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Lixin Wu

Heinz-Otto Kreiss

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Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

STABILITY OF THE DIFFERENCE APPROXIMATIONS FOR THE PARABOLIC INITIAL BOUNDARY VALUE PROBLEMS

LIXIN WU AND HEINZ-OTTO KREISS

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ABSTRACT.

For semi-discrete approximations to the initial boundary value problems of the parabolic equations we prove that the Kreiss condition is necessary and sufficient for the stability in the generalized sense. Moreover, we prove that the stability in the generalized sense is equivalent to the stability in the semigroup sense. Also, we show that the stability of totally discretized approximations generated from *the locally stable methods of line* follows from the stability of the semi-discrete problem.

1. CONTINUOUS PROBLEMS

We consider the numerical solution of the following second order parabolic equation

$$\mathbf{u}_t = A\mathbf{u}_{xx} + B\mathbf{u}_x + C\mathbf{u} + \mathbf{F} := P\mathbf{u} + \mathbf{F}, \quad x \geq 0, t \geq 0, \tag{1}$$

where $\mathbf{u}(x,t), \mathbf{F}$ are vector functions with n components, and A and B are $n \times n$ constant matrices. Without loss of generality, A is assumed to be a constant diagonal matrix,

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}, \quad a_i \geq \delta > 0.$$

The solution of (1) is uniquely determined if we prescribe initial values

$$\mathbf{u}(x,0) = \mathbf{f}(x), \qquad x \geq 0, \tag{2}$$

and boundary conditions

$$L_1\mathbf{u}(0,t) + L_2\mathbf{u}_x(0,t) = \mathbf{g}, \qquad t \geq 0, \tag{3}$$

where $L_i, i = 1, 2$ are $n \times n$ matrices and $\text{Rank}(L_1, L_2) = n$.

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Around the stability issues of the difference approximations for parabolic initial boundary value problem, there are only a handful of publications[4][5] [1]. Overall speaking, people's interests on the stability of parabolic problems seem to be much less than their interests on that of the hyperbolic problems. This situation could be because of the better behaviors of the solutions of parabolic problems. However, if in addition to the stability we require the solutions to be dissipative, then we must be more cautious in choosing boundary conditions. In fact, the well-posedness of the parabolic initial boundary value problems also requires the estimates of first order derivatives in terms of data[2]. This is a major difference between the well-posedness of parabolic problems and hyperbolic problems. In his pioneer work[5] on the stability of totally discretized approximations of parabolic IBV problems, Osher discovered a sufficient condition for the estimates of solution and its first order derivative with maximum norm. But his estimates have the disadvantage that they are time dependent, which is weaker than the estimates we expect. For a model semi-discrete problem, Kreiss obtained a necessary and sufficient stability condition which achieved complete parallelism to the continuous problem[1]. His results are going to be generalized in this paper.

Throughout this paper we assume the existence of the solutions for the continuous parabolic problems. There are two definitions of stability for problem (1,2,3). The most usual one is the stability in the semigroup sense.

Definition 1.1. *The parabolic problem (1,2,3) is stable if there exist η_0, K such that when $\boldsymbol{F}=0, g=0$, the solution satisfies*

$$\|u(t)\|^2 + \int_0^t \|e^{\eta_0(t-\tau)} u_x\|^2 d\tau \le K e^{2\eta_0 t}\|u(0)\|^2. \tag{4}$$

Here η_0 and K are universal constants.

Note that $\|\cdot\|$ is the usual L^2 norm in the half space. For the solutions of inhomogeneous equations, the estimates can be obtained by Duhamel's principle. If the Cauchy problem has energy estimates and the boundary condition is of Frederich's type, then the estimate (4) for the IBV problem can be easily derived from energy estimates. For more general boundary conditions, the next definition is more convenient for analysis.

Definition 1.2. *The parabolic problem (1,2,3) is stable if there exists $\eta_0 \ge 0$ such that when $\boldsymbol{f}=0, \boldsymbol{g}=0$, the solution satisfies*

$$\int_0^\infty \|e^{-\eta t} u\|^2 dt + \int_0^\infty \|e^{-\eta t} u_x\|^2 dt \le K(\eta) \int_0^\infty \|e^{-\eta t} \boldsymbol{F}\|^2 dt \tag{5}$$

for $\eta \ge \eta_0$. Here $K(\eta) \to 0$ as $\eta \to 0$.

Note that the homogeneity in initial condition can always be achieved by introducing new variable $\mathbf{u} \leftarrow \mathbf{u} - e^{-t}\mathbf{f}$.

There will be no difficulties to show that semigroup stability implies Definition 1.2, thus Definition 1.2 is called the definition of stability in the generalized sense. Both stability definitions have the property that they are stable against lower order perturbations. For the semigroup definition, this property follows from the use of

Duhamel's principle and Grondwall inequality[6]. For the stability in the generalized sense we will give a proof of the property as follows. With the Parseval's equality, (5) is equivalent to

$$\|\hat{\mathbf{u}}\| + \|\hat{\mathbf{u}}_x\| \le K(\eta)\|\hat{\mathbf{F}}\|, \tag{6}$$

where $\hat{\mathbf{u}}, \hat{\mathbf{F}}$ are the Laplace transforms of $\mathbf{u}, \mathbf{F}$. Suppose that there are lower order perturbations to P, which are denoted by $B_1\frac{\partial}{\partial x} + C_1$, then by (5) the solution of the perturbed equation satisfies

$$\begin{aligned} \|\hat{\mathbf{u}}\| + \|\hat{\mathbf{u}}_x\| &\le K(\eta)(\|B_1\hat{\mathbf{u}}_x + C_1\hat{\mathbf{u}} + \hat{\mathbf{F}}\|) \\ &\le c_p K(\eta)(\|\hat{\mathbf{u}}\| + \|\hat{\mathbf{u}}_x\|) + K(\eta)\|\hat{\mathbf{F}}\|. \end{aligned}$$

For sufficiently large η_0', we will have $c_p K(\eta) \le \frac{1}{2}, \eta \ge \eta_0'$, hence we obtain

$$\|\hat{\mathbf{u}}\| + \|\hat{\mathbf{u}}_x\| \le 2K(\eta)\|\hat{\mathbf{F}}\|, \quad \eta \ge \eta_0'.$$

Thus the bound of the solution of perturbed problem follows from that of the unperturbed problem. Hence, we can always lay off the lower order terms of the equations when we study the stability of the problems.

Next we will introduce some major results on the stability of continuous problems, which serve as guidance for our studies of discrete problems. We will speak in terms of an eigenvalue problem

$$\begin{aligned} &(s^{\frac{1}{2}}I - \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}\tfrac{\partial}{\partial x})\begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{z}} \end{pmatrix} = \mathbf{0} \\ &L_1\hat{\mathbf{u}}(0) + L_2 s^{\frac{1}{2}}\hat{\mathbf{z}}(0) = 0, \end{aligned} \tag{7}$$

which is obtained by introducing a new variable $\hat{\mathbf{z}} = s^{-\frac{1}{2}}\hat{\mathbf{u}}_x$ into the reduced homogeneous equation

$$\begin{aligned} &(sI - A\tfrac{\partial^2}{\partial x^2})\hat{\mathbf{u}} = 0, \\ &L_1\hat{\mathbf{u}}(0) + L_2\hat{\mathbf{u}}_x(0) = 0. \end{aligned}$$

Here is the definition of eigenvalues.

