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ON SINGULAR DIFFUSION EQUATIONS WITH APPLICATIONS TO SELF-ORGANIZED CRITICALITY

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Abstract

We consider solutions of the singular diffusion equation $u_t = (u^{m-1}u_x)_x$, $m \leq 0$, associated with the flux boundary condition $\lim_{x\to-\infty} u^{m-1}u_x = \lambda > 0$. The evolutions defined by this problem depend on both m and λ. We prove existence and stability of travelling wave solutions, parameterized by λ. Each travelling wave is stable in its appropriate evolution. These travelling waves are in L^1 for $-1 < m \leq 0$, but have non-integrable tails for $m \leq -1$. We also show that these travelling waves are the same as those for the scalar conservation law $u_t = -[f(u)]_x + u_{xx}$ for a particular nonlinear convection term $f(u) = f(u; m, \lambda)$.

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In this paper we prove the stability of travelling wave solutions of the family of singular diffusion equations:

$$u_t = (u^{m-1}u_x)_x \tag{0}$$

with $m \leq 0$. This equation has a one-parameter family of travelling wave solutions specified by the flux $u^{m-1}u_x$ at $x = -\infty$. These travelling waves belong to a one-parameter family of distinct evolutions defined by (0). Each travelling wave is stable in its particular evolution. These existence and stability results extend the work of Zhang ([Z1],[Z2]) who proved such results by different methods for $-1 < m \leq 0$. Although our proofs treat only equation (0) explicitly, we hope that it will be apparent that our methods are generally applicable to parabolic conservation laws. We also show in Appendix A that the travelling waves of (0) are identical to those for the scalar conservation law

$$u_t = -[f(u)]_x + u_{xx} \tag{1}$$

for a particular nonlinear choice of $f(u)$ depending on m and the asymptotic flux.

Our study of equation (0) is motivated by recent work on cellular automata and interacting particle systems which exhibit so-called "self-organized criticality" (SOC). We begin with a review of the physics which led to the singular diffusion equation (0).

Many dynamical systems in nature exhibit long-range spatial and temporal correlations. For example, spatial scale-invariance is observed in fractal geographical and topographical structures – mountain ranges, river basins, etc. [M], while temporal scale-invariance in the form of "$1/f$-like" power spectra is observed in such diverse phenomena as star flickers and earthquakes. The spatial scale invariance in particular is in marked contrast to the typical behavior of equilibrium systems, where such invariance is only achieved by tuning a parameter (e.g. the temperature) to a critical value. Bak, Tang and Weisenfeld [BTW] coined the term self-organized criticality to describe scale-invariance that occurs in dynamical systems without the explicit tuning of a parameter.

As a paradigm of self-organized critical behavior, Bak, Tang and Weisenfeld introduced a cellular automaton which they called the sandpile model [BTW]. A one-dimensional "limited local" version of the model is defined as follows [KNWZ]: Two positive integers, z_c and n, are prescribed. To each site $i = 1, \dots, L$, one assigns a non-negative integer $h(i)$ representing the height of sand at i. The slope of the sandpile at i is then $z(i) = h(i) - h(i+1)$. The system is assumed to be closed at the left boundary ($i = 0$) and open at the right ($i = L+1$). A single grain of sand is dropped onto a randomly chosen site and the resulting system is examined. If the slope at any site i exceeds z_c, then n grains of sand fall from i onto the site $i+1$. This may, of course, cause the slope to exceed z_c at adjacent sites. Again the system is examined, and, if necessary, n grains fall to the right from any site at which the slope now exceeds z_c. The process continues until all slopes are less than z_c, which is accomplished either with or without the loss of sand from the right boundary. The entire event caused by dropping a single grain is called an avalanche, and the number of sites from which sand has fallen is called the size of the avalanche. After an avalanche, another grain is dropped onto a randomly chosen site, another avalanche occurs, and the process continues. Numerical results ([BTW],[KNWZ]) indicate a power law distribution in the sizes of avalanches – the type of distribution found at the critical point of traditional equilibrium systems. Related quantities are also observed to have power law behavior, thereby defining a set of critical exponents and scaling relations ([TB],[KNWZ]).

Sites for which $z(i) \leq z_c - n$ form the boundaries of avalanches [CCGS1]. If the distribution of the sizes of avalanches obeys a power law, then one expects that the mean density of avalanche boundaries should also tend to zero with a power law in the size of the system. In [CCGS2] numerical evidence was obtained which indicated that a properly scaled form of the density of avalanche boundaries in the sandpile model obeys a singular diffusion equation of the form (0) on a finite interval with $m = -3$. The steady state solution of this equation with the appropriate boundary conditions has the mean value of u vanishing with a power law. This provides an explanation of self-organized criticality in sandpiles.

The sandpile automaton is defined as a short-range model. However, viewed only at the sequence of stopping times at which avalanches occur, the sandpile model is an effectively long-range model for the interaction of avalanche boundaries. A long-range interacting particle system in which the particles qualitatively mimic the behavior of avalanche boundaries was also considered in [CCGS2]. The analogues of the sandpile parameters z_c and n for the particle system are h_c and 1. As with sandpiles, a non-negative integer $h(i)$ is assigned to each site $i = 1, \dots, L$.

