UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

Relaxation Effects and the Evolution of a Mush

R. N. Hills
P. H. Roberts

October 1991
CAM Report 91-23

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

Relaxation effects and the evolution of a mush

R. N. Hills
AND
P. H. Roberts

Mathematics Department, Heriot-Watt University, Edinburgh, U.K.

Department of Mathematics, University of California,
Los Angeles, CA 90024, U.S.A.

ABSTRACT

An earlier paper [1] was concerned with the evolution of a layer of stagnant water and ice that contains sources of heat and is thermally insulated at one end and maintained at a sub-zero temperature at the other. There it was assumed that the melting/freezing processes occur instantaneously and it was shown that under certain circumstances, depending on the degree of heating and the position of the initial interface, a transient region of mixed phase develops. In this paper the effect of allowing a non-zero relaxation time for the phase change is studied. It is shown that this effect is passive in the sense that the pattern of phase evolution [1] remains substantially unchanged.

When the relaxation time is nonzero, the sharp front that appears in classical theory (where the relaxation time is zero) becomes fuzzy or "blurred". The structure of this blurred front is analyzed for two choices of the law of relaxation.

1. Introduction.

The term mushy zone usually applies to a (finite) region in which two phases, solid and liquid, are in intimate contact throughout, simultaneously coexisting as a fine mixture. It is known that for pure substances such regions only occur when volumetric heating is present. In a recent paper [1], the present authors have studied, in the limit of large Stefan number, the evolution of a layer of stagnant water and ice to the final equilibrium state from an arbitrary initial configuration in which the interface between the phases is sharp. Specifically, the system studied lies in the one-dimensional region $x \in [0,1]$ with the left-hand end maintained at the sub-zero (non-dimensional) temperature, $u = -1$, but with the right-hand end thermally isolated and the freezing temperature for the phase change being $u = 0$. The degree of volumetric heating, r per unit mass per unit time, determines not only the final equilibrium state of the system but also a critical point $s_c = \sqrt{2/r}$. If $r > 2$ then s_c lies within [0,1] and the behaviour of the phase evolution critically depends on the position of the initial sharp interface $x = s_i$ relative to s_c. If $s_i \leq s_c$ then the system is able to accommodate the heating without the appearance of a mushy zone and the phase interface remains sharp, continually moving towards the equilibrium position $x = s_\infty$. At that position the heat input exactly balances the heat extracted to maintain the prescribed temperature of the lower boundary. If, however, $s_i > s_c$ a transient mushy zone appears. For a one-dimensional system this is the only physically acceptable solution: a solution involving superheated ice cannot occur (see [1]). Governing the evolution of the mush there are two intrinsic time scales, one associated with latent heat and the other with thermal diffusion. Since the Stefan number is assumed large, the former is the slow time scale, while the latter is the scale of fast processes. The mush is always formed by an ice/mush interface appearing and initially moving left on the fast time scale but it actually approaches the critical position $x_i = s_c$ with increasing slowness. Ultimately the mush vanishes when this leading edge is overhauled by the mush/liquid trailing edge. Thereafter ice and water are again separated by a sharp interface that moves on the slow timescale to $s = s_\infty$.

In [1] the melting/freezing processes were assumed instantaneous but in [2], [3] the present authors developed a theory of mixed phase regions in which these processes took a characteristic time, τ, to complete. They demonstrated by means of a very simple similarity solution that phase transitions that were "sharp" in the conventional ($\tau = 0$) theory were "blurred" when $\tau \neq 0$, that is, the transition from pure water to ice occurred within a front whose thickness increased monotonically with τ. The morphology of the front is characterized by the mass fraction, $\phi(\mathbf{x}, t)$, of ice at point $\mathbf{x}$ at time t. Since ϕ lies between 0 and 1 within the front, Hills and Roberts called it a "mixed phase region" and also a "mush". In this paper, in which we attempt to solve some of the cases discussed in [1] but for the $\tau \neq 0$ theory, we prefer to call such regions "blurred fronts" to distinguish them from the extended mushy zones of [1]. Our motive in undertaking this generalization is to demonstrate that the theory of [2] and [3] is mathematically complete and physically robust, that is it closes the problem and leads to sensible solutions. We find that, although the fine detail is altered when the theory of [2] and [3] is used, substantially the pattern of phase evolution presented in [1] remains unchanged. Corresponding to the case $s_i \leq s_c$ the blurred front mimics the motion of the sharp front in the $\tau = 0$ case. In the case analogous to $s_i > s_c$, a finite mushy zone again develops as previously described in [1] but with the detail of the temperature and mass fraction field slightly modified.