Definition 1.3. *s is an eigenvalue if there exists a nontrivial vector $\hat{w} = (\hat{u}, \hat{z})^T$ such that $(s, \hat{w})$ satisfies the following conditions:*

(1) $(s^{\frac{1}{2}}I - \begin{pmatrix} 0 & A \\ I & 0 \end{pmatrix}\frac{\partial}{\partial x})\hat{w} = \mathit{0}, \quad x \ge 0,$

(2) $Re(s) \ge 0,$

(3) $L_1\hat{u}(0) + L_2 s^{\frac{1}{2}}\hat{z}(0) = 0,$

(4) *when* $s \ne 0, \quad \|\hat{w}\|_2 < \infty.$

s is a generalized eigenvalue if condition 1, 2 and 3 are satisfied, and condition 4 is replaced with

(4) *when* $s = 0$, $\hat{w}(x, s) = \lim_{\epsilon\to 0}\hat{w}(x, s+\epsilon)$, *where* $Re(\epsilon) \ge 0$ *and* $\hat{w}(x, s+\epsilon)$ *satisfies condition 1 with s replaced by $s + \epsilon$.*

In the case of generalized eigenvalue, we have $\|\hat{\mathbf{w}}\|_2 = \infty$. We now can state[1].

Theorem 1.1 (Kreiss). *The problem (1,2,3) is well-posed if and only if there is no eigenvalue or generalized eigenvalue to its corresponding eigenvalue problem.*

The eigenvalue condition in above theorem is referred as the Kreiss' condition (of parabolic version). With some techniques, including the ones developed in this paper, one can show that the Kreiss' condition is also sufficient for the semigroup stability, thus there is

Theorem 1.2. *For problem (1,2,3), the stability definition* 1.1 *is equivalent to definition* 1.2.

For the sake of the simplicity, this paper will be limited to one dimensional problems. Under some conditions, our results also apply to multidimensional problems.

2. Semi-discrete approximations

We will solve (1,2,3) with *The methods of line.* The first step of the methods is spatial discretizations, which result in consistent ordinary differential equations or what are called the semi-discrete approximations, and the second step is the employment of standard numerical methods for solving ordinary differential equations. In this section we will study the well-posedness of the semi-discrete approximations only. For that purpose, we introduce a mesh with size $h = \Delta x > 0$, and, with the notation $\mathbf{u}_\nu(t) \approx \mathbf{u}(\nu h, t)$, approximate the equations (1,2,3) by consistent semi-discrete scheme of the form

$$\begin{cases} \frac{d\mathbf{u}_j(t)}{dt} = (AD_+D_-(I+R_1) + BR_2D_+ + C)\mathbf{u}_j(t) + \mathbf{F}_j(t), \\ \mathbf{u}_j(0) = \mathbf{f}_j, \qquad\qquad j = 1, 2, \dots \\ L_1\mathbf{u}(0) + L_2D_+\mathbf{u} = \mathbf{g} \end{cases} \tag{8}$$

where

$$R_1 = \sum_{\mu=-r}^{p} a_\mu E^\mu, \quad R_2 = \sum_{\mu=-s}^{q} b_\mu E^\mu, \quad p \geq q, \quad r \geq s,$$

and $L_i, i = 1, 2$ now represent

$$L_i\mathbf{u}_0 = \sum_{j=1}^{q} B_j^{(i)}\mathbf{u}_{j-r-1}, \quad i = 1, 2$$

and

$$\text{Rank}(B_1^{(1)}, \dots, B_q^{(1)}, B_1^{(2)}, \dots, B_q^{(2)}) = n(r+1)$$

The solution space of the IBV problem (8) is $l^{2,n}(1,\infty)$, which is defined by

$$l^{2,n}(-M, N) = \{\mathbf{u} = \{\mathbf{u}_j\}_{-M}^{N}, \mathbf{u}_j \in \mathbf{C}^n \mid \|\mathbf{u}\|_{-M,N} < \infty\}$$

where the norm comes from the associated inner product

$$(\mathbf{u}, \mathbf{v})_{-M,N} = \sum_{-M}^{N} \mathbf{u}_j^T \bar{\mathbf{v}}_j h,$$

thus

$$\|\mathbf{u}\|_{-M,N}^2 = (\mathbf{u}, \mathbf{u})_{-M,N}.$$

For the sake of convenience, the indices of the norm and inner product of $l^{2,n}(1,\infty)$ will be omitted. And we write $l^2(-M, N)$ for $l^{2,1}(-M, N)$.

As a basic assumption, we require the semi-discrete approximation (8) to inherit the parabolicity of the continuous problems.

Assumption 2.1. *for any* $u \in l^2(-\infty, \infty)$,

$$Re(u, R_1 u)_{-\infty,\infty} \leq (1 - q_0)\|u\|^2_{-\infty,\infty}, \quad 0 \leq q_0 < 1.$$

For both semi-discrete and totally discretized approximations, we have the discrete versions of the stability in the semigroup sense

Definition 2.1. *The parabolic problem (8) is stable if there exist universal constants* η_0 *and* $K > 0$ *such that when* $\boldsymbol{F} = 0, g = 0$, *the solution satisfies*

$$\|u(t)\|^2 + \int_0^t \|e^{\eta_0(t-\tau)} D_+ u\|^2 d\tau \leq K e^{2\eta_0 t}\|u(0)\|^2. \tag{9}$$

And the stability in the generalized sense

Definition 2.2. *The parabolic problem (8) is stable if there exists* η_0 *such that when* $f = 0, g = 0$, *the solution satisfies*

$$\int_0^\infty \|e^{-\eta t} u\|^2 dt + \int_0^\infty \|e^{-\eta t} D_+ u\|^2 dt \leq K(\eta) \int_0^\infty \|e^{-\eta t} \boldsymbol{F}\|^2 dt \tag{10}$$

for $\eta \geq \eta_0$. *Here* $K(\eta) \to 0$ *as* $\eta \to \infty$.

In terms of the Laplace transformed variables, (10) reads

$$\|\hat{\mathbf{u}}\| + \|D_+ \hat{\mathbf{u}}\| \leq K(\eta)\|\hat{\mathbf{F}}\|. \tag{11}$$

Like their continuous counterparts, Definition 2.2 is implied by Definition 2.1. With the same arguments as for the continuous problems, we can show that these two definitions are stable against lower order perturbations. Thus, we can throw away lower order terms in the equations, and, consequently, manipulate with a single scalar equation in the stability analysis.

