UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

Schwarz Alternating and Iterative Refinement Methods for Mixed Formulations of Elliptic Problems, Part I: Algorithms and Numerical Results

Tarek P. Mathew

January 1992

CAM Report 92-04

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

SCHWARZ ALTERNATING AND ITERATIVE REFINEMENT METHODS FOR MIXED FORMULATIONS OF ELLIPTIC PROBLEMS, PART I: ALGORITHMS AND NUMERICAL RESULTS.

January 22, 1992

TAREK P. MATHEW *

Abstract. We describe sequential and parallel algorithms based on the Schwarz alternating method for the solution of mixed finite element discretizations of elliptic problems using the Raviart-Thomas finite element spaces. These lead to symmetric indefinite linear systems and the algorithms have some similarities with the traditional block Gauss-Seidel or block Jacobi methods with overlapping blocks. The indefiniteness requires special treatment.

The sub-blocks used in the algorithm correspond to problems on a coarse grid and some overlapping subdomains and is based on a similar partition used in an algorithm of Dryja and Widlund for standard elliptic problems. If there is sufficient overlap between the subdomains, the algorithm converges with a rate independent of the mesh size, the number of subdomains and discontinuities of the coefficients.

Extensions of the above algorithms to the case of local grid refinement is also described. Convergence theory for these algorithms will be presented in a subsequent paper.

Key Words. Schwarz alternating method, domain decomposition, elliptic equations, mixed finite elements, block Gauss-Seidel methods, block Jacobi methods, iterative refinement methods, parallel algorithms

AMS(MOS) subject classifications: 65N30, 65F10

1. Introduction. The Schwarz alternating method was introduced more than 120 years ago by H. A. Schwarz [25] as a technique for proving the existence of solutions to certain elliptic problems on complicated geometries by iteratively solving it on subregions. Since then, the method has been extended to nonlinear problems, the Stokes problem and has recently proven to be a suitable divide and conquer technique to solve a wide class of problems, [15, 16, 14, 4, 7, 1, 2, 22, 26].

Our purpose of this paper is to extend these algorithms to *mixed formulations* of elliptic problems [23] which lead to large, sparse, symmetric indefinite linear systems. These mixed formulations of elliptic problems are useful in certain applications where good approximations to the derivatives of the primary unknown in the differential equation is required. In its matrix version, these algorithms have similarities with generalizations of the standard block Gauss-Seidel and block Jacobi methods with overlapping blocks and some further modifications for the indefiniteness.

Our studies were initiated by a recent paper of Glowinski and Wheeler [10] which proposed a non-overlapping domain decomposition algorithm for this problem, with a rate of convergence, which was independent of the coefficient variations, but which depended mildly on the mesh parameters. With the use of overlapping blocks based on an appropriate choice of overlapping subregions and a coarse grid, the algorithms, described here, converge at a rate that is independent of mesh parameters and coefficient variations.

In § 2, we describe the mixed formulation of elliptic problems and its finite element

* Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90024-1555.
This work was supported in part by the National Science Foundation under Grant NSF-CCR-8903003, and in part by the Army Research Office under Grant DAAL 03-91-G-0150.

discretization. In § 3, we describe the classical Schwarz algorithm and a highly parallel version of it for symmetric, positive definite linear systems arising from standard discretizations of elliptic problems. The corresponding algorithms and the modifications necessary for solving the symmetric, indefinite linear systems arising from the mixed case are described in § 4. Numerical results on the convergence of these methods for various mesh and subdomain sizes and for problems having highly discontinuous coefficients are also presented in § 4. In § 5, we discuss sequential and parallel iterative refinement algorithms for solving problems on grids with local refinement. In a subsequent paper, we will provide theoretical estimates on the rates of convergence of the algorithms described here.

2. An Elliptic Neumann Problem and its Mixed Finite Element Discretization. We consider the following problem for unknown p on a polygonal domain $\Omega \subset R^2$ with $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$:

$$\left\{ \begin{array}{rcll} -\nabla \cdot (a(x,y)\nabla p) & = & f & \text{in } \Omega \\ \vec{n} \cdot (a\nabla p) & = & g & \text{in } \partial\Omega. \end{array} \right. \tag{1}$$

Here $\vec{n}$ is the outward normal to $\partial\Omega$ and $a(x,y)$ denotes a 2×2 symmetric positive definite matrix function with $L^\infty(\Omega)$ entries satisfying

$$\xi^T a(x,y)\xi \geq \alpha \|\xi\|^2, \qquad \text{for a.e. } (x,y) \in \Omega,$$

for a positive constant α. By applying the divergence theorem to the elliptic equation, we see that:

$$\int_\Omega f dxdy = -\int_\Omega \nabla \cdot (a(x,y)\nabla p) dxdy = -\int_{\partial\Omega} (a(x,y)\nabla p) \cdot \vec{n} ds = -\int_{\partial\Omega} g ds,$$

from which we obtain the compatibility condition

$$\int_\Omega f dxdy + \int_{\partial\Omega} g ds = 0.$$

When this condition is satisfied, this equation has a solution p which is unique up to a constant. Throughout the rest of this paper, we will assume, without loss of generality, that $g = 0$, and that f has mean value zero.

In this paper, we assume that we are primarily interested in computing the velocity $\vec{u} = -a(x,y)\nabla p$. In an application to reservoir simulation, p represents the pressure of the fluid, $a(x,y)$ the permeability properties of the medium and $\vec{u}$ the Darcy velocity of the fluid. If the coefficients $a(x,y)$ are discontinuous, the gradient of the pressure will also be discontinuous in general, but the velocity $\vec{u} = -a(x,y)\nabla p$ will still be smooth in most applications. In such cases, if a good approximation to the velocity is required, approximating p and numerically differentiating it leads to loss of accuracy. If instead, the following *mixed formulation* of (1) having both p and $\vec{u}$ as unknowns is discretized, then it yields good approximations to the velocity $\vec{u}$.

$$\left\{ \begin{array}{rcll} \vec{u} & = & -a(x,y)\nabla p & \text{in } \Omega \quad \text{Darcy's law} \\ \nabla \cdot \vec{u} & = & f & \text{in } \Omega \quad \text{Conservation of mass} \\ \vec{n} \cdot \vec{u} & = & -g & \text{in } \partial\Omega \quad \text{Flux boundary condition} \end{array} \right. \tag{2}$$

2.1. Weak form of equations and its finite element discretization. We obtain a mixed finite element discretization of (2), by introducing a weak form and then restricting the trial and test functions to finite dimensional spaces. To obtain the weak form for Darcy's law, we multiply $a(x,y)^{-1}\vec{u} + \nabla p = 0$ by sufficiently smooth velocity test functions $\vec{v}$ with zero flux (normal component) on $\partial\Omega$ and integrate. The term $\int_\Omega \nabla p \cdot \vec{v} dxdy$ can be integrated by parts to give $-\int_\Omega p \nabla \cdot \vec{v} dxdy$, since the boundary terms are zero. Similarly, we multiply the conservation of mass equation by test functions $q \in L^2(\Omega)$ and integrate. We obtain:

$$
\begin{array}{ll}
\text{Find } \vec{u} \in H_0(div,\Omega) \text{ and } p \in L^2(\Omega) \text{ such that} & \\
\int_\Omega \vec{u}^T a(x,y)^{-1}\vec{v} dxdy + \int_\Omega p(\nabla \cdot \vec{v}) dxdy = 0, & \forall \vec{v} \in H_0(div,\Omega) \\
\int_\Omega q(\nabla \cdot \vec{u}) dxdy = \int_\Omega f q dxdy, & \forall q \in L^2(\Omega) \ .
\end{array} \tag{3}
$$

Here the appropriate function space for the velocity is $H_0(div,\Omega)$, where

$$H(div,\Omega) = \{(v_1,v_2) \in (L^2(\Omega))^2 : \nabla \cdot \vec{v} \in L^2(\Omega)\},$$

is equipped with the norm

$$\|\vec{v}\|^2_{H(div,\Omega)} \equiv \|\vec{v}\|^2_{L^2} + \|\nabla \cdot \vec{v}\|^2_{L^2},$$

and

$$H_0(div,\Omega) \equiv \{\vec{v} \in H(div,\Omega) : \vec{v} \cdot \vec{n} = 0 \text{ on } \partial\Omega\},$$

see Raviart and Thomas [23]. The appropriate space for the pressure is $L^2(\Omega)$, and the pressure unique only up to a constant.