2. Illustrative problem.

Within the region $0 \leq x \leq 1$ the solid and liquid phases of H_2O are separated by a thin region of mixed phase that occupies $[s_1(t), s_2(t)]$. The ice lies to the left of this region (smaller x) while on its right there is pure water. Throughout [0,1] a uniform distribution of heat sources is present that, in the absence of heat conduction and latent heat, would increase the temperature, u, at the rate $r > 0$. The difference in the densities of ice and water is ignored so that questions of convection and of volume preservation do not arise. The plane $x = 1$ is thermally isolated while at $x = 0$ the temperature is maintained at $u = -1$. The temperature in the pure phase regions ($1 \equiv$ ice, $2 \equiv$ water) satisfies

$$u_{\alpha t} = u_{\alpha xx} + r, \qquad r = 2/s_c^2 \qquad (\alpha = 1, 2), \tag{2.1}$$

subject to

$$u_1(0, t) = -1, \qquad u_{2x}(1, t) = 0. \qquad (2.2, 3)$$

In the blurred front (subscript 3) (see [2])

$$u_{3t} = u_{3xx} + \lambda\phi_t + r, \tag{2.4}$$

$$\tau\phi_t + u_3 = 0. \tag{2.5}$$

In (2.4), $\lambda = L/c_pT_M$ denotes the Stefan number, where L is the latent heat, c_p the specific heat and T_M a reference temperature on the absolute scale. In what follows we shall assume that $\lambda \gg 1$ and that the dimensionless characteristic time of the melting/freezing process, τ, satisfies $\tau \ll 1$. Within the mixed phase region of thickness $\delta(t)$,

$$\delta(t) = s_2(t) - s_1(t), \tag{2.6}$$

the mass fraction ϕ varies but it is continuous at the edges so that there is no absorption or emission of latent heat there. Thus

$$[\![u]\!] = 0, \quad [\![u_x]\!] = 0, \qquad \text{at } x = s_1(t), s_2(t), \qquad (2.7,\ 8)$$
$$\phi(s_1, t) = 1, \qquad \phi(s_2, t) = 0. \qquad (2.9,\ 10)$$

Here $[\![\psi]\!]$ denotes the jump at the front, e.g. $[\![\psi]\!] = \psi_3 - \psi_1$ at $x = s_1$. Finally we make the assumption that the temperature is zero on the leading edge of the front. Thus, if the front moves to the right as a freezing front, we have

$$u(s_2, t) = 0, \qquad \text{if } \dot{s}_2 > 0, \qquad (2.11\text{f})$$

while if it moves to the left as a melting front

$$u(s_1, t) = 0, \qquad \text{if } \dot{s}_1 < 0. \qquad (2.11\text{m})$$

These leave unsettled what to assume if $\dot{s}_1 > 0$ and $\dot{s}_2 < 0$, and how to deal with the apparent over-determination that arises if $\dot{s}_1 < 0$ and $\dot{s}_2 > 0$. We regard these as bizarre cases not allowed by the theory, although we cannot prove this. Certainly they do not arise below, where $\dot{s}_1$ and $\dot{s}_2$ invariably have the same sign.

Although latent heat is not released or removed at the boundaries of the mixed phase region, equivalent amounts appear or disappear volumetrically throughout its interior as reflected by the term $\lambda\phi_t$ in (2.4). The conditions (2.7)–(2.11) replace the Stefan jump condition of the conventional theory (see [1], equation (2.4)).