2.1. Stability in the generalized sense. We now study the reduced problem of (8)

$$\begin{cases} \frac{du_j(t)}{dt} = D_+ D_-(I + R_1)u_j(t) + F_j(t), & j = 1, 2, \ldots, \\ u_j(0) = f_j, & \\ L_1^I u_0(t) = g^I, \quad L_1^{II} u_0(t) + L_2^{II} D_+ u_0(t) = g^{II}. & \end{cases} \tag{12}$$

Here,

$$Rank(\begin{pmatrix} L_1^I & 0 \\ L_1^{II} & L_2^{II} \end{pmatrix}) = r + 1.$$

Note that consistency requires

$$\sum_{\mu=-r}^{p} a_\mu = 0.$$

To discuss the stability in the semigroup sense we let $f = 0, g = 0$. Then the Laplace transform of (12) reads

$$\begin{cases} s\hat{u}_j = D_+ D_-(I + R_1)\hat{u}_j + \hat{F}_j, \quad j = 1, 2, \ldots. \\ L_1^I \hat{u}_0 = 0, \quad L_1^{II}\hat{u}_0 + L_2^{II} D_+ \hat{u}_0 = 0. \end{cases} \tag{13}$$

The corresponding characteristic equation of (13) is

$$\tilde{s} = \frac{(\kappa-1)^2}{\kappa}(1+\sum_{\mu=-r}^{p} a_\mu \kappa^\mu), \quad \tilde{s} = sh^2,$$

and its roots are characterized in the following lemma.

Lemma 2.1. *For $\tilde{s} \neq 0$ and $Re(\tilde{s}) \geq 0$, the characteristic equation has exactly $r+1$ roots, counted according to their multiplicity, with $|\kappa| < 1$. For $\tilde{s} = 0$, it has exactly r roots inside the unit circle, and two roots on the unit circle such that $\kappa_1 = \kappa_2 = 1$. Moreover, around $\tilde{s} = 0$, κ_1 and κ_2 are not analytic in $\tilde{s}$, but in $\tilde{s}^{\frac{1}{2}}$.*

PROOF: Assume there is a root $\kappa = e^{i\xi}$, ξ real. Then

$$\tilde{s} = \frac{(e^{i\xi}-1)^2}{e^{i\xi}}(1+\sum_{\mu=-r}^{p} a_\mu e^{i\xi\mu}) = -4\sin^2\frac{\xi}{2}(1+\sum_{\mu=-r}^{p} a_\mu e^{i\xi\mu}). \tag{14}$$

From Assumption 2.1 we know that

$$Re\{(1+\sum_{\mu=-r}^{p} a_\mu e^{i\xi\mu})\} > 0.$$

When $Re(\tilde{s}) \geq 0$, (14) only hold for $\tilde{s} = 0$, thus $\kappa = 1$, which is a double root. Note that $\frac{d\tilde{s}}{d\kappa}|_{\kappa=1} = 0$ but $\frac{d\tilde{s}^{\frac{1}{2}}}{d\kappa}|_{\kappa=1} \neq 0$, thus the double root $\kappa = 1$ are analytic in $\tilde{s}^{\frac{1}{2}}$ instead of $\tilde{s}$. When $\tilde{s} \to \infty$, the first approximation of the characteristic is

$$\tilde{s} = a_{-r}\kappa^{-r-1}.$$

Hence, there are exactly $r+1$ roots with $|\kappa| < 1$ when $\tilde{s} \neq 0$ $\square$.

The corresponding eigenvalue problem of (13) is

$$\begin{cases} s\phi_j = D_+D_-(I+R_1)\phi_j, & j = 1,2,\ldots. \\ L_1^I\phi_0 = 0, \quad L_1^{II}\phi_0 + L_2^{II}D_+\phi_0 = 0. \end{cases} \tag{15}$$

From Lemma 2.1 we have[1]

Lemma 2.2. *The problem (13) is not stable in the generalized sense, if its eigenvalue problem (15) has an eigenvalue s_0 with $Re(s_0) > 0$ for some $h = h_0$.*

PROOF: The general solution of the difference equation is

$$\phi_j = \sum_{\alpha=1}^{l}\sum_{\beta=0}^{m_\alpha-1} c_{\alpha\beta}P_{\alpha\beta}(j)\kappa_\alpha^j,$$

where m_α is the multiplicity of κ_α, $\sum_{\alpha=1}^{m} m_\alpha = r+1$, $P_{\alpha\beta}(j)$ is arbitrary polynomial in j with degree exactly equal to β, and $c_{\alpha\beta}$ is the parameter determined by the boundary conditions. Therefore s is an eigenvalue, if

$$L_1^I\phi_0 = 0, \quad L_1^{II}\phi_0 + L_2^{II}D_+\phi_0 = 0.$$

These conditions are linear for the coefficients $c_{\alpha\beta}$, and its determinant is a function of $\tilde{s} = sh^2$. Any eigenvalue s_0 with $Re(s_0) > 0$ for some mesh of size h_0 will generate eigenvalue $s = s_0h_0^2/h^2$ for the mesh of size h, whose real part becomes arbitrarily large for $h \to 0$. Thus problem (13) is unstable $\square$

Parallel to the definition of eigenvalue for the continuous problem, we will use

$$
(16) \qquad \left\{ \begin{array}{ll} s^{\frac{1}{2}}\phi_j = D_-(I+R_1)\psi_j, & j = 1,2,\dots \\ s^{\frac{1}{2}}\psi_j = D_+\phi_j, & \\ L_1^I\phi_0 = 0, \quad L_1^{II}\phi_0 + L_2^{II}s^{\frac{1}{2}}\psi_0 = 0 & \end{array} \right.
$$

for the definition of eigenvalue or generalized eigenvalue of (13). Note that any eigenvalue or generalized eigenvalue $\tilde{s}$ to (16) is also an eigenvalue or generalized eigenvalue to (15), but the converse is not true. The Kreiss condition is defined in the say way as in the continuous problems. Now we come to our first major result.

Lemma 2.3. *A difference approximation (12) is stable if the Kreiss condition is satisfied for (16).*

We will prove this lemma by solving (13), which can be rewritten as an one-step scheme. Introducing $\hat{z} = s^{-1/2}\hat{D}_+\hat{u}$, we write (13) as

$$
(17) \qquad \left\{ \begin{array}{ll} s^{1/2}\hat{z}_\mu = D_+\hat{u}_\mu, & \mu = 1,2,\dots, \\ s^{1/2}\hat{u}_\mu = D_-(I+R)\hat{z}_\mu + s^{-1/2}\hat{F}_\mu, & \\ L_1^I\hat{u}_0 = 0, \quad L_1^{II}\hat{u}_0 + s^{1/2}L_2^{II}\hat{z}_0 = 0, & \end{array} \right.
$$

or

$$
(18) \qquad \begin{array}{ll} \hat{u}_{\mu+1} = \hat{u}_\mu + \tilde{s}^{\frac{1}{2}}\hat{z}_\mu, & \mu = 1,2,\dots \\ \hat{z}_{\mu+p} = \sum_{-r-1}^{p-1}\alpha_j\hat{z}_{\mu+j} + \tilde{s}^{\frac{1}{2}}\beta_1\hat{u}_\mu + \gamma h^2\tilde{s}^{-\frac{1}{2}}\hat{F}_\mu, & \\ L_1^I\hat{u}_0 = 0, \quad hL_1^{II}\hat{u}_0 + \tilde{s}^{\frac{1}{2}}L_2^{II}\hat{z}_0 = 0. & \end{array}
$$

