Discrete Multi-Resolution Analysis and Generalized Wavelets

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Abstract.

In this paper we consider a situation where we are given a finite number of values which represent sampling of weighted-averages of a function $f(x)$ corresponding to a uniform grid. We show that if the weight function $\varphi(x)$ satisfies a dilation equation, there is a discrete multi-resolution analysis of these values corresponding to a diadic coarsening of the grid. We introduce a reconstruction procedure R which predicts $f(x)$ from its discrete weighted-averages to any desired order of accuracy and is conservative in the sense that weighted-averaging of R reproduces the given input data. Our formulation allows for adaptive data-dependent reconstruction techniques in which R is a nonlinear functional of the input data.

At each level of resolution k we use the reconstruction R to predict $f(x)$ and its weighted-averages at the $(k-1)$ level, which is the next finer level of resolution. We define $Q_k(x; f)$, the k-th scale component of $f(x)$ to be the difference between the reconstruction of $f(x)$ at level $(k-1)$ to that of level k and $\{d_j^k\}$, the k-th scale coefficients of $f(x)$ to be the weighted-averages of Q_k on the finer grid. We show that the given input data can be reconstructed from knowledge of the scale coefficients $\{d_j^k\}$ for all k and the weighted averages of $f(x)$ at the coarsest grid. This observation leads to an efficient data-compression technique.

On the functional side, $f(x)$ can be reconstructed to the accuracy of the finest grid from knowledge of the scale components $Q_k(x; f)$ for all k and the reconstruction of $f(x)$ from the coarsest grid. When R is data-independent we show that each scale-component Q_k can be represented in a basis of linearly independent generalized wavelets. This leads to representation of $f(x)$ in a multi-resolution basis which is the union of these generalized wavelets for all levels of resolution.

In this framework the original wavelets are obtained from a particular choice of reconstruction technique, namely taking R to be the projection of f into the linear span of all dilates and translates of $\varphi(x)$. This is a restrictive coupling between the approximation technique R and the sense of averaging φ, which is unnecessary from the point of view of numerical analysis.

Introduction and Overview.

In this paper we consider a situation where we are given a finite number of values

$$\{\bar{f}_j^0\}_{j=1}^{N_0}, \quad N_0 = 2^{n_0},$$

which represent a sampling of weighted-averages of a periodic function $f(x)$ (unless otherwise specified) corresponding to a uniform partition of $[0,1]$, i.e.

$$x_j^0 = j \cdot h_0, \quad 0 \leq j \leq N_0, \quad h_0 = 1/N_0,$$
$$\bar{f}_j^0 = \langle f, \frac{1}{h_0}\varphi\left(\frac{x - x_j^0}{h_0}\right)\rangle.$$

and

$$\int \varphi(x)dx = 1$$

Here $\langle\ ,\ \rangle$ denotes the Euclidean inner product and $\varphi(x)$ is the weight function. Introducing the nested grids

$$\{\{x_j^k\}_{j=1}^{N_k}\}_{k=1}^{L}, \quad x_j^k = j \cdot h_k, \quad h_k = 2^k h_0, \quad N_k = \frac{1}{h_k}, \quad L < n_0,$$

and the corresponding scaled quantities

$$\bar{f}_j^k = \langle f, \varphi_j^k \rangle, \quad \varphi_j^k(x) = \frac{1}{h_k}\varphi\left(\frac{x - x_j^k}{h_k}\right), \quad 1 \leq j \leq N_k, \quad 0 \leq k \leq L,$$

we say that

$$\{\{\bar{f}_j^k\}_{j=0}^{N_k}\}_{k=0}^{L}$$

constitute a discrete multi-resolution analysis if knowledge of the discrete values at level k determines the corresponding values at level $k+1$. Assuming linearity of this relation we show in section 1 that $\varphi(x)$ has to satisfy a dilation equation

$$\varphi(x) = 2\sum_{\ell} \alpha_\ell \varphi(2x - \ell).$$

We refer the reader to [1] and [8] for review of the history of this field and its terminology.

This definition of discrete multi-resolution analysis implies that the values at level k includes all larger scales. How can we get information about the scale present in $f(x)$ at each locality? In section 2 we introduce a reconstruction procedure $R(x; \bar{f}^k)$ which predicts $f(x)$ from knowedge of $\{\bar{f}_j^k\}_{j=1}^{N_k}$ to any desired order of accuracy, and is conservative in the sense that weighted averaging of R reproduces the given input data, i.e.

$$\langle R(\cdot; \bar{f}^k),\ \varphi_j^k \rangle = \bar{f}_j^k, \quad 1 \leq j \leq N_k.$$

