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Schwarz Alternating and Iterative Refinement Methods for Mixed Formulations of Elliptic Problems, Part II: Convergence Theory

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Our early results for iterative refinement algorithms in the mixed case similarly showed independence of the mesh sizes, but not of the number of refinement levels, see [16]. By reducing the mixed problem to an equivalent standard discretization, we now obtain optimal bounds as in Dryja and Widlund [20, 3] also for the mixed finite element case. We also derive quantitative bounds for some iterative refinement algorithms using strengthened Cauchy inequalities.

In § 2, we describe the *mixed formulation of elliptic problems* and its discretization based on Raviart-Thomas spaces. In § 3, we provide an abstract framework for the Schwarz and iterative refinement algorithms of [15]. In § 4, we establish optimal convergence rates of the Schwarz methods. In § 5, we discuss locally refined grids, the stability of mixed discretizations on locally refined grids, and provide qualitative and quantitative bounds for the iterative refinement algorithms.

2. A mixed formulation for an elliptic Neumann problem. Consider the following problem on a polygonal domain $\Omega \subset R^2$

$$(1) \qquad \begin{cases} -\nabla \cdot (a(x,y)\nabla p) &= f \quad \text{in } \Omega \\ \vec{n} \cdot (a\nabla p) &= g \quad \text{in } \partial\Omega. \end{cases}$$

Here $\vec{n}$ is the outward normal to $\partial\Omega$, $a(x,y)$ a 2×2 symmetric positive definite matrix function with $L^\infty(\Omega)$ entries satisfying

$$\xi^T a(x,y)\xi \geq \alpha ||\xi||^2, \qquad \text{for a.e. } (x,y) \in \Omega,$$

for a positive constant α. We assume the inhomogeneous terms $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ satisfy the compatibility condition:

$$\int_\Omega f dxdy + \int_{\partial\Omega} g ds = 0.$$

The solution p is then unique up to a constant. Without loss of generality, we assume $g = 0$.

2.1. A mixed formulation of the problem, and its discretization. In many computations, such as in porous flow, the quantity of primary interest is the velocity $\vec{u} = -a(x,y)\nabla p$. Accurate approximations to $\vec{u}$ can be obtained with a mixed formulation using the pressure p and velocity $\vec{u}$ as unknowns:

$$(2) \qquad \begin{cases} \vec{u} &= -a(x,y)\nabla p & \text{in } \Omega & \text{Darcy's law} \\ \nabla \cdot \vec{u} &= f & \text{in } \Omega & \text{Conservation of mass} \\ \vec{n} \cdot \vec{u} &= -g & \text{in } \partial\Omega & \text{Flux boundary condition} \end{cases}$$

The corresponding weak form, see [18, 15], is given by:

$$(3) \qquad \begin{array}{lll} \text{Find } \vec{u} \in H_0(div,\Omega) \text{ and } p \in L^2(\Omega) \text{ such that} & & \\ \int_\Omega \vec{u}^T a(x,y)^{-1}\vec{v} dxdy + \int_\Omega p(\nabla \cdot \vec{v}) dxdy &= 0, & \forall \vec{v} \in H_0(div,\Omega) \\ \int_\Omega q(\nabla \cdot \vec{u}) dxdy &= \int_\Omega fq dxdy, & \forall q \in L^2(\Omega). \end{array}$$

The function space $H_0(div,\Omega) \subset H(div,\Omega)$ for the velocity $\vec{u}$ is defined by

$$H_0(div,\Omega) \equiv \{\vec{v} \in H(div,\Omega) : \vec{v} \cdot \vec{n} = 0 \text{ on } \partial\Omega\},$$

where $H(div,\Omega) = \{(v_1, v_2) \in (L^2(\Omega))^2 : \nabla \cdot \vec{v} \in L^2(\Omega)\}$, is equipped with the norm

$$\|\vec{v}\|^2_{H(div,\Omega)} \equiv \|\vec{v}\|^2_{L^2} + \|\nabla \cdot \vec{v}\|^2_{L^2},$$

see Raviart and Thomas [18]. The appropriate space for the pressure is $L^2(\Omega)$. Let

$$A(u,v) \equiv \int_\Omega u^T a(x,y) v dxdy, \quad \forall u, v \in H(div,\Omega), \tag{4}$$

and let

$$B(u,q) \equiv \int_\Omega (\nabla \cdot u) q dxdy, \quad u \in H(div,\Omega), \quad q \in L^2(\Omega). \tag{5}$$

A discretization of (3) is obtained by replacing the function spaces by finite dimensional subspaces $V^h \subset H_0(div,\Omega)$ and $Q^h \subset L^2(\Omega)$, respectively. In particular, we choose V^h and Q^h to be the lowest order Raviart-Thomas finite element spaces [18] which are briefly described in § 2.2. The discrete problem leads to a symmetric, indefinite linear system:

$$\begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} \begin{bmatrix} u_h \\ p_h \end{bmatrix} = \begin{bmatrix} W_h \\ F_h \end{bmatrix}. \tag{6}$$

The Schwarz and iterative refinement methods for solving it are described in [15].

2.2. The Raviart-Thomas spaces on a triangular grid. For simplicity, we only describe the lowest order Raviart-Thomas spaces on a *quasi-uniform* triangular grid τ^h on Ω with elements of size h.

Definition. A family of triangulations $\{\tau^h\}$ of Ω is said to be *quasi-uniform*, if there exists positive constants c_*, c^*, such that for all triangles K and for all h,

$$c_* \sigma_K \geq h_K \geq c^* h,$$

where σ_K is the diameter of the largest inscribed circle inside K, h_K is the diameter of K and $h = \max h_K$.

The lowest order Raviart-Thomas velocity space $V^h(\Omega)$ consists of piecewise linear vector functions which have constant normal component (flux) on the edges of the triangles, see [18]. For instance, on the unit reference triangle $\widehat{K}$ with vertices

$$\hat{a}_1 = (0,0); \;\; \hat{a}_2 = (1,0); \;\; \hat{a}_3 = (0,1),$$

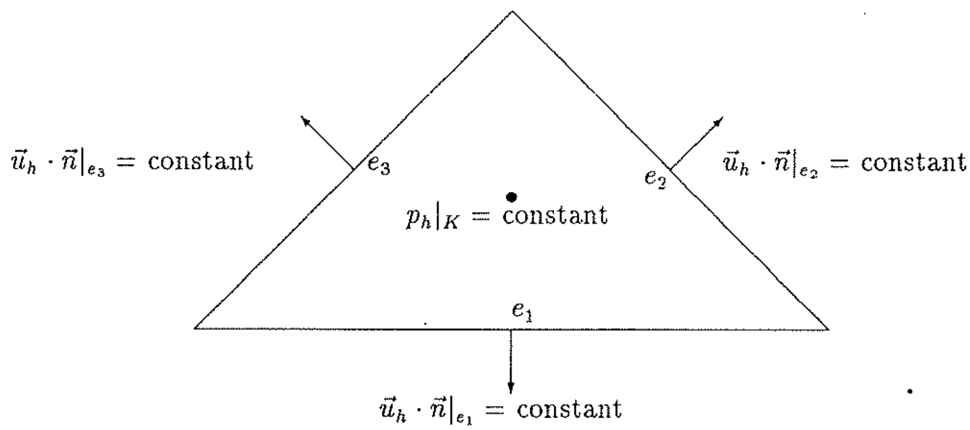
the velocities have the following form:

$$V_h(\widehat{K}) \equiv \{ \begin{pmatrix} a + b\hat{x}_1 \\ c + b\hat{x}_2 \end{pmatrix} : \forall a, b, c \in R; \text{ where } \hat{x} = (\hat{x}_1, \hat{x}_2) \in \widehat{K}\}.$$

For a general triangle $K \in \tau^h$, see Fig. 1, with vertices a_1, a_2, a_3, it is possible to map $\widehat{K}$ onto K, so that $F_K(\hat{a}_i) = a_i$, for $i = 1,2,3$, using an invertible affine linear map $F_K : \widehat{K} \longrightarrow K$ of the form $F_K(\hat{x}) \equiv B_K \hat{x} + b_K$. Here B_K is a 2×2 invertible matrix and b_K is a 2-vector, see [2, 7] and the following map between vector valued functions $\vec{\hat{v}}$ on $\widehat{K}$ and $\vec{v}$ on K,

$$\vec{v}(x) \equiv (1/J_K) B_K \vec{\hat{v}} \circ F_K^{-1}(x) \text{ and } \vec{\hat{v}}(\hat{x}) \equiv J_K B_K^{-1} \vec{v} \circ F_K(\hat{x}),$$

FIG. 1. *Lowest order Raviart-Thomas spaces on a triangle K.*



is valid with $J_K \equiv det(B_K)$. Such maps preserves the normal components on the edges. The velocity space on K, $V_h(K)$ is defined by this map:

$$V_h(K) \equiv \{\vec{v} : \vec{v} \longleftrightarrow \hat{\vec{v}} \in V_h(\widehat{K})\},$$

and $V^h(\Omega)$ is defined by

$$V_h(\Omega) \equiv \{\vec{v} \in H_0(div, \Omega) : \vec{v}|_K \in V_h(K)\}.$$

The pressure space $Q^h(\Omega)$ consists of piecewise constant functions, constant in each triangle K. These Raviart-Thomas spaces lead to a stable discretization with $O(h)$ discretization error.