For adequately large $|s|$, the term $hL_1^{II}\hat{u}_0$ can be treated as an $O(h)$ perturbation to the boundary condition and hence can be ignored in the discussions. We introduce vectors

$$
\begin{array}{l} \mathbf{y}_\mu = (\hat{u}_\mu, \hat{z}_{\mu+p-1}, \hat{z}_{\mu+p-2}, \dots, \hat{z}_{\mu-r-1})^T, \\ \mathbf{g}_\mu = (0, \gamma\hat{F}_\mu, 0, \dots, 0)^T, \end{array}
$$

then (18) becomes

$$
(19) \qquad \mathbf{y}_{\mu+1} = (M_0 + \tilde{s}^{\frac{1}{2}}M_1)\mathbf{y}_\mu + h^2\tilde{s}^{-\frac{1}{2}}\mathbf{g}_\mu, \quad \mu = 1,2,\dots,
$$

here M_0 and M_1 are constant matrices. With the help of the above equations, boundary conditions in (18) become an underdetermined linear system

$$
(20) \qquad D(\tilde{s}^{\frac{1}{2}})\mathbf{y}_1 = h^{\frac{3}{2}}\tilde{s}^{-\frac{1}{2}}g, \quad |g| \le K\|\mathbf{g}\|.
$$

Here, $D(\tilde{s}^{\frac{1}{2}})$ is an $(r+1)\times n$ matrix, analytic in $\tilde{s}^{\frac{1}{2}}$ for $Re(\tilde{s}) \ge 0$, and $\mathrm{Rank}(D(\tilde{s}^{\frac{1}{2}})) = r+1$. The eigenvalues of $M_0 + \tilde{s}^{\frac{1}{2}}M_1$ are the roots of the characteristic equation given in Lemma 2.1. Moreover, M_0 has the form

$$
M_0 = \begin{pmatrix} 1 & 0 \\ 0 & M_0^I \end{pmatrix}.
$$

Therefore the next Lemma follows directly.

Lemma 2.4. *There exists a transformation $T = T(\tilde{s}^{\frac{1}{2}})$, analytic in $\tilde{s}^{\frac{1}{2}}$, such that*

$$T^{-1}(M_0 + \tilde{s}^{\frac{1}{2}} M_1)T = \begin{pmatrix} M_{11} & \\ & M_{22} \end{pmatrix},$$

here M_{11} is of order $r+1$ and has eigenvalues $|\kappa_j| \leq 1$ and M_{22} is of order $p+1$ and has eigenvalues $|\kappa_j| \geq 1$. Moreover, given any $\delta > 0$, there are $\epsilon > 0$ such that the block matrices have the following properties:

(I). When $|\tilde{s}^{\frac{1}{2}}| \geq \delta$

$$M_{11}^* M_{11} \leq 1 - \epsilon, \quad M_{22}^* M_{22} \geq 1 + \epsilon.$$

(II). When $|\tilde{s}^{\frac{1}{2}}| \leq \delta$,

$$M_{11}^* M_{11} \leq 1 - \epsilon |\tilde{s}^{\frac{1}{2}}|, \quad M_{22}^* M_{22} \geq 1 + \epsilon |\tilde{s}^{\frac{1}{2}}|.$$

For the new variables

$$\phi_\nu = T^{-1}\mathbf{y}_\nu \ , \quad \nu = 1, 2, \ldots,$$

we have the diagonalized one-step scheme

$$\phi_{\nu+1} = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} \phi_\nu + h^2 \tilde{s}^{-\frac{1}{2}} T^{-1} \mathbf{g}_\nu \qquad \nu = 1, 2, \ldots, \tag{21}$$

and the boundary conditions of (21) is rewritten as

$$D_1(\tilde{s}^{\frac{1}{2}})\phi_1^I = h^{\frac{3}{2}} \tilde{s}^{-\frac{1}{2}} g + D_2(\tilde{s}^{\frac{1}{2}})\phi_1^{II}, \tag{22}$$

where $D_1(\tilde{s}^{\frac{1}{2}})$ and $D_2(\tilde{s}^{\frac{1}{2}})$ are analytic in $\tilde{s}^{\frac{1}{2}}$. The general solution ϕ_ν with $\|\phi\|_h < \infty$ is given by

$$\begin{aligned} \phi_j^I &= h^2 \tilde{s}^{-\frac{1}{2}} \textstyle\sum_{\nu=1}^{j-1} M_{11}^{j-\nu-1} (T^{-1}\mathbf{g})_\nu^I + M_{11}^{j-1} \phi_1^I, \\ \phi_j^{II} &= -h^2 \tilde{s}^{-\frac{1}{2}} \textstyle\sum_{\nu=j}^{\infty} M_{22}^{j-\nu-1} (T^{-1}\mathbf{g})_\nu^{II}, \end{aligned} \qquad j \geq 2. \tag{23}$$

With discrete Minkowski inequality we can prove

Lemma 2.5. *The solutions of (21) satisfy the estimates*

$$\|\phi^I\| \leq \frac{const.h|s^{-\frac{1}{2}}|}{1 - \|M_{11}\|} \|(T^{-1}g)^I\| + \left(\frac{h}{1 - \|M_{11}\|^2}\right)^{1/2} |\phi_1^I|,$$

$$\|\phi^{II}\| \leq \frac{const.h|s^{-\frac{1}{2}}| \, \|M_{22}^{-1}\|}{1 - \|M_{22}^{-1}\|} \|(T^{-1}g)^{II}\|,$$

$$|\phi_1^{II}| \leq \frac{h|s^{-\frac{1}{2}}|}{(h(1 - \|M_{22}^{-1}\|^2))^{1/2}} \|(T^{-1}g)^{II}\|.$$

With boundary condition (22), we have for ϕ

$$\|\phi^I\| \le \frac{h|s^{-\frac{1}{2}}|}{1-\|M_{11}\|}\|(T^{-1}\mathbf{g})^I\| + \frac{const.h|s^{-\frac{1}{2}}|\,\|D_1^{-1}\|}{(1-\|M_{11}\|)^{1/2}(1-\|M_{22}^{-1}\|)^{1/2}}\|(T^{-1}\mathbf{g})^{II}\|,$$

$$\|\phi^{II}\| \le \frac{h|s^{-\frac{1}{2}}|\,\|M_{22}^{-1}\|}{1-\|M_{22}^{-1}\|}\|(T^{-1}\mathbf{g})^{II}\|.$$

Transforming back to $\hat{u}_\nu, \hat{z}_\nu$ we have

$$\|\hat{u}\| + \|\hat{z}\| \le \frac{const\|D_1^{-1}\|}{|s|}\|\hat{F}\|.$$

Hence, if $\|D_1^{-1}\| \le K$ for all $\tilde{s}^{\frac{1}{2}}$ with $\mathrm{Re}(\tilde{s}) \ge 0$, the solution is bounded by the data in the desired way and Lemma 2.3 follows. In addition, we have

Lemma 2.6. *For sufficiently large $|\tilde{s}|$, we have the boundary estimates*

$$|\phi_j| \le \frac{K_j h^{\frac{1}{2}}}{|s|^{\frac{1}{2}}}\|\hat{F}\|, \quad j = 1, 2, \dots, \tag{24}$$

where K_j depends on j only.