Associated with the volumetric heating there is a critical point, $x = s_c$, where

$$s_c = \sqrt{2/r}. \tag{2.12}$$

The case $r > 2$ is of greatest interest for then s_c falls within [0,1] and the final steady state has a sharp phase interface at $x = s_\infty$ with the temperature distribution (see [1], §2)

$$u_\infty = -(x - s_\infty)(x - \bar{s}_\infty)/s_c^2, \tag{2.13}$$

where

$$\left.\begin{matrix} s_\infty \\ \bar{s}_\infty \end{matrix}\right\} = 1 \mp (1 - s_c^2)^{1/2}, \tag{2.14}$$

and so $s_c > s_\infty$, $s_\infty \leq 1$, $\bar{s}_\infty > 1$.

We shall assume that initially the ice/blurred front boundary is situated at $s_1(0) = s_i$ with the front thickness being $\delta(0) = O(n^{-1})$, where $n^2 = \lambda/\tau \gg 1$. We shall consider only the limit $\lambda \to \infty$. The system then possesses the 'fast', $O(1)$, intrinsic time scale associated with thermal diffusion, a fast relaxaton time scale $O(\tau)$, and the 'slow', $O(\lambda)$, intrinsic scale of the latent heat processes. On the slow scale, heat conduction (and relaxation) processes are effectively instantaneous. We shall show that across the front the temperature change is $O(n^{-1})$. Then, to leading order, an initial temperature distribution in the pure phase regions quickly evolves on the $O(1)$ time scale to

$$u_1 = -(x - s_1)(x - \bar{s}_1)/s_c^2, \qquad u_2 = (x - s_2)(2 - x - s_2)/s_c^2, \tag{2.15, 16}$$

where $\bar{s}_1 = s_c^2/s_1$ and thereafter changes in these regions take place on the slow $O(\lambda)$ time scale.

We shall reconsider three situations discussed in [1]:

(a) Case $r > 2$ and $s_i < s_\infty$ (or $r < 2$ and $s_i < 1$) In the $\tau = 0$ theory, no mushy zone develops in either of these cases. The sharp phase surface, $x = s(t)$, moves slowly towards $x = s_\infty$ as a freezing front; $s(t)$ increases monotonically to $x = 1$ (if $r < 2$) or to s_∞ (if $r > 2$). The motion is governed by the differential equation

$$\lambda \dot{s} = \Delta(s) \equiv (s - s_\infty)(s - \bar{s}_\infty)/s s_c^2, \tag{2.17}$$

that arises from the Stefan condition. When $\tau \neq 0$, the sharp interface is replaced by a blurred front and from (2.4) and (2.5) we find to leading order that

$$\phi_{3txx} = n^2 \phi_{3t} + 2/\tau s_c^2. \tag{2.18}$$

The general solution that satisfies (2.9) and (2.10) is

$$\phi_3(x,t) = f(x) - \frac{2t}{\lambda s_c^2} + a(t)\frac{\sinh n(s_2 - x)}{\sinh n\delta} + b(t)\frac{\sinh n(x - s_1)}{\sinh n\delta}, \quad (2.19)$$

where

$$a(t) = 1 - f(s_1) + 2t/\lambda s_c^2, \qquad b(t) = -f(s_2) + 2t/\lambda s_c^2, \qquad (2.20)$$

and $f(x)$ is an arbitrary function that still has to be determined. Since $n\delta = O(1)$, to leading order the temperature distribution follows from (2.5) as

$$u_3 = -\frac{\lambda}{n}\left[a(t)\frac{\cosh n(s_2 - x)}{\sinh n\delta}\,\dot{s}_2 - b(t)\frac{\cosh n(x - s_1)}{\sinh n\delta}\,\dot{s}_1\right]. \qquad (2.21)$$

From condition (2.8) and the pure phase temperature distributions (2.15), (2.16), we have

$$\lambda a(t)\dot{s}_2 = (s_c^2 - s_1^2)/s_1 s_c^2, \qquad \lambda b(t)\dot{s}_1 = 2(1 - s_2)/s_c^2, \qquad (2.22,\ 23)$$

but since $s_2 - s_1 = O(n^{-1})$ we deduce from these [and (2.20)] that the leading edge s_2 satisfies (2.17) to leading order (as indeed does the trailing edge s_1). Thus the blurred region evolves in much the same way as the sharp interface in the $\tau \neq 0$ case. From (2.20) and (2.23) we have

$$f(s_2) = \frac{2t}{\lambda s_c^2} - \frac{2(1 - s_2)}{s_c^2 \Delta(s_2)}. \qquad (2.24)$$