A Galerkin approximation is obtained by replacing the function spaces by finite dimensional subspaces $V^h \subset H_0(div,\Omega)$ and $Q^h \subset L^2(\Omega)$, respectively. In particular, we choose V^h and Q^h to be the Raviart-Thomas finite element spaces [23], which satisfies the uniform *inf sup* condition, a condition that is required for a stable discretization. The discrete problem becomes:

$$
\begin{array}{ll}
\text{Find } u_h \in V^h \text{ and } p_h \in Q^h \text{ such that} & \\
\int_\Omega u_h^T a(x,y)^{-1} v_h dxdy + \int_\Omega p_h(\nabla \cdot v_h) dxdy = 0, & \forall v_h \in V^h \\
\int_\Omega q_h(\nabla \cdot u_h) dxdy = \int_\Omega f q_h dxdy, & \forall q_h \in Q^h \ .
\end{array} \tag{4}
$$

We expand the solution $u_h = \sum_{i=1}^n u_i \Phi_i$ and $p_h = \sum_{i=1}^m p_i \chi_i$, where $\Phi_1, \cdots, \Phi_n$ and $\chi_1, \cdots, \chi_m$ denote a basis for V_h and Q_h respectively. Choosing test functions from this basis, we obtain a linear system for the coefficients $(u_h, p_h) = (u_1, \ldots, u_n, p_1, \ldots, p_m)$:

$$\begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} \begin{bmatrix} u_h \\ p_h \end{bmatrix} = \begin{bmatrix} W_h \\ F_h \end{bmatrix}. \tag{5}$$

Here A_h is a symmetric, positive definite matrix with $A_{ij} = \int_\Omega {\Phi_i}^T a(x,y)^{-1} \Phi_j dxdy$, B_h is an approximation to the divergence map with $B_{ij} = \int_\Omega \chi_i \nabla \cdot \Phi_j dxdy$, and its transpose B_h^T is an approximation to the gradient. The right hand side $W_h = 0$ and F_h is a vector with $F_i = \int_\Omega q\chi_i dxdy$. Since $\int_\Omega v \cdot \nabla p = 0$ for constant p, it follows that $(1, \ldots, 1)^T$ belongs to the null space of B_h^T. Because $\int_\Omega f dxdy = 0$, it follows that $F_h^T(1, \ldots, 1)$ is zero and the linear system is consistent.

Remark. We will interchangeably use matrix and function notations; thus (u_h, p_h) will denote either functions or its vector representation with respect to the basis.

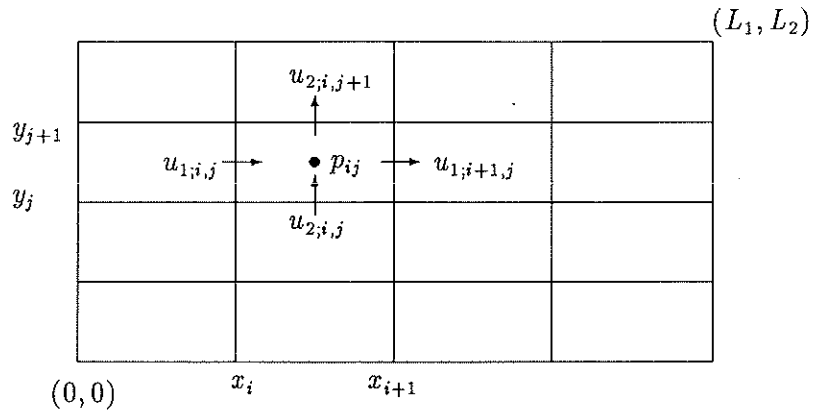


FIG. 1. *Lowest order Raviart-Thomas unknowns for element K_{ij} in a rectangular grid.*

2.2. The lowest order Raviart-Thomas spaces on a rectangular grid. For simplicity, we describe only the lowest order case on a rectangular grid, leading to $O(h)$ convergence for both the velocity and pressure components, see [23]. The domain $\Omega = [0, L_1] \times [0, L_2]$ is partitioned into an $n_1 \times n_2$ rectangular grid with elements of area $h_1 h_2$ where $h_i = L_i/n_i$, for $i = 1, 2$, see Figure 1. We define the nodes $(x_i, y_j) = (ih_1, jh_2)$ for $i = 0, \cdots, n_1$ and $j = 0, \cdots, n_2$. The ijth rectangular element is $K_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ for $i = 0, \cdots, n_1 - 1$, and $j = 0, \cdots, n_2 - 1$.

The velocity space V_h is defined by first order piecewise polynomial vector functions $u_h = (u_1, u_2)$ whose flux $\vec{n} \cdot u_h$ is continuous across elements (both components of the velocity need not be continuous across elements). More specifically, the components have the form:

$$u_1(x, y) = u_{1;i,j} \frac{(x_{i+1} - x)}{(x_{i+1} - x_i)} + u_{1;i+1,j} \frac{(x - x_i)}{(x_{i+1} - x_i)} \quad \text{for } (x, y) \in K_{ij},$$

and

$$u_2(x, y) = u_{2;i,j} \frac{(y_{j+1} - y)}{(y_{j+1} - y_j)} + u_{2;i,j+1} \frac{(y - y_j)}{(y_{j+1} - y_j)} \quad \text{for } (x, y) \in K_{ij},$$

where $u_{1;i,j} = u_1\left(x_i, (y_j + y_{j+1})/2\right)$ and $u_{2;i,j} = u_2\left((x_i + x_{i+1})/2, y_j\right)$ denote the value of u_1 and u_2 on the midpoints of the edges $x_i \times [y_j, y_{j+1}]$ and $[x_i, x_{i+1}] \times y_j$, respectively. Since $(0, 1)$ is normal to each horizontal edge, $u_h \cdot (0, 1) = u_2(x, y)$ is continuous across horizontal boundaries and constant on each horizontal edge. Similarly, $u_h \cdot (1, 0) = u_1(x, y)$ is continuous across vertical boundaries and constant on each vertical edge. Thus, the velocity unknowns consists of the values $u_{1;i,j}$ on each vertical edge $x_i \times [y_j, y_{j+1}]$ and $u_{2;i,j}$ on each horizontal edge $[x_i, x_{i+1}] \times y_j$.

The pressure space Q^h consists of piecewise constant functions p_h with unknowns p_{ij} associated with the center of each cell K_{ij}.

If the coefficient matrix $a(x, y)$ is diagonal, then A_h is a block diagonal matrix with tridiagonal blocks, provided all the vertical edges are ordered sequentially along each horizontal strip of Ω and all the horizontal edges are ordered sequentially along each vertical strip of Ω. For example, if $a(x, y) = I$, then the diagonal blocks of A_h are of the form $(h_1 h_2/6)\text{tridiag}(1, 4, 1)$. In case of general coefficients $a(x, y)$, numerical

integration rules are needed to evaluate the stencil. The stencil associated with B_h on each rectangular element K_{ij} is:

$$(B_h u_h)_{ij} = h_1[u_{2;i,j+1} - u_{2;i,j}] + h_2[u_{1;i+1,j} - u_{1;i,j}].$$

The stencil for B_h^T at a vertical edge ij is:

$$\left(B_h^T p_h\right)_{ij} = h_2[p_{i-1,j} - p_{i,j}],$$

and at a horizontal edge ij:

$$\left(B_h^T p_h\right)_{ij} = h_1[p_{i,j-1} - p_{i,j}].$$

2.3. Properties of the discrete problem and a method for solving it. System (5) is by construction symmetric, indefinite and singular with $B_h(1,\ldots,1)^T = 0$. The matrix A_h is symmetric, positive definite, and its condition number is proportional to the variation in $a(x,y)$, and uniform in h. A nice analysis of the bounds for the range of negative and positive eigenvalues of (5) can be found in [24].

System (5) is most often solved by iterative methods. The standard conjugate gradient method is not directly applicable, but gradient type methods based on *regularization* or *penalty* can be employed, but require the solution of systems of the form $A + \rho B^T C B$ for some choice of regularization matrix C and penalty parameter ρ, see [8, 27, 9]. Alternatively, the preconditioned Minimum Residual method can be applied, which does not involve the selection of parameters; see [24] for the analysis of such a preconditioned system.

We now describe a simple technique, which reduces (5) to two positive definite problems. This procedure is employed as a solver for the subproblems in § 4. From the first block row of (5), we obtain that u_h satisfies

$$A_h u_h = W_h - B_h^T p_h,$$

which determines u_h in terms of p_h. Substituting the velocity into the divergence constraint, we obtain a reduced equation for the pressure p_h:

$$Sp_h = F_h - B_h A_h^{-1} W_h, \qquad \text{where } S \equiv -B_h A_h^{-1} B_h^T. \tag{6}$$

S is symmetric, positive definite with null vector $(1,\cdots,1)^T$ and is spectrally equivalent to a discretization of (1). It is usually expensive to compute, requiring the solution of one linear system with coefficient matrix A_h for each column of S. This can be avoided if equation (6) is solved by the conjugate gradient method which requires only matrix-vector products of S. Neglecting the null eigenvalue of S, the condition number of S is $O(h^{-2})$, as for standard elliptic problems.

Since S is ill-conditioned, preconditioning is required for large problems. Preconditioners, such as ILU see [21], or $MINV$ see [3] can be applied indirectly to a finite difference approximation of (1), and used as preconditioner for S, see [28]. However, in the tests reported in § 4.5 we used only diagonal scaling as the preconditioner. Note that if $a(x,y)$ is diagonal, A_h is tridiagonal and can be inverted in $O(n)$ operations, where n is the size of A_h.

3. Schwarz methods for standard formulations of elliptic problems. Before we describe the Schwarz methods for the mixed problems of § 4, we illustrate the sequential and parallel versions of the Schwarz algorithm for solving a symmetric, positive definite linear system arising from the standard discretization of an elliptic problem.

3.1. Multiplicative Schwarz method. We consider

$$\left\{ \begin{array}{rcll} Lu = -\nabla \cdot (a(x,y)\nabla u) & = & f & \text{in } \Omega \\ u & = & 0 & \text{in } \partial\Omega \end{array} \right. , \tag{7}$$

which yields a symmetric, positive definite linear system:

$$A_h u_h = f_h, \tag{8}$$

for some suitable finite difference or finite element discretization.

