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THE SEMIGROUP STABILITY OF THE DIFFERENCE APPROXIMATIONS FOR INITIAL BOUNDARY VALUE PROBLEMS

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ABSTRACT.
We will show that, for semi-discrete approximations and two-step totally discretized approximations to the initial-boundary value problems of linear hyperbolic equations, Kreiss' condition is a sufficient condition for the semigroup stability (or l_2 stability). This result proves that the semigroup stability is contained by the strong stability used in the well-known GKS theory for the difference approximations.

1. INTRODUCTION

Consider the following first order one dimensional hyperbolic equations

$$\frac{\partial \mathbf{u}}{\partial t} = A\frac{\partial \mathbf{u}}{\partial x} + C\mathbf{u} + \mathbf{F} \tag{1}$$

in the quarter plane $\Omega = \{(x,t) \mid x,t \geq 0\}$. Here $\mathbf{u}(x,t) = (u^{(1)}(x,t),\ldots,u^{(n)}(x,t))'$ and $\mathbf{F} = (F^{(1)}(x,t),\ldots,F^{(n)}(x,t))'$ are vector functions, A and C are $n\times n$ constant matrices so that A is diagonal,

$$A = \begin{pmatrix} A^I & 0 \\ 0 & A^{II} \end{pmatrix}, \quad \text{with } A^I < 0, A^{II} > 0,$$

A^I, A^{II} are $l \times l$ and $(n-l)\times(n-l)$ diagonal matrices respectively. The solution is uniquely determined[4][7] if we prescribe the initial conditions

$$\mathbf{u}(x,0) = \mathbf{f}(x), \qquad x \geq 0 \tag{2}$$

and the boundary conditions

$$\mathbf{u}^I(0,t) = S\mathbf{u}^{II}(0,t) + \mathbf{g}(t), \qquad t \geq 0, \tag{3}$$

where $\mathbf{u}^I$ and $\mathbf{u}^{II}$ are the partition of $\mathbf{u}$ according to that of A, and S is a $l\times(n-l)$ matrix.

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We assume the well-posedness of the above initial-boundary value problem (IBV for short) in the following sense.

Assumption 1.1. *The IBV problem (1,2,3) is well-posed in the sense:*

1) *For a dense set of smooth data there is a smooth solution.*
2) *The solutions of the homogeneous equations ($F = 0$) and homogeneous boundary condition ($g = 0$) satisfies*

$$\|u(\cdot,t)\| \le Ke^{\eta_0 t}\|u(\cdot,0)\|, \tag{4}$$

where η_0 and K are universal constants.

(4) is interpreted as the stability in the semigroup sense. Note that $\|\cdot\|$ is the usual L^2 norm in the half space. For the solutions of inhomogeneous equations, the estimates can be obtained by Duhamel's principle. The theory of well-posedness for the continuous IBV problems is well developed. Readers can find results from [1][4][7][12] and other sources, we will simply not get into it in this paper.

The definition of stability in the semigroup sense (4) will also be used for later the difference approximations to (1,2,3). We will show that, for semi-discrete approximations and two-step totally discretized approximations to the initial-boundary value problems of linear hyperbolic equations, Kreiss' condition is a sufficient condition for the semigroup stability (or l_2 stability). This result proves that the semigroup stability is contained by the strong stability used in the well-known GKS theory for the difference approximations. For the semigroup stability of the semi-discrete approximations, our theory is perhaps unique. For totally discretized approximations, our results overwrite and generalize the other two classical theories by Kreiss[5][6] and Osher[11].

2. Semi-discrete Approximations

2.1. Prerequisite. *The methods of line* are nowadays very popular methods to solve for the approximate solutions of hyperbolic equations. The first step of the methods is spatial discretizations, which result in consistent ordinary differential equations or what are called the semi-discrete approximations, and the second step is the employment of standard numerical methods for solving ordinary differential equations. In this section we will study the well-posedness of the semi-discrete approximations only. For that purpose, we introduce a mesh with size $h = \Delta x > 0$, and, with the notation $\mathbf{u}_\nu(t) \approx \mathbf{u}(\nu h, t)$, approximate the equations (1) by a consistent semi-discrete scheme of the form

$$\frac{d\mathbf{u}_\nu(t)}{dt} = Q\mathbf{u}_\nu(t) + \mathbf{F}_\nu(t), \quad \nu = 1, 2, \dots \tag{5}$$

$$Q = \sum_{j=-r}^{p} A_j E^j,$$

where A_j are linear functions of A and C. Initial conditions follow naturally from (2), i.e,

$$\mathbf{u}_\mu(0) = \mathbf{f}_\mu, \quad \mu = 1, 2, \dots. \tag{6}$$

Boundary conditions, both natural and artificial, are put in the form

$$\mathbf{u}_\mu(t) = \sum_{j=1}^{q} C_{\mu j} E^j \mathbf{u}_\mu(t) + \mathbf{g}_\mu(t), \quad \mu = -r+1, -r+2, \ldots, 0. \tag{7}$$

For our later reference, we call (5) the *basic scheme* of the semi-discrete approximation.

The solution space of the IBV problem (5,6,7) is $l^{2,n}(1,\infty)$, which is defined by

$$l^{2,n}(-M,N) = \{\mathbf{u} = \{\mathbf{u}_j\}_{-M}^{N}, \mathbf{u}_j \in \mathbf{C}^n \mid ||\mathbf{u}||_{-M,N} < \infty\}$$

where the norm comes from the associated inner product

$$(\mathbf{u},\mathbf{v})_{-M,N} = \sum_{-M}^{N} \mathbf{u}_j^T \bar{\mathbf{v}}_j h,$$

thus

$$||\mathbf{u}||^2_{-M,N} = (\mathbf{u},\mathbf{u})_{-M,N}.$$

For the sake of convenience, the indices of the norm and inner product of $l^{2,n}(1,\infty)$ will be omitted. And we write $l^2(-M,N)$ for $l^{2,1}(-M,N)$.

To the well-posedness of the semi-discrete problems, the solvability is never a problem as we can always obtain the formal solutions. The stability of the problems, therefore, becomes our main concern. Naturally we will only be interested in those special discretizations which result in stable semi-discrete approximations for the corresponding Cauchy problems.