Noting that, since s_2 is a monotonically increasing function, we may introduce the inverse function, $T(s_2)$, to s_2, i.e. such that $s_2(T(x)) \equiv x$. Then (2.24) gives

$$f(x) = \frac{2}{\lambda s_c^2}T(x) - \frac{2(1 - x)}{s_c^2 \Delta(x)}. \qquad (2.25)$$

Next we turn to condition (2.11f). From (2.21)–(2.23), to leading order, we find

$$2s_1(1 - s_2)\cosh n\delta = s_c^2 - s_1^2, \qquad (2.26f)$$

and, given $s_2(t)$ satisfies (2.17), this equation will determine the position of the trailing edge, $s_1(t) = s_2 - \delta$, of the blurred front: since $s_2 - s_1 = O(\delta)$ we have

$$s_1 = s_2 - \frac{1}{n}\cosh^{-1}\left[\frac{s_c^2 - s_2^2}{2s_2(1-s_2)}\right].$$

We see that $s_1 = s_2 = s_\infty$ is a solution expressing the fact that the blurred front becomes sharp in the final equilibrium state.

Using (2.26f), we see that the temperature field (2.21) can be written, to leading order, as

$$u_3 = -\frac{2(1-s_2)}{ns_c^2}\sinh n(s_2 - x). \tag{2.27f}$$

As expected $u_3 < 0$ within the front. It only remains to satisfy (2.7) at $x = s_1$. This can be accomplished by making a small $O(n^{-1})$ correction to the outer solution (2.15).

(b) Case $s_\infty < s_i < s_c$ In the $\tau = 0$ theory, a sharp phase surface $x = s(t)$ moves to the left as a melting front; $s(t)$ decreases monotonically to s_∞. The analysis of this case parallels that of (a) above with the labels 1 and 2 of the leading and trailing edges interchanged. In place of (2.26f) we have

$$(s_c^2 - s_1^2)\cosh n\delta = 2s_1(1-s_2), \tag{2.26m}$$

so that $s_2 = s_1 + n^{-1}\cosh^{-1}[(2s_1(1-s_1)/(s_c^2 - s_1^2)]$. In place of (2.27f)

$$u_3 = \frac{(s_c^2 - s_1^2)}{ns_1 s_c^2}\sinh n(x - s_1). \tag{2.27m}$$

As expected, $u_3 > 0$ within the front. The addition of a small $O(n^{-1})$ constant to u_2 takes care of (2.7) at $x = s_2$.

(c) Case $s_i > s_c$ In the $\tau = 0$ theory, the initially sharp phase interface moves rapidly to the left until the condition

$$u_{1x}(s_1, t) = 0 \tag{2.28}$$

is satisfied for the first time. Then a mushy zone forms by the leading edge moving left towards $x = s_c$ with condition (2.28) holding continually as it does so. Indeed, the detailed motion of the leading edge is determined by solving (2.1), (2.2) and (2.28). This is done in [1] using numerical methods. At the beginning the leading edge moves on the fast time scale but, as the critical point $s = s_c$ is approached, the front moves increasingly slowly. For the case when $s_i < 1$, the trailing edge proceeds left on the slow time scale and eventually the mush disappears when the trailing edge overhauls the leading edge in the neighbourhood of the critical point $x = s_c$. Then the last stage of the evolution from $x = s_c$ to $x = s_\infty$ proceeds as for the sharp front described in case (b).

For $\tau \neq 0$ the picture is substantially the same. The sharp front of the first stage of the $\tau = 0$ evolution is replaced by a thin blurred front. The mush begins to evolve only when the condition (2.28) is first satisfied. Since the motion of the ice/mush interface is essentially determined by the ice region contracting so that (2.11m) and (2.28) are satisfied, the leading edge moves as in the $\tau = 0$ case. The point here is that (2.11) and (2.28) are already obeyed at $x = s_1$ by the $\tau = 0$ solution given in [1].