Let $\Omega_1', \cdots, \Omega_N'$ denote some overlapping cover of Ω, with corresponding coefficient matrices A_i'. Then one iteration of the multiplicative Schwarz method consists of sequential correction on each of the overlapping subregions Ω_i'. More precisely, we define R_i to be the restriction map of nodal values on Ω to the nodal values in the interior of Ω_i':

$$(R_i u_h)(x_k) = u_h(x_k), \quad \text{for } x_k \in \Omega_i'.$$

Then R_i^T is an extension by zero of nodal values in the interior of Ω_i' to Ω. The coefficient matrices for the subproblems can be expressed as $A_i' \equiv R_i A_h R_i^T$. Let $tol \ll 1$ be a given tolerance. The iterates are defined by

Begin
Let k = 0
While $\|A_h u^k - f_h\| > tol$ do
For $i = 1, \ldots, N$
    Compute the residual $\tilde{f}_i$
        $\tilde{f}_i \equiv (f_h - A_h u^{k+\frac{i-1}{N}})$
    Solve subproblem
        $w_i = R_i^T A_i^{-1} R_i \tilde{f}_i$
    Update Solution
        $u^{k+\frac{i}{N}} \equiv u^{k+\frac{i-1}{N}} + w_i$
endFor
endWhile
End

The iterates of the algorithm satisfy:

$$\|u^k - u_h\|_{A_h} \leq \rho^k \|u^0 - u_h\|_{A_h},$$

for some $0 < \rho < 1$, where $\|v\|_{A_h} \equiv (v^T A_h v)^{1/2}$ is equivalent to the $H^1(\Omega)$ norm, see [15], [1], [19]. For various choices of overlapping subregions along with a coarse grid problem, the rate of convergence ρ has been shown to be independent of mesh parameters, [1, 4].

Remark. Note that one iteration consists of a sweep over all the subregions Ω_i' in some sequential order. This method can be parallelized by *coloring* the subdomains, see Fig. 2, and solving simultaneously on disjoint subdomains of the same *color*.

3.2. Additive Schwarz method. The additive Schwarz method to solve (8) is a highly parallel variant of the Schwarz alternating method in which all the subproblems are solved simultaneously; see Dryja and Widlund [4]. It generally requires conjugate gradient acceleration [11].

The method can be viewed as a preconditioning technique; see [4], [12], and in the following, we describe the action of the inverse of the preconditioner M to solve (8). Given a forcing term f, the action of the inverse of the preconditioner: $M^{-1}f$, based on subproblems on the overlapping subregions Ω_i', is:

$$M^{-1}f \equiv \sum_{i=1}^{N} R_i^T A_i^{-1} R_i f.$$

Note that each term in the sum can be computed in parallel. Equivalently, the system

$$M^{-1}Au_h = M^{-1}f,$$

can be solved using the conjugate gradient algorithm [11] with the standard Euclidean inner product replaced by the A inner product. The CG iterates satisfy:

$$\|u^m - u\|_A \leq 2\left(1 - \frac{2}{\sqrt{\kappa}+1}\right)^m \|u^0 - u\|_A,$$

where κ is the quotient of the maximum and minimum eigenvalues of $M^{-1}A$, see [11]. As for the multiplicative version, with the use of a coarse grid problem and overlapping subdomains, κ has been shown to be independent of the mesh parameters [4].

4. Schwarz methods for the mixed finite element discretizations. The Schwarz methods for the symmetric positive definite case (8) carries over, with some modifications, to the singular, symmetric indefinite case (5). As before, solution to (5) is found by iterative correction on subregions, with the requirement that the *flux of the velocity be continuous* across subdomain boundaries. No boundary conditions are applied to the pressure. However, in order for the local Neumann subproblems to be well posed, the flux boundary condition must be compatible with the local divergence constraint. In matrix terms, two issues need to be addressed in order to define a Schwarz method having subproblems of the form

$$\begin{bmatrix} * & * & * & * \\ * & A_i & B_i^T & * \\ * & B_i & 0 & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} * \\ u_i \\ p_i \\ * \end{bmatrix} = \begin{bmatrix} * \\ W_i \\ F_i \\ * \end{bmatrix}, \tag{9}$$

after some suitable reordering of (5). They are:

1. F_i must have mean value zero, since the local matrix B_i will be singular with $(1, \cdots, 1)^T$ spanning its null space. This corresponds to the requirement that the local flux boundary conditions be compatible with the local divergence constraint.
2. Due to the non-uniqueness of the local pressure solution p_i, its mean value on the subregion is arbitrary and should be suitably prescribed in order to compute a globally defined pressure p_h.

These and other issues are discussed in § 4.2, § 4.3 and § 4.4. Numerical tests of the algorithms are presented in § 4.5.

4.1. Coarse grid and subdomain problems. We assume that Ω is partitioned into subdomains $\Omega_1, \cdots, \Omega_N$ of diameter $O(H)$, which forms a shape regular coarse grid triangulation τ^0. A fine mesh τ^h consisting of shape regular elements of size $O(h)$, is obtained by refinement of the subdomains Ω_i. An overlapping covering is obtained by extending each subdomain Ω_i to a larger region Ω_i' by including all fine grid elements at a distance βH or less from Ω_i, for a fixed fraction $0 < \beta < 1$, see Figure 2.

Let V^0 and Q^0 denote the Raviart-Thomas velocity and pressure spaces on the coarse mesh τ^0. On each subdomain Ω_i let $V_i = V^h \cap H_0(div, \Omega_i)$ and let $Q_i = Q^h \cap L^2(\Omega_i)$ denote the fine grid Raviart-Thomas velocity and pressure spaces restricted to Ω_i. Similarly, on each extended subdomain Ω_i', we let $V_i' = V^h \cap H_0(div, \Omega_i')$ and $Q_i' = Q^h \cap L^2(\Omega_i')$.

We use $R_0^T : V^0 \times Q^0 \longrightarrow V^h \times Q^h$, to denote the standard interpolation map from the coarse grid to the fine grid. For example, for the rectangular grid of § 2.2, with rectangular subdomains, each fine grid vertical edge value is obtained by linearly interpolating the two adjacent coarse grid vertical edge values; similarly for the fine grid horizontal edges. The fine grid pressure value in an element is the same as the coarse grid pressure restricted to that element. $R_0 : V^h \times Q^h \longrightarrow V^0 \times Q^0$ will denote the restriction onto the coarse grid.

On the subdomains, we use $R_i^T : V_i \times Q_i \longrightarrow V^h \times Q^h$, to denote the extension by zero from Ω_i to Ω. Its transpose $R_i : V^h \times Q^h \longrightarrow V_i \times Q_i$, denotes the restriction map, where:

$$R_i\,(v_h, q_h)(x, y) = (v_h(x, y), q_h(x, y)) \qquad \text{for } (x, y) \in \Omega_i.$$

Analogously, $R_i'^T : V_i' \times Q_i' \longrightarrow V^h \times Q^h$, will denote the extension map on the extended subdomains Ω_i', while $R_i' : V^h \times Q^h \longrightarrow V_i' \times Q_i'$, will denote the restriction map.

The submatrices on the various grids can now be expressed easily in terms of (5) and the restriction and extension maps. The coarse grid matrix is

$$L_0 \equiv \begin{bmatrix} A_0 & B_0^T \\ B_0 & 0 \end{bmatrix} = R_0 \begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} R_0^T. \tag{10}$$

The coefficient matrices on the subdomains are

$$L_i \equiv \begin{bmatrix} A_i & B_i^T \\ B_i & 0 \end{bmatrix} = R_i \begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} R_i^T, \tag{11}$$

and on the extended subdomains,

$$L_i' \equiv \begin{bmatrix} A_i' & B_i'^T \\ B_i' & 0 \end{bmatrix} = R_i' \begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} R_i'^T, \tag{12}$$

and these are easily seen to be submatrices of (5).

4.2. Structure of the Schwarz algorithms for the mixed problem. As mentioned earlier, in order to apply the Schwarz methods to the symmetric, indefinite problem (5), two issues need to be addressed. One concerns the compatibility of the local flux boundary conditions on each $\partial\Omega_i$ with the local divergence constraint:

$$\int_{\Omega_i} f(x, y)dxdy = \int_{\partial\Omega_i} u_h \cdot \vec{n} ds.$$

The other concerns the computation of a global pressure solution from the results of local Neumann problems. These two issues will be treated in three steps.

