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On the Stability Definition of Difference Approximations for the Initial Boundary Value Problem

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On the stability definition of difference approximations for the initial boundary value problem

by

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1. The continuous problem. Consider the Cauchy problem for a first order system of linear partial differential equations with constant coefficients

$$\begin{aligned}\frac{\partial u}{\partial t} &= \sum_{j=1}^{s} A_j \frac{\partial u}{\partial x_j} + Bu + F, \qquad -\infty < x < \infty,\\ &=: Pu + F, \qquad j = 1, 2, \ldots, s, \quad t \geq 0,\\ u(x, 0) &= f.\end{aligned} \tag{1.1}$$

For numerical purposes there is essentially only one satisfactory way to define wellposedness.

Definition 1.1: *The Cauchy problem is well posed if*

1) *For a dense set of smooth data there is a smooth solution.*
2) *The solutions of the homogeneous equations ($F \equiv 0$) satisfy an energy estimate*

$$\|u(\cdot, t)\| \leq K e^{\alpha(t-t_1)} \|u(\cdot, t_1)\|. \tag{1.2}$$

Here $\|\cdot\|$ denotes the usual L_2-norm and K, α are universal constants.

Estimates for the inhomogeneous system are obtained by Duhamel's principle. Other definitions, like the original one by Hadamard [2] and Petrovskii [7] have the disadvantage that they are not stable against lower order perturbations. This is essential, if one wants to localize the problem and use the principle of frozen coefficients. In fact, for systems with constant coefficients Yamaguti and Kasahara [11] have shown that our definition is the weakest definition, which is stable against lower order perturbations.

Let us now consider (1.1) in the halfplane R_- defined by $x_1 \geq 0, -\infty < x_j < \infty, \quad j = 2, \ldots, s$.

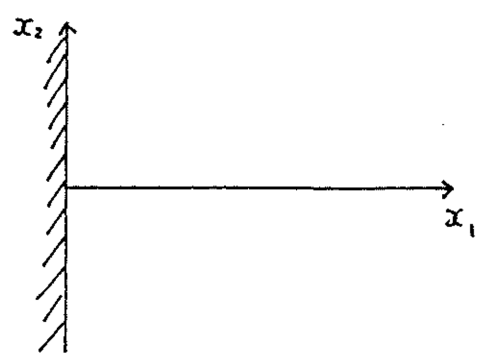

Fig. 1.1

Then we have to augment (1.1) by boundary conditions consisting, for example, of linear relations between the components of u

$$Su = g, \qquad x_1 = 0. \tag{1.3}$$

By a suitable change of variables we can always make the boundary conditions homogeneous.

Therefore we can use Definition 1.1 also for the halfplane problem. $\|\cdot\|$ now denotes the L_2-norm over the halfplane and the estimate (1.2) is required to hold for the case $F \equiv g \equiv 0$.

The definition is very satisfactory for symmetric hyperbolic systems with boundary conditions of Friedrich's type. In this case one can derive the energy estimates by integration by parts. For more general boundary conditions and non-symmetric hyperbolic systems another approach has turned out to be more powerful. We have the

Lopatinsky condition. *Consider (1.1)–(1.3) with $F \equiv g \equiv 0$. The initial boundary value problem is not well posed, if we can find a sequence of solutions of type*

$$u(x,t) = e^{s^{(j)}t + i<\omega_-^{(j)}, x_->}\varphi^{(j)}(x_1), \qquad \|\varphi^{(j)}\|^2 = \int_0^\infty |\varphi^{(j)}|^2 dx_1 = 1, \tag{1.4}$$

with $\text{Real}_{j\to\infty} s^{(j)} = \infty$. *Here* $x_- = (x_2, \ldots, x_s)$ *and* $\omega_-^{(j)} = (\omega_2^{(j)}, \ldots, \omega_s^{(j)})$ *are constant real vectors and* $s^{(j)}$ *are complex numbers.*

To find necessary algebraic conditions for the problem to be well posed we introduce (1.4) into the homogeneous equations (1.1)–(1.3) and obtain the eigenvalue problem

$$\begin{aligned} \big(s - P(i\omega_-, \partial/\partial x)\big)\varphi = 0, \\ S\varphi = 0, \quad \|\varphi\| < \infty. \end{aligned} \tag{1.5}$$

Thus we can phrase Theorem 1.1 in the form

Theoerem 1.1: *A necessary condition for wellposedness is that there is a real constant η_0 such that (1.5) has no eigenvalues with* Real $s > \eta_0$.