The particle system evolves continuously in time according to the following two symmetrical rules: If $h(i) \geq h_c$, then at rate 1 (I) a particle is transferred from i to the first site j to the right of i such that $h(j) < h_c$;(II) a particle is transferred from i to the first site j to the left of i such that $h(j) < h_c$. It was proved [CGST] that the hydrodynamic limit of this interacting particle system on the torus has a density which obeys a singular diffusion equation of the form (0) with $m = -2$. Finally, it was proved [CGST] that particle systems defined as above with transition rates which are non-increasing functions of $|i-j|$ also give rise to singular diffusion equations with other values of $m \geq -2$.

The travelling waves we will study here describe the time-dependence of the density of avalanche boundaries or particles when there is a nonzero flux through the open system.

As an alternative to the study of SOC in cellular automata, several groups have investigated differential equations which describe the fluctuations of a conserved quantity ([HK],[GLMS],[GLS]). Specifically, these groups studied diffusion equations of the form (1) driven by noise terms. These groups were motivated by the fact that the solutions of such equations exhibit scale-invariant behavior similar to that observed in sandpiles and related discrete models. According to these analyses, SOC can already be understood on the level of the linear Langevin equation: algebraic decay of spatial correlations is a reflection in space of the well-known algebraic decay in time for system with conservative dynamics.

To our knowledge, there has been no direct connection yet established between the studies of SOC in discrete systems and in driven diffusion systems. It is therefore of some interest to note that particular *un*driven analogues of such diffusion equations have the same travelling wave solutions as the singular diffusion equation, which, at least in one case, is rigorously known to be the hydrodynamic limit of a discrete SOC model. This is consistent with the fact that the critical exponents of models obeying the singular diffusion equation (0) are determined by the order m of the singularity [CCCGS2], whereas the critical exponents of the driven diffusion equation are apparently determined by the non-linear convection term $f(u)$ ([HK],[GLMS],[GLS]). In Appendix A, we derive the convection term $f(u)$ such that (0) and (1) have the same travelling waves. We also derive the asymptotics of these solutions. It is worth noting that for $m \leq -1$ (and therefore in the two cases of interest discussed above), the tails of the travelling waves are not integrable. It is for this reason that the previous results of Zhang ([Z1],[Z2]) do not extend to these cases. Finally, it is also worth noting that for $-1 < m \leq 0$, there is a stable self-similar solution ([ERV], [Z3]) in addition to the travelling waves. This solution has zero flux at $x = -\infty$ and belongs to yet another distinct evolution. However, the self-similar solution does not extend to $m \leq -1$.

Travelling wave solutions $u(x,t) = \varphi(x - st)$ to $u_t = (u^{m-1}u_x)_x$ satisfy

$$-s\varphi' = (\varphi^{m-1}\varphi')'. \tag{2}$$

Imposing the conditions $\lim_{x\to-\infty}\varphi(x) = 0$ and $\lim_{x\to-\infty}\varphi^{m-1}(x)\varphi'(x) = \lambda > 0$, and integrating (2) from $-\infty$ to x, we have

$$-s\varphi(x) = \varphi^{m-1}(x)\varphi'(x) - \lambda. \tag{3}$$

Imposing $\lim_{x\to\infty}\varphi(x) = 1$ and $\lim_{x\to\infty}\varphi'(x) = 0$, it follows that $s = \lambda$. Thus

$$x = \frac{1}{\lambda}\int_{\varphi(0)}^{\varphi(x)} \frac{\varphi^{m-1}}{1-\varphi}d\varphi \tag{4}$$

This gives positive solutions satisfying the boundary conditions if and only if $0 < \varphi(0) < 1$ and $m \leq 0$. From (3) and (4) one sees that φ is strictly increasing and $1 - \varphi(x) \leq (1 - \varphi(0))e^{-\lambda x}$. When $m = 0$, one also has $\varphi(x) \leq \varphi(0)e^{\lambda x}$. When $m < 0$, (4) implies that

$$\lim_{x\to-\infty}\varphi(x)(m\lambda x)^{-\frac{1}{m}} = 1.$$

Hence, by the mean value theorem

$$\begin{aligned}|\varphi(x+x_0) - \varphi(x)| &= |x_0\varphi'(x^*)| \\ &\leq C\varphi^{1-m}(x^*) \leq C'|x|^{\frac{1}{m}-1}\end{aligned}$$

for $x << 0$. Thus for any x_0

$$\int_{\mathbf{R}}(\varphi(x+x_0) - \varphi(x))dx < \infty. \tag{5}$$

For definiteness, henceforth we will let $\varphi(x)$ be the travelling wave profile with $\varphi(0) = \frac{1}{2}$. Defining

$$\begin{aligned}\mathcal{D}_{x_0} =&\{f \in C^\infty(\mathbf{R}) : \varphi(x - x_0) \leq f(x) \leq \varphi(x + x_0) \text{ for } x \in \mathbf{R} \\ &\text{and } f(x) - \varphi(x - x_0) \text{ has compact support}\} \text{ and} \\ \mathcal{D} =& \bigcup_{x_0 \in \mathbf{R}_+} \mathcal{D}_{x_0},\end{aligned}$$

we can formulate the results of this article.

Theorem 1: *For $f \in \mathcal{D}$ the initial value problem $u_t = (u^{m-1}u_x)_x$, $u(x,0) = f(x)$ has a classical solution such that the solution operator $[S(t)f](x) = u(x,t)$ extends to an $\mathcal{L}^1$-contraction semigroup on the $\mathcal{L}^1$-closure of $\mathcal{D}$ i.e.*

$$\int_{\mathbf{R}} |[S(t)f](x) - [S(t)g](x)|dx \leq \int_{\mathbf{R}} |f(x) - g(x)|dx$$

for $t \geq 0$. Moreover, travelling waves remain travelling waves, $[S(t)\varphi(\cdot + x_0)](x) = \varphi(x + x_0 - \lambda t)$.