Assumption 2.1. *The operator Q for the Cauchy problem of the semi-discrete equation (5) is semi-bounded, i.e, there exists a real constant η_0 such that*

$$(u,Qu)_{-\infty,\infty} + (Qu,u)_{-\infty,\infty} \le 2\eta_0(u,u)_{-\infty,\infty}.$$

The immediate consequence of semi-boundedness of Q is, with $\mathbf{F}=0$,

$$||\mathbf{u}(t)||_{-\infty,\infty} \le Ce^{\eta_0 t}||\mathbf{u}(0)||_{-\infty,\infty}$$

In the prospect of the convergence of the approximate solutions to the exact solutions, we also define the stability for the semi-discrete problems in the semigroup sense as that used for the continuous problems.

Definition 2.1. *The discrete problem (5,6,7) is stable if the solutions of the homogeneous equations ($\mathbf{F}=0$) with homogeneous boundary conditions ($g=0$) satisfy an energy estimate*

$$||u(\cdot,t)|| \le Ke^{\eta_0 t}||u(\cdot,0)||,$$

where η_0, K are universal constants.

The estimate for inhomogeneous equations can be obtained with Duhamel's principle,

$$||\mathbf{u}(\cdot,t)|| \le Ke^{\eta_0 t}(||\mathbf{u}(\cdot,0)||, + \int_0^t ||\mathbf{F}(\cdot,\tau)||d\tau).$$

Next we will introduce the major results of the semi-discrete version of the classical GKS theory[13]. upon which our theory will be based. In the GKS theory, the following stability definition, which is named the stability in the strong sense, is adapted:

Definition 2.2. *The discrete problem (5,6,7) is stable if for $\eta > \eta_0$, the solutions of the problems with homogeneous initial value ($f = 0$) satisfy*

$$\begin{aligned} &\int_0^\infty (|u(0,t)|_B^2 + (\eta - \eta_0)\|u(\cdot,t)\|^2)e^{-2\eta t}\,dt \\ &\leq K\int_0^\infty (|g|_B^2 + \tfrac{1}{\eta-\eta_0}\|F(\cdot,t)\|^2)e^{-2\eta t}\,dt, \end{aligned} \tag{8}$$

where η_0, K are universal constants.

The terms with index B are boundary norms defined by

$$|\mathbf{u}|_B = \sum_{j=0}^{-r+1} |\mathbf{u}_j|, \quad |\mathbf{u}_j| = \sum_{i=1}^{n} |u_j^{(i)}|. \tag{9}$$

The necessary and sufficient condition for the strong stability is determined by an eigenvalue problem, which is obtained by taking the Laplace transform on the homogenized equations of (5,6,7):

$$\begin{aligned} &s\hat{\mathbf{u}}_\nu = Q\hat{\mathbf{u}}_\nu, \qquad Re(s) \geq 0, \quad \nu = 1, 2, \ldots \\ &\hat{\mathbf{u}}_\mu = \textstyle\sum_{j=1}^{q} C_{\mu j}\hat{\mathbf{u}}_{\mu+j}, \quad \mu = -r+1, -r+2, \ldots, 0. \end{aligned} \tag{10}$$

The eigenvalues and generalized eigenvalues of (10) are defined below.

Definition 2.3. *Let $C = 0$. s is an eigenvalue if it satisfies the following conditions:*

(1) $(sI - Q)\hat{u} = 0$,
(2) $Re(s) \geq 0$,
(3) $\hat{u}_\mu = \sum_{j=1}^{q} C_{\mu j}\hat{u}_{\mu+j}, \quad \mu = -r+1, -r+2, \ldots, 0$,
(4) *when* $Re(s) > 0, \quad \|u\|_2 < \infty$.

s is a generalized eigenvalue if condition 1,2 and 3 are satisfied, and condition 4 is replaced with

(4) *when $Re(s) = 0$, $\hat{u}_j(s) = \lim_{\epsilon\to 0} \hat{u}_j(s+\epsilon)$, where $Re(\epsilon) > 0$ and $u_j(s+\epsilon)$ satisfies (1) with s replaced by $s + \epsilon$.*

We can now state

Theorem 2.1 (Strikwerda). *The approximations (5,6,7) is stable according to Definition 2.2 if and only if (10) has no eigenvalue nor generalized eigenvalue on the half plane $Re(s) \geq 0$.*

The eigenvalue condition in the above theorem is usually referred as the Kreiss' condition. Sometimes it is more convenient to use the following interpretation of the Kreiss condition[13].

Lemma 2.1 (Strikwerda). *For the semi-discrete approximations, Kreiss condition is equivalent to that, if $F = 0, f = 0$,*

$$|\hat{u}|_B \leq K|\hat{g}|_B. \tag{11}$$

There are certain assumptions in the GKS theory which have to be imposed in this paper as well. In fact, we have found no semi-discrete approximation which is an exception to these assumptions.

Assumption 2.2. *The basic scheme (5) is either dissipative or nondissipative, i.e, the roots of the characteristic equation*

$$det \mid sI - \hat{Q}(i\xi) \mid = 0, \quad \hat{Q}(i\xi) = \sum_{j=-r}^{p} A_j e^{ij\xi} \tag{12}$$

satisfy either

$$Re(s) < 0, \quad 0 < |\xi| \le \pi,$$

or

$$Re(s) = 0, \quad |\xi| \le \pi.$$

Finally in this section we claim that we can reduce our stability study to scalar problems. The definition of stability in the semigroup sense has the property that it is stable against lower order perturbations. More specifically, with Duhamel's principle and the Gronwall's inequality, we can show(see [9] for instance)

Lemma 2.2. *Suppose the solution of infinite system*

$$\frac{du}{dt} = Qu$$

satisfies energy estimate

$$||u(t)|| \le Ke^{\eta_0 t}||u(0)||.$$

Let H be any bounded linear operator with

$$||H|| \le \beta,$$

then the solution of the perturbed system

$$\frac{dw}{dt} = (Q + H)w$$

satisfies

$$||w(t)|| \le Ke^{\gamma t}||w(\cdot, 0)||, \quad \gamma = \eta_0 + K\beta$$

Thus, lower order terms play no role in stability analysis and hence they will be laid off from our discussions. Once all lower order terms are ignored, the equations become decoupled (except the boundary conditions). Under such circumstance, all our discussions and assertions on a single scalar equation are formally the same as those on a system of equations. For the sake of simplicity, we will proceed with a single equation in subsequent sections, and indicate results for systems of equations accordingly.