At each level of resolution k we use the reconstruction R to predict $f(x)$ and its weighted averages at the next finer level of resolution $(k-1)$ and define $Q_k(x; f)$, the k-th scale component of $f(x)$, to be

$$Q_k(x; f) = R(x; \bar{f}^{k-1}) - R(x; \bar{f}^k)$$

and $\{d_j^k\}$, the k-th scale coefficients of $f(x)$, to be the weighted-averages of Q_k on the finer grid. Thus

$$R(x; \bar{f}^0) = R(x; \bar{f}^L) + \sum_{k=1}^{L} Q_k(x; f)$$
$$d_j^k = \langle Q_k(\cdot; f),\ \varphi_j^{k-1}\rangle = \bar{f}_j^{k-1} - \langle R(\cdot; \bar{f}^k),\ \varphi_j^{k-1}\rangle.$$

d_j^k so defined measures our ability to predict $\bar{f}_j^{k-1}$ from our knowledge of $\bar{f}^k$. When we fail, i.e. d_j^k is large, this could be either because of inadequacy of the approximation scheme or because there is a new scale of $f(x)$ at level $(k-1)$ which is not predictable by any approximation method. In order to reduce the approximation error component in d_j^k we have to allow the use of adaptive (data-dependent) approximation schemes which are nonlinear functionals of the input data. We show in section 2 that the input data $\bar{f}^0$ can be reconstructed exactly from knowledge of

$$\{\bar{f}^L, (d^L, \ldots, d^1)\}.$$

Once we remove the redundancy which is inherent to this representation, we get efficient data-compression algorithms in which adaptive approximations can be used.

In section 3 we examine the compactly supported orthonormal wavelet bases of Daubechies [1] and the associated data-compression algorithm of Mallat [7]. We show that in the context of this paper, wavelets correspond to a particular method of reconstruction R, namely taking R to be the orthogonal projection into the linear span of $\{\varphi_j^k\}$, i.e.

$$R(x; \bar{f}^k) = \sum_{j=1}^{N_k} \bar{f}_j^k \varphi_j^k(x).$$

In this paper we assume that the choice of $\varphi(x)$ is dictated by the nature of the computational problem and therefore it is considered to be given. From this point of view the choice of "reconstruction via projection" which is associated with wavelets is not necessarily the best method of approximation. In sections 4 and 5 we consider multi-resolution analysis of pointvalues and cell-averages corresponding to $\varphi(x)$ being Dirac's-δ and the box function, respectively. In section 6 we return to the general case and show that if the reconstruction procedure is data-independent and projective in the sense that

$$R(x; \hat{f}^{k-1}) \equiv R(x; \bar{f}^k),$$

where

$$\hat{f}_j^{k-1} = \langle R(\cdot; \bar{f}^k),\ \varphi_j^{k-1} \rangle, \quad 1 \leq j \leq N_{k-1},$$

then $R(x, \bar{f}^0)$ can be represented in a multi-resolution basis of "generalized wavelets" $\{\{\bar{\psi}_j^k\}_{j=1}^{N_k}\}_{k=1}^{L}$

$$R(x; \bar{f}^0) = R(x; \bar{f}^L) + \sum_{k=1}^{L} \sum_{j=1}^{N_k} \hat{d}_j^k(f) \bar{\psi}_j^k(x),$$

where $\hat{d}_j^k(f)$ are the k-th scale coefficients of $f(x)$ corresponding to grid points with odd indeces. This grouping of terms in the representation of the approximation scheme enables us to intelligently reduce its dimesnionality by dropping terms with negligible coefficients $\hat{d}_j^k(f)$.

Finally in section 7 we present a modified data-dependent encoding procedure which keeps track of this truncation procedure and generates modified coefficients $\tilde{d}_j^k(f)$. Using these modified coefficients in the expansion above or in the decoding procedure, yields a finest-grid approximation (level 0) which is accurate to an arbitrarily specified tolerance.

1. Multi-Resolution Analysis

In this section we review the concept of multi-resolution analysis due to Meyer and Mallat, except that here we consider the discrete case in a finite domain. Therefore we associate the various levels of resolution to grids rather than to function spaces as was done in the original development.

We consider the interval $0 \leq x \leq 1$ and its partition into $N = 2^n$ intervals of size $h = 1/N = 2^{-m}$ by $x_j = j \cdot h, \quad j = 0, \ldots, N$. To simplify our presentation let us consider a periodic function $f(x)$ with period of 1, $f \in L^2[0,1]$, and assume that f is discretized on this grid by

$$\bar{f}_j = \langle f, \frac{1}{h}\varphi(\frac{x}{h} - j) \rangle, \quad j = 1, \ldots, N \tag{1.1}$$

where $\langle \, , \, \rangle$ is the Euclidean inner product and $\varphi(x)$ is a function of compact support satisfying

(1.2) $$\int \varphi(x)dx = 1.$$

Thus $\{\bar{f}_j\}$ are "averages" of $f(x)$ over support of size h around x_j with the "weight function" $\frac{1}{h}\varphi\left(\frac{x-x_j}{h}\right)$.