Next, with different formulation, we will prove

Lemma 2.7. *Kreiss condition holds for (16) if the approximation (12) is stable.*

Instead of z, we use $\hat{u}' = hD_+\hat{u}$ as a new variable. Then (13) is rewritten as

$$\begin{cases} \hat{u}'_\mu = \hat{u}_{\mu+1} - \hat{u}_\mu, & \\ hs\hat{u}_\mu = D_-(I+R)\hat{u}'_\mu + h\hat{F}_\mu, & \mu = 1, 2, \dots, \\ L_1^I\hat{u}_0 = 0, \quad hL_1^{II}\hat{u}_0 + L_2^{II}\hat{u}'_0 = 0, & \end{cases} \tag{25}$$

or, after dropping the $O(h)$ terms in the boundary conditions,

$$\begin{cases} \hat{u}'_\mu = \hat{u}_{\mu+1} - \hat{u}_\mu, & \\ \hat{u}'_{\mu+p} = \sum_{-r-1}^{p-1}\alpha_j\hat{u}'_{\mu+j} + \tilde{s}\beta_1\hat{u}_\mu + \gamma h^2\hat{F}_\mu, & \mu = 1, 2, \dots, \\ L_1^I\hat{u}_0 = 0, \quad L_2^{II}\hat{u}'_0 = 0. & \end{cases} \tag{26}$$

Similarly, by introducing the vectors

$$\begin{aligned} \mathbf{y}_\mu &= (\hat{u}_\mu, \hat{u}'_{\mu+p-1}, \dots, \hat{u}'_{\mu-r-1})^T, \\ \mathbf{g}_\mu &= (0, \gamma\hat{F}_\mu, 0, \dots, 0)^T, \end{aligned}$$

we end up with an one-step scheme

$$\mathbf{y}_{\mu+1} = M\mathbf{y}_\mu + h^2\mathbf{g}_\mu, \quad \mu = 1, 2, \dots,$$

where

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \beta_1\tilde{s} & \alpha_{p-1} & \dots & \alpha_1 & \alpha_0 & \alpha_{-1} & \dots & \alpha_{-r-1} \\ 0 & 1 & 0 & & \dots & \dots & & 0 \\ 0 & 0 & 1 & 0 & & \dots & & 0 \\ \vdots & & \dots & \dots & & \ddots & \ddots & \vdots \\ 0 & & \dots & \dots & & 0 & 1 & 0 \end{pmatrix}.$$

The boundary conditions become

$$D(\tilde{s})\mathbf{y}_1 = h^{\frac{3}{2}}g, \quad |g| \leq const.\|\mathbf{g}\|.$$

Matrix M has the following properties:

Lemma 2.8. *There is a transformation matrix $T = T(\tilde{s})$, analytic in $\tilde{s}$, satisfying*
(I). When $|\tilde{s}| \geq \delta$,

$$T^{-1}MT = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix},$$

where M_{11} is of order $r+1$, M_{22} is of order $p+1$, and

$$M_{11}^*M_{11} \leq 1-\epsilon, \quad M_{22}^*M_{22} \geq 1+\epsilon.$$

Moreover, for any integer N, there is constant $K(N) > 0$ such that for $j = 1, 2, \dots, N$

$$(M_{11}^j)^*M_{11}^j \geq K(N)(1-\epsilon)^j, \quad (M_{22}^{-j})^*M_{22}^{-j} \geq K(N)(1-\epsilon)^j.$$

(II). When $|\tilde{s}| < \delta$,

$$T^{-1}MT = \begin{pmatrix} \tilde{M}_{11} & & \\ & \tilde{M}_{12} & \\ & & \tilde{M}_{22} \end{pmatrix},$$

where $\tilde{M}_{11}$ is of order r, $\tilde{M}_{22}$ is of order p, and

$$\tilde{M}_{11}^*\tilde{M}_{11} \leq 1-\epsilon, \quad \tilde{M}_{22}^*\tilde{M}_{22} \geq 1+\epsilon.$$

There is a constant $K(N)$ dependent on N only such that for $j = 1, 2, \dots, N$

$$(\tilde{M}_{11}^j)^*\tilde{M}_{11}^j \geq K(N)(1-\epsilon)^j, \quad (\tilde{M}_{22}^{-j})^*\tilde{M}_{22}^{-j} \geq K(N)(1-\epsilon)^j.$$

Moreover,

$$\tilde{M}_{12} = \begin{pmatrix} \kappa_1 & 1+\alpha(\tilde{s}) \\ & \kappa_2 \end{pmatrix},$$

where $\alpha(\tilde{s})$ is analytic in $\tilde{s}$ and $\alpha(0) = 0$. $\kappa_1(0) = \kappa_2(0) = 1$ and

$$\bar{\kappa}_1\kappa_1 \leq 1 - \epsilon|\tilde{s}^{\frac{1}{2}}|, \quad \bar{\kappa}_2\kappa_2 \geq 1 + \epsilon|\tilde{s}^{\frac{1}{2}}|.$$

The partition of $T^{-1}MT$ for $|\tilde{s}| < \delta$ follows from the fact that, in a neighborhood of $\tilde{s} = 0$, κ_1 and κ_2 are analytic in $\tilde{s}^{\frac{1}{2}}$, instead of $\tilde{s}$.

We will prove Lemma 2.7 by the method of contradiction. We begin with the assumption that $\tilde{s} = 0$ is a generalized eigenvalue. Introduce

$$\begin{aligned} \phi_\mu &= T^{-1}\mathbf{y}_\mu, \\ G_\mu &= T^{-1}\mathbf{g}_\mu, \end{aligned}$$

then the equation for ϕ is

$$\phi_{\mu+1} = \begin{pmatrix} \tilde{M}_{11} & & \\ & \tilde{M}_{12} & \\ & & \tilde{M}_{22} \end{pmatrix} \phi_\mu + h^2G_\mu, \quad \mu = 1, 2, \dots, \tag{27}$$

$$D_1(\tilde{s})\phi_1^I = h^{3/2}g + D_2(\tilde{s})\phi_1^{II}.$$

We will show that we can choose a vector function G such that the solution of (26) does not satisfy the estimate (11) for some particular $\hat{F}$. Note that for any function G, we can reduce (27) to (26) with some $\hat{F}$ such that

$$\|\hat{F}\| \leq K\|G\|.$$

For our purpose, we consider the special forcing function of the form

$$\begin{array}{ll} G_\mu^I = 0, & \mu = 1, 2, \ldots, \infty, \\ G_\mu^{II} = 0, & \mu = 1, 2, \ldots, s \end{array}$$