Step 1. In order to provide compatible local flux boundary conditions, we compute a velocity $u_h^\star$ which satisfies the divergence constraint

$$B_h u_h^\star = F_h,$$

and use this as the initial iterate in step 2. All subsequent corrections in step 2 will be divergence free and hence zero flux boundary conditions will be compatible with the zero divergence constraint. To find such a $u_h^\star$, we first compute a discrete velocity $u_0^\star$ by solving the original problem on the coarse grid:

$$\begin{bmatrix} u_0^\star \\ p_0 \end{bmatrix} \equiv R_0^T \begin{bmatrix} A_0 & B_0^T \\ B_0 & 0 \end{bmatrix}^{-1} R_0 \begin{bmatrix} W_h \\ F_h \end{bmatrix}. \tag{13}$$

The coarse grid solution $u_0^\star$ does not necessarily satisfy the fine grid divergence constraint $B_h u_0^\star - F_h = 0$. However, $B_h u_0^\star - F_h$ has mean value zero on each Ω_i since in the weak form

$$q_0^T F_h - q_0^T B_0 u_0^\star = \int_\Omega q_0 (F_h - \nabla \cdot u_0^\star) dx dy = 0, \quad \forall q_0 \in Q^0,$$

and since in particular for the piecewise constant characteristic function $\chi_{\Omega_i} \in Q^0$

$$\int_{\Omega_i} \nabla \cdot u_0^\star dx dy = \int_{\partial\Omega_i} \vec{n} \cdot u_0^\star ds = \int_{\Omega_i} F_h dx dy.$$

Now, since $F_h - \nabla \cdot u_0^\star$ has mean value zero in each Ω_i, the zero flux boundary condition on each $\partial\Omega_i$ is compatible with $F_h - \nabla \cdot u_0^\star$ on Ω_i, and the following subproblems are therefore well posed:

$$\begin{bmatrix} u_i^* \\ q_i^* \end{bmatrix} \equiv R_i^T L_i^{-1} R_i \begin{bmatrix} W_h - A_h u_0^\star \\ F_h - B_h u_0^\star \end{bmatrix}.$$

We define

$$u_h^\star = u_0^\star + u_1^\star + \cdots + u_N^\star,$$

which by construction satisfies $B_h u_h^\star = F_h$.

Step 2. The solution to (5) can be written

$$\begin{bmatrix} u_h \\ p_h \end{bmatrix} = \begin{bmatrix} u_h^\star \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{u}_h \\ p_h \end{bmatrix},$$

where the correction satisfies

$$\begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_h \\ p_h \end{bmatrix} = \begin{bmatrix} W_h - A_h u_h^\star \\ 0 \end{bmatrix}, \tag{14}$$

since $F_h - B_h u_h^\star = 0$. Therefore $\tilde{u}_h = u_h - u_h^\star$ is divergence free.

In order to compute $\tilde{u}_h$, we will apply either of two iterative procedures based on solving subproblems so that the velocity correction remains divergence free in each

iteration. The first algorithm is based on the multiplicative Schwarz method of § 3.1. It converges as a fixed point iteration and will be described in § 4.3. The second algorithm is based on the additive Schwarz method of § 3.2. It requires conjugate gradient acceleration for convergence and is discussed in § 4.4.

In both algorithms, the velocity v_h in the corrections (v_h, q_h), will be divergence free, i.e., $B_h v_h = 0$. Thus,

$$\begin{bmatrix} v_h \\ q_h \end{bmatrix}^T \begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} \begin{bmatrix} v_h \\ q_h \end{bmatrix} = v_h^T A_h v_h > 0, \tag{15}$$

which shows that the problem of determining $\tilde{u}_h$ is symmetric positive definite.

Step 3. The pressure $(0, p_h)$ satisfies:

$$\begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} \begin{bmatrix} 0 \\ p_h \end{bmatrix} = \begin{bmatrix} B_h^T p_h \\ 0 \end{bmatrix}, \tag{16}$$

where $B_h^T p_h = W_h - A_h u_h$ is computed once the velocity $u_h = \tilde{u}_h + u_h^\star$ is known. When local Neumann problems, with zero flux boundary conditions, are solved by restricting this residual to the subregions Ω_i', the solution will have the form $(0, p_i')$. Both $(0, p_h)$ and $(0, p_i')$ will be solutions to the Neumann problem on Ω_i' , and so the pressure p_i' will differ from the global pressure p_h restricted to Ω_i' only by a constant, due to uniqueness for the local Neumann problem on Ω_i'. Thus, corresponding to each extended subdomain Ω_i', there exists a constant c_i such that

$$p_i' = p_h + c_i \quad \text{ in } \Omega_i'.$$

From this it follows that

$$p_i' = p_j' + c_j - c_i \quad \text{ in } \Omega_i' \cap \Omega_j'.$$

Starting with any extended subdomain, say Ω_1', we determine these constants in the overlapping subdomains to obtain a consistent pressure in Ω. Details are given in § 4.3 and § 4.4.

Remark. In each Schwarz iteration, we will solve local problems with a zero divergence constraint. Such discrete velocities v_h are also divergence free in the pointwise sense. This can be seen by using that for the Raviart-Thomas spaces, the divergence map takes the velocity space V^h *onto* Q^h. From this it follows that by choosing $q_h = \nabla \cdot v_h$, if

$$\int_\Omega q_h (\nabla \cdot v_h) dx = 0, \quad \forall q_h \in V^h \text{ then } \int_\Omega |\nabla \cdot v_h|^2 dx = 0.$$

4.3. A multiplicative Schwarz method for mixed finite elements. We now describe the three steps of the multiplicative Schwarz method to solve (5) based on the subregions Ω_i, Ω_i' and the coarse grid. The iteration is continued till the residual is smaller than *tol*.

1. Compute $u_h^\star$ such that $B_h u_h^\star = F_h$ in step 1a and 1b.

1a. Solve the coarse grid problem:

$$\begin{bmatrix} u_0^* \\ p_0^* \end{bmatrix} = R_0^T L_0^{-1} R_0 \begin{bmatrix} 0 \\ F_h \end{bmatrix}.$$

1b. Solve the problems on each subdomain Ω_i:

$$\begin{bmatrix} u_i^* \\ q_i^* \end{bmatrix} \equiv R_i^T L_i^{-1} R_i \begin{bmatrix} W_h - A_h u_0^* \\ F_h - B_h u_0^* \end{bmatrix}.$$

Define

$$u_h^* = u_0^* + u_1^* + \cdots + u_N^*.$$

2. Let $(\tilde{u}^0, p^0) \equiv (u_h^*, 0)$ and let $k = 0$
While $\|W_h - A_h \tilde{u}^k - B_h^T p^k\| > tol$ do:
For $i = 0, \ldots, N$
2a. Compute the correction $\delta\tilde{u}^{k+\frac{i+1}{N+1}}$ to the velocity using the current residual:

$$\begin{bmatrix} \delta\tilde{u}^{k+\frac{i+1}{N+1}} \\ q_i' \end{bmatrix} = {R_i'}^T (L_i')^{-1} R_i' \begin{bmatrix} W_h - A_h \tilde{u}^{k+\frac{i}{N+1}} \\ 0 \end{bmatrix}.$$

2b. Update the velocity $\tilde{u}^{k+\frac{i+1}{N+1}}$ using this correction:

$$\tilde{u}^{k+\frac{i+1}{N+1}} \equiv \delta\tilde{u}^{k+\frac{i+1}{N+1}} + \tilde{u}^{k+\frac{i}{N+1}}.$$

3a. If $i \geq 2$, determine a constant c_i so that:

$$\overline{q_i'(x) + c_i} = \overline{q_{i-1} + c_{i-1}}, \quad \text{on } \Omega_i' \cap \Omega_{i-1}',$$

where $c_1 = 0$.
endFor
3b. For $i = 1, \cdots, N$ let

$$p^k \equiv \begin{cases} q_i' + c_i & \text{in } \Omega_i' \\ p^k & \text{in } \Omega - \Omega_i' \end{cases}$$

3c. Normalize the pressure to have mean value zero on Ω:

$$p^k \equiv p^k - \frac{(1, \cdots, 1) \cdot p^k}{\|(1, \cdots, 1)\|^2} (1, \cdots, 1)^T.$$

endWhile

Remark. During each iteration, we may sequentially solve the coarse grid problem and each of the extended subdomain problems in any order. One choice is to use the standard *lexicographic* ordering of the subdomains. Another possibility is to partition the extended subdomains Ω_i' into C distinct colors, so that no two extended subdomains having the same color intersect, see Figure 2. Then we may concurrently solve the local subproblems on different subdomains of the same color.

The following result will be proved in [19], [18].

THEOREM 4.1. *If the extended subdomains $\Omega_1', \ldots, \Omega_N'$ are kept fixed and the mesh size h is varied, then there exists a constant $0 \leq \rho < 1$, independent of h, such that*

$$\|\tilde{u}^k - u_h\|_{A_h} \leq \rho^k \|\tilde{u}^0 - u_h\|_{A_h}.$$

ρ will depend on H if the coarse mesh is not included in the iteration. Based on a idea used in a recent paper of Ewing and Wang [6], we are now able to prove that ρ is independent of H when a coarse grid is used. Numerical tests of § 4.5 confirms that when a coarse grid is used, the convergence factor ρ is independent of h, H and the variation in the coefficients.

4.4. Additive Schwarz method for mixed problems. In step 2 of the preceding algorithm, the divergence free component $\tilde{u}_h = u_h - u_h^\star$ was updated sequentially on each of the extended subdomains Ω_i' or simultaneously on subdomains of the same color. In the additive version, all the subproblems can be solved simultaneously. As for the previous algorithm, the additive algorithm consists of three steps.

Step 1. Compute $u_h^\star$ satisfying the constraint $B_h u^* = F_h$ as follows:

1a. Solve the coarse grid problem:

$$\begin{bmatrix} u_0^* \\ p_0^* \end{bmatrix} = R_0^T L_0^{-1} R_0 \begin{bmatrix} W_h \\ F_h \end{bmatrix}.$$

1b. For $i = 1, \ldots, N$ compute:

$$\begin{bmatrix} u_i^* \\ q_i^* \end{bmatrix} = R_i^T L_i^{-1} R_i \begin{bmatrix} W_h - A_h u_0^* \\ F_h - B_h u_0^* \end{bmatrix},$$

and define $u_h^\star = u_0^\star + u_1^\star + \cdots + u_N^\star$.