The stability of travelling waves takes the following form.

Theorem 2: *Let f be in the L^1-closure of $\mathcal{D}$ and let x_0 be determined by $\int_{\mathbf{R}}(f(x) - \varphi(x + x_0))dx = 0$. Then $S(t)f$ is asymptotic to $\varphi(x + x_0 - \lambda t)$, in the L^1-sense, i.e.*

$$0 = \lim_{t\to\infty} \int_{\mathbf{R}} |S(t)f - \varphi(x + x_0 - \lambda t)|dx.$$

Remark 1. For $m > -1$, the L^1-closure of $\mathcal{D}$ is easily seen to be equal to $\{f : 0 \leq f(x) \leq 1 \text{ and } \int_{\mathbf{R}} |f(x) - \chi_{[0,\infty)}(x)|dx < \infty\}$. Thus, while the sets $\mathcal{D}_{x_0}$ for different λ's are disjoint, their closures are identical when $m > -1$. However, the semigroups $S(t)$ are always distinct for different λ's: it is only the φ in the definition of $\mathcal{D}_{x_0}$ and its translates that propagate as travelling waves under the semigroup $S(t)$ in Theorem 1.

Using (A.7) from Appendix A, one can give a similar characterization of the L^1-closure of $\mathcal{D}$ for $m \leq -1$: $\{f : 0 \leq f(x) \leq 1 \text{ and } \int_{\mathbf{R}} |f(x) - \chi_{[0,\infty)}(x) - \varphi_m(x)|dx < \infty\}$, though in this case the closures for different λ's are disjoint.

Proof of Theorem 1. For initial data $f \in \mathcal{D}_{x_0}$ we will construct classical solutions of (0) as limits of the solutions $u_{N,M}$ of the following mixed problems:

$$(6)\qquad \begin{cases} u_t = (u^{m-1}u_x)_x \text{ on } [-N,M]\times[0,T], \\ u(x,0) = f(x) \text{ on } [-N,M], \\ \text{where we assume that } N \text{ and } M \text{ are large enough that} \\ \text{the support of } f - \varphi(\cdot - x_0) \text{ is contained in } [-N,M], \\ u^{m-1}(-N,t)u_x(-N,t) = \varphi^{m-1}(-N-x_0-\lambda t)\varphi'(-N-x_0-\lambda t) \\ \qquad\qquad \text{and} \\ u(M,t) = \varphi(M-x_0-\lambda t). \end{cases}$$

In other words $u_{N,M}$ solves (0) on $-N \le x \le M$ with the boundary conditions of flux at $x = -N$ and value at $x = M$ equal to the corresponding data for the travelling wave $\varphi(x - x_0 - \lambda t)$. Since $f(x) - \varphi(x - x_0) \ge 0$, we can replace (6) by a nonsingular parabolic problem by choosing a smooth positive function $b(x,t,u)$ on $[-N,M]\times[0,T]\times\mathbb{R}$ such that $b(x,t,u) = u^{m-1}$ when $u \ge \varphi(x - x_0 - \lambda t)$, and replacing (6) by

$$(6')\qquad \begin{cases} u_t = (b(x,t,u)u_x)_x \text{ on } [-N,M]\times[0,T], \\ u(x,0) = f(x) \text{ on } [-N,M] \\ b(x,t,u)u_x = \varphi^{m-1}\varphi' \text{ at } x = -N \\ \qquad\qquad \text{and} \\ u = \varphi \text{ at } x = M. \end{cases}$$

The existence of a unique classical solution to (6$'$) is a standard result. For Dirichlet boundary conditions it is given by Theorem 6.1 and for flux boundary conditions by Theorem 7.4 in Chapter V of [LSU]. The extension to Dirichlet conditions at $x = M$ and flux conditions at $x = -N$ is immediate.

Since $\varphi(y+x_0) > \varphi(y-x_0)$ and $\varphi^{m-1}(y+x_0)\varphi'(y+x_0) < \varphi^{m-1}(y-x_0)\varphi'(y-x_0)$ for $x_0 > 0$ (see (2) for the second inequality), it follows from the maximum principle that

$$(7)\qquad \varphi(x - x_0 - \lambda t) \le u_{N,M}(x,t) \le u_{N,M+1}(x,t) \le \varphi(x + x_0 - \lambda t)$$

for $(x,t) \in [-N,M] \times [0,T]$. The details of this argument are given in Appendix B. Note that the first inequality in (7) shows that $u_{N,M}$ satisfies (6) as well as (6′), and that $u_{N,M}$ exists for $t \in [0,\infty)$.

Next one observes that the solutions $u_{N,M}$ satisfy the contraction inequality

$$(8) \qquad \int_{-N}^{M} |u_{N,M}(x,t) - v_{N,M}(x,t)| dx \le \int_{-N}^{M} |u_{N,M}(x,0) - v_{N,M}(x,0)| dx.$$

This is also a standard result (see [K]). One can verify it by using (6) to eliminate time derivatives from

$$\int_{-N}^{M} (u_t - v_t) \text{Arctan}\left(\frac{u-v}{\varepsilon}\right) dx,$$

integrating by parts and letting $\varepsilon \to 0$.