We construct a set of nested $(L+1)$ grids $0 \le k \le L$ of size $h_k = 2^k h$ with $N_k = \frac{1}{h_k}$ intervals by

(1.3a) $$x_j^k = j \cdot h_k, \qquad j = 0, \ldots, N_k.$$

Thus $k = 0$ is the original grid, which is the finest in the hierarchy, and

(1.3b) $$x_j^k = x_{2j}^{k-1}.$$

Thus the $(k-1)$-th grid is formed from the k-th grid by dividing each of its intervals into two; this is done by adding a partition point x_{2j-1}^{k-1} in the middle of the interval $[x_{j-1}^k, x_j^k]$.

With each of the grids we associate a discretization $\{\bar{f}_j^k\}_{j=1}^{N_k}$ of the function $f(x)$

(1.4a) $$\bar{f}_j^k = \langle f, \frac{1}{h_k}\varphi(\frac{x}{h_k} - j)\rangle \equiv \langle f, \varphi_j^k\rangle, \quad j = 1, \ldots, N_k.$$

Each k represents a different level of resolution of the function $f(x)$, which is determined by two factors: (1) f is averages over support of size h_k. (2) $\bar{f}^k$ is sampled with a spacing of h_k, i.e.

(1.4b) $$\bar{f}_j^k = \bar{f}^k(x_j^k)$$

where

(1.4c) $$\bar{f}^k(y) = \langle f, \frac{1}{h_k}\varphi(\frac{\cdot - y}{h_k})\rangle,$$

is the "sliding average" of $f(y)$ with size h_k. It seems to us that it is the frequency of the sampling which is the dominant factor in determining the level of resolution of $f(x)$.

The set of values $\left\{\{\bar{f}_j^k\}_{j=1}^{N_k}\right\}_{k=0}^{L}$ is called multi-resolution analysis of $f(x)$, if for each k the knowledge of $\{\bar{f}_j^k\}_{j=1}^{N_k}$ determines the values of the next level $\{\bar{f}_j^{k+1}\}_{j=1}^{N_{k+1}}$. This means that the k-th level of resolution contains the information of all larger scales of variation in the levels $\ell = k+1, \ldots, L$.

Let us assume that the relation between $\bar{f}^k$ and $\bar{f}^{k+1}$ is linear, i.e.

$$\bar{f}_j^{k+1} = \sum_\ell \alpha_\ell \bar{f}_{2j+\ell}^k. \tag{1.5}$$

It follows from (1.5) and the definition (1.4a) that

$$\langle f, \varphi_j^{k+1} - \sum_\ell \alpha_\ell \varphi_{2j+\ell}^k \rangle = 0$$

for all $f \in L_2[0,1]$, and therefore

$$\frac{1}{h_{k+1}} \varphi\left(\frac{x}{h_{k+1}} - j\right) = \frac{1}{h_k} \sum_\ell \alpha_\ell \varphi\left(\frac{x}{h_k} - 2j - \ell\right);$$

taking $y = \frac{x}{h_{k+1}} - j$ in the above identity we get for all y

$$\varphi(y) = 2\sum_\ell \alpha_\ell \varphi(2y - \ell). \tag{1.6}$$

Hence for relation (1.5) to hold, $\varphi(y)$ has to satisfy the dilation equation (1.6). At this point we refer the reader to the excellent review paper [8] by G. Strang. Let us assume that $\varphi(y)$ has a Fourier transform $\hat{\varphi}(\xi)$. We note that (1.2) implies

$$\hat{\varphi}(0) = 1 \tag{1.7a}$$

and that the dilation equation (1.6) implies

$$\hat{\varphi}(\xi) = M\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right) \tag{1.7b}$$

where

$$M(\xi) = \sum_\ell \alpha_\ell e^{i\ell\xi}. \tag{1.7c}$$

(Note that $M(0) = \sum \alpha_\ell = 1$).

It follows therefore that formally

$$\hat{\varphi}(\xi) = \prod_{m=1}^{\infty} M(\xi/2^m), \tag{1.7d}$$

and thus $\varphi(x)$ is determined uniquely by the dilation equation and the requirement (1.2). However, as pointed out by Daubechies [1], the "function" $\varphi(x)$ defined by (1.7) tends to have a fractal nature and in order to ensure some smoothness we have to impose additional conditions on $M(\xi)$ (1.7c).