Step 2. To determine the velocity $\tilde{u}_h \equiv u_h - u_h^\star$, solve the following equivalent positive definite problem

$$P\tilde{u}_h = g_h, \tag{17}$$

using the conjugate gradient method. The matrix vector product is defined

$$Pv_h = \sum_{i=0}^{N} P_i v_h,$$

where for $i = 0, \cdots, N$:

$$\begin{bmatrix} P_i v_h \\ q_i' \end{bmatrix} \equiv {R_i'}^T \begin{bmatrix} A_i' & {B_i'}^T \\ B_i' & 0 \end{bmatrix}^{-1} R_i' \begin{bmatrix} A_h v_h \\ 0 \end{bmatrix}.$$

This requires the solution of $N+1$ subproblems. Though $\tilde{u}_h$ is not known explicitly, the forcing function g_h can be computed as follows:

$$\begin{bmatrix} g_h \\ q \end{bmatrix} \equiv \sum_{i=0}^{N} {R_i'}^T \begin{bmatrix} A_i' & {B_i'}^T \\ B_i' & 0 \end{bmatrix}^{-1} R_i' \begin{bmatrix} W_h - A_h u_h^\star \\ 0 \end{bmatrix}.$$

Note that by construction, g_h is divergence free since each term $P_i\tilde{u}_h$ is divergence free. In addition, the matrix vector product with P yields divergence free vectors. Thus all iterates in the CG method remain divergence free. The matrix P is symmetric, positive definite only in the A_h inner product, and we therefore use the inner product $x^T A_h y$ in the CG method. Once $\tilde{u}_h$ is computed, we define

$$u_h = u_h^\star + \tilde{u}_h.$$

Step 3. To compute the pressure p_h, we first solve:

$$\begin{bmatrix} 0 \\ q_i' \end{bmatrix} \equiv {R_i'}^T (L_i')^{-1} R_i' \begin{bmatrix} W_h - A_h u_h \\ 0 \end{bmatrix}.$$

For $i = 2, \cdots, N$, choose c_i such that:

$$q_i' + c_i = q_{i-1}' + c_{i-1} \text{ in } \Omega_i' \cap \Omega_{i-1},$$

where $c_1 \equiv 0$. Define

$$p_h = q_i + c_i \quad \text{on } \Omega_i.$$

The following result is proved in [18].

THEOREM 4.2. *If the extended subdomains $\Omega_1', \ldots, \Omega_N'$ are kept fixed, then there exists a constant $0 < C$ independent of h such that*

$$\kappa(P) \leq C.$$

Here $\kappa(P) \equiv \max(R(x))/\min(R(x))$ and $R(x) \equiv (x^T A_h P x / x^T A_h x)$.

The convergence bound can be improved to make ρ independent of H and h when the coarse mesh is included in the iteration. This improved bound will also be proved in [18].

4.5. Numerical results. In this Section, we present some experimental results for the algorithms described in § 4.3 and § 4.4. Tests were conducted to determine how the convergence rates of the various algorithms depend on the mesh size h, subdomain size H, amount of overlap, and jumps J in the coefficients. We chose $\Omega = [0,1]^2$, and the coefficients $a(x,y)$ of the elliptic operator were taken to be a scalar piecewise constant function $a(x,y)$, with a jump across $x = \frac{1}{2}$:

$$a(x,y) = \begin{cases} 1 & 0 \leq x < 0.5 \\ J & 0.5 \leq x \leq 1 \end{cases} \tag{18}$$

The parameter J was varied between 1 and 10^{-6}. An $n \times n$ rectangular fine grid was chosen and an $n_s \times n_s$ coarse grid was chosen with $h = 1/n$ and $H = 1/n_s$. We assume that n_s divides n. The number of velocity and pressure unknowns are listed in Table 2 for various values of n. The lowest order Raviart-Thomas elements were used. To obtain the extended subdomains Ω_i', we introduce an integer parameter n_o and group neighboring elements within a distance of n_o elements from Ω_i, as in Fig. 2. The overlap ratio $n_o n_s / n$ represents the overlap compared to H.

The *four color* and the *standard lexicographic* multiplicative Schwarz algorithm were tested, as well as the additive Schwarz algorithm. The subproblems were solved using the Schur complement based method described in § 2.3, in which the Schur complement system was solved by a conjugate gradient method with a diagonal preconditioner.

In Table 1, we list the convergence factor

$$\rho \equiv \left(\|e^n\|/\|e^0\|\right)^{1/n},$$

and the number of iterations, ITN, required to reduce the relative error by a factor 10^{-5}. The exact solutions were randomly chosen using the uniform distribution on $(-2,2)$.

Conclusions. The numerical results indicate that the convergence factor of all the algorithms is independent of the jumps J in coefficients, the mesh size h, the

FIG. 2. *Subdomains Ω_i, Overlapping Subdomains Ω_i' and Four Colors.*

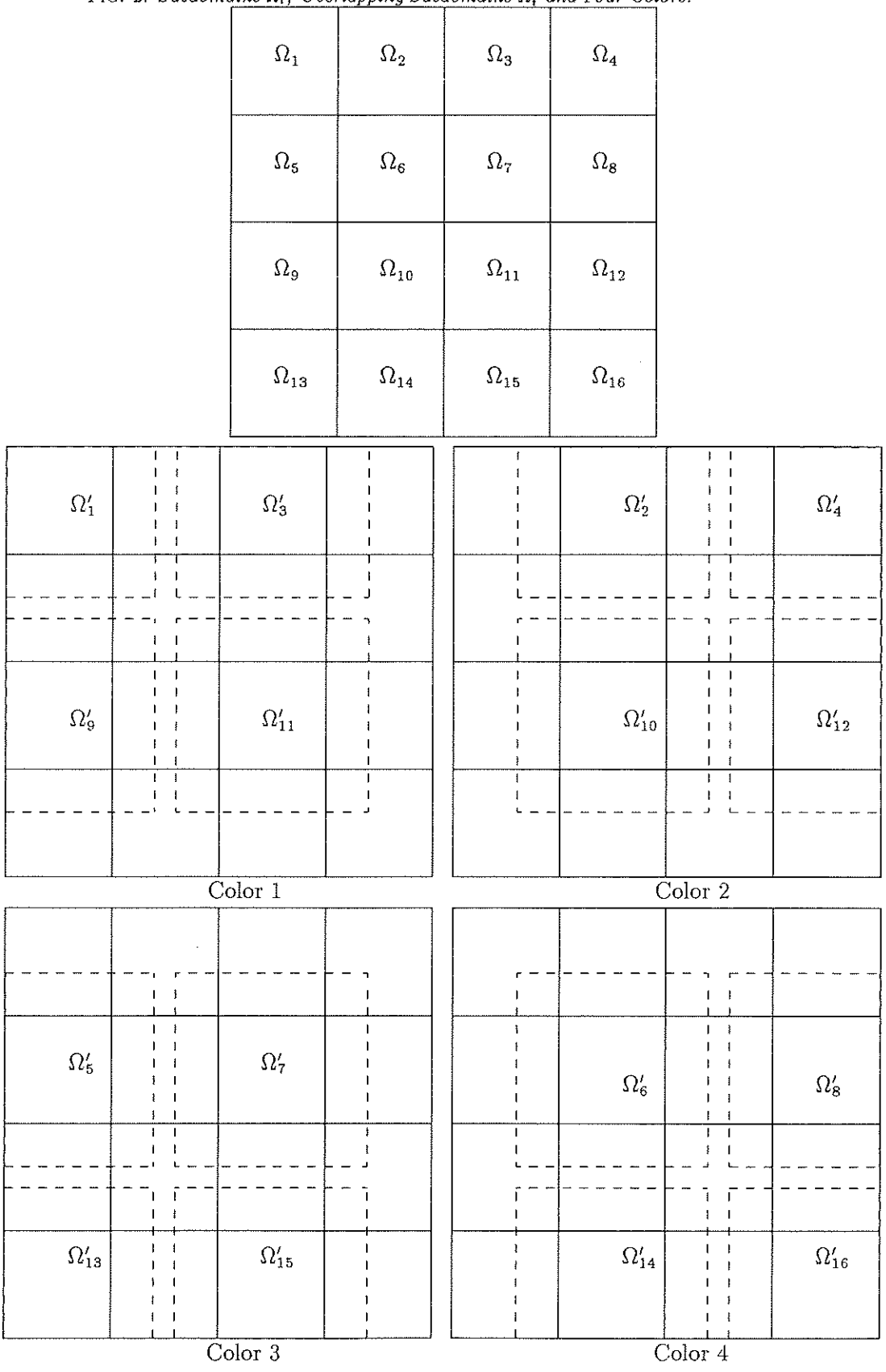


TABLE 1
NUMERICAL RESULTS.