Since (7) implies $u_{N,M}(x,t)$ is a bounded increasing sequence in M, $u_{N,M}$ converges pointwise to u_N for $(x,t) \in [-N,\infty) \times [0,\infty)$. Since $\varphi(x + x_0 - \lambda t) - \varphi(x - x_0 - \lambda t)$ goes to zero like $\exp(-\lambda x)$ as $x \to \infty$, the dominated convergence theorem implies that, defining $u_{N,M}(x,t) = \varphi(x - x_0 - \lambda t)$ for $x > M$,

$$\int_{-N}^{\infty} (u_N(x,t) - u_{N,M}(x,t)) dx \to 0$$

as $M \to \infty$, for each $t \in [0,\infty)$.

Estimate (7) shows that for fixed N the $u_{N,M}$ are uniformly bounded above and away from zero. Thus, writing (0) as the linear equation

$$u_t = a(x,t) u_{xx} + b(x,t) u_x$$

where $a(x,t) = (u(x,t))^{m-1}$ and $b(x,t) = (m-1)(u(x,t))^{m-2} u_x(x,t)$, one can apply linear regularity theory for parabolic equations, simply using the fact that a given regularity for the coefficients implies a higher regularity for the solution (cf. Theorem 10 of Chapter 3 of [F]) to increase the regularity of the coefficients. Using the results of Chapter 3 of [F], one then concludes that the functions $u_{N,M}$ are smooth, and the u_N are smooth with bounded derivatives of all orders on

$[-N, \infty] \times [0, T]$. Moreover, extending the coefficients $a_N = (u_N)^{m-1}$ and $b_N = (m-1)u_N^{m-2}(u_N)_x$ to all of $\mathbb{R} \times [0, T]$ and representing ψu_N, where $\psi = 1$ for $x > -N + 2$ and $\psi = 0$ for $x < -N + 1$, in terms of the fundamental solution for the resulting equation, one sees that $\lim_{x \to \infty} (u_N)_x(x, t) = 0$ uniformly on $[0, T]$ (see Theorem 11 of Chapter 1 of [F]). This has the immediate consequence,

$$(9) \qquad \int_{-N}^{\infty} (u_N(x,t) - \varphi(x - x_0 - \lambda t))dx = \int_{-N}^{\infty} (u_N(x, 0) - \varphi(x - x_0))dx.$$

Finally, since (7) also implies

$$(10) \qquad \varphi(x - x_0 - \lambda t) \leq u_N(x,t) \leq \varphi(x + x_0 - \lambda t),$$

the linear regularity results of Chapter 3 of [F] also imply for any r, s

$$(11) \qquad \sup_{[-L,L] \times [0,T]} |D_t^r D_x^s u_N(x,t)| \leq C_{L,T}$$

for $L < N$, where $C_{L,T}$ is independent of N.

Let $\mathcal{L}_{x_0}$ be the closure of $\mathcal{D}_{x_0}$ in the L^1-topology, and let $\mathcal{A}_{x_0}$ be a countable subset of $\mathcal{D}_{x_0}$ which is dense in $\mathcal{L}_{x_0}$. Since (11) implies that any subsequence of the u_N's has a subsequence which converges uniformly on compact subsets of $\mathbb{R} \times [0, \infty)$ together with all of its derivatives, by the Cantor diagonal argument we can choose a sequence $\{N_k\}_{k=1}^{\infty}$ such that u_{N_k} converges uniformly on compact subsets of $\mathbb{R} \times [0, \infty]$ together with all of its derivatives *for each initial data* $f \in \mathcal{A}_{x_0}$ for all x_0 in a sequence $\mathcal{S}$ tending to ∞. Note that (5) and (10) imply that the $u_{N_k}(\cdot, t)$ converge in $L^1(\mathbb{R})$ as well by the dominated convergence theorem. For $f \in \mathcal{A}_{x_0}, x_0 \in \mathcal{S}$, we define $S(t)f$ in Theorem 1 by $[S(t)f](x) = \lim_{k \to \infty} u_{N_k}(x, t)$. Then $[S(t)f](x)$ is a classical solution of (0) which inherits the properties:

i) $\varphi(x - x_0 - \lambda t) \leq u(x, t) \leq \varphi(x + x_0 - \lambda t)$
from (10),

ii) $\int_{\mathbb{R}} (u(x,t) - \varphi(x - x_0' - \lambda t))dt = \int_{\mathbb{R}} (f(x) - \varphi(x - x_0'))dx$,
for all $x_0' \in \mathbb{R}$, from (9) and the identity

$$\int_{\mathbb{R}} (\varphi(x - x_0' - \lambda t) - \varphi(x - x_0 - \lambda t))dx = \int_{\mathbb{R}} (\varphi(x - x_0') - \varphi(x - x_0))dx,$$

and

iii) $\quad \int_{\mathbb{R}} |u_1(x,t) - u_2(x,t)| dt \le \int_{\mathbb{R}} |f_1(x) - f_2(x)| dx$

from (8).