3. An abstract framework for the Schwarz methods. The multiplicative or additive Schwarz algorithm to solve a symmetric, positive definite linear equation in a Hilbert space can be viewed as a procedure for residual correction within certain subspaces, see [12, 1, 17, 21, 4]. The rate of convergence is then expressible in terms of certain partitioning and interaction properties of these subspaces. Here, we summarize the main results that will be used in this paper. An abstract form of the Schwarz methods is described in § 3.1 and § 3.2, and its application to mixed finite element discretizations is summarized in § 4.

Let X denote a finite dimensional Hilbert space with inner product $(.,.)$ and consider the following symmetric, positive definite problem:

$$\text{Find } u_h \in X \text{ such that } A(u_h, v_h) = W(v_h), \quad \forall v_h \in X, \tag{7}$$

where $A(.,.)$ is symmetric, coercive bilinear form satisfying

$$C_1(v_h, v_h) \le A(v_h, v_h) \le C_2(v_h, v_h),$$

for positive constants C_1 and C_2 and $W(.)$ is a bounded linear functional satisfying:

$$|W(v_h)| \le C\|v_h\|, \quad \forall v_h \in X.$$

For $i = 0, \cdots, N$, let $X_i \subset X$ be subspaces whose sum spans X:

$$X = X_0 + \cdots + X_N.$$

We assume that the following partition and interaction properties hold. They will be used to estimate the rate of convergence of the Schwarz algorithms.
Partition assumption. There exists a positive constant C_0 such that for any $u_h \in X$ there is a partition

$$u_h = u_0 + u_1 + \cdots + u_N, \qquad \text{with } u_i \in X_i,$$

satisfying

$$\sum_{i=0}^{N} A(u_i, u_i) \leq C_0 A(u_h, u_h).$$

Remark. Such a C_0 always exists for any finite dimensional X provided $X_0 + \cdots + X_N = X$, see [12]. What is important is to establish an estimate which is independent of, or grows very slowly when $h \to 0$.

Following [21], we also use the following
Interaction parameters. Let $\mathcal{E}$ denote a symmetric matrix of dimension $N+1$ whose entries $0 \leq \epsilon_{ij} \leq 1$, are the constants in the following strengthened Cauchy inequalities between spaces X_i and X_j:

$$|A(u_i, u_j)| \leq \epsilon_{ij} |A(u_i, u_i)|^{1/2} |A(u_j, u_j)|^{1/2}, \quad \forall u_i \in X_i \text{ and } u_j \in X_j.$$

Let $I_0 \subset \{0, \cdots, N\}$ be a subset of indices and let $i_0 \equiv |I_0|$ be the number of elements in the set I_0. Also let,

$$n_0 \equiv \max_{j \notin I_0} \sum_{i \notin I_0} |\epsilon_{ij}|.$$

The constant $i_0 + n_0$ represents the interaction between the various subspaces X_i ; if all the subspaces are mutually orthogonal, then $i_0 + n_0 = 1$, while if all the spaces interact strongly, then $i_0 + n_0 \approx N$.

For each subspace X_i, and $u_h \in X$, we let $P_i u_h$ denote the orthogonal projection of u_h onto X_i:

$$P_i u_h \in X_i \text{ such that } A(P_i u_h, v_h) = A(u_h, v_h) \quad \forall v_h \in X_i. \tag{8}$$

3.1. Multiplicative Schwarz algorithm. The multiplicative Schwarz algorithm to solve (7) is given as follows, in terms of the subspaces $X_0, \cdots, X_N$, and the projections P_i. Let $u^0 \in X$ be an initial guess:
Set $k = 0$
While $\|W(.) - A(u^k, .)\| \geq tol$ do
For $i = 0, \cdots, N$ define

$$u^{k+\frac{i+1}{N+1}} \equiv u^{k+\frac{i}{N+1}} + P_i \left(u_h - u^{k+\frac{i}{N+1}} \right).$$

endFor
endWhile

Note that $w_i \equiv P_i \left(u_h - u^{k+\frac{i}{N+1}} \right)$ can be computed without explicit knowledge of u_h since the right hand side in

$$A(w_i, v_h) = A(u_h - u^{k+\frac{i}{N+1}}, v_h) = W(v_h) - A(u^{k+\frac{i}{N+1}}, v_h), \quad \forall v_h \in X_i,$$

is known. It can be easily verified that the kth iterate u^k of the multiplicative Schwarz algorithm satisfies

$$(u_h - u^k) = (I - P_N)\cdots(I - P_0)(u_h - u^{k-1}), \tag{9}$$

and thus recursively

$$\|u_h - u^k\|_A \le \rho^k \|u_h - u^0\|_A,$$

where

$$\rho \equiv \|(I - P_N)\cdots(I - P_0)\|_A.$$

The following Lemma of [1, 21] is a generalization of a result in [12] relating the convergence factor to the partition and interaction parameters C_0 and $i_0 + n_0$.

LEMMA 3.1. *Suppose that the spaces X_i satisfy the partition assumption with constant C_0, and let i_0 and n_0 denote the interaction parameters. Then,*

$$\rho \equiv \|(I - P_N)\cdots(I - P_0)\|_A \le 1 - \frac{1}{C_0(1 + i_0 + n_0)^2}. \tag{10}$$

3.2. Additive Schwarz method. The additive Schwarz preconditioned system to solve (7), based on the subspaces $X_0, \cdots, X_N$ is given by

$$(P_0 + P_1 + \cdots + P_N)\, u_h = g_h, \tag{11}$$

where g_h can be computed without explicit knowledge of u_h. If the conjugate gradient method is used to solve (11) , then by standard estimates, see [11], the mth iterate u^m satisfies:

$$\|u_h - u^m\|_A \le 2\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^m \|u_h - u^0\|_A, \tag{12}$$

where $\kappa = \lambda_{max}/\lambda_{min}$, is the condition number of $P_0 + \cdots + P_N$:

$$\lambda_{min} \le \frac{A\left((P_0 + \cdots + P_N)u_h, u_h\right)}{A\left(u_h, u_h\right)} \le \lambda_{max}, \quad \forall u_h \in X.$$

The following Lemma of [21] is a generalization of a result in [4].

LEMMA 3.2. *Suppose the spaces X_i satisfy the partition assumption with constant C_0 and let i_0 and n_0 denote the interaction parameters. Then, the condition number of the preconditioned system $\kappa(P_0 + \cdots + P_N)$ satisfies*

$$\kappa(P_0 + \cdots + P_N) \le C_0(i_0 + n_0).$$

4. Applications to Schwarz algorithms for mixed finite element discretizations. The Schwarz methods of § 3 were described for symmetric, positive definite problems, and so are not directly applicable to symmetric, indefinite problems such as (6). However, if the velocity in (6) is divergence free, then the pressure can be eliminated implicitly, and the symmetric, indefinite problem can be reduced to a symmetric, positive definite problem for the divergence free velocity, thereby enabling application of the results of § 3. We briefly describe this procedure, and provide estimates for the partition and interaction parameters C_0, i_0 and n_0 for the specific choice of subspaces and norms appropriate for the mixed formulation case.