h^{-1} = n	SUB-DOMAINS	OVERLAP size	OVERLAP ratio	JUMP J^{-1}	FOUR COLOR SEQUENTIAL		LEXICOGRAPHIC SEQUENTIAL		PARALLEL	
	$n_s * n_s$	n_0	$n_o n_s/n$		ρ	ITN	ρ	ITN	ρ	ITN
16	2 * 2	1	1/8	1	0.17	7	0.16	6	0.34	8
16	2* 2	1	1/8	10^6	0.16	7	0.15	6	0.43	10
16	2 * 2	2	1/4	1	0.05	5	0.06	4	0.26	7
16	2* 2	2	1/4	10^6	0.05	4	0.05	4	0.38	9
16	4 * 4	1	1/4	1	0.06	5	0.08	5	0.34	9
16	4* 4	1	1/4	10^6	0.06	5	0.09	5	0.41	10
16	4 * 4	2	1/2	1	0.03	4	0.05	4	0.34	9
16	4* 4	2	1/2	10^6	0.03	4	0.06	5	0.37	9
16	8 * 8	1	1/2	1	0.05	4	0.11	6	0.34	9
16	8* 8	1	1/2	10^6	0.05	4	0.13	6	0.35	9
24	4* 4	1	1/6	1	0.09	5	0.09	5	0.38	10
24	4* 4	1	1/6	10^6	0.10	5	0.10	5	0.41	10
24	4* 4	2	1/3	1	0.05	4	0.06	4	0.34	9
24	4* 4	2	1/3	10^6	0.05	4	0.08	5	0.34	9
24	8* 8	1	1/3	1	0.05	4	0.10	5	0.35	9
24	8* 8	1	1/3	10^6	0.05	4	0.12	6	0.36	9
32	4* 4	1	1/8	1	0.16	7	0.16	6	0.42	10
32	4* 4	1	1/8	10^6	0.16	7	0.17	7	0.44	11
32	4* 4	2	1/4	1	0.06	5	0.08	5	0.33	8
32	4* 4	2	1/4	10^6	0.06	5	0.09	5	0.37	9
32	8* 8	1	1/4	1	0.06	5	0.10	5	0.35	9
32	8* 8	1	1/4	10^6	0.06	5	0.12	6	0.37	9
32	8* 8	2	1/2	1	0.04	4	0.11	6	0.35	9
32	8* 8	2	1/2	10^6	0.04	4	0.12	6	0.35	9
40	4* 4	1	1/10	1	0.25	9	0.25	8	0.44	11
40	4* 4	1	1/10	10^6	0.26	9	0.26	9	0.48	12
40	4* 4	2	1/5	1	0.07	5	0.07	5	0.36	9
40	4* 4	2	1/5	10^6	0.07	5	0.10	5	0.39	9
40	8* 8	1	1/5	1	0.07	5	0.09	5	0.37	9
40	8* 8	1	1/5	10^6	0.07	5	0.10	5	0.39	10
40	8* 8	2	2/5	1	0.04	4	0.10	5	0.33	9
40	8* 8	2	2/5	10^6	0.04	4	0.11	6	0.34	9
40	10 * 10	1	1/4	1	0.06	5	0.10	5	0.36	9
40	10 * 10	1	1/4	10^6	0.06	5	0.11	6	0.38	10
40	10 * 10	2	1/2	1	0.04	4	0.12	6	0.35	9
40	10 * 10	2	1/2	10^6	0.04	4	0.13	6	0.36	9
40	20 * 20	1	1/2	1	0.07	5	-	-	0.35	9
60	10 * 10	1	1/6	10^6	0.09	5	-	-	-	-
60	10 * 10	2	1/3	1	0.07	5	-	-	-	-

TABLE 2
Number of unknowns.

n	16	24	32	40	60
Velocity unknowns	480	1104	1984	3120	7080
pressure unknowns	256	576	1024	1600	3600

subdomain size H, and that it depends only mildly on the overlap $n_o n_s/n$. The results also indicate that the additive version requires about twice the number of iterations as the multiplicative version.

Remark. In testing problems with large jump ratio $1/J$, we found that the relative error of the computed pressure was larger than the relative error of the velocity by a factor on the order of $1/J$. We observed this phenomenon, even when direct methods were used to solve (5). We attribute this to round-off error, see also [24].

5. Iterative refinement methods for mixed discretizations. Iterative refinement methods are algorithms to solve linear systems arising from problems with local grid refinement. In this section, we discuss iterative methods to solve symmetric, indefinite systems like (5) arising from locally refined meshes. The algorithms discussed are modifications of corresponding algorithms for the symmetric, positive definite case described in Hart and McCormick [13], McCormick and Thomas [20], Mandel and McCormick [17], Dryja and Widlund [29]. In the symmetric, positive definite case the sequential algorithm is referred to as the Fast Adaptive Composite grid method (FAC), while the parallel algorithm is referred to as the Asynchronous Fast Adaptive Composite grid (AFAC) method, see [17], [29].

5.1. Repeated local refinement. We consider a polygonal domain $\Omega = \Omega_0 \subset R^2$, which is successively refined in nested subregions Ω_i, $i = 1, \cdots, N$,

$$\Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_{N-1} \supset \Omega_N;$$

see Fig. 3 for the case $N = 2$. The composite grid is defined as follows. Ω_0 is initially triangulated by a *regular, quasi-uniform* mesh of size h_0, denoted $\tau^{h_0}(\Omega_0)$. Each element in the subregion Ω_1 is refined, for instance by connecting the midpoints of the edges of each element resulting in four subelements of size $h_1 = h_0/2$. The triangulation of Ω_1 is denoted by $\tau^{h_1}(\Omega_1)$. This procedure is repeated on all the nested subregions, with $\tau^{h_i}(\Omega_i)$ denoting the *quasi-uniform* refinement of elements of $\tau^{h_{i-1}}(\Omega_{i-1})$ in subregion Ω_i, with $h_i = h_{i-1}/2$. The composite grid $\tau^{h_0,\ldots,h_N}(\Omega_0, \ldots, \Omega_N)$ is defined to be the union of all elements, see Fig. 3, i.e.,

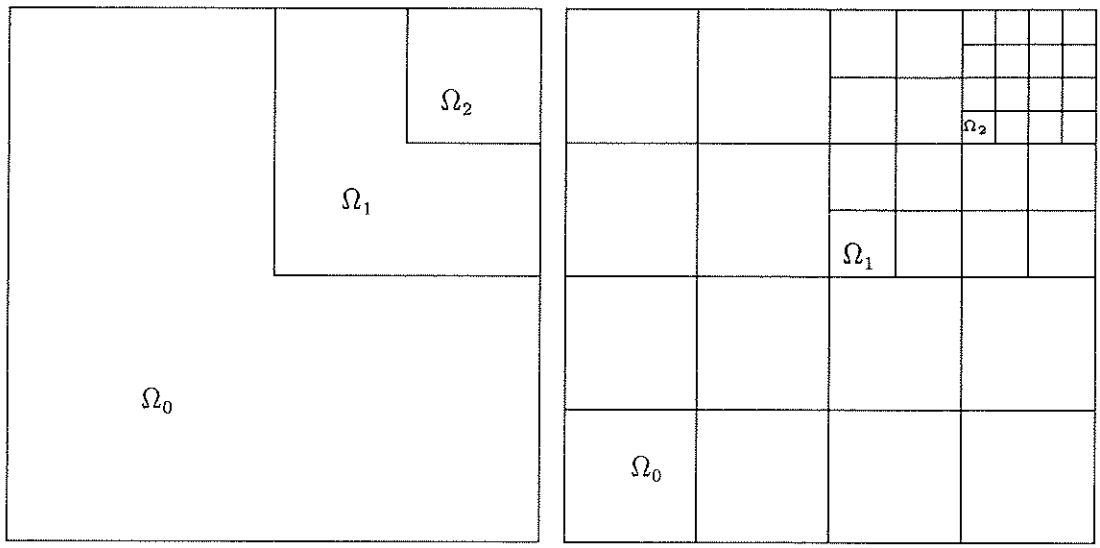
$$\tau^{h_0,\ldots,h_N}(\Omega_0, \ldots, \Omega_N) \equiv \cup_{i=0}^{N} \tau^{h_i}(\Omega_i).$$

In order to defne the finite elment functions on the composite mesh, the extra degrees of freedom on the edges on $\partial\Omega_i$ will be treated as *slave variables.* The composite Raviart-Thomas velocity space $V^{h_0,\ldots,h_N}$ is defined as:

$$V^{h_0,\ldots,h_N} \equiv V^{h_0}(\Omega_0) + \cdots + V^{h_N}(\Omega_N),$$

where $V^{h_i}(\Omega_i) \subset H_0(div, \Omega_i)$ is the standard Raviart-Thomas velocity space on $\tau^{h_i}(\Omega_i)$, with zero velocity flux on $\partial\Omega_i$. Such velocities can be continuously extended by zero

FIG. 3. *Nested subregions with repeated local refinement.*



outside Ω_i with the resulting function in $H_0(div, \Omega)$. Similarly, the composite pressure space $Q^{h_0,\ldots,h_N}$ is defined as:

$$Q^{h_0,\ldots,h_N} \equiv Q^{h_0}(\Omega_0) + \cdots + Q^{h_N}(\Omega_N),$$

where $Q^{h_i}(\Omega_i) \subset L^2(\Omega_i)$ is the standard Raviart-Thomas pressure space on $\tau^{h_i}(\Omega_i)$. The functions in $Q^{h_i}(\Omega_i)$ can be extended by zero outside Ω_i with the resulting function in $L^2(\Omega)$.