For the blurred front of case (b) we sought a solution in which $n \gg 1$ but $n\delta = O(1)$, but now with a finite mushy zone developing we should properly look for a solution for which $n \gg 1$ but $\delta = O(1)$. We shall therefore treat all terms that are of order $\exp(-n\delta)$ as zero. The mass fraction is again determined by (2.18) and the general solution satisfying the end conditions (2.9) and (2.10) is [cf. (2.19)]

$$\phi_3(x,t) = f(x) - \frac{2t}{\lambda s_c^2} + a(t)e^{-n(x-s_1)} + b(t)e^{-n(s_2-x)}, \qquad (2.29)$$

with $a(t)$ and $b(t)$ being given by (2.20). The temperature distribution follows from (2.5) as [cf. (2.21)]

$$u_3(x,t) = -\frac{\lambda}{n}\left[a(t)e^{-n(x-s_1)}\dot{s}_1 - b(t)e^{-n(s_2-x)}\dot{s}_2\right] + \frac{2}{n^2 s_c^2}, \qquad (2.30)$$

where we have omitted $O(n^{-2})$ terms that are exponentially small away from $x = s_1$ or $x = s_2$. To leading order u_3 is exponentially small across the zone and, in particular, condition (2.7) is satisfied to dominant order. From the condition (2.8) at $x = s_1$ we find that $a(t) = 0$ so that, by again introducing the inverse function $T(s_1)$, we have

$$f(x) = 1 + 2T(x)/\lambda s_c^2. \qquad (2.31)$$

At the trailing edge the boundary condition (2.8) gives

$$\lambda\left\{1 + \frac{2}{\lambda s_c^2}\left[T(s_2) - t\right]\right\}\dot{s}_2 = -\frac{2(1-s_2)}{s_c^2}, \qquad (2.32)$$

which is precisely the equation governing the evolution of the trailing edge in the $\tau = 0$ case.

The final forms for the temperature and mass fraction in the mushy zone become

$$u_3 = -\frac{\lambda}{n}\left\{1 - \frac{2}{\lambda s_c^2}\left[t - T(s_2)\right]\right\}\dot{s}_2 e^{-n(s_2 - x)} + \frac{2}{n^2 s_c^2}, \tag{2.33}$$

$$\phi_3 = 1 - \frac{2}{\lambda s_c^2}\left[t - T(x)\right] - \left\{1 - \frac{2}{\lambda s_c^2}\left[t - T(s_2)\right]\right\} e^{-n(s_2 - x)}. \tag{2.34}$$

Thus, while the fine detail of these fields are altered for the $\tau \neq 0$ case, the motion of the leading and trailing edges of the mushy zone are substantially the same as for the non-relaxing case.

3. Structure of the general blurred front.

Guided by cases (a) and (b) of Section 2, we can now understand better the structure and motion of a blurred front in the general, three-dimensional case. The equations to be solved are [cf. (2.1), (2.4), (2.5), (2.7)–(2.11)]

$$u_{\alpha t} = \nabla^2 u_\alpha + r, \qquad (\alpha = 1,2), \tag{3.1}$$

$$u_{3t} = \nabla^2 u_3 + \lambda\phi_t + r, \tag{3.2}$$

$$\tau\phi_t + u_3 = 0, \tag{3.3}$$

where (1,2,3) again refer to (ice, water, mush), together with

$$[\![u]\!] = 0; \qquad [\![\mathbf{n}_\alpha \cdot \nabla u]\!] = 0, \qquad \text{on } S_1, S_2, \tag{3.4, 5}$$

$$\phi = 1, \quad \text{on } S_1; \qquad \phi = 0, \quad \text{on } S_2, \tag{3.6, 7}$$

and

$$u = 0 \quad \begin{cases} \text{on } S_1, & \text{if } U_1 < 0, \qquad (3.8\text{f}) \\ \text{on } S_2, & \text{if } U_2 > 0. \qquad (3.8\text{m}) \end{cases}$$

Here S_1 and S_2 denote the edges of the blurred front adjacent to solid and liquid, $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to those edges in the direction from solid to liquid, and U_1 and U_2 are the velocities of those edges along the normals.