The Schwarz algorithms described in [15] to solve (6) on a quasi-uniform grid $\tau^h(\Omega)$ are based on two levels of discretization, a coarse grid $\tau^H(\Omega)$ and the fine grid $\tau^h(\Omega)$ obtained by successive refinement of the coarse grid. The coarse grid elements $\Omega_1, \cdots, \Omega_N$, also referred to as subdomains, are extended to form overlapping subdomains $\Omega'_1, \cdots, \Omega'_N$ with $distance(\partial\Omega_i, \partial\Omega'_i) \geq \beta H$, where $0 < \beta \leq 1$. The Raviart-Thomas velocity spaces on the coarse grid and extended subdomains Ω'_i are denoted $V^H(\Omega)$ and $V^h(\Omega'_i)$, respectively.

The algorithms consist of three steps. In step *one*, a discrete velocity u_h^* satisfying

$$B_h u_h^* = F_h,$$

is computed. The divergence free velocity correction $\tilde{u}_h = u_h - u_h^*$ in (6) then satisfies:

$$\text{(13)} \qquad \begin{array}{lcll} \multicolumn{4}{l}{\text{Find } \tilde{u}_h \in V^h(\Omega) \text{ and } p_h \in Q^h(\Omega) \text{ such that}} \\ A(\tilde{u}_h, v_h) + B(v_h, p_h) & = & W_h(v_h) - A(u_h^*, v_h), & \forall v_h \in V^h(\Omega) \\ B(\tilde{u}_h, q_h) & = & 0 & \forall q_h \in Q^h(\Omega) \ . \end{array}$$

By choosing divergence free test vectors v_h in equation (13), we obtain that $\tilde{u}_h$ satisfies:

$$\text{(14)} \qquad \text{Find } \tilde{u}_h \in X \text{ such that } A(\tilde{u}_h, v_h) = W_h(v_h) - A(u_h^*, v_h) \quad \forall v_h \in X,$$

where

$$X \equiv V^h(\Omega) \cap H_0(div^0, \Omega).$$

The problem for determining the divergence free velocity $\tilde{u}_h$ is symmetric, positive definite since the bilinear form $A(.,.)$ is symmetric, positive definite on X:

$$\text{(15)} \qquad A(v_h, v_h) = \int_\Omega v_h^T a(x,y) v_h dxdy \geq \alpha \int_\Omega v_h^T v_h dxdy = \alpha \|v_h\|^2_{H(div,\Omega)},$$

where α is the minimum eigenvalue of $a(x,y)$ on Ω.

In the *second* step, the divergence free velocity correction $\tilde{u}_h = u_h - u_h^*$ is computed using either the multiplicative or additive Schwarz method. The subspaces in the Schwarz methods are given by

$$\begin{array}{l} X_0 = V^H(\Omega'_i) \cap H_0(div^0, \Omega) \subset X \\ X_i = V^h(\Omega'_i) \cap H_0(div^0, \Omega'_i) \subset X, \quad i = 1, \cdots, N, \end{array}$$

and the projection $P_i v_h$ onto X_i, implemented in [15], is given by

$$\begin{bmatrix} P_i v_h \\ q_i \end{bmatrix} \equiv R_i^T \begin{bmatrix} A_i & B_i^T \\ B_i & 0 \end{bmatrix}^{-1} R_i \begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} \begin{bmatrix} v_h \\ 0 \end{bmatrix},$$

where R_i is the restriction map onto Ω'_i for $i = 1, \cdots, N$ and onto the coarse grid for $i = 0$.

In the *third* step, the pressure p_h is computed.

Remark. A discrete divergence free velocity w_h is divergence free in the sense of distributions. To prove this, we use the property that the divergence operator maps the velocity space $V^h(\Omega)$ *onto* $Q^h(\Omega)/R$. Choosing $q_h = \nabla \cdot w_h$, in the discrete divergence constraint

$$\int_\Omega q_h(\nabla \cdot w_h)dxdy = 0, \quad \forall q_h \in Q^h(\Omega),$$

we obtain

$$\int_\Omega |\nabla \cdot w_h|^2 dxdy = 0.$$

We now describe some general results about divergence free functions.

4.1. Some properties of divergence free functions. We describe tools for analyzing the convergence of the algorithms. Many of them are given in a recent paper of Ewing and Wang [6] and are presented for completeness. We also describe extensions of their results suitable for additive algorithms. The whole analysis centers on a scalar stream function representing divergence free functions.

Let $H^1(\Omega)$ be the standard Sobolev space

$$H^1(\Omega) \equiv \{u \in L^2(\Omega) : \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(\Omega)\},$$

equipped with norm $\|u\|_{H^1(\Omega)} \equiv \left(\int_\Omega \left(|u|^2 + |\frac{\partial u}{\partial x}|^2 + |\frac{\partial u}{\partial y}|^2\right) dxdy\right)^{1/2}$, and let $H_0^1(\Omega)$ be its subspace with zero boundary values on $\partial\Omega$. Let $S^h(\Omega) \subset H_0^1(\Omega)$ be the space of continuous, piecewise linear functions on the given triangulation $\tau^h(\Omega)$. The following result establishes the existence of a stream function for divergence free functions in two dimensions.

LEMMA 4.1. *If $\vec{v} \in H(div, \Omega)$ is a two dimensional divergence free vector function, then there exists a scalar stream function $\psi(x, y) \in H^1(\Omega)$ with*

$$\vec{v} = (v_1, v_2) = (\psi_y, -\psi_x) = \textbf{Curl } \psi.$$

Proof. See [6]. □

Remark 1. Let u be a divergence free function with the associated stream function $u = (\psi_y, -\psi_x)$, let $\vec{n} = (n_1, n_2)$ be a normal to an edge, and $\vec{\tau} = (-n_2, n_1)$ be the tangent. Then on the edge

$$u \cdot \vec{n} = (\psi_y, -\psi_x) \cdot (n_1, n_2) = (\psi_x, \psi_y) \cdot (-n_2, n_1) = \frac{\partial \psi}{\partial \tau},$$

where $\partial\psi/\partial\tau$ is the tangential derivative. Thus, in particular, if $\vec{u}$ has zero flux on $\Gamma \subset \partial\Omega$, then ψ will be constant on Γ.

The following result stated in [6], relates the divergence free velocities in the lowest order Raviart-Thomas space X on the triangular grid to the space of continuos, piecewise linear functions $S^h(\Omega)$.

LEMMA 4.2. *If $v_h \in X = V^h(\Omega) \cap H_0(div^0, \Omega)$, then there exists a scalar function $\psi_h \in S^h(\Omega)$ with*

$$\textbf{Curl } \psi_h = v_h.$$

Conversely, if $\xi_h \in S^h(\Omega)$, then **Curl** $\xi_h \in X$.

Proof. By Lemma 4.1, there exists a scalar stream function with **Curl** $\psi_h = v_h$. We now show that ψ_h is piecewise linear. Since $\partial\psi/\partial\tau = v_h \cdot \vec{n} = Const$ on the edges of each triangle, it follows that ψ is linear on each edge. Since the lowest order divergence free Raviart-Thomas velocities are piecewise constant, ψ_h is piecewise linear on each triangle. Since $\partial\psi_h/\partial\tau = 0$ on $\partial\Omega$, ψ_h can be chosen to be zero on $\partial\Omega$ and thus $\psi_h \in S^h(\Omega)$.