2.2. Kreiss' Condition and the Semigroup Stability. Our fundamental technique here is to construct a set of special boundary conditions to form an auxiliary problem whose solution at every line $x = x_j$ can be bounded in terms of the initial values,

$$\int_0^\infty e^{-2\eta_0 t}|u_j(t)|^2 dt \le c_j \|u(0)\|^2, \quad j = 1, 2, \dots,$$

where c_j depends on j only, and then subtract that auxiliary problem from the original one, (5,6,7). In this way the original problems with inhomogeneous initial data are reduced to the problems with inhomogeneous boundary conditions but homogeneous initial data. Then, Lemma 2.1 and energy estimates will lead to our results.

As we have explained previously, we only need to consider the scalar problem

$$(13) \qquad \begin{cases} \frac{du_j(t)}{dt} = Qu_j(t), & j = 1, 2, \dots, \\ u_j(0) = f_j, \\ Bu_0(t) = 0, \end{cases}$$

where

$$Q = \frac{1}{h}\sum_{j=-r}^{p} a_\mu E^j, \quad \text{with } a_{-r} \ne 0, a_p \ne 0,$$

is the difference approximation to $a\frac{\partial}{\partial t}$, and the operator B represents a set of boundary conditions of the form

$$u_\mu(t) = \sum_{j=1}^{q} \beta_{\mu j} u_{\mu+j}, \qquad -r+1 \le \mu \le 0,$$

which make Q well-defined in $l^2(1,\infty)$.

For later discussions we define a one-to-one mapping $I : l^2(1,\infty) \to l_0^2(-\infty,\infty)$ by

$$(Iu)_j = \begin{cases} u_j & j = 1, 2, \dots, \\ 0 & j \le 0. \end{cases}$$

Recalling Assumption 2.1 we know that for any $u \in l^2(-\infty,\infty)$,

$$Re(u, Qu)_{-\infty,\infty} \le \eta_0(u,u)_{-\infty,\infty},$$

from which we have

Theorem 2.2. *There exists boundary operator B_0 such that for all $u \in l^2(1,\infty)$ satisfying $B_0 u_0(t) = 0$ the following inequality holds*

$$Re(u, Qu) \le \eta_0(u,u) - \sum_{j=1}^{r} |u_j|^2.$$

PROOF: We translate the boundary conditions, which are not yet determined, into a single vector $u^b \in l^2(-\infty,\infty)$,

$$(u^b(t))_j = \begin{cases} u_j^b(t) & \text{for } -r+1 \le j \le 0, \\ 0 & \text{others.} \end{cases}$$

Then, the inner products in $l^2(1,\infty)$ can be expressed by that in $l^2(-\infty,\infty)$:

$$\begin{aligned}(u,Qu) &= (Iu, Q(Iu+u^b))_{-\infty,\infty}\\ &= (Iu,QIu)_{-\infty,\infty}+(Iu,Qu^b)_{-\infty,\infty}.\end{aligned}$$

So we have, after taking the real parts of each term,

$$\begin{aligned}Re(u,Qu) &\le \eta_0(u,u)+Re(Iu,Qu^b)_{-\infty,\infty}\\ &= \eta_0(u,u)+Re(u,Qu^b)_{1,r}\\ &= \eta_0(u,u)+Re\{U^*Q_1U^b\},\end{aligned}$$

where

$$U=(u_1,u_2,\ldots,u_r)^T,$$
$$U^b=(u^b_{-r+1},u^b_{-r+2},\ldots,u^b_0)^T,$$

and Q_1 is a nonsingular triangular matrix

$$Q_1=\begin{pmatrix} a_{-r} & a_{-r+1} & a_{-r+2} & \ldots & a_{-1} & a_0\\ 0 & a_{-r} & a_{-r+1} & a_{-r+2} & \ldots & a_{-1}\\ 0 & 0 & a_{-r} & a_{-r+1} & \ldots & a_{-2}\\ & \vdots & & \vdots & & \vdots\\ 0 & 0 & 0 & 0 & 0 & a_{-r}\end{pmatrix}.$$

Hence if we choose

$$U^b=-Q_1^{-1}U, \tag{14}$$

then we will get

$$Re(u,Qu)_{1,\infty}\le\eta_0(u,u)-\sum_{j=1}^{r}|u_j|^2.$$

The boundary operator corresponding to (14) is called B_0 □

With the special boundary operator B_0 the solution of (13) satisfies

$$e^{-2\eta_0 t}\|u(t)\|^2+\int_0^t\sum_{j=1}^{r}e^{-2\eta_0\tau}|u_j|^2d\tau=\|u(0)\|^2. \tag{15}$$

If we introduce notation

$$\langle u_j,u_j\rangle=\int_0^\infty e^{-2\eta_0 t}|u_j(t)|^2dt,$$

then (15) yields

$$\langle u_j,u_j\rangle\le\|u(0)\|^2,\quad j=1,\ldots,r.$$

Next we will show that with the special boundary operator B_0 we can obtain estimates for any u_j. Considering $u_j, j=1,\ldots,r$, as known, we treat $u_j, j=r+1,\ldots,\infty$, as the solutions of the following system

$$\begin{cases}\frac{du_j}{dt}=Qu_j(t), & j=r+1,r+2,\ldots\\ u_j(0)=f_j, & \\ u_\mu=u_\mu,\quad 1\le\mu\le r\quad\text{given.}\end{cases}$$

We split $u_j(t), j \geq r+1$ into

$$u_j(t) = v_j + (u_j(t) - v_j(t)), \quad j = r+1, r+2, \dots,$$

where $v_j(t), j \geq r+1$ satisfy

$$\begin{cases} \frac{dv_j}{dt} = Qv_j(t), & j = r+1, r+2, \dots \\ v_j(0) = f_j, & \\ B_0 v_r = 0, & \end{cases}$$

and $w_j(t) := u_j(t) - v_j(t), j = r+1, r+2, \dots$, satisfy

$$\begin{cases} \frac{dw_j}{dt} = Qw_j(t), & j = r+1, r+2, \dots \\ w_j(0) = 0, & \\ w_\mu = u_\mu - v_\mu, \quad 1 \leq \mu \leq r. & \end{cases} \tag{16}$$