Property iii) implies that for $t \ge 0$, $S(t)$ has a unique extension to $\mathcal{L}_{x_0}$ satisfying i) a.e. in x, ii) and iii). However, we need to show that $S(t)$ is well-defined on $\cup_{x_0 \in \mathcal{S}} \mathcal{L}_{x_0}$. If $f \in \mathcal{L}_{x_0} \cap \mathcal{L}_{\tilde{x}_0}$ with $\tilde{x}_0 > x_0$, we choose $f_n \in \mathcal{D}_{x_0}$ such that f_n converges to f pointwise a.e. and in $L^1(\mathbb{R})$, and then choose $\tilde{f}_n \in \mathcal{D}_{\tilde{x}_0}$ such that $\tilde{f}_n \le f_n$ on $\mathbb{R}$, and $\tilde{f}_n = f_n$ on the support of $f_n(x) - \varphi(x - x_0)$ and $[-n, n]$. Hence, $\tilde{f}_n$ also converges to f pointwise a.e. and in $L^1(\mathbb{R})$. If we let $u_n(x,t)$ and $\tilde{u}_n(x,t)$ denote the solutions of (0) that we have constructed for these two sequences of initial data, it follows by the maximum principle argument used to establish (7) that

$$u_n(x,t) \ge \tilde{u}_n(x,t), \tag{12}$$

see Appendix B. Thus, letting $u(\cdot,t)$ and $\tilde{u}(\cdot,t)$ denote the limits of these sequences in $L^1(R)$ we have

$$u(x,t) \ge \tilde{u}(x,t) \text{ a.e. in } x \text{ for all } t.$$

However,

$$\int_{\mathbb{R}} (u(x,t) - \varphi(x - x_0 - \lambda t)) dx = \int_{\mathbb{R}} (f(x) - \varphi(x - x_0)) dx$$
$$= \int_{\mathbb{R}} (\tilde{u}(x,t) - \varphi(x - x_0 - \lambda t)) dx$$

Thus $u(x,t) = \tilde{u}(x,t)$ a.e. in x, and it follows that $S(t)$ is well-defined on $\cup \mathcal{L}_{x_0}$

Since $\mathcal{L}_{x_0} \subset \mathcal{L}_{\tilde{x}_0}$ for $\tilde{x}_0 > x_0$, $S(t)$ is a family of contraction operators on $\cup_{x_0 \in \mathcal{S}} \mathcal{L}_{x_0}$, and hence has a unique extension to a family of contractions on $\overline{\cup_{x_0 \in \mathcal{S}} \mathcal{L}_{x_0}} = \bar{\mathcal{D}}$.

Finally, to see that $S(t)$ is a semigroup we need to show that for $f \in \mathcal{D}_{x_0}, x_0 \in \mathcal{S}$, $S(t+s)f = S(t)(S(s)f)$. Note that $S(s)f \notin \mathcal{D}_{x_0}$ in general, but we do have $S(s)f \in C^\infty(\mathbb{R})$ and

$$\varphi(x - x_0 - \lambda s) \le [S(s)f](x) \le \varphi(x + x_0 - \lambda s),$$

so that choosing $\tilde{x}_0 \in \mathcal{S}$, $\tilde{x}_0 > x_0 + \lambda s$, we can choose $\tilde{f}_n \in \mathcal{D}_{\tilde{x}_0}$ such that $\tilde{f}_n < S(s)f$ and $\tilde{f}_n$ converges to $S(s)f$ pointwise a.e. and in $L^1(\mathbf{R})$. If we let $\tilde{u}_n(x,t)$ be the solution of (0) that we have constructed with initial data $\tilde{f}$, the argument from the maximum principle used to prove (7) gives

$$\tilde{u}_n(x,t) \leq [S(t+s)f](x), \tag{13}$$

see Appendix B. Since $\tilde{u}_n(\cdot,t)$ converges in L^1 to $S(t)(S(s)f)$, and

$$\int_{\mathbf{R}} ([S(t)(S(s)f](x) - \varphi(x - x_0 - \lambda(s+t)))dx =$$
$$\int_{\mathbf{R}} ([S(s)f](x) - \varphi(x - x_0 - \lambda s))dx =$$
$$\int_{\mathbf{R}} ([S(t+s)f](x) - \varphi(x - x_0 - \lambda(s+t)))dx,$$

we conclude $S(t)(S(s)f) = S(t+S)f$. ■

In proving Theorem 2 we will make use of the construction of $S(t)f$ for $f \in \mathcal{D}$ using the sequences of classical solutions $u^{N,M}$. This proof is a variation on the method used by Newman and Ralston in [N], [R].

Proof of Theorem 2: Since $S(t)$ is a contraction semigroup on $\bar{\mathcal{D}}$, it suffices to prove Theorem 2 for $f \in \mathcal{A}_{x_0}, x_0 \in \mathcal{S}$. We will set $v(x,t) = u(x+\lambda t, t) - \varphi(x - x_0)$ so that $v(\cdot,t)$ is in $L^1(\mathbf{R})$ and, when u is a travelling wave, v does not depend on t. Then Theorem 2 becomes the following statement:

Given $f \in \mathcal{A}_{x_0}$, let $v(x,t)$ be the solution of

$$v_t = \left(\frac{(\varphi(x - x_0) + v)^m}{m} - \frac{\varphi(x - x_0)^m}{m} \right)_{xx} + \lambda v_x, \tag{14}$$

with initial data $v(x,0) = f(x) - \varphi(x - x_0) \in C_0^\infty(\mathbf{R})$, constructed as the limit of the sequence $u_{N_K,M}(x+\lambda t, t) - \varphi(x - x_0) \equiv v_{K,M}(x,t)$. Then $v(\cdot,t)$ converges to $\varphi(\cdot - x_0') - \varphi(\cdot - x_0)$ in $L^1(\mathbf{R})$ as $t \to \infty$, where x_0' is determined by

$$\int_{\mathbf{R}} (f(x) - \varphi(x - x_0))dx = \int_{\mathbf{R}} (\varphi(x - x_0') - \varphi(x - x_0))dx.$$

Actually, since $S(t)$ is a contraction semigroup, it suffices to show that $v(\cdot, t_j) \to \varphi(\cdot - x_0') - \varphi(\cdot - x_0)$ for some sequence $t_j \to \infty$.