R. Hersch [3] has shown that one can solve the initial boundary value problem, if there is no eigenvalue with Real $s > \eta_0$. Without restriction we can assume that the initial data are homogeneous, i.e.,

$$u(x, 0) \equiv 0. \tag{1.6}$$

Otherwise we introduce a new dependent variable by $w = u - e^{-t}f(x)$. Let u be a smooth solution of the initial boundary value problem with $\sup_t e^{-\eta_0 t}\|u(\cdot,t)\| < \infty$ and denote by

$$\hat{u}(x_1, s, \omega_-) = \int_0^\infty \int_R e^{-(st + i<\omega_-, x_->)}u(x,t)dx_- dt, \qquad \eta > \eta_0, \tag{1.7}$$

its Fourier-Laplace transform with respect to x_- and t. It satisfies

$$\begin{aligned} \big(s - P(\omega_-, \partial/\partial x_1)\big)\hat{u} &= \hat{F}, \qquad \text{Real } s > \eta_0, \\ S\hat{u}(0,s,\omega_-) &= \hat{g}, \qquad \|\hat{u}(\cdot,s,\omega_-)\| < \infty. \end{aligned} \tag{1.8}$$

Conversely, if there are no eigenvalues with Real $s > \eta_0$, then one can solve (1.8) and obtains a solution of the initial boundary value by inverting the transform. One can also estimate the solution in terms of F, g. However, in general there is a loss of derivatives and the estimate is not stable with respect to lower order terms, i.e., one can change B in (1.1) such that the eigenvalue condition ceases to be fulfilled. Therefore one has to strengthen the eigenvalue condition. H. Kreiss [4], R. Sakamoto [9] have required that for $\eta > \eta_0$ the solutions of (1.8) satisfy an estimate of type

$$|\hat{u}(0,s,\omega_-)|^2 + (\eta-\eta_0)^2\|\hat{u}(\cdot,s,\omega_-)\|^2 \le K\big(|\hat{g}|^2 + \|\hat{F}(\cdot,s,\omega_-)\|^2\big). \tag{1.9}$$

Here η_0, K are universal constants, which do not depend on g, F, s, ω_-. For a large class of hyperbolic problems (which includes strictly hyperbolic systems) a rather complete theory has been developed (see also M.S. Agronovich [1]).

In particular one has proved

Theorem 1.1: *Consider the eigenvalue problem (1.5) with $B \equiv 0$. The estimate (1.9) is valid if and only if there are no eigenvalues or generalized eigenvalues for* Real $s \ge 0$.

Also, J. Rauch [8] has shown that (1.9) implies wellposedness according to Definition 1.1.

Unfortunately, (1.9) is more restrictive than the estimate of Definition 1.1. Consider, for example, the wave equation

$$u_{tt} = u_{xx} + u_{yy}$$

written as a first order system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_y + F$$

with boundary conditions

$$u = v + g.$$

Its solutions do not satisfy the estimate (1.9), but for $g \equiv F \equiv 0$ they satisfy the estimate (1.2).

We will weaken (1.9) to the resolvent condition

Definition 1.2: *Assume that $g = 0$. We say that the solutions of (1.8) satisfy the resolvent condition, if there are constants K and η_0 such that*

$$(\eta-\eta_0)\|\hat{u}(\cdot,s,\omega_-)\| \le K\|\hat{F}(\cdot,s,\omega_-)\|. \tag{1.10}$$

The inverted Fourier-Laplace transform shows that (1.10) is equivalent with

$$(\eta - \eta_0) \int_0^\infty e^{-\eta t} \|u(\cdot,t)\|^2 dt \leq K \int_0^\infty e^{-\eta t} \|F(\cdot,t)\|^2 dt, \qquad \eta > \eta_0, \tag{1.11}$$

and we define

Definition 1.3: *Consider the initial boundary value problem (1.1)–(1.3) with* $f \equiv g \equiv 0$. *We say that it is well posed in a generalized sense, if for a dense set of smooth* F *there is a smooth solution, which satisfies the estimate (1.11).*

There are no dificulties to prove that the definition is stable against lower order perturbations. Thus we can treat systems with variable coefficients. Also problems in general domains can be reduced to halfplane problems.