The converse follows trivially. □

As a consequence of this stream function representation, the bilinear form $A(.,.)$ on X is equivalent to a bilinear form $\tilde{A}(.,.)$ on $S^h(\Omega)$. Let **Curl** $\psi_h = w_h$ and **Curl** $\xi_h = v_h$. Then, by substitution, we obtain

$$A(\textbf{Curl } \psi_h, \textbf{Curl } \xi_h) = \tilde{A}(\psi_h, \xi_h),$$

where the bilinear form $\tilde{A}(.,.)$ is given by

$$\tilde{A}(\psi_h, \xi_h) \equiv \int_\Omega (\nabla\psi_h)^T \tilde{a}(x,y)(\nabla\xi_h)dxdy, \tag{16}$$

with a symmetric, positive definite coefficient matrix $\tilde{a}(x,y)$ given by

$$\tilde{a}(x,y) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} a(x,y)^{-1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{17}$$

This equivalence between the bilinear forms $A(.,.)$ and $\tilde{A}(.,.)$ leads to an equivalence between orthogonal projections in X and $S^h(\Omega)$. For $i = 1, \cdots, N$, let

$$S^i \equiv S^h(\Omega) \cap H_0^1(\Omega_i'),$$

and let S^0 denote the space of continuous piecewise linear functions on the coarse grid $\tau^H(\Omega)$. For each i, let P_i^A denote the orthogonal projection onto X_i in the $A(.,.)$ inner product, and let $P_i^{\tilde{A}}$ denote the orthogonal projection onto S^i in the $\tilde{A}(.,.)$ inner product.

LEMMA 4.3. *Let $v_h \in X$ with $v_h =$ **Curl** ψ_h for $\psi_h \in S^h(\Omega)$. Then,*
1).

$$P_i^A v_h = P_i^A \textbf{ Curl } \psi_h = \textbf{ Curl } P_i^{\tilde{A}} \psi_h.$$

2).

$$\frac{A\left((P_0^A + \cdots + P_N^A)v_h, v_h\right)}{A(v_h, v_h)} = \frac{\tilde{A}\left((P_0^{\tilde{A}} + \cdots + P_N^{\tilde{A}})\psi_h, \psi_h\right)}{\tilde{A}(\psi_h, \psi_h)}.$$

Thus $\kappa(P_0^A + \cdots + P_N^A) = \kappa(P_0^{\tilde{A}} + \cdots + P_N^{\tilde{A}})$.
3).

$$\|(I - P_N^A)\cdots(I - P_0^A)\|_A = \|(I - P_N^{\tilde{A}})\cdots(I - P_0^{\tilde{A}})\|_{\tilde{A}}.$$

Proof. To verify 1), let $\psi_i \in S^i$ be the stream function with **Curl** $\psi_i = P_i^A v_h$, and let $w_h \in X_i$ be test functions with $w_i =$**Curl** ξ_i, where $\xi_i \in S^i$. Then, applying the definition of projection, we obtain

$$A(P_i^A v_h, w_i) = A(v_h, w_i), \quad \forall w_i \in X_i,$$

which is equivalent to

$$\tilde{A}(\psi_i, \xi_i) = \tilde{A}(\psi_h, \xi_i), \quad \forall \xi_i \in S^i,$$

by the equivalence between the bilinear forms. Thus, $\psi_i = P_i^{\tilde{A}} \psi_h$ and **Curl** $P_i^{\tilde{A}} \psi_h = P_i^A \mathbf{Curl} \psi_h$.

2) Equivalence of the two Rayleigh quotients follows from 1), since

$$(P_0^A + \cdots + P_N^A)\mathbf{Curl}\psi = \mathbf{Curl}(P_0^{\tilde{A}} + \cdots + P_N^{\tilde{A}})\psi,$$

and since $A(\mathbf{Curl}\ \psi_h, \mathbf{Curl}\ \psi_h) = \tilde{A}(\psi_h, \psi_h)$.

3) Equivalence of the two error propagation maps also follows from 1) by using $A(\mathbf{Curl}\ \psi_h, \mathbf{Curl}\ \psi_h) = \tilde{A}(\psi_h, \psi_h)$ and

$$(I - P_N^A) \cdots (I - P_0^A)\mathbf{Curl}\psi = \mathbf{Curl}(I - P_N^{\tilde{A}}) \cdots (I - P_0^{\tilde{A}})\psi,$$

and by applying the definition of the norm. □

We end this section, with a partition Lemma for the mixed finite element case described in [6].

THEOREM 4.4 (EWING AND WANG). *Let $u_h \in X = V^h(\Omega) \cap H_0(div, \Omega)$. For the given collection of subspaces $X_i \subset X$, $i = 1, \cdots, N$ based on the overlapping subdomains Ω_i' with overlap ratio β, and the coarse grid space $X_0 \subset X$ based on the coarse triangulation $\tau^H(\Omega)$, there exists a partition*

$$u_h = u_0 + u_1 + \cdots + u_N,$$

with $u_i \in X_i$, $i = 0, \cdots, N$, satisfying

$$\sum_{i=0}^{N} A(u_i, u_i) \leq C_0 A(u_h, u_h).$$

The constant $C_0 > 0$ is independent of H and h, but mildly dependent on β.

Proof. Let $u_h \in X$ be represented $u_h = \mathbf{Curl}\ \psi_h$, where $\psi_h \in S^h(\Omega)$. By a standard partition lemma for the piecewise linear finite element space $S^h(\Omega)$, see Dryja and Widlund [4], there is a partition

$$\psi_h = \psi_0 + \cdots + \psi_N, \quad \text{with } \psi_i \in S^i,$$

satisfying

$$\sum_{i=0}^{N} \tilde{A}(\psi_i, \psi_i) \leq C_0 \tilde{A}(\psi_h, \psi_h),$$

with a positive constant C_0 independent of H and h provided the overlap β is fixed. The partition Lemma follows by defining

$$u_i = \mathbf{Curl}\ \psi_i \in X_i,$$

and using the equivalence between the norms. □

Remark 1. If the coarse grid space is not used, then there is a partition

$$u_h = u_1 + \cdots + u_N, \quad \text{with } u_i \in X^i,$$

satisfying

$$\sum_{i=1}^{N} A(u_i, u_i) \leq (C/H^2) A(u_h, u_h),$$

where C is independent of H and h for fixed overlap β. This follows from the corresponding result for $S^h(\Omega)$, see [4].

4.2. Convergence rates of the Schwarz methods for mixed formulations. An optimal bound for the multiplicative Schwarz method in the mixed case was recently established by Ewing and Wang [6]. We describe this briefly and include a simple extension of their results to the additive Schwarz algorithm.

The velocity iterates $\tilde{u}^k$ obtained by applying either of the Schwarz methods to solve mixed finite element problem (13) are divergence free. By application of the results of § 3.1, the iterates of the multiplicative Schwarz algorithm satisfy:

$$\|\tilde{u}_h - \tilde{u}^k\|_A \le \rho^k \|\tilde{u}_h - \tilde{u}^0\|_A,$$

where $\rho = \|(I - P_N^A)\cdots(I - P_0^A)\|_A$.

THEOREM 4.5 (EWING AND WANG). *For the collection of overlapping subdomains $\Omega_1', \cdots, \Omega_N'$ and the coarse grid, the convergence factor ρ of the multiplicative Schwarz algorithm is independent of H and h for fixed overlap β.*

Proof. By Lemma 3.1 of § 3.1, the convergence factor satisfies

$$\rho \le 1 - \frac{1}{C_o(i_0 + n_0)^2},$$

where C_0, $i_0 + n_0$ are the partition and interaction parameters, respectively, for the spaces X_i. By the Ewing-Wang partition Lemma 4.4, C_0 is independent of H and h for fixed overlap β. $i_0 + n_0$ is also bounded independent of H and h. See [6] for the details. □

Next, we consider the additive Schwarz algorithm.

THEOREM 4.6. *For the choice of overlapping subdomains $\Omega_1', \cdots, \Omega_N'$ and the coarse grid, the condition number of the additive Schwarz preconditioned system satisfies:*

$$\kappa\left(P_0^A + \cdots + P_N^A\right) \le C,$$

where C is independent of H and h for fixed overlap ratio β.

Proof. By Lemma 3.2 of § 3, the condition number of the additive Schwarz method satisfies

$$\kappa\left(P_0^A + \cdots + P_N^A\right) \le C_0(i_0 + n_0),$$

where C_0 and $i_0 + n_0$ are the partition and interaction parameters, respectively, for the subspaces X_i of X. By Lemma 4.4 of § 4.1, C_0 is independent of H and h, for fixed overlap β. To estimate i_0 and n_0, we let $I_0 = \{0\}$. Then $i_0 = |I_0| = 1$. Since for standard triangulations, each triangular subdomain Ω_i' intersects with at most thirteen neighbors, for $i, j \ne 0$, there are at most 13 non-zero entries ϵ_{ij} for in each row for $i, j \ne 0$. Since $|\epsilon_{ij}| \le 1$,

$$n_0 = \max_{i \notin I_0} \sum_{j \notin I_0} \epsilon_{ij} \le 13,$$

and so $i_0 + n_0 = 14$. The result now follows. □

Remark. If the coarse grid space is omitted, then the condition number of the additive Schwarz algorithm satisfies

$$\kappa \le C_1/H^2,$$

and the convergence factor of the multiplicative Schwarz algorithm satisfies:

$$\rho \leq 1 - \frac{H^2}{C_2},$$

where C_1 and C_2 are independent of H and h.