The analysis hinges on the assumption

$$n \to \infty, \qquad \delta \to 0, \qquad n\delta = O(1). \tag{3.9}$$

The leading and trailing edges of the front are then close together and are asymptotically coincident in the limit $n \to \infty$. To leading order $\mathbf{n}_1$ and $\mathbf{n}_2$ are the same (=$\mathbf{n}$, say); let it define the Oξ axis of a locally Euclidean coordinate system which meets S_α at P$_\alpha$ ($\alpha = 1,2$). Although $\delta(\eta,\zeta)$ is a function of position we may, in the neighbourhood of P$_1$ and P$_2$, treat S_1 and S_2 as flat, and δ as a constant, $\delta(0,0)$. In the limit (3.9),

$$\frac{\partial}{\partial\xi} \gg \frac{\partial}{\partial\eta}, \quad \frac{\partial}{\partial\zeta},$$

so that (3.1), (3.2) and (3.5) assume forms analogous to (2.1), (2.4) and (2.8) with $\partial/\partial x$ replaced by $\partial/\partial\xi$, and with S_1 and S_2 given locally by $\xi = s_1(t)$ and $\xi = s_2(t)$.

There is no necessity to follow Section 2 by supposing that the Stefan number is large and that there are two time scales, although we require that τ be small compared with thermal and latent heat time scales. As in Section 2 [see (2.21)] we obtain

$$u_3 = -\frac{\lambda}{n}\left[a(t)\frac{\cosh n(s_2-\xi)}{\sinh n\delta}U_2 - b(t)\frac{\cosh n(\xi - s_1)}{\sinh n\delta}U_1\right], \qquad (3.10)$$

from which

$$\left(\mathbf{n}\cdot\nabla u_3\right)_{S_1} = \lambda a U_2, \qquad \left(\mathbf{n}\cdot\nabla u_3\right)_{S_2} = \lambda b U_1. \qquad (3.11,12)$$

Since $U_1 = U_2$ to leading order ($= U$, say), we see from (2.20) that

$$\left[\mathbf{n}\cdot\nabla u_3\right]_{S_2}^{S_1} = \lambda U. \qquad (3.13)$$

By (3.5) this is

$$\lambda U = \left[\mathbf{n}\cdot\nabla u_1\right]_{S_1} - \left[\mathbf{n}\cdot\nabla u_2\right]_{S_2}, \qquad (3.14)$$

which is essentially the Stefan jump condition, which holds even though latent heat is released or absorbed volumetrically throughout the blurred front rather than at a sharp surface as when $\tau = 0$, and $s_i < s_c$ or $r < 2$.

As in cases (2.27f) or (2.27m) there is an $O(n^{-1})$ discontinuity in u at the trailing edge, whose removal requires a tiny adjustment to the solution behind the trailing edge.

4. A more general relaxation law.

So far we have, as in [3], concentrated on the simplest of all possible relaxation laws, in which the relaxation time, τ, is constant if ϕ lies within (0,1) but which is infinite if $\phi = 0$ or $\phi = 1$. Reality is perhaps better modelled by a law such as

$$\tau = \tau_0/\phi(1-\phi), \qquad (4.1)$$

which holds for all ϕ, including 0 and 1. The disadvantage of such a law is, of course, the greater analytic complexity it creates. Its advantage is first that there are no sharp leading or trailing edges to a blurred front. The front can be located by a single surface, e.g. that on which $u = 0$. Such a front resembles more an internal boundary layer which smoothly matches to the surrounding liquid and solid. Second (and because of this), assumptions such as (2.11) and (3.8) become redundant; ambiguities, such as the one arising below (2.11) and at corresponding points in the argument of Section 3, do not arise.