One can also express (1.10) as an eigenvalue condition. We allow certain eigenvalues or generalized eigenvalues with Real $s = 0$. Definition 1.3 is not more restrictive than Definition 1.1, i.e., if the estimate (1.1) holds, then (1.11) follows. We conjecture that the definitions are equivalent. We conjecture also that Definition 1.3 is the weakest definition, which is stable against lower order perturbations.

2. Semidiscrete approximations. In this section we will discretize the space derivatives but keep time continuous. We will explain our results with help of an example. For simplicity we consider (1.1) in two space dimensions and assume that $B \equiv 0$. We begin with the Cauchy problem. Let $h > 0$ be the mesh size and introduce gridpoints by

$$x_\nu = h(\nu_1, \nu_2), \quad \nu_j = 0, \pm 1, \pm 2, \ldots .$$

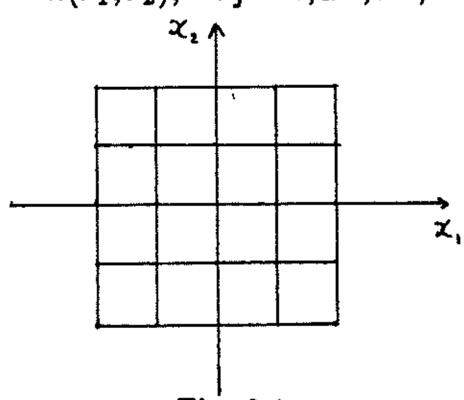

Fig. 2.1

We introduce also the translation operators E_j by

$$E_1 f(x_1, x_2) = f(x_1 + h, x_2), \qquad E_2 f(x_1, x_2) = f(x_1, x_2 + h).$$

Then the usual backward, forward and centered approximations of $\partial/\partial x_j$ can be written as

$$hD_{-j} = I - E_j^{-1}, \quad hD_{+j} = E_j - I, \quad 2hD_{0j} = E_j - E_j^{-1}.$$

Now we can write down our approximation of the differential equations.

$$\begin{aligned} \frac{dv_\nu}{dt} &= (A_1 D_{01} + A_2 D_{02})v_\nu + F_\nu, \\ v_\nu(0) &= f_\nu. \end{aligned} \tag{2.1}$$

Corresponding to Section 1 there is only one satisfactory way to define the stability of (2.1). We introduce a discrete L_2-scalar product and norm

$$(v,w)_h = \sum_\nu < v_\nu, w_\nu > h^2, \qquad \|v\|_h^2 = (v,v)_h,$$

and define

Definition 2.1: *Consider (2.1) with $F \equiv 0$. We call the approximation stable if there are universal constants $\bar\alpha$, $\bar K$, such that*

$$\|v(\cdot,t)\|_h \le \bar K e^{\bar\alpha(t-t_1)} \|v(\cdot,t_1)\|_h.$$

If A_1, A_2 are symmetric matices, then summation by parts gives us

$$\frac{d}{dt}\|v\|_h^2 \le 0 \tag{2.2}$$

and stability follows. ($\bar K = 1,\ \bar\alpha = 0$).

In general, if (2.2) holds for a difference approximation, then we say that it satisfies an energy estimate.

Now we consider the halfplane problem. In this case we use (2.1) for $x_1 > 0$ and determine v for $x_1 = 0$ by boundary conditions

$$Lv = g, \quad L_1 D_+^p v = 0, \quad x_1 = 0. \tag{2.3}$$

Here we have augmented the boundary conditions (1.3) by extrapolation conditions consisting of linear relations between the components of $D_+^p v$. We assume that we can use (2.3) to express $v(0,x_2,t)$ in terms of g and v in interior points $x_1 > 0$.

Now we define the discrete L_2-scalar product and norm by summing over all interior gridpoints $x_1 > 0$.

As in Section 1 there is an eigenvalue problem connected with the halfplane problem.

$$\begin{aligned} s\hat v &= (A_1 D_{01} + A_2 D_{02})\hat v, &\qquad x_1 > 0, \\ L\hat v &= 0, \quad L_1 D_+^p \hat v = 0, &\qquad x_1 = 0. \end{aligned} \tag{2.4}$$

All the definitions and theorems in Section 1 have counterparts for the semidiscrete problem. For example, if we consider (2.1),(2.3) with $F \equiv g \equiv 0$, then we can use Definition 2.1 to define stability. Recently Lixin Wu [6] has proved the semidiscrete version of J. Rauch's result.