5. Iterative refinement methods for mixed formulations. Two kinds of iterative refinement algorithms for solving mixed formulations of elliptic problems on locally refined grids are described in [15]. In this Section, we first establish the stability of the mixed finite element discretization on locally refined grids, and then provide qualitative and quantitative bounds for both kinds of iterative refinement algorithms.

We recall the construction of the composite refined grid in [15]. A quasi-uniform grid $\tau^{h_0}(\Omega_0)$ of size h_0 on $\Omega_0 = \Omega$, is successively refined locally on a sequence of nested subregions Ω_i:

$$\Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_{N-1} \supset \Omega_N.$$

All elements of $\tau^{h_0}(\Omega_0)$ in Ω_1 are refined uniformly, and the locally refined grid on Ω_1 is denoted $\tau^{h_1}(\Omega_1)$, with elements of size $h_1 = h_0/2$. Similarly, for $i = 2, \cdots, N$, all elements of $\tau^{h_{i-1}}(\Omega_{i-1})$ in Ω_i are refined uniformly and this refined grid on Ω_i is denoted $\tau^{h_i}(\Omega_i)$, with elements of size $h_i = h_{i-1}/2$. The composite grid $\tau^{h_0,\ldots,h_N}$ is defined by:

$$\tau^{h_0,\ldots,h_N} \equiv \cup_{i=0}^{N-1} \tau^{h_i}(\Omega_i - \Omega_{i+1}) \cup \tau^{h_N}(\Omega_N),$$

and the Raviart-Thomas velocity space on $\tau^{h_0,\cdots,h_N}$ by

$$V^{h_0,\ldots,h_N} \equiv V^{h_0}(\Omega_0) + \cdots + V^{h_N}(\Omega_N);$$

see [5, 15]. Each $V^{h_i}(\Omega_i) \subset H_0(div, \Omega_i)$, and so has zero flux on $\partial\Omega_i$. The composite pressure space is defined by

$$Q^{h_0,\ldots,h_N} \equiv Q^{h_0}(\Omega_0) + \cdots + Q^{h_N}(\Omega_N).$$

The discretization of (3) based on the composite grid spaces $V^{h_0,\cdots,h_N} \times Q^{h_0,\cdots,h_N}$ leads to a linear system of the form:

$$\begin{bmatrix} A_{0,\ldots,N} & B^T_{0,\ldots,N} \\ B_{0,\ldots,N} & 0 \end{bmatrix} \begin{bmatrix} u_{0,\ldots,N} \\ p_{0,\ldots,N} \end{bmatrix} = \begin{bmatrix} W_{0,\ldots,N} \\ F_{0,\ldots,N} \end{bmatrix}. \tag{18}$$

5.1. Stability of the composite grid discretization. The discretization of (3) is stable provided two conditions hold, see [7, 18]. One is that the bilinear form $A(.,.)$ is coercive in the subspace of divergence free functions, $V^{h_0,\cdots,h_N} \subset H_0(div^0, \Omega)$, i.e., there exists a positive constant C independent of h_i such that

$$A(w_h, w_h) \geq C\|w_h\|^2_{H(div,\Omega)}, \quad \forall w_h \in V^{h_0,\cdots,h_N} \text{ with } \nabla \cdot w_h = 0.$$

This coercivity condition is satisfied for all choices of velocity subspaces since for a divergence free function w,

$$\alpha\|w\|^2_{H(div,\Omega)} = \alpha \int_\Omega w^T w dx dy \leq \int_\Omega w^T a(x,y) w dx dy = A(w,w),$$

where α is the uniform lower bound for the eigenvalues of the coefficient matrix $a(x,y)$ and is independent of h_i. The bilinear form $B(.,.)$ must also satisfy an *inf sup* condition, see [7], i.e., there is a positive constant β independent of h such that

$$\sup_{v_h \in V^h} \frac{|B(v_h, q_h)|}{\|v_h\|_{H(div,\Omega)}} \geq \beta \|q_h\|_{L^2(\Omega)}, \quad \forall q_h \in Q^h(\Omega).$$

According to [18, 7], it is sufficient to show that given $q_h \in Q^h/R$, there exists $v_h \in V^h$ such that

$$\nabla \cdot v_h = q_h,$$

and such that

$$\|v_h\|_{H(div,\Omega)} \leq (1/\beta)\|q_h\|_{L^2(\Omega)}.$$

This condition is proved for the Raviart-Thomas elements on quasi-uniform meshes, see [18]. On the composite mesh, we have the following result.

THEOREM 5.1. *There is a positive constant $\beta(\Omega_0, \cdots, \Omega_N)$ independent of $h_0, \cdots, h_N$ such that for any $q_h \in Q^{h_0,\cdots,h_N}$ having mean value zero, there exists $v_h \in V^{h_0,\cdots,h_N}$ satisfying*

$$\nabla \cdot v_h = q_h,$$

and

$$\|v_h\|_{H(div,\Omega)} \leq \beta \|q_h\|_{L^2(\Omega)}.$$

Proof. For $i = 0, \cdots, N$, let T_i denote the L^2 projection onto $Q^{h_0} + \cdots + Q^{h_i}$. Then $T_N = I$, and we can partition

$$q_h = T_0 q_h + (T_1 - T_0)q_h + \cdots + (T_N - T_{N-1})q_h.$$

By this construction, each $q_i = (T_i - T_{i-1})q_h$ is orthogonal in $L^2(\Omega)$ to all other q_j. Since the functions in each $Q^{h_j}(\Omega_j)$ are discontinuous across element edges, it also follows that $q_i = (T_i - T_{i-1})q_h$ has support in Ω_i and hence $q_i \in Q^{h_i}(\Omega_i)$. Since χ_{Ω_i}, the characteristic function of Ω_i, is in $Q^{h_{i-1}}(\Omega_{i-1})$, and since q_i is orthogonal to $Q^{h_i}(\Omega_i)$, it follows that each q_i has mean value zero on Ω_i.

Since the *inf sup* condition holds for the mixed discretizations on each of the local quasi-uniform meshes $\tau^{h_i}(\Omega_i)$ on Ω_i, there exists a positive constant $\beta_i(\Omega_i)$ such that for any $q_i \in Q^{h_i}(\Omega_i)/R$, there is $v_i \in V^{h_i}(\Omega_i)$ such that $\nabla \cdot v_i = q_i$ and satisfying

$$\|v_i\|_{H(div,\Omega_i)} \leq \beta_i(\Omega_i)\|q_i\|_{L^2(\Omega_i)}.$$

Choosing $q_i = (T_i - T_{i-1})q_h$, and defining

$$v_h \equiv v_0 + \cdots + v_N,$$

we obtain,

$$\nabla \cdot v_h = q_o + \cdots + q_N = q_h,$$

and

$$\|v_h\|_{H(div,\Omega)} \le \sum_{i=0}^{N} \|v_i\|_{H(div,\Omega_i)} \le \sum_{i=0}^{N} \beta_i(\Omega_i)\|q_i\|_{L^2(\Omega_i)}.$$

By the Schwarz inequality, the right hand side is bounded by

$$\max(\beta_0,\ldots,\beta_N)\sqrt{N+1}\left(\sum_{i=0}^{N}\|q_i\|_{L^2(\Omega)}^2\right)^{1/2} = \max(\beta_0,\ldots,\beta_N)\sqrt{N+1}\|q_h\|_{L^2(\Omega)}.$$

Note that the constants $\beta_0,\ldots,\beta_N$ are independent of $h_0,\ldots,h_N$. Therefore, the composite Raviart-Thomas spaces satisfy an *inf sup* condition, with a constant independent of $h_0,\ldots,h_N$. It is mildly dependent on the number of levels N of refinement. □

5.2. Convergence rates of iterative refinement algorithms. Both the multiplicative and additive iterative refinement algorithms described in [15] to solve (18) consist of three steps. In the first step a discrete velocity u^* satisfying $B_{0,\ldots,N}u^* = 0$ is computed. In the second step, the divergence free correction to the velocity $\tilde{u}_{0,\cdots,N} = u_{0,\cdots,N} - u^*$, is computed by either a multiplicative or an additive Schwarz type iteration. In the third step, the pressure $p_{0,\cdots,N}$ is computed.