(which also makes the boundary condition homogeneous). Then the solution of (27) is given by

$$(28) \qquad \begin{cases} \tilde{\phi}_j^I = \tilde{M}_{11}^{j-1}\tilde{\phi}_1^I, \\ \phi_{j,r+1}^I = \kappa_1^{j-1}\phi_{1,r+1}^I + \sum_{\mu=1}^{j-1}\kappa_1^{j-\mu-1}\phi_{\mu,1}^{II}, \\ \phi_{j,1}^{II} = -h^2\sum_{\mu=j}^{\infty}\kappa_2^{j-\mu-1}G_{\mu,1}^{II}, \\ \tilde{\phi}_j^{II} = -h^2\sum_{\mu=j}^{\infty}\tilde{M}_{22}^{j-\mu-1}\tilde{G}_\mu^{II}, \end{cases} \qquad j = 1, 2, \ldots, \infty,$$

where $\tilde{\phi}^I, \tilde{\phi}^{II}$ are defined according to the partition of $\tilde{M}_{11}$ and $\tilde{M}_{22}$,and $\phi_{r+1}^I, \phi_1^{II}$ are defined according to the partition of $\tilde{M}_{12}$, respectively.

If the boundary condition is independent of the scheme, then we must have

$$\operatorname{Rank}(D_1, D_2) = r + 1.$$

Because $\tilde{s} = 0$ is an generalized eigenvalue, D_1^{-1} has a pole at $\tilde{s} = 0$. Thus for some l, m, $1 \leq l \leq r+1$, $1 \leq m \leq p+1$, there is

$$|\phi_{1,l}^I| \geq \frac{K}{|\tilde{s} - \tilde{s}_0|}|\phi_{1,m}^{II}|.$$

To construct a special forcing function, we denote

$$M_{22} = \begin{pmatrix} \kappa_2 & \\ & \tilde{M}_{22} \end{pmatrix},$$

and select ϵ_0 such that

$$G_\mu^{II} = (1+\epsilon_0)^{-\mu}M_{22}^\mu \rho e_m, \quad \|M_{22}\| \leq 1 + \epsilon_0,$$

where e_m is the m^{th} column of the identity matrix $I_{(p+1)\times(p+1)}$, then

$$\|G^{II}\|^2 = \sum_{\mu=1}^{\infty}|(1+\epsilon_0)^{-\mu}M_{22}^\mu \rho e_m|^2 h \leq K_1\rho^2 h,$$

and

$$|\phi_{1,m}^{II}| = h^2|\sum_{\mu=1}^{\infty}M_{22}^{-\mu}(1+\epsilon_0)^{-\mu}M_{22}^\mu \rho e_m| \geq \frac{h^2\rho}{\epsilon_0} \geq const.h^{\frac{3}{2}}\|G^{II}\|.$$

As the result,

$$\begin{aligned} |\phi_{1,l}^{I}| &\geq \frac{K}{|\tilde{s}-\tilde{s}_0|}|\phi_{1,m}^{II}| \\ &\geq \frac{const.}{h^2|s-s_0|}h^{\frac{3}{2}}\|G^{II}\| \\ &\geq \frac{const.}{h^{\frac{1}{2}}|s-s_0|}\|G\|. \end{aligned}$$

We have to treat two cases $l=1$ and $l\neq 1$ separately.

(I). $l\neq r+1$. We have

$$\begin{aligned} \|\tilde{\phi}^I\|^2 &= \sum_{j=1}^{\infty}|\tilde{M}_{11}^{j-1}\tilde{\phi}_1^I|^2 h \\ &\geq K(N)\sum_{j=1}^{N}(1-\epsilon)^{2(j-1)}|\phi_1^I|^2 h \\ &\geq \frac{const.}{|s-s_0|^2}\|G\|^2. \end{aligned}$$

(II). $l=r+1$. According to Lemma 2.8, we have

$$\begin{aligned} \|\phi_{r+1}^{I}\| &\geq (\frac{h}{1-|\kappa_1|^2})^{\frac{1}{2}}|\phi_{1,r+1}^{I}| - \frac{1}{1-|\kappa_1|}\|\phi_1^{II}\| \\ &\geq (\frac{h}{1-|\kappa_1|^2})^{\frac{1}{2}}\frac{const.}{h^{\frac{1}{2}}|s-s_0|}\|G^{II}\| - \frac{h^2}{(1-|\kappa_1|)^{\frac{1}{2}}(|\kappa_2|-1)^{\frac{1}{2}}}\|G^{II}\| \\ &\geq \frac{const.}{h^{\frac{1}{2}}|s-s_0|}\|G^{II}\|. \end{aligned}$$

Correspondingly, $\mathbf{y}$ will have the lower bounds

$$\|\mathbf{y}\| \geq \begin{cases} \frac{const.}{|s-s_0|}\|G\|, & l\neq r+1, \\ \frac{const.}{h^{\frac{1}{2}}|s-s_0|}\|G\|, & l=r+1, \end{cases}$$

Hence there must be

$$\|D_+\hat{u}\| \geq \begin{cases} \frac{const.}{h|s-s_0|}\|G\|, & l\neq r+1, \\ \frac{const.}{h^{\frac{3}{2}}|s-s_0|}\|G\|, & l=r+1. \end{cases}$$

Thus, the problem is unstable if $\tilde{s}=0$ is an eigenvalue.

Next we consider the case that $D_1^{-1}(\tilde{s})$ has a pole at $\tilde{s}_0$ with $|\tilde{s}_0|>\delta$. This case is simpler because matrix M can be diagonalized into two diagonal blocks in the neighborhood of $\tilde{s}_0$. Again we are able to find a forcing function G such that

$$\|D_+\hat{u}\| \geq \frac{const.}{h|s-s_0|}\|G\|.$$

The proof resembles the previous discussions for the case $l\neq r+1$. Thus far Lemma 2.7 is proved.

The previous results can easily be generalized to system of equations (8). In terms of the eigenvalue problem

$$
\left\{
\begin{array}{ll}
s^{\frac{1}{2}}\Phi_j = AD_-(I+R_1)\Psi_j, & j=1,2,\ldots \\
s^{\frac{1}{2}}\Psi_j = D_+\Phi_j, & \\
L_1^I\Phi_0 = 0, \quad L_1^{II}\Phi_0 + L_2^{II}s^{\frac{1}{2}}\Psi_0 = 0, &
\end{array}
\right.
\tag{29}
$$

we have

Theorem 2.1. *The difference approximation (8) is stable according to Definition 2.2 if and only if the Kreiss condition is satisfied for (29).*