We construct a "Lyapunov functional" $[H(v)](t)$ as follows. Writing (14) as

$$v_t = (a(x,v))_{xx} + \lambda v_x, \tag{15}$$

we want to find b and F such that

$$(a(x,v))_x + \lambda v \equiv b(x,v)(F(x,v))_x. \tag{16}$$

Then, setting

$$G(x,v) = \int_0^v F(x,s)ds$$

and

$$[H(v)](t) = \int_{\mathbf{R}} G(x, v(x,t))dx,$$

when v is a solution of (15), *formal* computation shows

$$\frac{dH}{dt} = -\int_{\mathbf{R}} b(x,v)((F(x,v))_x)^2 dx.$$

Thus, when $b > 0$, H is decreasing on the orbits of (15), and one expects that solutions will tend to functions v such that $\frac{dH}{dt} = 0$, i.e. level curves of $F(x,v)$. Since we expect solutions to tend to the (zero speed) travelling waves for (15),

$$z(x,s) = \varphi(x - x_0 + s) - \varphi(x - x_0),$$

we define $F(x,v)$ by

$$F(x, z(x,s)) = s. \tag{17}$$

Because $z(x,s)$ is strictly increasing in s, (17) determines F as a smooth function on $\{(x,v) : 0 \le v < 1 - \varphi(x - x_0),\ x \in \mathbf{R}\}$. Since for each $s > 0$ $z(x,s)$ is a stationary travelling wave, we have

$$(a(x, z(x,s)))_x + \lambda z(x,s) \equiv 0, \tag{18}$$

and differentiating (17) w.r.t. x gives

$$F_x(x,z) + F_v(x,z)z_x \equiv 0. \tag{19}$$

Eliminating z_x in (18) and (19), we have

$$a_v F_x \equiv (\lambda z + a_x)F_v. \tag{20}$$

The identity (16) is equivalent to:

$$a_x + \lambda v = bF_x \quad \text{and} \quad a_v = bF_v.$$

Thus, provided that $F_v \neq 0$, (20) implies that (16) holds with the F defined in (17) and $b = a_v(F_v)^{-1}$. Differentiating (17) with respect to s, we have $F_v z_s = 1$. Thus (16) holds with

$$b(x, z(x,s)) = (a(x, z(x,s)))_s = (\varphi(x - x_0 + s))^{m-1}\varphi'(x - x_0 + s), \tag{21}$$

and we see that $b(x,v) > 0$. We can now construct the formal Lyapunov functional $H(v)$.

The first step in using $H(v)$ to prove Theorem 2 is showing that $H(v)$ is finite on v corresponding to $f \in \mathcal{D}_{x_0}$. Since $F(x,v)$ is increasing in $v, G(x,v) \leq vF(x,v)$, and we have $0 \leq v(x,t) \leq \varphi(x + x_0) - \varphi(x - x_0) = z(x, 2x_0)$ by property i) of Theorem 1. Thus

$$\begin{aligned}
[H(v)] = \int_{\mathbf{R}} G(x, v(x,t))dx &\leq \int_{\mathbf{R}} v(x,t)F(x, z(x, 2x_0))dx \\
&\leq \int_{\mathbf{R}} (\varphi(x + x_0) - \varphi(x - x_0))(2x_0)dx < \infty.
\end{aligned}$$

We will not differentiate $[H(v)](t)$ with respect to t. However, it will be evident that the fact $H(v)$ is formally decreasing on orbits is the essential ingredient in the following more indirect argument. Since $H(v) \geq 0$

$$\begin{aligned}
&[H(v)](0) \geq [H(v)](0) - [H(v)](T) \\
&= \lim_{k\to\infty} \lim_{M\to\infty} \int_{-N_k - \lambda t}^{M - \lambda t} G(x, v^{k,M}(x,t))dx \Big|_{t=T}^{t=0}
\end{aligned}$$

by the Lebesgue dominated convergence theorem. Since $v^{k,M}$ is smooth,

$$
\begin{aligned}
&\int_{-N_k-\lambda t}^{M-\lambda t} G(x, v^{k,M}(x,t))dx\Big|_{t=T}^{t=0} \\
&= -\int_0^T dt \int_{-N_k-\lambda t}^{M-\lambda t} F(x, v^{k,M}(x,t))v_t^{k,M}(x,t)dx \\
&+ \lambda \int_0^T G(x, v^{k,M}(x,t))\Big|_{x=-N_k-\lambda t}^{x=M-\lambda t} dt \\
&= \int_0^T dt \int_{-N_k-\lambda t}^{M-\lambda t} b(x, v^{k,M}(x,t))((F(x, v^{k,M}(x,t)))_x)^2 dx \\
&+ \lambda \int_0^T (v^{k,M}(x,t)F(x, v^{k,M}(x,t)) - G(x, v^{k,M}(x,t)))\Big|_{x=-N_k-\lambda t} dt
\end{aligned}
\tag{22}
$$

In the last equality we need the boundary conditions from (6) and formulas (14) and (16) to simplify the boundary terms. Note that there is no contribution from $x = M - \lambda t$ because $F(x,0) = G(x,0) = 0$, and that the contribution from $x = -N_k - \lambda t$ is nonnegative since $G(x,v) \leq vF(x,v)$.

