thermal diffusivity, $\nu = \mu/\rho$ is the kinematic viscosity, $L$ is the vertical scale of the system and $\beta$ is typical of the temperature gradient applied across the system.

Accordingly, we assume that the pressure field has an expansion of the form

$$p = p^0 g + p^1 + p^2/g + \ldots, \tag{21}$$

and that for the fields $\mathbf{v}$, $T - T_r$ we have

$$\chi = \chi^1 + \chi^2/g + \ldots. \tag{22}$$

These expansions can equally be regarded as in rising powers of $\alpha_r$, since $g\alpha_r = O(1)$ by hypothesis. A corresponding expansion for the free surface assumes

$$f = f^2/g + \ldots, \tag{23}$$

expressing the fact that in the limit $g \to \infty$, gravity holds a free surface to an equipotential and that for the layer, to leading order, the free boundary becomes planar and horizontal.

Taking the reference state at $x_3 = 0$, the dominant terms of equation (14) give

$$0 = \rho_r\delta_{3i} - p^0_{,i}, \tag{24}$$

so that

$$p^0 g = \rho_r g x_3 + p_r, \tag{25}$$

and to leading order $\dot{p} = \rho_r g v_3^1$. To leading order the condition $(16)_1$ yields $\pi = p_r$, while (18) gives $v_3^1 = 0$ on $x_3 = f$. Combining this with the remaining condition (16) we have

$$v^1_{1,3} = v^1_{2,3} = v^1_3 = 0, \qquad \text{on } x_3 = 0. \tag{26}$$

The dominant term in (13) gives

$$v^1_{i,i} = 0, \tag{27}$$

and we determine the leading order terms of the energy equation as

$$(\rho c_p)_r \dot{T}^1 - \rho_r g \alpha_r (T_r + T^1) v^1_3 = \rho_r Q + \kappa_r \nabla^2 T^1 + 2\mu_r d^1_{ij} d^1_{ij}. \tag{28}$$

The second term on the left-hand side arises from the adiabatic temperature gradient [Jeffreys (1930)]. The next order terms from linear momentum conservation are

$$\rho_r \dot{v}^1_i = -p^1_{,i} - \rho_r g \alpha_r T^1 \delta_{i3} + \mu_r v^1_{i,jj}. \tag{29}$$

Equations (27)–(29), together with appropriate conditions [e.g. specified $T^1$ together with (25) and $\mathbf{v} = 0$ on $x_3 = L$], constitute a closed system of equations determining $p^1$, $T^1$ and $v^1_i$. It contains the OB–approximation as we shall see, but it is more general. We shall call it the "generalized OB–system". The point here is that, because $g \to \infty$ in our limit, (27)–(29) still cling to some vestiges of compressibility. Note particularly that the adiabatic gradient,

$$\beta_{ad} = g(\alpha T / c_p)_r, \tag{30}$$

may be large or small compared with $\beta$.

**4.** To develop the OB–equations, we must assume that the layer is thin, in the sense that $\beta_{ad} \ll \beta$. Equivalently

$$\epsilon \to 0, \tag{31}$$

where $\epsilon$, the dissipation parameter, is given by

$$\epsilon = g \alpha_r L / (c_p)_r. \tag{32}$$

In short, in our double limit $g \to \infty$, $\alpha_r \to 0$, we must assume that $g\alpha_r \ll (c_p)_r / L$ even though $g\alpha_r = O(\nu_r k_r / T_r L^3)$; see (20). Thus $(c_p)_r \gg \nu_r k_r / T_r L^2$. This point was overlooked by Roberts (1967).

To develop consequences of (31) it is perhaps simplest to introduce dimensionless variables by

$$x_i \to L x_i, \quad t \to (L^2/\nu_r) t, \quad v^1_i \to (\nu_r / L) v_i, \quad p^1 \to (\rho_r \nu_r^2 / L^2) p$$
$$T_r \to \beta L \sigma_r T_r, \quad T^1 \to \beta L \sigma_r T, \quad Q \to [\beta (\nu c_p)_r / L] Q, \tag{33}$$

where $\sigma = \nu / k$ is the Prandtl number. Equation (27) is unchanged while (28) and (29) become

$$\sigma_r \left[ \dot{T} - \epsilon (T_r + T) v^1_3 \right] = Q + T_{,ii} + 2(\sigma_r \epsilon / R) d_{ij} d_{ij}, \tag{34}$$
$$\dot{v}_i = -RT\delta_{i3} - p_{,i} + v_{i,jj}. \tag{35}$$

In the limit (31), the last effect of compressibility on the left-hand side of (34) disappears. Simultaneously, the viscous regeneration of heat, the final term in (34), no longer affects the energy budget. Returning to the dimensional form, we recover the Boussinesq system (dropping the superscript 1):

$$\boldsymbol{\nabla}\cdot\mathbf{v} = 0, \tag{36}$$

$$\dot{T} = k\nabla^2 T + q, \tag{37}$$

$$\dot{\mathbf{v}} = -\nabla\varpi - \alpha_r T\mathbf{g} + \nu\nabla^2\mathbf{v}, \tag{38}$$

where $\varpi = p/\rho$ and $q = Q/c_p$.

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