From Theorem 2.2 and (15) we have

$$\langle v_j, v_j \rangle \leq \|u(0)\|^2_{r+1,\infty} \leq \|u(0)\|^2, \quad j = r+1, \dots, 2r.$$

Thus we only need to estimate $w_j, j = r+1, \dots, 2r$. Taking Laplace transform on (16) for $s = \eta + i\xi$ with $\eta > \eta_0$, we end up with

$$\begin{cases} s\hat{w}_j = Q\hat{w}_j, & j = r+1, r+2, \dots \\ \hat{w}_\mu = \hat{u}_\mu - \hat{v}_\mu, & \mu = 1, \dots, r. \end{cases} \tag{17}$$

Its corresponding eigenvalue problem reads

$$s\hat{w}_j = Q\hat{w}_j, \quad j = r+1, r+2, \dots, \tag{18}$$

$$\hat{w}_\mu = 0, \quad \mu = 1, \dots, r. \tag{19}$$

The characteristic equation of (18) is

$$sh = \sum_{j=-r}^{p} a_j \kappa^j, \tag{20}$$

Its roots are the continuous functions of $\tilde{s} = sh$. Let $\kappa_\alpha = \kappa_\alpha(s), \ 1 \leq \alpha \leq l$, whose multiplicity are $m_\alpha = m_\alpha(s)$ accordingly, be those roots of (20) lying inside the unit circle when $Re(\tilde{s}) > 0$, then the general solution of (18) in $l^2(1, \infty)$ is given by

$$\hat{w}_j = \sum_{\alpha=1}^{l} \sum_{\beta=0}^{m_\alpha - 1} c_{\alpha\beta} P_{\alpha\beta}(j) \kappa_\alpha^j,$$

where $P_{\alpha\beta}(j)$ are arbitrary polynomials in j with degree exactly equal to β, and $c_{\alpha\beta}$ are parameters determined by boundary conditions (19), which now read

$$\sum_{\alpha=1}^{l} \sum_{\beta=0}^{m_\alpha - 1} c_{\alpha\beta} P_{\alpha\beta}(\mu) \kappa_\alpha^\mu = 0, \quad \mu = 1, \dots, r. \tag{21}$$

Note that the number of these roots is equal to the number of boundary conditions, i.e,

$$\sum_{\alpha=1}^{m} m_\alpha = r,$$

thus the number of the parameters $c_{\alpha\beta}$ is exactly the number of equations in (21). We will show that there is only the trivial solution to (19) with a technique introduced by Goldberg and Tadmor[2].

Lemma 2.3. *The eigenvalue problem (18) satisfies the Kreiss condition.*

PROOF: We make a special selection of $P_{\alpha\beta}$ as

$$P_{\alpha\beta}(\mu) = \kappa_\alpha^{-1-\beta}\beta! \begin{pmatrix} \mu-1 \\ \beta \end{pmatrix}.$$

Then (19) reads

$$\sum_{\alpha=1}^{l}\sum_{\beta=0}^{m_\alpha-1} \beta! \begin{pmatrix} \mu-1 \\ \beta \end{pmatrix} \kappa_\alpha^{\mu-1-\beta} c_{\alpha\beta} = 0, \quad \mu = 1, \dots, r,$$

i.e, if we let $\nu = \mu - 1$,

$$\sum_{\alpha=1}^{l}\sum_{\beta=0}^{m_\alpha-1} \frac{\partial^\beta \kappa_\alpha^\nu}{\partial \kappa_\alpha^\beta} c_{\alpha\beta} = 0, \quad , \nu = r-1, r-2, \dots, 0.$$

The coefficient matrix of above system is

$$J = [B(\kappa_1, m_1), \dots, B(\kappa_l, m_l)],$$

where, for $i = 1, \dots, l$,

$$B(k_i, m_i) = \left[\begin{pmatrix} \kappa_i^{r-1} \\ \kappa_i^{r-2} \\ \vdots \\ 1 \end{pmatrix}, \frac{\partial}{\partial \kappa} \begin{pmatrix} \kappa_i^{r-1} \\ \kappa_i^{r-2} \\ \vdots \\ 1 \end{pmatrix}, \dots, \frac{\partial^{m_i-1}}{\partial \kappa^{m_i-1}} \begin{pmatrix} \kappa_i^{r-1} \\ \kappa_i^{r-2} \\ \vdots \\ 1 \end{pmatrix} \right]_{\kappa=\kappa_i}.$$

Let $\mathbf{c} = (c_1, c_2, \dots, c_r)^T$ be a vector such that

$$J\mathbf{c} = 0,$$

which means

$$\{\sum_{\mu=0}^{r-1} c_\mu \frac{\partial^j}{\partial \kappa^j} \kappa^\mu\}_{\kappa=\kappa_i} = 0, \quad 0 \le j \le m_i - 1, \qquad 1 \le i \le l,$$

i.e,

$$\frac{\partial^j}{\partial \kappa^j} \{\sum_{\mu=0}^{r-1} c_\mu \kappa^\mu\}_{\kappa=\kappa_i} = 0, \quad 0 \le j \le m_i - 1, \qquad 1 \le i \le l.$$

From the above relations we conclude that the polynomial

$$P(\kappa) := \sum_{\mu=0}^{r-1} c_\mu \kappa^\mu$$

has r roots, $\kappa_j, 1 \le j \le m_i - 1,\ 1 \le i \le j$. As $P(\kappa)$ is of degree $r-1$, this means $P(\kappa) \equiv 0$. So, $c_\mu = 0, \mu = 1, \dots, r-1$. Hence we know that the coefficient matrix J must be nonsingular, and the Kreiss condition is thus satisfied □

According to Lemma 2.1, we can estimate the solution of (17) in terms of the boundary data,

$$|\hat{w}_j| \le const. \sum_{\mu=1}^{r} |\hat{w}_\mu|, \quad j = r+1, r+2, \dots$$