The multiplicative iterative refinement algorithm in the mixed formulation case is based on an iterative refinement algorithm for standard formulations, in which case the algorithm is often referred to as the Fast Adaptive Composite grid method (FAC) by Mandel and McCormick[14]. We will also refer to the multiplicative iterative refinement algorithm in the mixed finite element case as the FAC algorithm. The FAC algorithm corresponds to the multiplicative Schwarz algorithm based on the spaces

$$\begin{aligned} X &= V^{h_0,\cdots,h_N} \cap H_0(div^0,\Omega), \\ X_i &= V^{h_i}(\Omega_i) \cap H_0(div^0,\Omega_i), \end{aligned} \tag{19}$$

with projections P_i^A onto X_i defined in terms of the $A(.,.)$ inner product. These projections are computed in [15] by solving symmetric, indefinite linear systems:

$$\begin{bmatrix} P_i^A v_h \\ q_i \end{bmatrix} \equiv R_i^T \begin{bmatrix} A_i & B_i^T \\ B_i & 0 \end{bmatrix}^{-1} R_i \begin{bmatrix} A_{0,\ldots,N} & B_{0,\ldots,N}^T \\ B_{0,\ldots,N} & 0 \end{bmatrix} \begin{bmatrix} v^h \\ 0 \end{bmatrix},$$

where R_i is the standard restriction map onto $V^{h_i}(\Omega_i) \times Q^{h_i}(\Omega_i)$.

The stream functions corresponding to the divergence free spaces X_i lie in $S^{h_i}(\Omega_i)$ and thus the stream functions corresponding to X lie in the composite space

$$S^{h_0,\cdots,h_N} \equiv S^{h_0}(\Omega_0) + \cdots + S^{h_N}(\Omega_N).$$

Let $P_i^{\tilde{A}}$ denote the projection onto $S^{h_i}(\Omega_i)$ in the $\tilde{A}(.,.)$ inner product, defined by (16). Since

$$P_i^A \mathbf{Curl}\ \psi_h = \mathbf{Curl}\ P_i^{\tilde{A}}\psi_h,$$

we have the following result.

THEOREM 5.2. *There exists a positive constant C independent of $h_0,\cdots,h_N$ and N such that*

$$\rho = \|(I - P_N^A)\cdots(I - P_0^A)\|_A \le 1 - \frac{1}{C}.$$

Proof. Using the fact that $P_i^A \mathbf{Curl}\ \psi_h = \mathbf{Curl} P_i^{\tilde{A}} \psi_h$, it follows that,

$$\rho = \|(I - P_N^A) \cdots (I - P_0^A)\|_A = \|(I - P_N^{\tilde{A}}) \cdots (I - P_0^{\tilde{A}})\|_{\tilde{A}}. \tag{20}$$

The latter, however, corresponds to the standard error propagation map for the FAC algorithm on $S^{h_0,\cdots,h_N}$ and there is a positive constant C independent of h_i and N such that

$$\|(I - P_N^{\tilde{A}}) \cdots (I - P_0^{\tilde{A}})\|_{\tilde{A}} \le 1 - \frac{1}{C},$$

see Dryja and Widlund [20]. □

Next, we describe the convergence bounds for the two additive refinement algorithms considered in [15]. The first algorithm, to determine $\tilde{u}_{0,\cdots,N}$, is based on the subspaces

$$X_i = V^{h_i}(\Omega_i) \cap H_0(div^0, \Omega_i), \quad i = 0, \cdots, N,$$

and are the same as for the multiplicative algorithm. The corresponding preconditioned problem is

$$P^{(1)} \tilde{u}_{0,\cdots,N} \equiv \left(P_0^A + \cdots + P_N^A\right) \tilde{u}_{0,\cdots,N} = g^{(1)}.$$

THEOREM 5.3. *There exists a positive constant C independent of $h_0, \cdots, h_N$, and N such that the condition number of the first additive refinement algorithm satisfies*

$$\kappa\left(P^{(1)}\right) \le CN.$$

Proof. Since each $P_i^A \mathbf{Curl}\ \psi_h = \mathbf{Curl}\ P_i^{\tilde{A}} \psi_h$, for $\psi_h \in S^{h_0,\cdots,h_N}$, we can apply the same proof as in Lemma 4.3 and obtain

$$\kappa\left(P^{(1)}\right) = \kappa\left(P_0^A + \cdots + P_N^A\right) = \kappa\left(P_0^{\tilde{A}} + \cdots + P_N^{\tilde{A}}\right).$$

The latter represents the additive refinement algorithm in $S^{h_0,\cdots,h_N}$ based on the finite element spaces $S^{h_0}, \cdots, S^{h_N}$, and is known to have a condition number

$$\kappa\left(P_0^{\tilde{A}} + \cdots + P_N^{\tilde{A}}\right) \le CN,$$

for a constant C independent of h_i and N, provided the ratio of the areas $|\Omega_i|/|\Omega_{i-1}|$ is uniformly bounded, see [20]. □

The second additive refinement algorithm in the mixed finite element case, described in [15], is based on an additive refinement algorithm for standard formulations of elliptic problems, in which case the algorithm is sometimes referred to as the asynchronous fast adaptive composite grid method (AFAC). We will also refer to the second additive iterative refinement algorithm in the mixed finite element case as the AFAC algorithm. It is based on projections P_i^A onto

$$V^{h_i}(\Omega_i) \cap H_0(div^0, \Omega_i), \quad \text{for } i = 0, \cdots, N,$$

and $P^A_{i,i-1}$ onto

$$V^{h_{i-1}}(\Omega_i) \cap H_0(div^0, \Omega_i), \quad \text{for } i = 1, \cdots, N.$$

The preconditioned system for the divergence free velocity $\tilde{u}_{0,\cdots,N}$ is given by

$$P^{(2)} \tilde{u}_{0,\cdots,N} = g^{(2)},$$

where

$$P^{(2)} \equiv P^A_0 + \left(P^A_1 - P^A_{1,0}\right) + \cdots + \left(P^A_N - P^A_{N,N-1}\right).$$

For $i = 1, \cdots, N$, note that $P^A_i - P^A_{i,i-1}$ is the projection onto the space

$$\left[V^{h_i}(\Omega_i) \cap H_0(div^0, \Omega_i)\right] \cap \left[V^{h_{i-1}}(\Omega_i) \cap H_0(div^0, \Omega_i)\right]^{\perp},$$

where $[V^{h_{i-1}}(\Omega_i) \cap H_0(div^0, \Omega_i)]^{\perp}$ is the orthogonal complement of $[V^{h_{i-1}}(\Omega_i) \cap H_0(div^0, \Omega_i)]$ in the $A(.,.)$ inner product.

The second additive refinement algorithm, thus corresponds to an additive Schwarz algorithm on X with subspaces

$$X_i \equiv \left[V^{h_i}(\Omega_i) \cap H_0(div^0, \Omega_i)\right] \cap \left[V^{h_{i-1}}(\Omega_i) \cap H_0(div^0, \Omega_i)\right]^{\perp}, \quad i = 1, \cdots, N,$$

and

$$X_0 \equiv V^{h_0}(\Omega_0) \cap H_0(div^0, \Omega_0).$$

The projections $P^A_i - P^A_{i,i-1}$ for $i = 1, \cdots, N$, are computed in [15] by solving the systems:

$$\begin{bmatrix} (P^A_i - P^A_{i,i-1})v_h \\ q_i \end{bmatrix} \equiv R^T_i L^{-1}_i R_i L_{0,\ldots,N} \begin{bmatrix} v_h \\ 0 \end{bmatrix} - R^T_{i,i-1} L^{-1}_{i,i-1} R_{i,i-1} L_{0,\ldots,N} \begin{bmatrix} v_h \\ 0 \end{bmatrix},$$

where $R_{i,i-1}$ denotes the restriction map onto the space $V^{h_{i-1}}(\Omega_i) \times Q^{h_{i-1}}(\Omega_i)$ with an associated coefficient matrix $L_{i,i-1}$, R_i is the restriction map onto $V^{h_i}(\Omega_i) \times Q^{h_i}(\Omega_i)$ with an associated coefficient matrix L_i. $L_{0,\cdots,N}$ represents the coefficient matrix (18) on the composite grid. $P^A_0 v_h$ is computed as $P^A_0 v_h$ in the multiplicative iterative refinement algorithm.