Theorem 2.1: *Assume that the difference approximation satisfies an energy estimate for the Cauchy problem. If the eigenvalue problem (2.4) has no eigenvalues or generalized eigenvalues for* Real $s \geq 0$, *then the approximation is stable, according to Definition 2.1 for the initial boundary value problem.*

As in Section 1 the eigenvalue condition can be too restrictive. For a detailed discussion we refer to N. Trefethen [10]. Corresponding to Section 1 we will therefore introduce the concept of *Stability in a generalized sense.* We assume that $g \equiv f \equiv 0$. Then we can use the boundary conditions (2.3) to eliminate $v(0, x_2, t)$ from (2.1) and obtain an (infinite) system of ordinary differential equations

$$\frac{d\mathbf{v}}{dt} = Q\mathbf{v} + \mathbf{F}, \qquad \mathbf{v}(0) = 0. \tag{2.5}$$

Here $\mathbf{v}$ consists of v in the interior gridpoints $x_1 > 0$. (In actual calculations the system is finite dimensional because more boundary conditions are added to restrict the computational domain.)

We define

Definition 2.2: *We call the approximation (2.5) stable in a generalized sense, if there are constants K, η_0 such that for all $\mathbf{F}$ with $e^{-\eta_0 t}\|\mathbf{F}\|_h^2 < \infty$ and all $0 < h \leq h_0$, $\eta > \eta_0$*

$$(\eta - \eta_0)\int_0^\infty e^{-\eta t}\|\mathbf{v}\|_h^2 dt \leq K\int_0^\infty e^{-\eta t}\|\mathbf{F}\|_h^2 dt. \tag{2.6}$$

We can Laplace transform (2.5) and obtain the resolvent equation

$$(sI - Q)\tilde{\mathbf{v}} = \tilde{\mathbf{F}}. \tag{2.7}$$

By Parseval's relation, (2.6) is equivalent with the resolvent condition

$$(\eta - \eta_0)\|\tilde{\mathbf{v}}\|_h \leq K\|\tilde{\mathbf{F}}\|_h, \qquad \eta > \eta_0$$

i.e.,

$$\|(sI - Q)^{-1}\|_h \leq \frac{K}{\eta - \eta_0} \tag{2.8}$$

Fourier transforming the resolvent equation with respect to y we obtain for our example

$$\begin{aligned} &(sI - A_1 D_{01} + i\frac{\sin\omega_2 h}{h}A_2)\hat{v} = \hat{F}, && x_1 > 0, \\ &L\hat{v} = 0, \quad L_1 D_+^p \hat{v} = 0, && x_1 = 0. \end{aligned} \tag{2.9}$$

(2.9) is a system of ordinary difference equations with constant coefficients, which can be solved explicitly. Therefore we can, at least in principle, decide whether the estimate (2.8) holds.

It is now also clear how to proceed for more general approximations. We can always write the semidiscrete approximation in the form (2.5). Then we define the stability by (2.6), which leads to the resolvent condition (2.8).

3. Runge-Kutta methods. We consider now the system of ordinary differential equations (2.5) and want to solve it numerically by using a method of Runge-Kutta type. Therefore we introduce a timestep k and discretize time. We assume that Q does not depend on t. In this case the methods of Runge-Kutta type are of the form

$$\begin{aligned} v(t+k) &= L(kQ)v(t) + kG, \\ v(0) &= 0, \end{aligned} \tag{3.1}$$

where

$$L = \sum_{j=0}^{q} \alpha_j \frac{(kQ)^j}{j!}. \tag{3.2}$$

is a polynomial in kQ. (For simplicity of notation we write v instead of $\mathbf{v}$ and $\|\cdot\|$ instead of $\|\cdot\|_h$.) In particular, if the method is accurate of order p, then

$$\alpha_0 = \alpha_1 = \cdots = \alpha_p = 1$$

and for the standard Runge-Kutta methods of order $p \le 4$ we have

$$L = \sum_{j=0}^{p} \frac{(kQ)^j}{j!}, \qquad p \le 4. \tag{3.3}$$

As in the semidiscrete case we can again distinguish between different stability definitions. In particular, Lixin Wu [6] has generalized his results to totally discretized schemes. Here we shall restrict ourselves to stability in a generalized sense.

In the theory for numerical methods for ordinary differential equations one applies the method to the scalar differential equation

$$y' = \lambda y, \qquad \lambda = \text{ const.} \tag{3.4}$$

Then (3.1) becomes

$$v(t+k) = L(\lambda k)v(t), \tag{3.5}$$

and one denotes by

$$\Omega : (\text{the set of all complex } \mu = k\lambda \text{ with } |L| < 1).$$

Clearly Ω is an open set.