Many of the functions $\varphi(x)$ that are used in numerical analysis automatically satisfy a dilation equation. For example $\varphi = \delta(x)$, where δ is the Dirac distribution, satisfies

$$\varphi(x) = 2\varphi(2x) \Rightarrow \alpha_0 = 1; \tag{1.8a}$$

the box function $\varphi(x) = \begin{cases} 1 & -1 \le x < 0 \\ 0 & \text{otherwise} \end{cases}$

satisfies

$$\varphi(x) = \varphi(2x) + \varphi(2x-1) \Rightarrow \alpha_0 = \alpha_{-1} = \frac{1}{2}; \tag{1.8b}$$

the hat function $\varphi(x) = \begin{cases} 1+x & -1 \le x \le 0 \\ 1-x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

satisfies

$$\varphi(x) = \frac{1}{2}[\varphi(2x-1) + 2\varphi(2x) + \varphi(2x+1)] \Rightarrow \alpha_1 = \alpha_{-1} = \frac{1}{4}, \alpha_0 = \frac{1}{2}; \tag{1.8c}$$

the quadratic spline function

$$\varphi(x) = \begin{cases} (x+2)^2 & -2 \le x \le -1 \\ -2x^2 - 2x + 1 & -1 \le x \le 0 \\ (x-1)^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

satisfies

$$
\text{(1.8d)} \qquad \begin{aligned} \varphi(x) &= \frac{1}{4}[\varphi(2x-1)+3\varphi(2x)+3\varphi(2x+1)+\varphi(2x+2)] \\ &\Rightarrow \alpha_{-2}=\alpha_1=\frac{1}{8}, \quad \alpha_{-1}=\alpha_0=\frac{3}{8}. \end{aligned}
$$

All the functions $\varphi(x)$ in (1.8) form a hierarchy of function $\varphi^m(x)$ which is obtained by repeated convolutions with a characteristic function

$$
\text{(1.9a)} \qquad \varphi^{m+1}=\varphi^m * \chi_{[-1+s_m,s_m]}, \quad s_m=\frac{1}{2}[1-(-1)^m],
$$

with

$$
\text{(1.9b)} \qquad \varphi^0=\delta(x).
$$

Let α_ℓ^m denote the coefficients of the dilation equation (1.6) which is satisfied by φ^m. It is easy to see that

$$
\text{(1.9c)} \qquad \alpha_\ell^{m+1}=\frac{1}{2}(\alpha_\ell^m+\alpha_{\ell+(-1)^m}^m).
$$

The shift between $\chi_{[-1,0]}$ and $\chi_{[0,1]}$ keeps the coefficients α_ℓ^m as centered as possible around $\ell=0$, which is convenient for formulating boundary conditions.

2. Data Compression and Scale Analysis

In this section we use the multi-resolution analysis of $f(x)$ in order to decompose it into scales and to design data compression algorithms. To accomplish that we use a reconstruction procedure $R(x;\bar{f}^k)$ which approximates $f(x)$ from the discrete values $\{\bar{f}_j^k\}_{j=1}^{N_k}$, i.e.

$$
\text{(2.1a)} \qquad R(x;\bar{f}^k)=f(x)+O((h_k)^r) \text{ wherever } f(x) \text{ is smooth,}
$$

and is conservative in the sense that for all k

$$
\text{(2.1b)} \qquad \langle R(\cdot;\bar{f}^k),\ \varphi_j^k\rangle=\bar{f}_j^k.
$$

We note that by (1.5) - (1.6)

$$
\begin{aligned} \langle R(\cdot;\bar{f}^k),\ \varphi_j^{k+1}\rangle &= \langle R(\cdot;\bar{f}^k),\sum_k \alpha_\ell\varphi_{2j+\ell}^k\rangle \\ &= \sum_\ell \alpha_\ell\langle R(\cdot;\bar{f}^k),\ \varphi_{2j+\ell}^k\rangle=\sum_\ell \alpha_\ell\bar{f}_{2j+\ell}^k=\bar{f}_j^{k+1} \end{aligned}
$$

and therefore by induction

$$\langle R(\cdot;\bar{f}^k),\ \varphi_j^m\rangle = \bar{f}_j^m \text{ for } m \geq k. \tag{2.2}$$

We decompose $f(x)$ into scales by

$$R(x;\bar{f}^0) = R(x;\bar{f}^L) + \sum_{k=1}^{L} Q_k(x;f) \tag{2.3a}$$

where the k-th scale component of $f(x)$ is

$$Q_k(x;f) = R(x;\bar{f}^{k-1}) - R(x;\bar{f}^k). \tag{2.3b}$$

We observe from (2.2) that

$$\langle Q_k(\cdot;f),\ \varphi_j^m\rangle = 0 \text{ for } m \geq k, \tag{2.4a}$$

and for $m = k-1$ we get the k-th scale coefficients

$$d_j^{k-1} \equiv \langle Q_k(\cdot;f),\ \varphi_j^{k-1}\rangle = \bar{f}_j^{k-1} - \langle R(\cdot;\bar{f}^k),\ \varphi_j^{k-1}\rangle. \tag{2.4b}$$