The interior estimates

$$\sup_{[-L,L]\times[0,T]} |D_t^\alpha D_x^\beta v^{k,M}(x,t)| \leq C_{L,T} \text{ for } M, N_k > L,$$

imply that we can choose a subsequence of the $v^{k,M}$'s so that the integrands in (22) converge pointwise to $b(x,v)((F(x,v))_x)^2$, where v is the solution in (14). Thus by Fatou's lemma

$$\int_0^T dt \int_{\mathbf{R}} b(x, v(x,t))((F(x, v(x,t)))_x)^2 dx$$

is bounded independent of T. Thus there is a sequence $t_j \to \infty$ such that

$$\int_{\mathbf{R}} b(x, v(x,t_j))((F(x, v(x,t_j)))_x)^2 dx \to 0. \tag{23}$$

As we observed earlier the functions $v(x,t_j)$ also satisfy

$$0 \leq v(x,t_j) \leq z(x, 2x_0), \tag{24}$$

and, since $v(x,t_j)$ is the L^1-limit of $u_{N_k}(x+\lambda t_j, t_j) - \varphi(x - x_0)$, as $k \to \infty$,

$$\int_{\mathbf{R}} v(x,t_j)dx = \int_{\mathbf{R}} v(x,0)dx. \tag{25}$$

Properties (23), (24) and (25) imply that a subsequence of $\{v(\cdot,t_j)\}$ converges in $L^1(\mathbb{R})$ to $z(\cdot,s)$, where

$$\int_{\mathbb{R}} z(x,s)dx = \int_{\mathbb{R}} v(x,0)dx.$$

This argument is a simpler version of the one given in Section IV of [R]. From (21) and (24) it follows that

$$\begin{aligned} b(x,v(x,t_j)) &\geq \inf_{s\in[0,2x_0]} (\varphi(x-x_0+s))^{m-1}\varphi'(x-x_0+s) \\ &= (\varphi(x+x_0))^{m-1}\varphi'(x+x_0). \end{aligned}$$

Thus by (23)

$$\int_{-M}^{M} ((F(x,v(x,t_j))_x)^2 dx \to 0 \text{ as } j\to\infty \tag{26}$$

for all M. Since $F_v(x,z(x,s))z_s \equiv 1$, (24) implies

$$F_v(x,v(x,t_j)) \geq \inf_{s\in[0,2x_0]} \frac{1}{\varphi'(x-x_0+s)}.$$

Thus from (26) and the boundedness of $F_x(x,v(x,t_j))$ on $|x|<M$ it follows that

$$\int_{-M}^{M} |v_x(x,t_j)|^2 dx \leq C_M.$$

Hence we can choose a subsequence of $\{v(x,t_j)\}$, denoted by $\{v(x,t_k)\}$, which converges uniformly on compact subsets of $\mathbb{R}$ to $v_0(x)$. By (24) $\{v(\cdot,t_k)\}$ converges to $v_0(\cdot)$ in $L^1(\mathbb{R})$ as well, and (25) implies

$$\int_{\mathbb{R}} v_0(x)dx = \int_{\mathbb{R}} v(x,0)dx. \tag{27}$$

From (26) we have for any r

$$F(r,v_0(r)) - F(0,v_0(0)) = \lim_{k\to\infty}\int_0^r (F(x,v(x,t_k)))_x dx = 0.$$

Thus $(x,v_0(x))$ is a level curve of $F(x,v)$ satisfying (27) as claimed. ■

Appendix A

In this appendix we show that the travelling waves which we have been studying are identical to those for the conservation law

$$(\text{A-1}) \qquad u_t = -[f(u)]_x + u_{xx}$$

for a particular choice of $f(u)$. To this end, note that travelling wave solutions $u(x,t) = \varphi(x - st)$ to (A-1) satisfy

$$(\text{A-2}) \qquad -s\varphi' = -f'(\varphi)\varphi' + \varphi''.$$

Imposing the conditions $\lim_{x\to-\infty} \varphi = 0$ and $\lim_{x\to-\infty} \varphi' = 0$, and integrating (A-2) from $-\infty$ to x, we have

$$(\text{A-3}) \qquad -s\varphi = -f(\varphi) + \varphi'.$$

Imposing $\lim_{x\to\infty} \varphi = 1$ and $\lim_{x\to\infty} \varphi' = 0$, it follows that $s = f(1)$. The requirement $\varphi' > 0$ follows from the entropy condition

$$\frac{f(\varphi)}{\varphi} > f(1) \text{ for } 0 < \varphi < 1$$

which we now must impose. Solving (A-3), we have

$$(\text{A-4}) \qquad x = \int_{\varphi(0)}^{\varphi(x)} \varphi^{-1} \left[\frac{f(\varphi)}{\varphi} - f(1)\right]^{-1} d\varphi,$$

which gives positive increasing solutions for $0 < \varphi(0) < 1$.

The analogue of (A-3) for the singular diffusion travelling wave is given by equation (3), i.e.

$$(\text{A-5}) \qquad \varphi' = \lambda(1-\varphi)\varphi^{1-m}.$$

Comparing (A-3) with $s = f(1)$ to (A-5), and setting $f(1) = \lambda$, we see that the two systems have the same travelling waves if

$$(\text{A-6}) \qquad f(\varphi) = \lambda[\varphi + (1-\varphi)\varphi^{1-m}].$$

This is a nonlinear convection term (concave if $1 \geq \varphi \geq \frac{-m}{-m+2}$ and convex if $0 \leq \varphi \leq \frac{-m}{-m+2}$) which in the special case $m = 0$ is just (the negative of) a standard Burgers flux. For all $m \leq 0$, $f'(1) - \lambda = -\lambda \neq 0$ which corresponds to the fact that the solution is well-behaved as $\varphi \to 1$ (i.e. as $x \to \infty$). Indeed, (A-5) implies that to leading order, as $x \to \infty$

$$\varphi \sim 1 - ce^{-\lambda x}$$

with $c > 0$. On the other hand, for $m < 0$, $f'(0) - \lambda = 0$ which corresponds to the fact that the solution decays algebraically, not exponentially, as $\varphi \to 0$ (i.e. as $x \to -\infty$); the solution fails to be integrable for $m \leq -1$.