Let $P^{\tilde{A}}_i$ denote the projection onto $S^{h_i}(\Omega_i)$ in the inner product $\tilde{A}(.,.)$, corresponding to the projection P^A_i onto $V^{h_i}(\Omega_i) \cap H_0(div^0, \Omega_i)$. Similarly, let $P^{\tilde{A}}_{i,i-1}$ denote the projection onto $S^{h_{i-1}}(\Omega_i) \cap H^1_0(\Omega_i)$ corresponding to the projection $P^A_{i,i-1}$ onto $V^{h_{i-1}}(\Omega_i) \cap H_0(div^0, \Omega_i)$. We have the following optimal bound.

THEOREM 5.4. *There exists a constant C, independent of h_i and N, such that the condition number of the second additive refinement algorithm (AFAC) satisfies*

$$\kappa\left(P^A_0 + (P^A_1 - P^A_{1,0}) + \cdots + (P^A_N - P^A_{N,N-1})\right) \leq C.$$

Proof. It is easily verified that $P^A_i \mathbf{Curl}\ \psi_h = \mathbf{Curl}\ P^{\tilde{A}}_i \psi_h$, and that $P^A_{i,i-1}\mathbf{Curl}\ \psi_h = \mathbf{Curl}\ P^{\tilde{A}}_{i,i-1}\psi_h$, for $\psi_h \in S^{h_0,\cdots,h_N}$. By similar arguments as in Lemma 4.3, it follows that

$$\kappa\left(P^A_0 + (P^A_{0,1} + \cdots + P^A_{N-1,N})\right) = \kappa\left(P^{\tilde{A}}_0 + (P^{\tilde{A}}_{0,1} + \cdots + P^{\tilde{A}}_{N-1,N})\right).$$

It is shown in [3] that the latter then has a uniformly bounded condition number which is independent of h_i and N. □

5.3. Quantitative bounds for the convergence factors. Following Mandel and McCormick [14] and Maitre and Musy [13] for standard formulations of elliptic problems, quantitative bounds can be obtained for the mixed finite element multiplicative (FAC) and additive (AFAC) iterative refinement algorithms, in the two level case. The quantitative estimates for the multiplicative iterative refinement algorithm (FAC) can be extended to the case of several levels. These results are based on strengthened Cauchy inequalities. First, we consider the two level multiplicative refinement algorithm.

Definition. Let X_0 and X_1 denote two subspaces of X. The cosine of the angle, in the $A(.,.)$ inner product, between X_0 and X_1 is defined by:

$$cos(X_0, X_1) \equiv \sup_{u_0 \in X_0, u_1 \in X_1, u_0, u_1 \neq 0} \frac{|A(u_0, u_1)|}{\|u_0\|_A \|u_1\|_A}.$$

Note that $cos(X_0, X_1) \in [0, 1]$, and if $X_0 \cap X_1$ is nontrivial, then $cos(X_0, X_1) = 1$. If X_0 and X_1 are orthogonal in the $A(.,.)$ inner product, then $cos(X_0, X_1) = 0$. Furthermore, if $u_0 \in X_0, u_1 \in X_1$, then

$$|A(u_0, u_1)| \leq cos(X_0, X_1)\|u_0\|_A \|u_1\|_A. \tag{21}$$

If $cos(X_0, X_1) < 1$, then (21) is referred to as a strengthened Cauchy inequality. The following result relates the *cosine* of the angle between spaces to the convergence factor of the two level FAC algorithm.

LEMMA 5.5. *Let X_0 and X_1 be nontrivial subspaces of X. Then*

$$\rho(P_0^A P_1^A) = \|P_0^A P_1^A P_0^A\|_A = cos^2(X_0, X_1).$$

where P_0^A, P_1^A denotes the orthogonal projections onto X_0 and X_1 respectively, and $\rho(P_0^A P_1^A)$ denotes the spectral radius. Furthermore, if $X_0^* \subset X_0$ and $X_1^* \subset X_1$ are subspaces satisfying

$$X_o^* + X_1^* = X,$$

then the spectral radius of the error propagation map $(I - P_0^A)(I - P_1^A)$ satisfies

$$\rho\left((I - P_0^A)(I - P_1^A)\right) = cos^2(X_0^\perp, X_1^\perp) \leq cos^2(X_0^*, X_1^*).$$

Proof. See Mandel and McCormick [14]. □

We estimate the spectral radius of the two level FAC error propagation map $(I - P_0^A)(I - P_1^A)$, by chosing $X_0^* = X_0$, and

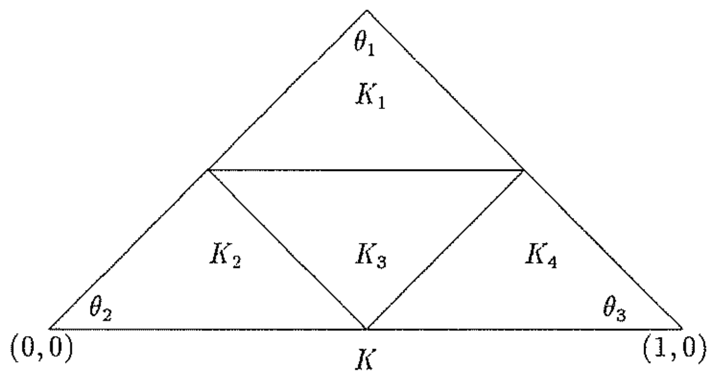
$$X_1^* = \{w = (I - \Pi^0)v : v \in X_1\} \subset X_1,$$

where Π^0 is the Raviart-Thomas interpolation map onto X_0, see [18]. $\Pi^0 v$ is the unique velocity in X_0 with a constant flux on each edge E_{ij} with the same mean value as the flux of v on E_{ij}, i.e.,

$$\int_{E_{ij}} (\Pi^0 v) \cdot \vec{n} ds = \int_{E_{ij}} v \cdot \vec{n} ds, \quad \forall i, j.$$

The interpolation operator preserves divergence free functions, i.e., $\Pi^0 v$ of a divergence free velocity is divergence free, see [16, 18]. To obtain an estimate for $cos(X_0^*, X_1^*)$,

FIG. 2. *Standard refinement of a triangle, used to compute the angle between coarse and fine mesh spaces.*



we work on each coarse mesh element separately, and sum over all the coarse mesh elements. Thus, suppose that for each element $K \in \tau^{h_0}(\Omega_1)$, there exists a non-negative constant $0 \leq \sigma_K \leq 1$, such that

$$|A_K(u,v)| \leq \sigma_K \sqrt{A_K(u,u)}\sqrt{A_K(v,v)}, \quad \forall u \in X_0, \text{ and } \forall v \in (I - \Pi^0)X_1.$$

Then

$$A(u,v) = \sum_K A_K(u,v) \leq \sum_K \sigma_K \sqrt{A_K(u,u)}\sqrt{A_K(v,v)}$$

$$\leq (\max_K \sigma_K)\sqrt{A(u,u)}\sqrt{A(v,v)}.$$

Thus,

$$cos(X_0, (I - \Pi^0)X_1) \leq (\max_K \sigma_K), \tag{22}$$

where σ_K is $cos(X_0|_K, (I - \Pi^0)X_1|_K)$.