Observe that d_j^{k-1} measures our success in using the reconstruction procedure R to predict $\bar{f}_j^{k-1}$ from our knowledge of $\bar{f}^k$. The scale coefficients d_j^{k-1} are defined for $j = 1, \ldots, N_{k-1}$ but only half of these values are independent. This can be seen form the fact that for $j = 1, \ldots, N_k$

$$\begin{aligned} \sum_\ell \alpha_\ell d_{2j+\ell}^{k-1} &= \sum_\ell \alpha_\ell \langle Q_k(\cdot;f), \varphi_{2j+\ell}^{k-1}\rangle \\ &= \langle Q_k(\cdot;f),\ \sum_\ell \alpha_\ell \varphi_{2j+\ell}^{k-1}\rangle = \langle Q_k(\cdot;f), \varphi_j^k\rangle = 0 \end{aligned} \tag{2.4c}$$

The relevant $\varphi(x)$ for multi-resolution analysis is a function of compact support for which $\frac{1}{h}\varphi\left(\frac{x}{h}\right)$ converges weakly to $\delta(x)$, Dirac's distribution. Consequently α_0 is expected to be significantly larger then $\alpha_{2\ell}$, $\ell \neq 0$. We assume now that the coefficients of (1.6) actually satisfy

$$\alpha_0 > \sum_{\ell \neq 0} |\alpha_\ell. \tag{2.5a}$$

This is certainly true for the family $\varphi^m(x)$ in (1.9) and for the compactly supported "orthonormal" $\varphi^r(x)$ of Daubechies (to be described in the next section). In this case it is possible to store the values of d_j^{k-1} with odd indices

$$\hat{d}_j^k = d_{2j-1}^{k-1}, \quad 1 \leq j \leq N_k \tag{2.5b}$$

and use relations (2.4c) in order to formulate a system of equations

$$\sum \alpha_{2\ell}\, d_{2j+2\ell}^{k-1} = -\sum_{\ell} \alpha_{2\ell-1} \hat{d}_{j+\ell}^k, \quad 1 \leq j \leq N_k \tag{2.5c}$$

for the unknowns $(d_2^{k-1}, d_4^{k-1}, \ldots, d_{N_k}^{k-1})$. Condition (2.5a) implies that the coefficient matrix of the system (2.5c) is diagonally dominant and hence invertible, for the periodic as well as for the free boundary case.

Let us denote the $2N_k \times N_k$ matrix that transfers $\{\hat{d}_j^k\}_{j=1}^{N_k}$ into $\{d_j^{k-1}\}_{j=1}^{N_{k-1}}$ by $\mathbf{D}$, i.e.

$$d^{k-1} = \mathbf{D} \cdot \hat{d}^k. \tag{2.5d}$$

For example, when φ is the box function (1.8b) we get from (2.5b)

$$d_{2j-1}^{k-1} + d_{2j}^{k-1} = 0 \tag{2.6a}$$

and thus $d^{k-1} = \mathbf{D} \cdot \hat{d}^k$ is expressed algorithmically by

$$\begin{cases} d_{2j-1}^{k-1} &= \hat{d}_j^k \\ d_{2j}^{k-1} &= -\hat{d}_j^k \end{cases} \qquad 1 \leq j \leq N_k \quad ; \tag{2.6b}$$

when φ is the hat function (1.8c) we get from (2.5b)

$$d_{2j-1}^{k-1} + 2d_{2j}^{k-1} + d_{2j+1}^{k-1} = 0, \quad 1 \leq j \leq N_k. \tag{2.7a}$$

Therefore we can compute $d^{k-1} = \mathbf{D} \cdot \hat{d}^k$ by

$$\begin{cases} d_{2j}^{k-1} = \hat{d}_j^k \\ d_{2j}^{k-1} = \frac{1}{2}(\hat{d}_j^k + \hat{d}_{j+1}^k) \end{cases} \qquad 1 \leq j \leq N_k \tag{2.7b}$$

Note that for $j = N_k$ we need to know $\hat{d}^k_{N_k+1}$. For periodic boundary conditions

(2.7c) $$\hat{d}^k_{N_k+1} = \hat{d}^k_1;$$

otherwise we compute $\hat{d}^k_{N_k+1}$ by extrapolation from the stored $\hat{d}^k$.

We turn now to discuss data compression of a sequence of numbers $\{c_j\}_{j=1}^{N_0}$. Let $f(x)$ be the smoothest function for which

$$\langle f, \varphi^0_j \rangle = c^0_j, \qquad 1 \leq j \leq N_0.$$

The data compression algorithm corresponds to the decomposition (2.3) for such a function $f(x)$, and its rate of compression depends strongly on the smoothness of $f(x)$. First we compute the multi-resolution analysis (1.4a) of $f(x)$, $\{\bar{f}^0, \bar{f}^1, \ldots, \bar{f}^2\}$ by (1.5), i.e.