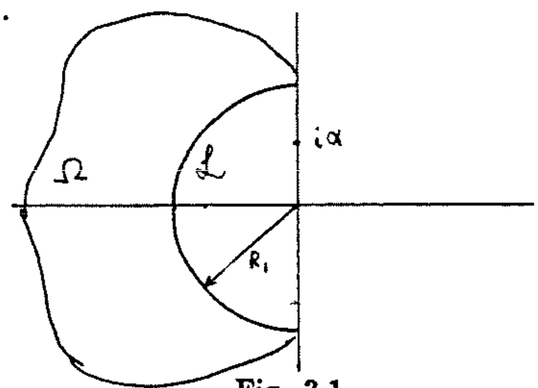

Fig. 3.1

Definition 3.1: *The method is called locally stable, if there is an $R_1 > 0$ such that the open halfcircle*

$$\mathcal{L}(R_1) =: \{\mu \text{ with } |\mu| < R_1, \ \text{Real}\, \mu < 0\}$$

is contained in Ω.

In an earlier paper [5] locally stable Runge-Kutta methods have been discussed. For $p = 3,4$ the standard methods are locally stable while for $p = 1,2$, they are not. Most of the higher order methods are not locally stable. An exception is the new fifth order method of Dormand-Prince.

Let us now apply the methods to systems and denote by $k\tau$ the eigenvalues of kQ. It is often argued that we need only to choose k so small that $k\tau \in \Omega$. For our purposes, where the system is infinite or for $k \to 0$ becomes infinite, such an argument is not correct. Consider, for example, the differential equation

$$u_t = -u_x + F, \qquad 0 \le x \le 1, \quad t \ge 0,$$

with boundary conditions

$$u(0,t) = 0,$$

and approximate it by the upwind scheme

$$\begin{aligned} v(x_\nu, t+k) =& v(x_\nu, t) - \frac{k}{h}\big(v(x_\nu,t) - v(x_{\nu-1},t)\big), \\ & x_\nu = \nu h, \quad \nu = 0,1,2,\dots,N, \quad Nh = 1, \end{aligned}$$

with boundary condition

$$v(0,t) = 0.$$

We can write the system in matrix form

$$\mathbf{v}(t+k) = \begin{pmatrix} 1-\frac{k}{h} & & & 0 \\ \frac{k}{h} & \ddots & \ddots & \\ & & \ddots & \ddots \\ 0 & & \frac{k}{h} & 1-\frac{k}{h} \end{pmatrix} \mathbf{v}(t).$$

The eigenvalue condition is satisfied for

$$k/h < 2.$$

However, we know from Fourier analysis that the method is not stable for $k/h > 1$.

Instead we will again define stability in such a way that it is equivalent with a resolvent condition. To derive the resolvent condition we need to Laplace transform (3.1). For that reason we define the solution of (3.1) for all t by the following procedure. $G(t)$ is a polynomial in F and we can assume that $G(t)$ is defined for all t. (Otherwise we define $G(t) = G(\nu k)$ for $\nu k \le t < (\nu+1)k$.) Also, we define

$$v(t) \equiv 0 \quad \text{for} \quad 0 \le t < k.$$

Then $v(t)$ is defined for all t and we can Laplace transform (3.1).

Observing that

$$\int_0^\infty e^{-st} v(t+k)dt = \int_0^\infty e^{-s(t-k)} v(t)dt = e^{sk} \int_0^\infty e^{-st} v(t)dt$$

the transformed equation (3.1) becomes

$$\big(zI - L(kQ)\big)\hat{v} = k\hat{G}, \qquad z = e^{sk}, \; s = i\xi + \eta. \tag{3.6}$$

Corresponding to Definition 2.2 we have

Definition 3.2: *We call the approximation (3.1) stable in a generalized sense for a sequence $k \to 0$, $h \to 0$, if there are constants K, η_0 such that for all G with $e^{-\eta_0 t}\|G\|^2 < \infty$, all k, h and all $\eta > \eta_0$*

$$(\eta - \eta_0)^2 \int_0^\infty e^{-2\eta t}\|v\|^2 dt \le K \int_0^\infty e^{-2\eta t}\|G\|^2 dt.$$

Parseval's relation gives us

Theorem 3.1: *The approximation (3.1) is stable in a generalized sense for a sequence $k \to 0$, $h \to 0$, if and only if the resolvent condition holds, i.e.,*

$$\|\big(e^{sk}I - L(kQ)\big)^{-1}\| \le \frac{K}{k(\eta - \eta_0)}, \qquad s = i\xi + \eta, \; \eta > \eta_0. \tag{3.7}$$

We will now show that under very mild conditions, (3.7) follows from the resolvent condition (2.8) of the semidiscrete approximation.