Let K denote a triangle in the coarse mesh $\tau^{h_0}(\Omega_1)$, with vertices a_1, a_2, a_3. Let the angles at the vertices be denoted by $\theta_1, \theta_2, \theta_3$, respectively, and consider the refinement obtained by dividing each coarse triangle K into four equivalent subtriangles $K_1, \ldots, K_4$, by connecting the midpoint of the edges of K, see Figure 2. Since each function in $V^h(K)$ is specified by the value of the normal component on the mid-point of edges, and since there are nine edges amongst the four subtriangles, there are nine degrees of freedom. However, due to the four divergence constraints this reduces to five linearly independent divergence free functions defined on X_1 restricted to K. Two of them belong to X_0. A basis for $X_0|_K$ is given by:

$$u_1 = (1,0) \text{ and } u_2 = (0,1).$$

The remaining three basis functions u_3, u_4 and u_5 are in $(I - \Pi^0)X_1|_K$ and are given by:

$$u_3 = \begin{cases} (\frac{1-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)}{2}) & \text{on } K_1 \\ (\frac{-1-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)}{2}) & \text{on } K_2 \\ (\frac{-1-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)}{2}) & \text{on } K_3 \\ (\frac{-1+\cos(\theta_2)}{2}, \frac{\sin(\theta_2)}{2}) & \text{on } K_4 \end{cases}$$

$$u_4 = \begin{cases} (\frac{\cos(\theta_3)-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)-\sin(\theta_3)}{2}) & \text{on } K_1 \\ (\frac{-\cos(\theta_3)-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)+\sin(\theta_3)}{2}) & \text{on } K_2 \\ (\frac{\cos(\theta_3)-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)-\sin(\theta_3)}{2}) & \text{on } K_3 \\ (\frac{\cos(\theta_3)+\cos(\theta_2)}{2}, \frac{\sin(\theta_2)-\sin(\theta_3)}{2}) & \text{on } K_4 \end{cases}$$

$$u_5 = \begin{cases} (\frac{\cos(\theta_3)+1}{2}, \frac{-\sin(\theta_3)}{2}) & \text{on } K_1 \\ (\frac{-\cos(\theta_3)-1}{2}, \frac{\sin(\theta_3)}{2}) & \text{on } K_2 \\ (\frac{\cos(\theta_3)-1}{2}, \frac{-\sin(\theta_3)}{2}) & \text{on } K_3 \\ (\frac{\cos(\theta_3)-1}{2}, \frac{-\sin(\theta_3)}{2}) & \text{on } K_4 \end{cases}$$

Computing the local stiffness matrix on K in the $A(.,.)$ inner product using the basis $u_1, \ldots, u_5$, we obtain a block partitioned matrix of the form:

$$\begin{bmatrix} M_K & G_K \\ G_K^T & N_K \end{bmatrix},$$

where

$$\begin{cases} (M_K)_{ij} &= A_K(u_i, u_j); \quad i,j = 1,2 \\ (G_K)_{ij} &= A_K(u_i, u_j); \quad i = 1,2; \ j = 3,4,5 \\ (N_K)_{ij} &= A_K(u_i, u_j); \quad i,j = 3,4,5. \end{cases}$$

By Lemma 5.5,

$$cos^2(X_0|_K, (I - \Pi^0)X_1|_K) = \rho\left(P_{X_0|_K} P_{X_1|_K}\right)$$

$$= \rho\left(\left(I - \begin{bmatrix} M_K^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} M_K & G_K \\ G_K^T & N_K \end{bmatrix}\right)\left(I - \begin{bmatrix} 0 & 0 \\ 0 & N_K^{-1} \end{bmatrix}\begin{bmatrix} M_K & G_K \\ G_K^T & N_K \end{bmatrix}\right)\right).$$

This spectral radius is independent of the order of multiplication of the two matrices since it equals $cos^2(X_0|_K, (I - \Pi^0)X_1|_K)$ for either ordering. It is easily seen that this spectral radius is the same as $\rho(M_K^{-1}G_K N_K^{-1}G_K^T)$ or $\rho(N_K^{-1}G_K^T M_K^{-1}G_K)$. For the Laplacian,

$$a(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A_K(u,v) = \int_K u^T v\,dx\,dy,$$

and we obtain,

$$M_K = \frac{|K|}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}; \quad G_K = \frac{|K|}{4}\begin{bmatrix} -1-\cos(\theta_2) & \cos(\theta_3)-\cos(\theta_2) & \cos(\theta_3)-1 \\ -\sin(\theta_2) & -\sin(\theta_2)-\sin(\theta_3) & -\sin(\theta_3) \end{bmatrix},$$

and

$$N_K = \frac{|K|}{4} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

where $|K|$ denotes the area of triangle K. By substitution, we obtain

$$M_K^{-1} G_K N_K^{-1} G_K^T = \frac{1}{4} \begin{bmatrix} 1 + \cos^2(\theta_2) + \cos^2(\theta_3) & \sin(\theta_2)\cos(\theta_2) - \sin(\theta_3)\cos(\theta_3) \\ \sin(\theta_2)\cos(\theta_2) - \sin(\theta_3)\cos(\theta_3) & \sin^2(\theta_2) + \sin^2(\theta_3) \end{bmatrix},$$

and its spectral radius ρ_K satisfies

$$\rho_K = \rho(M_K^{-1} G_K N_K^{-1} G_K^T) = \frac{\frac{3}{4} + \sqrt{\frac{9}{16} - \frac{4}{16} d(\theta_1, \theta_2)}}{2},$$

where

$$d(\theta_1, \theta_2) \equiv 2\sin^2(\theta_2)\cos^2(\theta_3) + 2\sin^2(\theta_3) + \frac{1}{2}\sin(2\theta_2)\sin(2\theta_3).$$

For an equilateral triangle K this gives $\rho_K \leq \frac{3}{8}$, and for a right triangle $\rho_K \leq \frac{1}{2}$. The constant σ_K in the local strengthened Cauchy inequality satisfies $\sigma_K^2 \leq \rho_K$. By applying Lemma 5.5 and equation (22), we obtain for the two level multiplicative iterative refinement algorithm

$$\rho \leq \max_K \sigma_K^2 \leq \begin{cases} 3/8 & \text{for all equilateral triangles} \\ 1/2 & \text{for all right triangles} \end{cases}$$

Note that if $a(x, y)$ is constant in each coarse triangle K, σ_K would still be the same. Hence, the convergence factor is independent of coefficient variations from element to element.

Remark 1. The preconditioned system corresponding to the symmetrized two-level FAC algorithm is $I - E_2^* E_2$, where $E_2 = (I - P_1^A)(I - P_0^A)$ denotes the error propagation map for the non-symmetrized two-level FAC algorithm. Since $E_2^* E_2$ is positive semi-definite, we obtain

$$\frac{I}{\kappa_2} \leq I - E_2^* E_2 \leq I,$$

where κ_2 is the condition number of the preconditioned system. If we let $\rho_2 = ||E_2^* E_2||_A^{1/2}$ denote the convergence factor of the two-level FAC algorithm, then

$$\kappa_2 \leq \frac{1}{1 - (\rho_2)^2}.$$

Remark 2. By a result in [15], it follows that the condition number κ_N of the symmetrized many-level FAC algorithm satisfies:

$$\kappa_N(I - E_N^* E_N) \leq \left[\frac{1}{1 - (\rho_2)^2}\right]^N,$$

where $\rho_2 = \|E_2^* E_2\|_A^{1/2}$ is the convergence factor of the two-level FAC algorithm. Thus, for a mesh containing only equilateral triangles, $\kappa_N \leq (\frac{64}{55})^N$, since $\rho \leq 3/8$, and for a mesh containing only right triangles, $\kappa_N \leq (\frac{4}{3})^N$, since $\rho_2 \leq 1/2$.

Remark 3. A quantitative bound for the spectral radius of the two-level AFAC algorithm can be obtained in terms of the spectral radius of the two-level FAC algorithm:

$$\rho\left(I - P^{(2)}\right) = \rho_2^{1/2} = \|(I - P_0^A)(I - P_1^A)\|_A^{1/2},$$

see Mandel and McCormick [14]. Here ρ_2 is the convergence factor of the two level FAC method. Thus, for a mesh containing only equilateral triangles, $\rho(I - P^{(2)}) \leq (\frac{3}{8})^{1/2}$, and for a mesh containing only right triangles, $\rho(I - P^{(2)}) \leq (\frac{1}{2})^{1/2}$.

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