We will now derive the asymptotics of that part of the solution which is not in L^1. Let k be the largest integer which is less than or equal to $-m$. We claim that for large negative x

$$\varphi(x) = \varphi_m(x) + \psi(x)$$

(A-7) with

$$\varphi_m(x) = (\lambda m x)^{\frac{1}{m}} + c_2(\lambda m x)^{\frac{2}{m}} + \cdots + c_k(\lambda m x)^{\frac{k}{m}}$$

and $\psi \in L^1$. This can be verified by substituting the above equations into (A-5), solving recursively for the coefficients c_j and bounding the remainder ψ in L^1.

Appendix B

To prove the first inequality in (7) one sets $\tilde{\varphi}(x,t) = \varphi(x - x_0 - \lambda t)$ and $w(x,t) = \tilde{\varphi}(x,t) - u(x,t)$, and chooses $B(x,t,u)$ so that $B_u(x,t,u) = b(x,t,u)$. Then, since u and $\tilde{\varphi}$ are solutions of (6′), w satisfies the linear parabolic equation

$$\text{(B-1)} \qquad w_t = \left(\frac{B(x,t,\tilde{\varphi}) - B(x,t,u)}{\tilde{\varphi} - u} w\right)_{xx} - \left(\frac{B_x(x,t,\tilde{\varphi}) - B_x(x,t,u)}{\tilde{\varphi} - u} w\right)_x .$$

Note that $B(x,t,u)$ is increasing so the coefficient of w_{xx} is strictly positive. Since $u(x,0) \geq \tilde{\varphi}(x,0)$, $e^{-\mu t}w$ for μ sufficiently large cannot have a positive maximum in $(-N,M) \times [0,T]$ by Theorem 4, Chapter 2 of [F]. Therefore, if $e^{-\mu t}w$ has a positive maximum in $[-N,M] \times [0,T]$, it must assume it at some point $(x,t) = (-N,t_0)$, $t_0 > 0$, and at no point in $(-N,M) \times [0,T]$. Thus by Theorem 14 of Chapter 2 of [F], $w_x(-N,t_0) < 0$. From the boundary conditions at $x = -N$, we have

$$\text{(B-2)} \qquad b(x,t,u)u_x = b(x,t,\tilde{\varphi})\tilde{\varphi}_x \quad \text{at} \quad (-N,t_0)$$

and, since u^{m-1} is a decreasing function of u, we may assume $b(x,t,u)$ is a decreasing function of u. Since by assumption $\tilde{\varphi}(-N,t_0) > u(-N,t_0)$ and we have shown $\tilde{\varphi}_x(-N,t_0) < u_x(-N,t_0)$, we have a contradiction to (B-2). This shows that w does not have a positive maximum on $[-N,M] \times [0,T]$ and proves the first inequality in (7).

The argument for the second inequality in (7) is precisely the same except that one now uses the first inequality to conclude that $u_{N,M+1}(M,t) \geq u_{N,M}(M,t) = \varphi(M - x_0 - \lambda t)$.

To prove the final inequality in (7) one uses the properties of φ referred to just before (7). Namely

$$\begin{aligned} &u_{N,M+1}^{m-1}(u_{N,M+1})_x\big|_{(-N,t)} \\ &= [\varphi^{m-1}\varphi_x](-N - x_0 - \lambda t) \\ &> [\varphi^{m-1}\varphi_x](-N + x_0 - \lambda t) \end{aligned}$$

and

$$\begin{aligned} u_{N,M+1}(M+1,t) &= \varphi(M + 1 - x_0 - \lambda t) \\ &< \varphi(M + 1 + x_0 - \lambda t). \end{aligned}$$

Thus, a positive maximum for $u_{N,M+1}(x,t) - \varphi(x + x_0 - \lambda t)$ on $[-N, M+1] \times [0, T]$ is contradictory as before.

The inequality (12) follows from the inequality

$$\tilde{u}_{n,N_k,M}(x,t) \leq u_{n,N_k,M}(x,t)$$

for k and M sufficiently large which is proven in the same manner as the last inequality (7).

Finally, the inequality (13) follows from

$$\tilde{u}_{n,N_k,M}(x,t) \leq u_{N_k,M}(x, t+s), \tag{B-3}$$

where $u_{N_k,M}(x,t)$ is the solution of (6), for k and M sufficiently large. Note that

$$(\tilde{u}_{n,N_k,M})^{m-1}(\tilde{u}_{n,N_k,M})_x\big|_{(-N_k,t)} = [\varphi^{m-1}\varphi_x](-N_k - \tilde{x}_0 - \lambda t)$$

and

$$(u_{N_k,M})^{m-1}(u_{N_k,M})_x\big|_{(-N_k,t)} = [\varphi^{m-1}\varphi_x](-N_k - x_0 - \lambda(s+t))$$

so that again the argument used to prove the last inequality in (7) proves (B-3).

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