Assume that the method is locally stable and assume that

$$\mu = i\alpha, \qquad |\alpha| < R_1, \qquad \text{(see Fig.3.1)}$$

does not belong to Ω. Then there is a real φ such that

$$L(i\alpha) = e^{i\varphi}, \qquad \varphi \text{ real.}$$

We make

Assumption 3.1: *If $\mu = i\alpha$, $|\alpha| < R_1$, is a solution of*

$$L(\mu) = e^{i\varphi}, \qquad \varphi \text{ real},$$

then there is no other purely imaginary root $\mu = i\beta$, $|\beta| < R_1$, with

$$L(\mu) = e^{i\varphi}.$$

For any consistent approximation the above condition is satisfied, if we restrict R_1 to be sufficiently small, because

$$L(\mu) = 1 + \mu + O(\mu^2).$$

It is also satified, if the approximation is dissipative, i.e., $\mu = i\alpha$, $0 < |\alpha| < R_1$, belongs to Ω.

Let $i\alpha$ be a root of the above type. Consider the perturbed equation

$$L(x) = e^{i(\varphi+\xi)+\eta}, \qquad \xi, \eta \text{ real}, \quad \eta > 0. \tag{3.8}$$

We want to show

Lemma 3.1: *$i\alpha$ is a simple root of $L(x) = e^{i\varphi}$. For sufficiently small $|i\xi + \eta|$ the corresponding root of (3.8) can be expanded into a convergent Taylor series*

$$\mu(i\xi + \eta) = i\alpha + \gamma(i\xi + \eta) + O(|i\xi + \eta|^2).$$

Here $\text{Real}\,\mu(i\xi + 0) \geq 0$, and $\gamma > 0$ is real and positive.

Proof: Assume that $d^\nu L(i\alpha)/dx^\nu = 0$, $\nu = 0, 1, \ldots, p-1$. For sufficiently small $(i\xi + \eta)$ (3.8) has p roots $x = i\alpha + \tau$ satisfying to first approximation

$$(d^p L(i\alpha)/dx^p)\tau^p = e^{i\varphi}(1 + i\xi + \eta).$$

If $p > 1$, then at least one of the roots will belong to Ω, which is a contradiction. Thus $p = 1$ and there is an expansion of the above form. We arrive at the same contradiction if $\text{Real}\,\mu(i\xi + 0) < 0$ for some $i\xi$ and therefore γ must be real. Observing that $\text{Real}\,\mu(i\xi + \eta)$ must be positive, it follows that $\gamma > 0$.

We will now prove

Theorem 3.2: *Assume that the Runge-Kutta method is locally stable and that the conditions of Assumption 3.1 are satisfied. If the semi-discrete approximation is stable in* *a generalized sense, then the totally discretized approximation is stable in the same sense, if*

$$\|kQ\| \le R < R_1. \tag{3.9}$$

Proof: We have to prove that the resolvent equation (3.6) of the totally discretized equation satisfies the estimate (3.7). For every z with $|z| > 1$ we can write (3.6) in the form

$$\Pi\big(\mu_j(z)I - kQ\big)\hat{w}_\nu = k\hat{G}.$$

The roots μ_j do not belong to Ω. There are three possibilities.

1) $|\mu_j(z)| - R > \delta > 0$, δ=constant. (3.9) implies

$$\|\big(\mu_j(z) - kQ\big)^{-1}\| \le \big(|\mu_j(z)| - R\big)^{-1} \le \delta^{-1}.$$

Let $\text{Real}\,\mu_j \le 0$. We know that μ_j does not belong to Ω and therefore there is a constant $\delta_1 > 0$ such that

$$|\mu_j(z)| - R \ge \delta_1 > 0.$$

Thus the above inequality holds for sufficiently small $\delta > 0$.