2.2. The semigroup stability. Now we consider stability of the problem

$$
\left\{
\begin{array}{ll}
\frac{du_j(t)}{dt} = D_+D_-(I+R_1)u_j(t) := Qu_j(t), & j=1,2,\ldots \\
u_j(0) = f_j, & \\
L_1^I u_0(t) = 0, \quad L_1^{II}u_0(t) + L_2^{II}D_+u_0(t) = 0 &
\end{array}
\right.
\tag{30}
$$

in the semigroup sense. We will show that under Assumption 2.1 we can actually construct a set of special boundary conditions which lead to estimates of the solutions at every line $x = x_j$. For convenience we use an abstract representation $Bu_0(t) = 0$ for the boundary conditions in (30), where B is thus a boundary operator. And we define a one-to-one mapping $I : l^2(1,\infty) \to l_0^2(-\infty,\infty)$ by

$$
(Iu)_j = \begin{cases} u_j & j = 1,2,\ldots, \\ 0 & j \le 0. \end{cases}
$$

We now prove

Theorem 2.2. *There exists boundary operator B_0 such that for all $u \in l^2(1,\infty)$ satisfying $B_0u_0(t) = 0$ the following inequality holds*

$$
Re(u,Qu) \le -q(D_+u, D_+u) - K\Big(\sum_{j=0}^{r}|D_+u_j|^2 + \sum_{j=0}^{-r}|D_+u^b|\Big),
$$

$$
|u_i| \le hK_j\sum_{j=0}^{r}|D_+u_j|, \quad i = 1,\ldots,r+1,
$$

$$
|u_i^b| \le hK_j\sum_{j=0}^{-r}|D_+u_j^b|, \quad i = -r,\ldots,0.
$$

Proof: We interpret the boundary conditions into a single vector $u^b \in l^2(-\infty,\infty)$:

$$
(u^b(t))_j = \left\{ \begin{array}{ll} u_j^b(t), & \text{for } -r \le j \le 0, \\ 0, & \text{others}, \end{array} \right.
$$

with $u_j^b(t), j = -r,\ldots,1$, not yet determined. Then we have

$$
\begin{aligned}
(u,Qu) &= (Iu, Q(Iu+u^b))_{-\infty,\infty} \\
&= (Iu, QIu)_{-\infty,\infty} + (Iu, Qu^b)_{-\infty,\infty} \\
&= (Iu, QIu)_{-\infty,\infty} - (D_+Iu, (I+R)D_+u^b)_{-\infty,\infty}.
\end{aligned}
$$

After taking the real parts of each term,

$$
\begin{aligned}
Re(u, Qu) &\le -q(D_+u, D_+u)_{-\infty,\infty} - Re(D_+Iu, (I+R)D_+u^b)_{-\infty,\infty} \\
&= -q(D_+u, D_+u)_{-\infty,\infty} - Re(D_+u, (I+R)D_+u^b)_{0,r} \qquad (31) \\
&= -q(D_+u, D_+u)_{-\infty,\infty} - Re\{U^* Q_1 U^b\},
\end{aligned}
$$

where

$$
\begin{aligned}
U &= (D_+u_0, D_+u_1, \dots, D_+u_r)^T, \\
U^b &= (D_+u^b_{-r}, D_+u^b_{-r+2}, \dots, D_+u^b_0)^T,
\end{aligned}
$$

and Q_1 is a nonsingular triangular matrix

$$
Q_1 = \begin{pmatrix}
a_{-r} & a_{-r+1} & a_{-r+2} & \dots & a_{-1} & a_0 \\
0 & a_{-r} & a_{-r+1} & a_{-r+2} & \dots & a_{-1} \\
0 & 0 & a_{-r} & a_{-r+1} & \dots & a_{-2} \\
\vdots & & \dots & & \ddots & \vdots \\
0 & & \dots & & 0 & a_{-r}
\end{pmatrix}.
$$

Hence if we choose

$$
U^b = Q_1^{-1} U, \tag{32}
$$

then we will get

$$
Re(u, Qu)_{1,\infty} \le -q(D_+u, D_+u)_{-\infty,\infty} - \sum_{j=0}^{r} |D_+u_j|^2.
$$

The remaining results follow from the relations

$$
\begin{aligned}
u_{j+1} &= u_j + hD_+u_j, \quad j = 1, \dots, r, \\
u_1 &= hD_+u_0,
\end{aligned}
$$

and

$$
\begin{aligned}
u^b_{-j} &= u^b_{-j+1} - hD_+u^b_{-j}, \quad j = 1, \dots, r \\
u^b_0 &= -hD_+u^b_0.
\end{aligned}
$$

We call the boundary operator corresponding to (32) B_0 □

For the special boundary operator B_0 we can get the energy estimate

$$
\|u(t)\|^2 + q\int_0^t \|D_+u(\tau)\|^2 d\tau \le \|u(0)\|^2
$$

and the estimates on the boundary

$$
\int_0^\infty \sum_{j=0}^{r} |D_+u_j|^2 dt = \|u(0)\|^2,
$$

$$
\int_0^\infty |u_j(t)| dt \le K_j h^2 \|u(0)\|^2, \quad -r \le j \le r+1.
$$

If we introduce notation

$$
\langle u_j, u_j \rangle = \int_0^\infty |u_j(t)|^2 dt,
$$

then we can show for the special operator B_0 there are estimates for any u_j.

Lemma 2.9. *For scheme (12) with special boundary condition B_0 we have*

$$\begin{aligned} \langle D_+u_j, D_+u_j\rangle &\le K_j(u^0,u^0) \\ \langle u_j, u_j\rangle &\le K_j h^2(u^0,u^0) \end{aligned} \qquad j\ge 1,$$

where c_j depends on j only.

The proof of above Lemma requires the following interpretation of the Kreiss' condition, which in turn can be obtained through similar arguments as those in the proof of Lemma 2.3.

Lemma 2.10. *For semi-discrete approximations (12), the Kreiss condition implies that, if $F=0, f=0$,*

$$|\hat{u}|_B \le K|\hat{g}|_B. \tag{33}$$

This lemma follows from the expression (23) or (28) of the solution of (12).

Now we can formulate one of our main results.

Theorem 2.3. *If (12) satisfies Kreiss condition, then it is stable in the semigroup sense.*

PROOF: Let v denote the solution of

$$\left\{\begin{array}{l} \frac{dv_j(t)}{dt} = Qv_j(t), \\ v_j(0) = f_j, \\ Bv_0(t) = 0. \end{array}\right. \qquad j=1,2,\ldots \tag{34}$$

Function $w=u-v$ satisfies

$$\left\{\begin{array}{l} \frac{dw_j(t)}{dt} = Qw_j(t), \\ w_j(0) = 0, \\ Bw_0(t) = -Bv_0(t). \end{array}\right. \qquad j=1,2,\ldots \tag{35}$$

Laplace transform on the above equation gives us

$$\left\{\begin{array}{l} s\hat{w}_j = Q\hat{w}_j, \quad j\ge 1, \\ B\hat{w}_0 = -B\hat{v}_0. \end{array}\right. \tag{36}$$

The Kreiss condition requires that the solution at the boundary can be estimated in terms of the data, i.e, for any integer p, there are constants c'_p, c_p such that

$$|D_+\hat{w}|^2_{B_p} = \sum_{j=1}^{p}|D_+\hat{w}_j|^2 \le c'_p\sum_{j=1}^{r}|(B_0D_+\hat{v}_0)_j|^2 \le c_p\sum_{j=1}^{p}|D_+\hat{v}_j|^2.$$

Hence,

$$\int_0^\infty |D_+w(t)|^2_{B_p}dt \le c_p\int_0^\infty\sum_{j=1}^{r}|D_+v_j(t)|^2dt \le const.\|u(0)\|^2.$$

Because

$$\frac{d(w,w)}{dt} = 2\mathrm{Re}(w,Qw) \le -q(D_+w,D_+w) + const.|D_+w|^2_{B_p}$$

for some p, we therefore obtain

$$||w(t)||^2 + q\int_0^t ||D_+w(t)||^2 dt \le const.||u(0)||^2,$$

and finally arrive at

$$||u(t)||^2 + q\int_0^t ||D_+u(t)||^2 dt \le const.||u(0)||^2.$$

This proves the theorem □

For the more general problem (8), we have

Theorem 2.4. *If (8) satisfies Kreiss condition, then it is stable in the semigroup sense.*

The following theorem on the relation between the two definitions is now apparent.