Hence, for $j = r+1, \dots, 2r$,

$$\begin{aligned} |\hat{u}_j| &\le |\hat{w}_j| + |\hat{v}_j| \\ &\le const. \sum_{i=1}^{r} (|\hat{w}_i| + |\hat{v}_j|). \end{aligned}$$

With Parseval equality, these inequalities lead us to the desired estimates

$$\langle u_j, u_j \rangle \le c_j \|u(0)\|^2, \quad j = r+1, \dots, 2r.$$

The approach used to derive the estimates of $\langle u_j, u_j \rangle, j = r+1, \dots, 2r$, can be used inductively to derive the estimates of all $\langle u_j, u_j \rangle$. Hence, we arrive at

Lemma 2.4. *For scheme (13) with special boundary condition B_0 we have*

$$\langle u_j, u_j \rangle \le c_j \|u(0)\|^2, \quad j \ge 1,$$

where c_j depends on j only.

Now we can formulate our major result about the semigroup stability of (13) in terms of the eigenvalue problem:

$$\left\{ \begin{array}{l} s\hat{u}_j = Q\hat{u}_j, \quad j \ge 1, \\ B\hat{u}_0 = 0. \end{array} \right. \tag{22}$$

Theorem 2.3. *If Q is semi-bounded for the Cauchy problem and the eigenvalue problem (22) satisfies the Kreiss condition, then the IBV problem (13) is stable in the semigroup sense.*

PROOF: Let v denote the solution of

$$\left\{ \begin{array}{ll} \frac{dv_j(t)}{dt} = Qv_j(t), & j = 1, 2, \dots, \\ v_j(0) = f_j & \\ Bv_0(t) = 0. & \end{array} \right. \tag{23}$$

From Lemma 2.4 we have

$$\langle v_j, v_j \rangle \le c_j \|u(0)\|^2, \quad j \ge 1.$$

Function $w = u - v$ satisfies

$$\left\{ \begin{array}{ll} \frac{dw_j(t)}{dt} = Qw_j(t), & j = 1, 2, \dots \\ w_j(0) = 0, & \\ Bw_0(t) = -Bv_0(t). & \end{array} \right. \tag{24}$$

Laplace transform on the above equation gives us

$$\left\{ \begin{array}{l} s\hat{w}_j = Q\hat{w}_j, \quad j \ge 1, \\ B\hat{w}_0 = -B\hat{v}_0. \end{array} \right. \tag{25}$$

The Kreiss condition implies that the solution at the boundary can be estimated in terms of the data, i.e, for any integer p, there are constants c'_p, c_p such that

$$|\hat{w}|^2_{B_p} = \sum_{j=1}^{p} |\hat{w}_j|^2 \leq c'_p \sum_{j=1}^{r} |(B\hat{v}_0)_j|^2 \leq c_p \sum_{j=1}^{p} |\hat{v}_j|^2.$$

Hence,

$$\int_0^\infty e^{-2\eta_0 t}|w(t)|^2_{B_p} dt \leq c_p \int_0^\infty e^{-2\eta_0 t} \sum_{j=1}^{r} |v_j(t)|^2 dt \leq const.\|u(0)\|^2.$$

Because

$$\frac{d(w,w)}{dt} = 2(w, Qw) \leq 2\eta_0(w,w) + const.|w|^2_{B_p}$$

for some p, we therefore obtain

$$\|w(t)\|^2 \leq const.e^{2\eta_0 t}\|u(0)\|^2,$$

and finally arrive at

$$\|u(t)\|_2 \leq \|v(t)\|_2 + \|w(t)\|_2 \leq const.e^{\eta_0 t}\|u(0)\|_2.$$

The result is hence proved □

The corresponding result for system of schemes follows.

Theorem 2.4. *If Q is semi-bounded for Cauchy problem, then the IBV problem (5,6,7) is stable in the semigroup sense if the Kreiss condition is satisfied.*

The previous results can be generalized to symmetric multi-dimensional hyperbolic equations. It is not hard to see, with the availability of the GKS theory for the difference approximations of multi-dimensional hyperbolic problems by Michalson[10], the derivation of the similar results can be proceeded parallelly, in no need of extra efforts.

3. Two-step Totally Discretized Approximations

3.1. Prerequisite. Now we consider the stability of the totally discretized approximations of the initial boundary value problems (1,2,3). Beside the space mesh with size $h = \Delta x > 0$, we define time step $k = \Delta t > 0$, and, with the notation $\mathbf{u}^n_\nu \approx \mathbf{u}(\nu h, nk)$, approximate the equations by a consistent two-step scheme of the form

$$\mathbf{u}^{n+1}_\nu = Q_0 \mathbf{u}^n_\nu + k\tilde{\mathbf{F}}^n_\nu, \quad \nu = 1, 2, \dots, \tag{26}$$

$$Q_0 = \sum_{j=-r}^{p} A_j E^j, \quad E\mathbf{u}_\nu = \mathbf{u}_{\nu+1},$$

where the $n \times n$ matrices A_j are polynomials in A and kC, and the n-vector $\tilde{\mathbf{F}}_\nu(t)$ is a smooth function of $\mathbf{F}$ and its derivatives. For later reference we call (26) a *basic scheme* for the totally discretized approximation. To solve it uniquely in the half space, we must specify the initial values

$$\mathbf{u}^0_\nu = \mathbf{f}^0_\nu, \quad \nu = 1, 2, \dots, \tag{27}$$

and boundary conditions

$$\mathbf{u}_\mu^n = S_\mu \mathbf{u}_1^n + \mathbf{g}_\mu^n, \quad \mu = -r+1, -r+2, \ldots, 0, \tag{28}$$

where

$$S_\mu = \sum_{j=0}^{q} C_j^\mu E^j$$

It is natural to require that all schemes considered here be stable in the semigroup sense for pure Cauchy problems, i.e, we need

Assumption 3.1. *For any two-level scheme (26), there is a CFL number $\lambda_0 > 0$ such that when $0 \le \lambda \le \lambda_0$, the solution of the corresponding Cauchy problem with $\tilde{F} = 0$ satisfies*

$$(u^n, u^n)_{-\infty,\infty} \le e^{2\eta_0 t}(u^0, u^0)_{-\infty,\infty} \tag{29}$$

for some real number η_0.