(i) set

(2.8a) $$\bar{f}^0_j = c_j, \qquad 1 \leq j \leq N_0 = N$$

(ii) calculate

(2.8b) $$\begin{cases} DO \quad k = 1, L \\ DO \quad j = 1, N_k \\ \bar{f}^k_j = \sum_\ell \alpha_\ell \bar{f}^{k-1}_{2j+\ell} \end{cases}$$

Next we calculate the scale coefficients by (2.4b)

(2.8c) $$\begin{cases} DO \quad k = 1, L \\ DO \quad j = 1, N_k \\ \hat{d}^k_j = \bar{f}^{k-1}_{2j-1} - \langle R(\cdot; \bar{f}^k), \varphi^{k-1}_{2j-1} \rangle. \end{cases}$$

At the end of this stage we have obtained c^{MR}, the multi-resolution representation of c,

(2.9a) $$c^{MR} = \{\bar{f}^L, (\hat{d}^L, \ldots, \hat{d}^1)\}.$$

From these data we can recover the exact values of c by reversing the operation (2.8c), i.e.

$$\text{(2.9b)} \qquad \begin{cases} DO \;\; k = L, 1 \\ d^{k-1} = \mathbf{D}\hat{\cdot} d^k \\ DO \;\; j = 1, N_{k-1} \\ \bar{f}_j^{k-1} = \langle R(\cdot; \bar{f}^k),\ \varphi_j^{k-1} \rangle + d_j^{k-1} \end{cases} .$$

Note that the $k - DO$ loop is done in reverse: $k = L, L-1, \dots, 1$ and that $\mathbf{D}$ is the matrix (2.5d).

The multi-resolution representation c^{MR} (2.9a) has exactly the same number of elements N as the original sequence c, since

$$\text{(2.10)} \qquad N_L + (N_L + \cdots + N_1) = N[2^{-L} + (2^{-L} + \cdots + 2^{-1})] = N.$$

Data compression can be achieved due to the possible smallness of elements in $(\hat{d}^L, \dots, \hat{d}^1)$. We recall that $\hat{d}_j^k$ (2.5b), (2.4b) is the error committed at x_{2j-1}^{k-1} in attempting to predict $\bar{f}_{2j-1}^{k-1}$ from $\bar{f}^k$, the discretization of f on the k-th grid. Therefore if f is properly resolved on the k-th grid at a certain locality, the coefficients $\hat{d}^\ell$, $\ell = k-1, \dots, 0$ corresponding to this locality will be small in absolute value.

Remark 2.1. Note that we have not assumed linearity of the reconstruction $R(\cdot; \bar{f}^k)$, and therefore we can use adaptive (= data dependent = nonlinear) techniques. Furthermore, for each k we can use a different reconstruction method $R_k(x; \bar{f}^k)$. Defining in (2.3b)

$$Q_k(x; f) = R_{k-1}(x; \bar{f}^{k-1}) - R_k(x; \bar{f}^k)$$

it is easy to see that the fundamental property (2.4) still holds.

Remark 2.2. The compression algorithm of this section enables us to specify the compression factor, but does not allow for a direct control over the quality of the decompressed data, i.e. the cumulative error at the finest grid. In section 7 we shall present a modification of this algorithm which will allow us to specify the quality of the decompressed data, but at the cost of losing direct control over the rate of compression.

Remark 2.3. When d_j^k is unacceptably large, this can be either due to the inadequacy of the reconstruction method or due to the fact that there is a truly new scale in this locality which is not predictable by any approximation method. In order to reduce the component of approximation error in the compressed data let us consider an invertible representation

$$\hat{d}_j^k = \sum_{m=1}^{N_k} \gamma_m^k \mu_m^k(x_{2j-1}^{k-1}), \quad 1 \leq j \leq N_k \tag{2.11a}$$

which we denote by G, i.e.

$$\gamma^k = G\hat{d}^k, \quad \hat{d}^k = G^{-1}\gamma^k; \tag{2.11b}$$

As an example let us consider a signal c which is a combination of a discontinuous piecewise-polynomial function and a high frequency sine wave. Taking R to be ENO reconstruction with subcell-resolution [5] we'll do the piecewise-polynomial part perfectly and $\hat{d}^k$ in this case will be the error of the ENO reconstruction in approximating the high frequency sine wave. Taking the RHS of (2.11a) to be Fourier collocation will result in a representation by γ^k which is more economical than the original $\hat{d}^k$.

Finally we truncate and quantize γ^k by some procedure H and denote its result by $\tilde{\gamma}^k$, i.e.

$$\tilde{\gamma}^k = H\gamma^k. \tag{2.11c}$$

Thus the encoding part of the compression algorithm is performed by (2.8) and

$$\{\hat{d}^L, \ldots, \hat{d}^1\} \overset{G}{\to} \{\gamma^L, \ldots, \gamma^1\} \overset{H}{\to} \{\tilde{\gamma}^L, \ldots, \tilde{\gamma}^1\}. \tag{2.12a}$$

The compessed data to be stored or transmitted is c^C,

$$c^C = \{\bar{f}^L, (\tilde{\gamma}^L, \ldots, \tilde{\gamma}^1)\}. \tag{2.12b}$$

The decoding part of the compression algorithm is then:

Set

$$\tilde{f}^L = \bar{f}^L, \tag{2.13a}$$

Calculate

$$\begin{cases} DO\ k = L, 1 \\ \tilde{d}^{k-1} = \mathbf{D}(G^{-1}\tilde{\gamma}^k) \\ DO\ j = 1, N_{k-1} \\ \tilde{f}_j^{k-1} = \langle R(\cdot; \tilde{f}^k),\ \varphi_j^{k-1}\rangle + \tilde{d}_j^{k-1}. \end{cases} \tag{2.13b}$$

Although it seems at first glance that the decompression procedure (2.9b), (2.13b) requires $N_{k-1} = 2N_k$ operations of reconstruction, we can combine the multi-resolution relation (1.5) with the conservation property (2.1b) in order to perform this calculation with only N_k operations of reconstruction. This will become obvious from the specific examples in this paper.

3. Compactly Supported Orthonormal Wavelets.

In this section we examine Mallat's multi-resolution analysis [7] with the compactly supported orthonormal wavelets of Daubechies [1] in the framework of section 2. Daubechies considers functions $\varphi(x)$ satisfying a dilation equation

$$\varphi(x) = 2\sum_{s=0}^{S} \alpha_s \varphi(2x - s) \tag{3.1}$$

for which $\{\varphi_j^k(x)\}$ in (1.4a) is an orthonormal set

$$\langle \varphi_i^k, \varphi_j^k \rangle = \delta_{i,j}; \tag{3.2}$$

here δ_{ij} is the Kronicker-δ. In terms of the Fourier symbol (1.7c)

$$M(\xi) = \sum_{s=0}^{S} \alpha_s \ell^{is\xi} \tag{3.3}$$

the orthogonality (3.2) can be expressed by the following condition on $M(\xi)$

$$|M(\xi)|^2 + |M(\xi + \pi)|^2 = 1 \tag{3.4a}$$

or equivalently as a condition on the coefficients $\{\alpha_s\}$

$$(3.4b) \qquad 4\sum_{s=0}^{S} \alpha_s \alpha_{s-2m} = \delta_{0,m}$$

(see [1], [8]).

In the context of this paper Mallat's multi-resolution algorithm can be described by (2.3) - (2.4) with the particular choice of reconstruction (2.1)

$$(3.5) \qquad R(x;\bar{f}^k) = (P_k f)(x) = \sum_{j=1}^{N_k} \bar{f}_j^k \varphi_j^k(x).$$

Here $\bar{f}_j^k = \langle f, \varphi_j^k \rangle$ (1.4a) and P_k is the orthogonal projection into the set V_k which is the linear span of $\{\varphi_j^k(x)\},\ 1 \le j \le N_k$.

The conservation property (2.1b) of the reconstruction (3.5)

$$(3.6a) \qquad \langle R(\cdot;\bar{f}^k),\ \varphi_j^k \rangle = \bar{f}_j^k$$

is a direct consequence of the orthonormality (3.2).

Strang [8] observes that the reconstruction (3.5) falls into the category of approximation by translates; based on this theory he shows that the accuracy requirement (2.5a),

$$(3.6b) \qquad R(x;\bar{f}^k) = f(x) + O((h_k)^r)$$

can be expressed by the requirement that $M(\xi)$ (3.3) has a zero of order r at $\xi = \pi$, i.e.

$$(3.7a) \qquad \frac{d^m}{d\xi^n} M(\xi)\Big|_{\xi=\pi} = 0, \quad 0 \le m \le r-1$$

or equivalently in terms of the coefficients $\{\alpha_s\}$,

$$(3.7b) \qquad \sum_{s=0}^{S} (-1)^s s^m \alpha_s = 0, \qquad 0 \le m \le r-1.$$

We recall from section 1 that (1.2) implies

$$M(0) = \sum_{s=0}^{S} \alpha_s = 1 \tag{3.8}$$

and that specifying the coefficients $\{\alpha_s\}$ determines $\varphi(x)$. In order to construct $\varphi(x)$ which satisfies the requirments (1.2), (3.2) and (3.6b) we have to find $\alpha_0, \ldots \alpha_s$ which satisfy equations (3.8), (3.7b) and (3.4b). Daubechies [1] has shown that given any r, there is a unique solution for $S = 2r - 1$ and actually calculated these sets of $2r$ coefficients for $r \leq 10$; let us denote the corresponding $\varphi(x)$ by φ^r. These $\varphi^r(x)$ have an inherent fractal nature, but their smoothness increases almost linearly with r,

$$\varphi^r \in C^{r(\mu-\varepsilon)} \tag{3.9}$$

with $\mu \approx 0.3$ for large r; e.g. $\varphi^2 \in C^{0.5-\varepsilon}$, $\varphi^4 \in C^{1.275}$, $\varphi^{10} \in C^{2.902}$. For $r \geq 2$ φ^r is not symmetric, has an oscillatory tail and r roots. These are interesting but certainly weird functions. Another unusual situation (from the point of view of numerical analysis) is that we get r-th order of accuracy with functions φ^r which have degree of smoothness much smaller than r.