Theorem 2.5. *Under Assumption* 2.1, *the stability definition* 2.1 *and* 2.2 *are equivalent for the semi-discrete problem (8).*

The above theorem indicates that for semidiscrete parabolic problems there is only one appropriate stability definition.

3. Stability of *the methods of line*

We now consider the numerical solution of (12) by standard methods for ordinary differential equations. Using the boundary conditions to eliminate $u_j(t), j = 0, -1, \ldots, -r$, we then obtain an infinite system of ordinary differential equations

$$\left\{ \begin{array}{l} \frac{du_j(t)}{dt} = \tilde{Q}u_j(t) + F_j(t), \\ u_j(0) = 0, \end{array} \right. \qquad j = 1, 2, \ldots. \tag{37}$$

The corresponding resolvent equation is

$$(sI - \tilde{Q})\hat{u} = \hat{F}.$$

We ignore the sign $\sim$ thereafter for convenience. The Kreiss condition of the corresponding eigenvalue problem implies the estimate

$$||\hat{u}|| + ||D_+\hat{u}|| \le K(\eta)||\hat{F}||. \tag{38}$$

Consider solving (37) with Runge-Kutta methods

$$\begin{array}{l} v(t+k) = \sum_{j=0}^{q} \alpha_j \frac{(kQ)^j}{j!} v(t) + kG, \quad \alpha_j = 1, \quad 0 \le j \le p \le q \\ v(0) = 0, \end{array} \tag{39}$$

and linear multistep methods

$$(I - k\beta_{-1}Q)v(t+k) = \sum_{j=0}^{r} (\alpha_j I + k\beta_j Q)v(t - jk) + kG(t - rk), \tag{40}$$

where G is a smooth function of F and $v(t), G(t)$ are step functions defined in the following way

$$v(t) = v(\nu k), \quad \text{for } \nu k \le t \le (\nu+1)k,$$
$$v(t) = 0, \quad \text{for } 0 \le t \le k.$$

The corresponding resolvent equations are

$$(zI - L(kQ))\hat{v} = k\hat{G}, \quad z = e^{sk}, \quad s = i\xi + \eta, \tag{41}$$

and

$$(L_1(z)I - kL_2(z)Q)\hat{v} = k\hat{G}, \quad z = e^{sk}, \quad s = i\xi + \eta, \tag{42}$$

respectively. Here

$$L_1(z) = z^{r+1} - \sum_{j=0}^{r} \alpha_j z^{r-j}, \quad L_2(z) = \beta_{-1} z^{r+1} + \sum_{j=0}^{r} \beta_j z^{r-j}.$$

Stability of above schemes requires that the solution satisfy

$$\|\hat{v}\| + q\|D_+\hat{v}\| \le K(\eta)\|\hat{G}\|, \quad \eta \ge \eta_0. \tag{43}$$

Parallel to our results on *the methods of line* for hyperbolic problems[3], we have

Theorem 3.1. *Assume that the Runge-Kutta method is locally stable. If the the semi-discrete approximation is stable, then the totally discretized approximation is stable as well, if*

$$\|kQ\| \le R < R_1. \tag{44}$$

PROOF: We have to prove that the resolvent equation (41) of the totally discretized equation satisfies the estimate (43). For every z with $|z| > 1$ we can write (41) in the form

$$\prod_{j=1}^{q} (\mu_j(z)I - kQ)\hat{v} = k\hat{G}.$$

The roots μ_j do not belong to Ω. Let $|\mu_1(z)| = \min_j |\mu_j(z)|$, then there must be

$$|\mu_j(z)| \ge R_1, \quad j = 2, \dots, p.$$

Thus $\hat{v}$ satisfies

$$(\mu_1(z)I - kQ)\hat{v} = k\prod_{j=2}^{p} (\mu_j(z)I - kQ)^{-1}\hat{G} := k\tilde{G}.$$

Because of (44),

$$\|\tilde{G}\| \le K\|\hat{G}\|.$$

There are three possibilities.

(1) $|\mu_1(z)| - R > \delta, \delta > 0$ constant. From Lemma 2.3, Lemma 2.6 and energy estimate we obtain (43).
(2) Re $\mu_j(z) \ge \delta_2, \delta_2 > 0$ constant. In this case (43) is directly resulted from (38).

(3) Re $\mu_j(z) > 0$ but for $k \to 0$ $\lim z = e^{i\phi}$, $\lim \mu_j(z) = i\alpha, \alpha, \phi$ real, $|\alpha| \le R$. Let

$$z = e^{i\phi + k(i\xi+\eta)}, \quad \phi, \xi, \eta \text{ real}.$$

By the Lemma 2.4 of [3]

$$\mu_j(z) = \mu_j(i\xi + 0) + \gamma k\eta + O(k^2(|\xi|\eta + \eta^2)). \tag{45}$$

Therefore again by (38) we obtain (43).

Combining the above estimates and observing that for a given $z = e^{i\phi}$ there is at most one root $\mu_j(z) = i\alpha$, $|\alpha| < R_1$, we obtain

$$\begin{aligned} &\|\hat{v}\| + \|D_+\hat{v}\| \\ \le &\begin{cases} const.K(\eta)\|\hat{G}\|, & \text{if one of the roots has the form (37)} \\ const.\|\hat{G}\|, & \text{otherwise} \end{cases} \end{aligned}$$

This proves the theorem □

The next theorem also follows from (38) in the similar approach as that in the proof of Theorem 2.8 in [3].

Theorem 3.2. *Assume that the linear multi-step method is locally stable. If the semi-discrete approximation is stable, then the totally discretized approximation is stable as well, if*

$$\|kQ\| \le R < R_1.$$

These two theorems assure us that *the locally stable methods* are reliable for solving the stable semidiscrete hyperbolic problems. We should point out here that for totally discretized approximations, it is not clear whether the stability in the generalized sense implies the stability in the semigroup sense.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT LOS ANGELES, CA 90024
E-mail: lwu@math.ucla.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA AT LOS ANGELES, CA 90024
E-mail: kreiss@math.ucla.edu