Remark: The solutions of a multistep scheme, which is stable in any sense, in general does not satisfy (29). This is why our discussion is limited on two-step methods.

Still, the semigroup stability is the definition of stability for the totally discretized schemes.

Definition 3.1. *The discrete problems (26,27,28) is stable if, with homogenized equations ($\tilde{F} = 0$) and the homogenized boundary conditions ($g = 0$), the solution satisfies an energy estimate*

$$\|u^n\| \le K e^{\eta_0 t} \|u^0\|, \qquad t = nk$$

where η_0, K are universal constants.

When the problem (26,27,28) is stable, the solution of the inhomogeneous equation can be estimated with Duhamel's principle

$$\|\mathbf{u}^n\| \le K e^{\eta_0 t}(\|\mathbf{u}^0\|, + \sum_{j=0}^{n} \|\tilde{\mathbf{F}}^j\| k), \quad t = nk.$$

Again we need to introduce an eigenvalue problem, which is obtained by taking the Laplace transform on the homogenized equations of (26,27,28):

$$z\hat{\mathbf{u}}_\nu = Q_0 \hat{\mathbf{u}}_\nu, \qquad z = e^{sk}, \quad \nu = 1, 2, \ldots \tag{30}$$

$$\hat{\mathbf{u}}_\mu = S_\mu \hat{\mathbf{u}}_1; \qquad \mu = -r+1, -r+2, \ldots, 0. \tag{31}$$

The eigenvalues and general eigenvalues are defined as

Definition 3.2. *Let $C = 0$. z is an eigenvalue if it satisfies the following conditions:*

(1) *(30,31) are satisfied,*
(2) $|z| \ge 1$,
(3) *when* $|z| > 1, \quad \|u\|_2 < \infty$.

z is a generalized eigenvalue if condition 1 and 2 are satisfied, and condition 3 is replaced with

(3) *when* $|z| = 1, \hat{u}_\nu(z) = \lim_{w \to z, |w|>1} \hat{u}_\nu(w)$, *where* $u_\nu(w)$ *satisfies (30) with* z *replaced by* w.

With the integration now representing the summation of the form

$$\int_0^\infty w(t)dt = \sum_{n=0}^\infty w(nk)k,$$

Definition 2.2, the definition of stability in strong sense, can also be used for the totally discretized schemes. For the strong stability we have[3]

Theorem 3.1 (Gustafsson, Kreiss and Sundström). *The approximation (26,27,28) is stable according to Definition 2.2 if and only if eigenvalue problem (30,31) has no eigenvalues nor generalized eigenvalues for* $|z| \geq 1$.

With boundary norms (9), the above theorem can be restated as[3]

Lemma 3.1 (Gustafsson, Kreiss and Sundström). *For discrete approximations, Kreiss condition is equivalent to that, if* $\boldsymbol{F} = 0, \boldsymbol{f} = 0$,

$$|\hat{v}|_B \leq K|\hat{g}|_B. \tag{32}$$

As in the semi-discrete case, certain assumptions have to be made, even we have found no exception to these assumptions.

Assumption 3.2. *The basic scheme (26) is either dissipative or nondissipative, i.e, the roots of the characteristic equation*

$$det \mid zI - \sum_{\sigma=0}^{s} \hat{Q}_\sigma(i\xi) z^{-\sigma k} \mid = 0, \quad \hat{Q}_\sigma(i\xi) = \sum_{j=-r}^{p} A_{j\sigma} e^{ij\xi} \tag{33}$$

satisfy either

$$|z(\xi)| < 1, \quad 0 < |\xi| \leq \pi,$$

or

$$|z(\xi)| = 1, \quad |\xi| \leq \pi.$$

For the same reasons as in the continuous and semi-discrete problems we realize that we only need to discuss with scalar problems without lower order terms. The same results for systems of equations with lower order terms will follow accordingly.

3.2. Kreiss' Condition and the Semigroup Stability. Consider the following totally discrete scheme

$$\begin{cases} u_j^{n+1} = Q_0 u_j^n, & 0 \le n \le \infty \\ u_j^0 = f_j, & \\ Bu_0^n = 0 & \end{cases} \qquad 1 \le j \le \infty \tag{34}$$

where

$$Q_0 = I + kQ, \quad Q = \frac{1}{h}\sum_{j=-r}^{p} a_j E^j, \quad a_{-r} \neq 0, a_p \neq 0$$

and a_j are the polynomials of $\lambda = k/h$. B is a boundary operator which represents a set of functions

$$u_\mu = \sum_{j=\mu+1}^{q} \beta_{\mu j} u_j, \quad -r+1 \le \mu \le 0$$

According to Assumption 3.1 the solution of the corresponding Cauchy problem of (34) satisfies, for $0 \le \lambda \le \lambda_0$,

$$(u^n, u^n)_{-\infty,\infty} \le e^{2\eta_0 nk}(u^0, u^0)_{-\infty,\infty}.$$

Similar to the semi-discrete problem, we define

$$\langle u_j, u_j \rangle = \sum_{n=1}^{\infty} e^{-2\eta_0 nk} |u_j(nk)|^2 k.$$

In addition we define a projection operator $P : l^2(-\infty,\infty) \to l_0^2(-\infty,\infty)$ by

$$(Pu)_j = \begin{cases} u_j & j = 1, 2, \dots, \\ 0 & j \le 0. \end{cases}$$

There exists the analogue of Theorem 2.2 for totally discrete schemes.