The discretization of (2) is obtained on the composite mesh by restricting the weak form (3) to the finite dimensional spaces $V^{h_0,\ldots,h_N}$ and $Q^{h_0,\ldots,h_N}$:

$$\begin{array}{l} \text{Find } u_h \in V_{h_0,\ldots,h_N}(\Omega),\ p_h \in Q_{h_0,\ldots,h_N}(\Omega) \text{ such that} \\ \int_\Omega u_h^T a(x,y)^{-1} v_h dxdy + \int_\Omega p_h(\nabla \cdot v_h) dxdy = 0, \qquad \forall v_h \in V_{h_0,\ldots,h_N}(\Omega) \\ \int_\Omega q(\nabla \cdot u_h) dxdy = \int_\Omega f q dxdy, \quad \forall q \in Q_{h_0,\ldots,h_{lN}}(\Omega). \end{array} \tag{19}$$

Discretization (19) can be shown to be stable, see [18], [5], and results in a symmetric indefinite linear system:

$$\begin{bmatrix} A_{0,\ldots,N} & B^T_{0,\ldots,N} \\ B_{0,\ldots,N} & 0 \end{bmatrix} \begin{bmatrix} u_{0,\ldots,N} \\ p_{0,\ldots,N} \end{bmatrix} = \begin{bmatrix} W_{0,\ldots,N} \\ F_{0,\ldots,N} \end{bmatrix}. \tag{20}$$

5.2. Multiplicative algorithms for solving the locally refined problem. Here we describe a multiplicative algorithm for solving (20) similar to the Fast Adaptive Composite grid method (FAC); see [17], [29]. For $i = 0, \cdots, N$, we define R_i^T as the standard extension (interpolation) map

$$R_i^T : V^{h_i}(\Omega_i) \times Q^{h_i}(\Omega_i) \longrightarrow V^{h_0,\ldots,h_N} \times Q^{h_0,\ldots,h_N},$$

which maps the nodal values in $V^{h_i}(\Omega_i) \times Q^{h_i}(\Omega_i)$ to the full grid by extending by zero outside Ω_i, and interpolating to finer grids defined on Ω_i. The restriction map is defined as its transpose:

$$R_i : \left(V^{h_0,\ldots,h_N} \times Q^{h_0,\ldots,h_N}\right)' \longrightarrow V^{h_i}(\Omega_i) \times Q^{h_i}(\Omega_i).$$

We denote

$$L_{0,\cdots,N} \equiv \begin{bmatrix} A_{0,\ldots,N} & B^T_{0,\ldots,N} \\ B_{0,\ldots,N} & 0 \end{bmatrix},$$

and define $L_i \equiv R_i^T L_{0,\cdots,n} R_i$, which forms the coefficient matrix corresponding to the refined problem on Ω_i.

The sequential algorithm consists of three steps:

Step 1. Compute the velocity $u^\star$ satisfying $B_{0,\ldots,N} u^\star = F_{0,\ldots,N}$:

$$\begin{pmatrix} u_0^* \\ p_0^* \end{pmatrix} \equiv R_0^T L_0^{-1} R_0 \begin{pmatrix} 0 \\ F_{0,\ldots,N} \end{pmatrix}$$

For $i = 1, \ldots, N$, compute

$$\begin{pmatrix} u_i^* \\ p_i^* \end{pmatrix} \equiv R_i^T L_i^{-1} R_i \left\{ \begin{pmatrix} 0 \\ F_{0,\ldots,N} \end{pmatrix} - L_{0,\ldots,N} \begin{pmatrix} \sum_{j=0}^{i-1} u_j^* \\ 0 \end{pmatrix} \right\}$$

and define $u^\star = u_0^* + \cdots + u_N^*$.

Step 2. Compute the divergence free component of the velocity $\tilde{u} \equiv u_{0,\ldots,N} - u^\star$, using the following iterative procedure. Let $\tilde{u}^0 \equiv u^\star$, $\tilde{p}^0 \equiv 0$, and set $k = 0$.

While $\|A_{0,\cdots,N}\tilde{u}^k + B_{0,\cdots,N}\tilde{p}^k - W_{0,\cdots,N}\| > tol$ do:
For $i = 0, \ldots, N$:

$$\begin{pmatrix} \tilde{u}^{k+\frac{i+1}{N+1}} \\ q_i \end{pmatrix} = \begin{pmatrix} \tilde{u}^{k+\frac{i}{N+1}} \\ 0 \end{pmatrix} + R_i^T L_i^{-1} R_i \left\{ \begin{pmatrix} 0 \\ F_{0,\ldots,N} \end{pmatrix} - L_{0,\ldots,N} \begin{pmatrix} \tilde{u}^{k+\frac{i}{N+1}} \\ 0 \end{pmatrix} \right\}$$

Step 3. Define the scalar $\overline{q_{\Omega_0}} \equiv 0$.
For $i = 1, \ldots, N$ define the scalars:

$$\overline{q_{\Omega_i}} \equiv \frac{\int_{\Omega_i} (q_{i-1} + \overline{q_{\Omega_{i-1}}}) dx}{\int_{\Omega_i} dx}$$

$$\tilde{p}^{k+1}(x) \equiv \; q_i(x) + \overline{q_{\Omega_i}} \quad x \in \Omega_i - \Omega_{i+1} \;,$$

where $\Omega_{N+1} \equiv \emptyset$, the empty set.
EndWhile

Note that all subproblems in step 2 are well posed since the initial iterate $\tilde{u}^0 = u^\star$ satisfies the discrete divergence constraint. The pressures q_i determined in step 2 are unique up to a constant and are assumed to have mean value zero. As the velocity iterates $\tilde{u}^k$ converge, the global pressure $\tilde{p}^k$ computed in Step 3 converges to $p_{0,\cdots,N}$, for the same reasons as in § 4.2.

THEOREM 5.1. *For a given choice of nested subregions $\Omega_0 \supset \Omega_1 \supset \ldots \supset \Omega_N$, there exists a constant $0 \leq \rho < 1$ independent of the mesh parameters $h_0, \ldots, h_N$ and N such that*

$$\|\tilde{u}_h - \tilde{u}^k\| \leq \rho^k \|\tilde{u}_h - \tilde{u}^0\|.$$

Here $\|v\| \equiv (\int_\Omega v^T a(x,y)^{-1} v dx dy)^{1/2}$.

Proof. See [19], [18]. □

We are able to provide some quantitative bounds for the convergence factor ρ in case there is just one level of refinement.

THEOREM 5.2. *For the lowest order Raviart-Thomas elements on a triangular grid, with one level of local grid refinement, and for coefficients $a(x,y)$ which are scalar piecewise constant functions (constant in the elements of the coarse grid $\tau^{h_0}(\Omega_0)$), the convergence factor for the FAC algorithm is*

$$\rho \leq 3/8$$

if all triangles are equilateral triangles and

$$\rho \leq 1/2$$

if all triangles are right angle triangles. ρ is independent of the coefficient variations.

Proof. See [19], [18]. □

Quantitative estimates for the case of many levels of refinement $N > 1$ are also available, but deteriorate with increasing N, see [19].

5.3. Parallel algorithms for solving local refinement problems. We will describe two parallel algorithms for solving (20) based on the corresponding algorithms for local refinement in the symmetric, positive definite case [17], [29]. One of the algorithms is based on the same local subproblems as in the FAC algorithm. However, the second algorithm requires additional subproblems, which we now define. For $i = 1, \ldots, N$, subproblems corresponding to the following finite element spaces $V^{h_{i-1}}(\Omega_i) \times Q^{h_{i-1}}(\Omega_i)$ are used, and we denote the corresponding local matrices by $L_{i,i-1}$, and the standard restriction and extension maps by $R_{i,i-1}$ and $R_{i,i-1}^T$, respectively.

As in the previous algorithms, there are three steps.

Step 1. Compute a velocity $u^\star$ satisfying: $B_{0,\ldots,N} u^\star = F_{0,\ldots,N}$ as follows:

Step 1a.

$$\begin{pmatrix} u_0^* \\ p_0^* \end{pmatrix} = R_0^T L_0^{-1} R_0 \begin{pmatrix} 0 \\ F_{0,\ldots,N} \end{pmatrix}$$

Step 1b. For $i = 1, \ldots, N$:

$$\begin{pmatrix} u_i^* \\ p_i^* \end{pmatrix} = R_i^T L_i^{-1} R_i \left\{ \begin{pmatrix} 0 \\ F_{0,\cdots,N} \end{pmatrix} - L_{0,\ldots,N} \begin{pmatrix} \sum_{j=0}^{i-1} u_j^* \\ 0 \end{pmatrix} \right\}$$

Define $u^\star = u_0^\star + u_1^\star + \cdots + u_N^\star$.

Step 2. To determine the correction $e^{0,\ldots,N} = u^{0,\ldots,N} - u^\star$, we form an equivalent new system

$$Pe^{0,\ldots,N} = g, \tag{21}$$

and solve it using the conjugate gradient algorithm [11]. Two choices for P will be described, $P^{(1)}$ and $P^{(2)}$, each involving the parallel solution of subproblems. The right hand side g is computed at the cost of one matrix vector product with P, without explicitly knowing $e^{0,\cdots,N}$. In solving either $P^{(1)}e^{0,\cdots,N} = g^{(1)}$ or $P^{(2)}e^{0,\cdots,N} = g^{(2)}$, the inner product $x^T A_{0,\cdots,N} y$ is used in the conjugate gradient method.