2) $\text{Real}\,\mu_j \ge \delta_2 > 0$, δ_2 constant > 0. In this case (2.8) tells us

$$\begin{aligned}\|\big(\mu_j(z) - kQ\big)^{-1}\| &= \frac{1}{k}\|\big(\frac{\mu_j(z)}{k} - Q\big)^{-1}\| \\ &\le \frac{1}{k}K\frac{k}{\delta_2} = \frac{K}{\delta_2}.\end{aligned}$$

3) $\text{Real}\,\mu_j(z) > 0$ but for $k \to 0$ $\lim z = e^{i\varphi}$, $\lim \mu_j(z) = i\alpha$, α, φ real, $|\alpha| \le R$. Let

$$z = e^{i\varphi + k(i\xi+\eta)}, \quad \varphi, \xi, \eta \text{ real.}$$

By Lemma 3.1

$$\mu_j(z) = \mu_j(i\xi + 0) + \gamma k\eta + O\big(k^2(|\xi|\eta + \eta^2)\big). \tag{3.10}$$

Therefore by (2.8)

$$\|\big(\mu_j(z) - kQ\big)^{-1}\| \le \frac{1}{k}\|\big(\frac{\mu_j(z)}{k} - Q\big)^{-1}\| \le \frac{K}{k(\eta - \eta_0)}.$$

Now we can prove the theorem. We want to show that

$$\|\big(zI - L(kQ)\big)^{-1}\| \le \text{ const. } \frac{1}{k(\eta - \eta_0)}.$$

If $|z| = e^{\eta k} > 2\|L(kQ)\|$, then

$$\|\big(zI - L(kQ)\big)^{-1}\| \le \big(|z| - \|L(kQ)\|\big)^{-1} \le \frac{1}{|z|} \le \frac{1}{\eta k}.$$

Thus we can restrict ourselves to $\eta k \leq$ constant. Combining the above estimates and observing that for a given $z = e^{i\varphi}$ there is at most one root $\mu_j(z) = i\alpha$, we obtain

$$\|(zI - L(kQ))^{-1}\| = \|\Pi(\mu_j(z) - kQ)^{-1}\|$$
$$\leq \begin{cases} \text{const.}\ \frac{1}{k(\eta-\eta_0)} & \text{if one of the roots has the form (3.10).} \\ \text{const.} & \text{otherwise.} \end{cases}$$

This proves the theorem.

4. Multistep methods. Instead of a Runge-Kutta method we now consider a multistep method

$$(I - k\beta_{-1}Q)v(t+k) = \sum_{j=0}^{r}(\alpha_j I + k\beta_j Q)v(t-jk) + k\,G(t-rk) \tag{4.1}$$

with real coefficients α_j, β_j. We want to show that stability in a generalized sense again follows from the resolvent condition of the semi-discrete problem. The resolvent equation becomes

$$(L_1(z)I - kL_2(z)Q)\hat{v} = k\hat{G}, \qquad z = e^{sk}, \quad s = i\xi + \eta,$$

and we have to prove that for $\eta > \eta_0$

$$\|(L_1(z)I - kL_2(z)Q)^{-1}\| \leq \frac{\text{const.}}{k(\eta - \eta_0)}. \tag{4.2}$$

Here

$$L_1 = z^{r+1} - \sum_{j=0}^{r}\alpha_j z^{r-j}, \quad L_2 = \beta_{-1}z^{r+1} + \sum_{j=0}^{r}\beta_j z^{r-j}.$$

We apply the method again to the scalar differential equation $y' = \lambda y$. Then we obtain the characteristic equation

$$L_1(z) - \mu L_2(z) = 0, \qquad \mu = \lambda k.$$

We make the usual assumptions for a multistep method.

Assumption 4.1: *The equations*

$$L_1(z) = 0, \qquad L_2(z) = 0$$

have no root in common.

$$\sum_{j=0}^{r}\alpha_j = 1, \qquad \sum_{j=-1}^{r}\beta_j = 1.$$

The roots z_j of $L_1(z) = 0$ with $|z_j| = 1$ are simple.