Theorem 3.2. *There exists a special boundary operator B_0 such that when $B = B_0$, the solution of (34) on the boundary satisfy*

$$\langle u_\mu, u_\mu \rangle \le c_\mu (u^0, u^0), \quad -r+1 \le \mu \le 1 \tag{35}$$

PROOF: we again interpret the boundary conditions as a vector $u_b^n \in l^2(-\infty,\infty)$:

$$(u_b^n)_j = \begin{cases} (u_b^n)_j, & -r+1 \le j \le 0, \\ 0, & \text{others.} \end{cases}$$

Then energy estimate proceeds as

$$\begin{aligned} &(u^{n+1}, u^{n+1}) = (Q_0 u^n, Q_0 u^n) \\ &= (PQ_0(Iu^n + u_b^n), PQ_0(Iu^n + u_b^n))_{-\infty,\infty} \\ &= (PQ_0 Iu^n, PQ_0 Iu^n)_{-\infty,\infty} + 2Re(PQ_0 Iu^n, PQ_0 u_b^n)_{-\infty,\infty} + (PQ_0 u_b^n, PQ_0 u_b^n)_{-\infty,\infty} \\ &\le e^{2\eta_0 k}(u^n, u^n) + 2Re(Q_0 Iu^n, Q_0 u_b^n)_{1,r} + (Q_0 u_b^n, Q_0 u_b^n)_{1,r} \end{aligned}$$

Note that

$$(Q_0 Iu^n)_j = (u^{n+1} - Q_0 u_b^n)_j, \quad 1 \le j \le r.$$

So we finally end up with an inequality

$$(36) \qquad \begin{aligned}(u^{n+1},u^{n+1}) &\leq e^{2\eta_0 k}(u^n,u^n)+2Re(u^{n+1},Q_0u_b^n)_{1,r}\\ &= e^{2\eta_0 k}(u^n,u^n)+2kRe(u^{n+1},Qu_b^n)_{1,r}.\end{aligned}$$

If we choose the following special boundary conditions

$$(37) \qquad \begin{aligned}&u_\mu^n=0, &&-r+2\leq\mu\leq 0,\\ &u_{-r+1}^n=-a_{-r}^{-1}u_1^{n+1},\end{aligned}$$

which amounts to solving for u_{-r+1}^n from

$$(1+\lambda)a_{-r}u_{-r+1}^n=-(Q_0Iu^n)_1,$$

then we get inequality

$$(u^{n+1},u^{n+1})+|a_{-r}u_{-r+1}^n|^2k\leq e^{2\eta_0 k}(u^n,u^n)$$

or

$$e^{-2\eta_0(n+1)k}(u^{n+1},u^{n+1})+e^{-2\eta_0(n+1)k}|a_{-r}u_{-r+1}^n|^2k\leq e^{-2\eta_0 nk}(u^n,u^n).$$

That means

$$\langle a_{-r}u_{-r+1},a_{-r}u_{-r+1}\rangle=\langle u_1,u_1\rangle\leq(u^0,u^0).$$

Naturally,

$$\langle u_\mu,u_\mu\rangle=0,\quad -r+2\leq\mu\leq 0.$$

We name the set of boundary conditions (37) by an operator B_0 □

Remark: There are more than one way to choose a special boundary operator. However, the above boundary operator has the advantage that

$$\langle (Q_0u_b^n)_j,(Q_0u_b^n)_j\rangle,\quad j=1,\ldots,r$$

are bounded independently of λ, which is used in a separated paper for discussing the estimates when $\lambda\to 0$.

Boundary operator B_0 gives us the estimate of u_1, and allows us to proceed to the estimates of u_j for $j\geq 2$. Consider the following problem

$$\begin{cases} u_j^{n+1}=Q_0u_j^n, & 0\leq n\leq\infty\\ u^0=f_j, & \\ u_\nu^n=u_\nu^n, & -r+2\leq\nu\leq 1.\end{cases}\qquad j=2,3,\ldots$$

We split $u_j^n, j\geq 2$ as

$$(38) \qquad u_j^n=v_j^n+(u_j^n-v_j^n),\qquad j=2,3,\ldots,$$

where $v_j^n, j\geq 2$ is the solution of

$$\begin{cases} v_j^{n+1}=Q_0v_j^n, & 0\leq n\leq\infty\\ v^0=f_j, & \\ B_0v_1^n=0, & -r+2\leq\nu\leq 1.\end{cases}\qquad j=2,3,\ldots,$$

As it is already shown in Theorem 3.2 that

$$\langle v_\mu,v_\mu\rangle\leq c_\mu(u^0,u^0),\quad -r+2\leq\mu\leq 2.$$

The last term of (38), which is denoted as $w_j^n := u_j^n - v_j^n (j \geq 2)$ for simplicity, satisfies

$$\begin{cases} w_j^{n+1} = Q_0 w_j^n, & j = 2, 3, \dots, \\ w_j^0 = 0, & \\ w_\mu^n = u_\mu^n - v_\mu^n, & -r+2 \leq \mu \leq 1. \end{cases}$$

The above equation can be solved by Laplace transform. Denoting the Laplace transforms of the step functions u_j, v_j and w_j as $\hat{u}_j, \hat{v}_j$ and $\hat{w}_j$, respectively, we have for $\hat{w}_j$

$$\begin{cases} z\hat{w}_j = Q_0 \hat{w}_j, & j = 2, 3, \dots \\ \hat{w}_\mu = \hat{u}_\mu - \hat{v}_\mu, & -r+2 \leq \mu \leq 1, \end{cases} \tag{39}$$

where $z = e^{sk}$. By the same arguments as those used in Lemma 2.3 we can prove

Lemma 3.2. *The corresponding eigenvalue problem of (39) satisfies Kreiss condition.*

Thus by Lemma 3.1 we get

$$\langle w_j, w_j \rangle \leq const. \sum_{-r+2}^{1} (\langle u_\mu, u_\mu \rangle + \langle v_\mu, v_\mu \rangle) \leq const.(u^0, u^0), \quad j \geq 2,$$

and therefore

$$\langle u_2, u_2 \rangle \leq const.(\langle v_2, v_2 \rangle + \langle w_2, w_2 \rangle) \leq const.(u^0, u^0).$$

Through the same approach as that for the estimate of u_2 we can get estimates for $u_j, j \geq 3$. Hence we obtain

Lemma 3.3. *For equation (34) with special boundary operator B_0, we have*

$$\langle u_j, u_j \rangle \leq c_j (u^0, u^0), \quad j \geq 1 \tag{40}$$

where c_j depends on j only.