The matrix vector product for the first choice $P^{(1)}$ is defined by:

$$\begin{pmatrix} P^{(1)}v^h \\ q_h \end{pmatrix} \equiv \sum_{i=0}^{l} R_i^T L_i^{-1} R_i L_{0,\cdots,N} \begin{pmatrix} v^h \\ 0 \end{pmatrix}.$$

The corresponding right hand side $g^{(1)}$ is obtained by:

$$\begin{pmatrix} g^{(1)} \\ q_h \end{pmatrix} \equiv \sum_{i=0}^{N} R_i^T L_i^{-1} R_i \begin{pmatrix} W_{0,\ldots,N} - A_{0,\ldots,N}u^\star \\ 0 \end{pmatrix}.$$

The second choice for P, denoted $P^{(2)}$ has its matrix vector product defined by:

$$P^{(2)}v^h \equiv \sum_{i=0}^{N} R_i^T L_i^{-1} R_i L_{0,\ldots,N} \begin{pmatrix} v^h \\ 0 \end{pmatrix} - \sum_{i=1}^{N} R_{i,i-1}^T L_{i,i-1}^{-1} R_{i,i-1} L_{0,\ldots,N} \begin{pmatrix} v^h \\ 0 \end{pmatrix}.$$

The corresponding right hand side $g^{(2)}$ is obtained by:

$$\begin{pmatrix} g^{(2)} \\ q_h \end{pmatrix} \equiv \sum_{i=0}^{N} R_i^T L_i^{-1} R_i \begin{pmatrix} W_{0,\ldots,N} - A_{0,\ldots,N}u^\star \\ 0 \end{pmatrix} - \sum_{i=1}^{N} R_{i,i-1}^T L_{i,i-1}^{-1} R_{i,i-1} \begin{pmatrix} W_{0,\ldots,N} - A_{0,\ldots,N}u^\star \\ 0 \end{pmatrix}.$$

Once $e^{0,\cdots,N}$ is computed, let

$$u_{0,\cdots,N} = u^\star + e_{0,\cdots,N}.$$

Step 3. After computing the velocity $u_{0,\cdots,N}$, we compute the global pressure $p_{0,\cdots,N}$. For $i = 1, \cdots, N$:

$$\begin{pmatrix} v_i \\ q_i \end{pmatrix} = R_i^T L_i^{-1} R_i \begin{pmatrix} W_{0,\ldots,N} - A_{0,\ldots,N}u_{0,\ldots,N} \\ 0 \end{pmatrix},$$

and $\overline{q_{\Omega_0}} \equiv 0$.
For $i = 1, \ldots, N$:

$$\overline{q_{\Omega_i}} \equiv \frac{\int_{\Omega_i}(q_{i-1} + \overline{q_{\Omega_{i-1}}})dx}{\int_{\Omega_i} dx}$$

For $i = 0, \ldots, N$:

$$p_{0,\cdots,N}(x) \equiv \; q_i(x) + \overline{q_{\Omega_i}} \quad \text{for } x \in \Omega_i - \Omega_{i+1} \;,$$

where $\Omega_{N+1} \equiv \emptyset$, the empty set.

THEOREM 5.3. *For a given choice of nested subregions $\Omega_0 \supset \Omega_1 \supset \ldots \supset \Omega_N$, there exists a constant $\rho < 1$ independent of the mesh parameters $h_0, \ldots, h_N$ such that*

$$\|u_h - \tilde{u}^k - u^*\| \leq 2\rho^k \|u_h - \tilde{u}^0 - u^*\|.$$

Here $\|v\| \equiv (\int_\Omega v^T a(x,y)^{-1} v dx dy)^{1/2}$, *and* $\tilde{u}^k$ *the* k*'th iterate in the conjugate gradient method. For* $P = P^{(1)}$, *the convergence factor* ρ *grows linearly with* N, *the number of levels of local refinement. For* $P = P^{(2)}$, *the convergence factor* ρ *is independent of the number of levels* N.

Proof. See [18], [29]. □

For the case $N = 1$, we are also able to prove quantitative bounds for the condition number $\kappa(P^{(2)})$.

REFERENCES

[1] J. H. Bramble, J. E. Pasciak, J. Wang, and J. Xu, *Convergence estimates for product iterative methods with applications to domain decompositions and multigrid*, Math. Comp. To appear.

[2] X. C. Cai and O. B. Widlund, *Domain decomposition algorithms for indefinite elliptic problems*, SIAM J. Sci. Statist. Comput., 13 (1992). To appear.

[3] P. Concus, G. H. Golub, and G. Meurant, *Block preconditioning for the conjugate gradient method*, SIAM Journal on Scientific and Statistical Computing, 6 (1985), pp. 220–252.

[4] M. Dryja and O. B. Widlund, *Some domain decomposition algorithms for elliptic problems*, in Proceedings of the Conference on Iterative Methods for Large Linear Systems held in Austin, Texas, October 1988, to celebrate the Sixty-fifth Birthday of David M. Young, Jr., Academic Press, Orlando, Florida, 1989., 1989.

[5] R. E. Ewing, R. D. Lazarov, T. F. Russel, and P. S. Vassilevski, *Local refinement via domain decomposition techniques for mixed finite element methods with rectangular raviart-thomas elements*, in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Philadelphia, 1990, SIAM.

[6] R. E. Ewing and J. Wang, *Analysis of multilevel decomposition iterative methods for mixed finite element methods*, tech. rep., R.A.I.R.O., University of Wyoming, 1991. Submitted.

[7] M. Fortin and R. Aboulaich, *Schwarz's decomposition method for incompressible flow problems*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Philadelphia, 1988, SIAM.

[8] V. Girault and P.-A. Raviart, *Finite element approximation of the Navier-Stokes equations: Theory and Algorithms*, Springer Verlag, 1986.

[9] R. Glowinski and P. L. Tallec, *Augmented Lagrangian Methods*, SIAM, 1990.

[10] R. Glowinski and M. F. Wheeler, *Domain decomposition and mixed finite element methods for elliptic problems*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Philadelphia, 1988, SIAM.

[11] G. H. Golub and C. F. V. Loan, *Matrix Computations*, Johns Hopkins Univ. Press, 1989. Second Edition.

[12] A. Greenbaum, C. Li, and Z. C. Han, *Parallelizing preconditioned conjugate gradient algorithms*, Computer Physics Communications, 53 (1989), pp. 295–309.

[13] L. Hart and S. McCormick, *Asynchronous multilevel adaptive methods for solving partial differential equations on multiprocessors: Computational analysis*, tech. rep., Comp. Math. Group, Univ. of Colorado at Denver, 1987. Submitted to Parallel Computing.

[14] P. L. Lions, *Interprétation stochastique de la méthode alternée de Schwarz*, C. R. Acad. Sci. Paris, 268 (1978), pp. 325 - 328.

[15] ———, *On the Schwarz alternating method. I.*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., Philadelphia, 1988, SIAM.

[16] ———, *On the Schwarz alternating method. II.*, in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Philadelphia, 1989, SIAM.

[17] J. Mandel and S. McCormick, *Iterative solution of elliptic equations with refinement: The two-level case*, in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Philadelphia, 1989, SIAM.

[18] T. P. Mathew, *Schwarz alternating and iterative refinement methods for mixed formulations of elliptic problems, part II: Theory.* Manuscript. To be submitted.

[19] ———, *Domain decomposition and iterative refinement methods for mixed finite element discretisations of elliptic problems*, Tech. Rep. 463, Computer Science Department, Courant Institute of Mathematical Sciences, September 1989. Courant Institute doctoral dissertation.

[20] S. McCormick and J. Thomas, *The fast adaptive composite grid (FAC) method for elliptic equations*, Math. Comp., 46 (1986), pp. 439–456.

[21] J. A. Meijerink and H. A. van der Vorst, *An iterative method for linear systems of which the coefficient matrix is a symmetric M-matrix*, Mathematics of Computation, 31 (137) (1977), pp. 148 – 162.

[22] K. Miller, *Numerical analogs of the Schwarz alternating procedure*, Numer. Math., 7 (1965), pp. 91–103.

[23] P. A. Raviart and J. M. Thomas, *A mixed finite element method for 2-nd order elliptic problems*, in Mathematical Aspects of Finite Element Methods, A. Dold and B. Eckmann, eds., Springer, 1975. Lecture Notes of Mathematics, Volume 606.

[24] T. Rusten and R. Winther, *A preconditioned iterative method for saddle point problems*, SIAM Journal of Scientific and Statistical Computing, (1990). Proceedings of Copper Mountain Conference on Iterative Methods.

[25] H. A. Schwarz, *Gesammelete Mathematische Abhandlungen*, vol. 2, Springer, Berlin, 1890, pp. 133–143. First published in Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich, volume 15, 1870, pp. 272–286.

[26] W. P. Tang, *Schwarz splitting and template operators*, PhD thesis, Department of Computer Science, Stanford University, 1988.

[27] R. Temam, *Navier-Stokes Equations*, North Holland, 1977.

[28] M. F. Wheeler and R. Gonzalez, *Mixed finite element methods for petroleum reservoir engineering problems*, in Computing Methods in Applied Sciences and Engineering, VI, R. Glowinski and J. L. Lions, eds., New York, 1984, North-Holland, pp. 639–658.

[29] O. B. Widlund, *Optimal iterative refinement methods*, in Domain Decomposition Methods, T. Chan, R. Glowinski, J. Périaux, and O. Widlund, eds., Philadelphia, 1989, SIAM.