The above assumption implies

$$L_1(1) = 0, \quad L_2(1) = 1, \quad L_1'(1) \neq 0. \tag{4.3}$$

As in the previous case there is an open domain Ω in the complex plane $\mu = \lambda k$ such that

$$L_1(z) - \mu L_2(z) \neq 0 \quad \text{for } |z| > 1.$$

We make now the same construction as earlier, which leads to

Assumption 4.2: *The approximation is locally stable, i.e., there exist a $R_1 > 0$ such that the open halfcircle*

$$|\mu| < R_1, \quad \textit{Real } \mu < 0, \quad \textit{belongs to } \Omega.$$

If $\mu = i\alpha$, α real, $|\alpha| \leq R_1$, does not belong to Ω, then there is a $z = e^{i\varphi}$, φ real, such that

$$L_1(e^{i\varphi}) - i\alpha L_2(e^{i\varphi}) = 0. \tag{4.4}$$

We make

Assumption 4.3: *$z = e^{i\varphi}$ is a simple root of*

$$L_1(z) - i\alpha L_2(z) = 0.$$

$L_1(z)$ has only simple roots near $z = 1$ and therefore the last assumption holds if we choose R_1 sufficiently small. Also

$$\begin{aligned} \frac{dR(e^{i\varphi})}{dz} &= \frac{L_2(e^{i\varphi})L_1'(e^{i\varphi}) - L_1(e^{i\varphi})L_2'(e^{i\varphi})}{L_2^2(e^{i\varphi})} \\ &= \frac{L_1'(e^{i\varphi}) - \alpha L_2'(e^{i\varphi})}{L_2(e^{i\varphi})} \neq 0. \end{aligned}$$

Let φ, α be the solution of (4.4) and replace $i\varphi$ by $i\varphi + i\xi + \eta$, $\eta > 0$. Then $z = e^{i\varphi + i\xi + \eta}$ satisfies

$$L_1(z) - \mu L_2(z) = 0$$

with

$$\mu(i\xi + \eta) = i\alpha + \gamma(i\xi + \eta) + O(|i\xi + \eta|^2).$$

In the same way as for Lemma 3.1 we obtain for sufficiently small $|i\xi + \eta|$

Lemma 4.1: *$\textit{Real}\,\mu(i\xi + 0) \geq 0$ and $\gamma > 0$ is real and positive.*

Now we can prove

Theorem 4.1: *Assume that the conditions of Assumptions 4.1 – 4.3 are satisfied and that the semi-discrete approximation is stable in a generalized sense. Then the totally discretized multistep method (4.1) is stable in the same sense, provided*

$$\|kQ\| \le R < R_1. \tag{4.5}$$

Proof: Let z_j denote the zeros of $L_2(z) = 0$. Clearly there is a neighbourhood $|z - z_j|$, where the estimate (4.2) holds. Therefore we can write the resolvent equation in the form

$$(R(z)I - kQ)\hat{v} = \frac{k}{L_2(z)}\hat{G}.$$

We know that $R(z)$ does not belong to Ω. There are three possibilities.

1) $|R(z)| - k\|Q\| \ge \delta$, $\delta > 0$ constant. Then

$$\|(R(z) - kQ)^{-1}\| \le \frac{1}{|R(z)| - \|kQ\|},$$

i.e.,

$$\|(L_1(z)I - kL_2(z)Q)^{-1}\| \le \frac{1}{(|R(z)| - \|kQ\|)|L_2(z)|}$$

and the desired estimate follows. The above inequality is satisfied, if δ is sufficiently small and $\operatorname{Real} R(z) \le 0$, because $R(z) \notin \Omega$. It is also satisfied, if $\beta_{-1} = 0$ and $|z|$ sufficiently large.

2) $\operatorname{Real} R(z) > \delta > 0$. By (2.8)

$$\|\big(R(z)I - kQ\big)^{-1}\| = k^{-1}\|\big(\frac{R(z)}{k} - Q\big)^{-1}\| \le \frac{K\,(k/\delta)}{k} = \frac{K}{\delta}.$$

Thus

$$\|\big(L_1(z)I - kL_2(z)Q\big)^{-1}\| \le \frac{K}{\delta|L_2(z)|},$$

and the desired estimate follows.

3) $\operatorname{Real} R(z) > 0$ but $\lim_{z \to z_0} R(z) = i\alpha$, $z_0 = e^{i\varphi}$, α, φ real, $|\alpha| \le R$. Let $z = e^{i\varphi + (i\xi + \eta)k}$. By Lemma 4.1

$$R(z) = \mu(i\xi + \eta) = i\alpha + \gamma k(i\xi + \eta) + O(k^2|i\xi + \eta|^2),$$
$$\operatorname{Real} R(z) \ge \gamma k\eta + O(|\xi|\eta + \eta^2).$$

Therefore

$$\|\big(R(z)I - kQ\big)^{-1}\| \le \frac{\text{const.}}{\gamma} \cdot \frac{1}{k(\eta - \eta_0)},$$

and the desired estimate is again valid.
This proves the theorem.

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