For (34) with general boundary conditions, we split its solution into

$$u_j^n = v_j^n + (u_j^n - v_j^n), \quad j = 1, 2, \dots,$$

where $v_j^n, j \geq 1$, is the solution of (34) with the special boundary condition $B_0 v_0^n = 0$, thus

$$\langle v_j, v_j \rangle \leq c_j (u^0, u^0), \quad j \geq 1, \tag{41}$$

and the last term of above equation, again denoted by w_j^n, satisfies

$$\begin{cases} w_j^{n+1} = Q_0 w_j^n, \quad 0 \leq n \leq \infty & j = 1, 2, \dots, \\ w_j^0 = 0 & \\ B w_0^n = -B v_0^n. & \end{cases}$$

Now we can prove

Theorem 3.3. *Under Assumption 3.1, the IBV problem (34) is stable in the semigroup sense if the Kreiss condition is satisfied.*

PROOF: From (36) we have

$$\begin{aligned} & e^{-2\eta_0(n+1)k}(w^{n+1}, w^{n+1}), \\ \leq\ & e^{-2\eta_0 nk}(w^n, w^n) + 2e^{-2\eta_0(n+1)k}(w^{n+1}, Q_0 w_b^n)_{1,r}, \\ \leq\ & e^{-2\eta_0 nk}(w^n, w^n) + const.e^{-2\eta_0(n+1)k}\sum_{j=1}^{r}(|w_j^{n+1}|^2 + |w_j^n|^2)k. \end{aligned}$$

If Kreiss condition holds, then by Lemma 3.1 and Lemma 3.3 there are constants $c_j, j = 1, 2, \ldots$ such that

$$(w_j, w_j) \leq c_j(u^0, u^0), \qquad j \geq 1.$$

Hence

$$(w^n, w^n) \leq const.e^{2\eta_0 nk}(u^0, u^0).$$

The bound for u^n comes from triangular inequality

$$\|u^n\| \leq \|v^n\| + \|w^n\| \leq const.e^{2\eta_0 nk}\|u^0\|.$$

This completes the proof □

For more general problems we have

Theorem 3.4. *Under Assumption* 3.1, *the IBV problem* (26,27,28) *is stable in the semigroup sense if the Kreiss condition is satisfied.*

For the schemes obtained by applying the Runge-Kutta methods to the semi-discrete approximations, the Kreiss condition is easier to be verified. A linear semi-discrete problems can be written as

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= \tilde{Q}\mathbf{u} + \mathbf{F} \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{aligned} \tag{42}$$

where $\tilde{Q}$ is a semi-infinite matrix which has taken into account of the boundary conditions. For convenience we will omit $\sim$ from now on. The Runge-Kutta type of methods are of the form

$$\begin{aligned} \mathbf{u}^{n+1} &= L(kQ)\mathbf{u}^n + k\mathbf{G}, \\ \mathbf{u}^0 &= \mathbf{u}_0, \end{aligned} \tag{43}$$

where

$$L(kQ) = \sum_{j=0}^{q}\frac{\alpha_j}{j!}(kQ)^j$$

is a polynomial in kQ. If the method is accurate of order p, then

$$\alpha_0 = \alpha_1 = \ldots = \alpha_p = 1.$$

The stability region of a Runge-Kutta method is defined by

$$\Omega = \{\mu \in C \mid |L(\mu)| < 1\}.$$

Clearly Ω is an open set. We will be interested in so-called *the locally stable methods*.

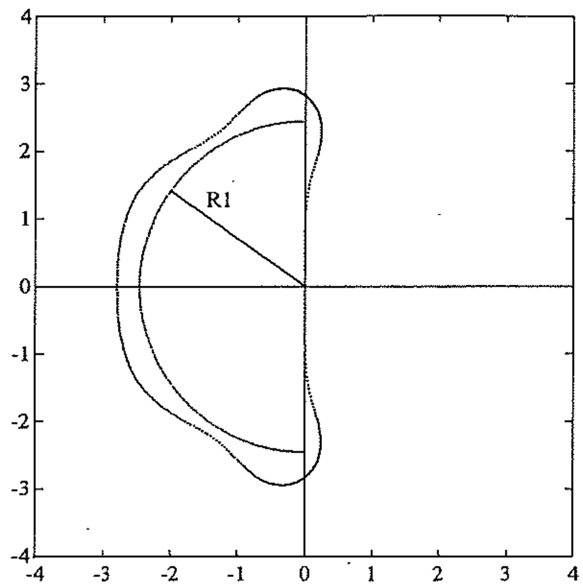

FIGURE 1. Stability region of the 4^{th} order Runge-Kutta method.

Definition 3.3. *The method is called locally stable, if there is an* $R_1 > 0$ *such that the open half circle*

$$\mathcal{L}(R_1) = \{\mu \mid |\mu| < R_1, Re(\mu) < 0\}$$

is contained in Ω.

The 4^{th} order Runge-Kutta method is an example of *the locally stable method*, whose stability region contains a half circle (see Figure 1). Please see [8] for more discussions on the algebraic properties of the Runge-Kutta methods.

We finish this paper with

Theorem 3.5. *Assume that the Runge-Kutta method is locally stable and that the condition of Assumption 3.1 are satisfied. If the semi-discrete approximation satisfies the Kreiss condition, then the Runge-Kutta methods (43) are stable in the semigroup sense as well, provided*

$$||kQ|| \leq R < R_1. \tag{44}$$

PROOF: We only need to verify that the Kreiss condition is satisfied under the given conditions. We have the decomposition

$$zI - L(kQ) = (z_1 I - kQ)(z_2 I - kQ)\ldots(z_p I - kQ).$$

For any z with $|z| \geq 1$, at most one of the $z_i, i = 1, 2, \ldots, p$, could be located inside the disk $|z| \leq R$. We assume $|z_1| \leq R$, then according to the definition of stability region there must be

$$Re(z_1) \geq 0.$$

As it is claimed for the semidiscrete problem,

$$(z_1 I - kQ)u = 0$$

can have only the trivial solution. It is also apparent that

$$(z_i I - kQ)u = 0, \quad i = 2, 3, \ldots, p$$

can also have only the trivial solution. Hence $zI - L(kQ)$ has no eigenvalue or generalized eigenvalue for $|z| \geq 1$, and the conclusion follows from the last theorem □

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