In the limit (3.9) we now have [cf. (2.4) and (2.5)]

$$u_t = u_{\xi\xi} + \lambda\phi_t + r, \tag{4.2}$$

$$\tau\phi_t + \phi(1-\phi)u = 0, \tag{4.3}$$

where we have omitted the subscript 0 from τ_0. Introduce a new variable v to replace ϕ:

$$\phi = \left(1 + e^{v/\tau}\right)^{-1}. \tag{4.4}$$

By (4.3) we now have

$$u = v_t, \tag{4.5}$$

so that by (4.2)

$$v_{t\xi\xi} = \frac{\lambda}{\tau}\frac{e^{v/\tau}}{\left(1+e^{v/\tau}\right)^2}\, v_t + v_{tt} - r. \tag{4.6}$$

Integration of (4.6) gives

$$v_{\xi\xi} = -\frac{\lambda}{1+e^{v/\tau}} + v_t - rt + \lambda f(\xi), \tag{4.7}$$

where $f(\xi)$ is at this stage arbitrary. In the limit $\tau \to 0$, (4.7) becomes

$$v_{\xi\xi} = v_t - rt + \lambda f(\xi), \qquad \text{if } v > 0, \tag{4.8L}$$

$$v_{\xi\xi} = v_t - rt + \lambda f(\xi) - \lambda, \qquad \text{if } v < 0, \tag{4.8S}$$

corresponding to pure liquid and solid respectively, as (4.4) shows. If $v = O(\tau)$ as $\tau \to 0$ however, ϕ takes values between 0 and 1, corresponding to states within a blurred front, and (4.7) must be used instead of (4.8).

In the limit $n^2 \equiv \lambda/\tau \gg 1$ of large Stefan number, the left-hand side of (4.2) is small. Then v_{tt} may be omitted from (4.6), and v_t from (4.7) and (4.8). Moreover, if a zero of v exists in the domain [at $\xi = s(t)$ say], i.e. if a front exists, its thickness is of order n^{-1} so that $f(\xi)$ is approximately $f_s(t) \equiv f\left(s(t)\right)$. We may then integrate (4.7) to obtain

$$\frac{1}{2\lambda\tau}\left(\frac{dv}{d\xi}\right)^2 = \ln\left(e^{-v/\tau}+1\right) + \left[f_s - \frac{rt}{\lambda}\right]\frac{v}{\tau} + g, \tag{4.9}$$

where $g(t)$ is an arbitrary function of integration. We now have

$$\xi - s(t) = \frac{1}{2^{1/2}n} \int_0^{v/\tau} \frac{dw}{\left[\ln(e^{-w}+1) + (f_s - rt/\lambda)w + g\right]^{1/2}}, \tag{4.10}$$

where $s(t)$ is (see above) the location of the zero of v. [Note: since v increases with ξ, we took the positive square root in (4.10).]

By differentiating (4.10) with respect to t we obtain approximately

$$-\dot{s}(t) = \frac{v_t}{(2\lambda\tau)^{1/2}} \frac{1}{\left[\ln(e^{-v/\tau}+1) + (f_s - rt/\lambda)(v/\tau) + g\right]^{1/2}},$$

or, by (4.9),

$$v_t + \dot{s}v_\xi \approx 0. \tag{4.11}$$

Since $v = 0$ on $\xi = s$, $D_t v = 0$ thereon, where D_t denotes differentiation with respect to the frame moving with the front at the point concerned. This confirms (4.11).

By (4.5) and the ξ-derivative of (4.11), we may re-interpret (4.8) as

$$u_\xi = -\dot{s}\left[\lambda f_s(t) - rt\right], \qquad \text{if } v > 0, \tag{4.12L}$$

$$u_\xi = -\dot{s}\left[\lambda f_s(t) - rt - \lambda\right], \qquad \text{if } v < 0. \tag{4.12S}$$

This shows how f_s may be determined from the gradient of u on either side of the blurred front. Stefan's law is recovered by subtraction.

Acknowledgement.

The research of one of the authors (P. H. R.) is sponsored by the U. S. Office of Naval Research under contract N00014–86–K–0691 with the University of California, Los Angeles.

References.

[1] Hills, R. N. and Roberts, P. H., "On the life cycle of a mush", (with appendix by A. M. Soward), *Stability Appl. Anal. Cts. Media* **1**, 1 (1991).

[2] Hills, R. N. and Roberts, P. H., "Relaxation effects in a mixed phase region: I. General theory", *J. Non Equilib. Thermodyn.* **12**, 169 (1987).

[3] Hills, R. N. and Roberts, P. H., "Relaxation effects in a mixed phase region: II. Illustrative examples, *J. Non Equilib. Thermodyn.* **12**, 183 (1987).