We turn now to examine the scale analysis and data compression which is associated with this particular choice of reconstruction (3.5), i.e. $R = P_k$. We recall from section 2 that $Q_k(x; f)$ (2.3b), (2.14) satisfies

$$\langle Q_k(\cdot; f),\ \varphi_j^m \rangle = 0 \text{ for } m \geq k. \tag{3.10a}$$

This property holds for any conservative reconstruction, including nonlinear ones. In terms of the function spaces V_k, (3.10a) can be expressed by

$$Q_k(\cdot; f) \perp V_m, \quad m \geq k. \tag{3.10b}$$

Since the reconstruction (3.5) is a linear operator, so is $Q_k(x; f)$ in (2.3); we denote it here by $(Q_k f)(x)$,

$$Q_k = P_{k-1} - P_k \tag{3.11a}$$

Clearly

$$Q_k f \in V_{k-1} \supset V_k. \tag{3.11b}$$

Let us denote the orthogonal complement of V_k in V_{k-1} by W_k, i.e.

$$V_{k-1} = V_k \oplus W_k. \tag{3.12}$$

Relation (3.10b) for $m = k$ together with (3.11b) shows that $Q_k f$ is the orthogonal projection of f into W_k.

Let us define the wavelets $\{\psi_j^k\}$ by

$$\psi(x) = 2 \sum_{s=-1}^{S-1} (-1)^s \alpha_{s+1} \varphi(2x + s) \tag{3.13a}$$

$$\psi_j^k(x) = \frac{1}{h_k} \psi\left(\frac{x}{h_k} - j\right), \quad 1 \leq j \leq N_k. \tag{3.13b}$$

It is easy to verify that due to the orthogonality (3.2), (3.4b) we get from (3.13) that

$$\langle \psi_j^k, \varphi_m^k \rangle = 0 \qquad \text{for all} \quad m, j \tag{3.14a}$$

$$\langle \psi_j^k, \psi_j^k \rangle = \delta_{jj'}, \tag{3.14b}$$

which shows that the wavelets $\{\psi_j^k\}$, $1 \leq j \leq N_k$, form an orthonormal basis of W_k and consequently

$$Q_k f = \sum_{m=1}^{N_k} \gamma_m^k \psi_m^k(x), \quad \gamma_m^k = \langle Q_k f, \psi_m^k \rangle. \tag{3.14c}$$

Using (3.10b) and (3.12) it follows that

$$\langle \psi_j^k, \varphi_{j'}^m \rangle = 0, \quad m \geq k, \tag{3.14d}$$

$$\langle \psi_j^k, \psi_{j'}^{k'} \rangle = \delta_{jj'} \cdot \delta_{kk'} \tag{3.14e}$$

The scale coefficients d_j^{k-1}, $1 \leq j \leq N_{k-1}$, in (2.4b) are

$$d_j^{k-1} = \langle Q_k f, \varphi_j^{k-1} \rangle = \bar{f}_j^{k-1} - \sum_{m=1}^{N_k} \bar{f}_m^k \langle \varphi_m^k, \varphi_j^{k-1} \rangle. \tag{3.15a}$$

In section 2 we have shown that always

$$\sum_{s=0}^{S} \alpha_s d_{2j+s}^{k-1} = 0, \quad 0 \leq j \leq N_k, \tag{3.15b}$$

and that for any decent $\varphi(x)$ (i.e. one which is a "good" approximation to the Dirac-δ in the sense of (2.15a)) we can store $\{d_{2j-1}^{k-1}\}$, $1 \leq j \leq N_k$, and use the relations (3.15b) to get $\{d_{2j}^{k-1}\}$, $1 \leq j \leq N_k$ by solving the system of linear equations (2.5c). Relation (3.15b) is a direct consequence of the dilation relation and the conservation property of the reconstruction (even nonlinear); it has nothing to do with the orthogonality (3.2). However when there is orthogonality, we can also remove the redundancy in d^{k-1} by using (3.14c), i.e.

$$d_j^{k-1} = \langle Q_k f, \varphi_j^{k-1} \rangle = \sum_{m=1}^{N_k} \langle \psi_m^k, \varphi_j^{k-1} \rangle. \tag{3.15c}$$

This enables us to represent the $N_{k-1} = 2N_k$ elements of d^{k-1} in terms of the N_k elements of γ^k. In this case it is convenient to express the data compresion algorithm (2.8) - (2.9) also in term of γ^k. The encoding part is obtained from (3.14c) by

$$\gamma_j^k = \langle Q_k f, \psi_j^k \rangle = \langle P_{k-1} f, \psi_j^k \rangle = \sum_{m=1}^{N_{k-1}} \bar{f}_m^{k-1} \langle \varphi_m^{k-1}, \psi_j^k \rangle; \tag{3.16a}$$

since

$$\langle \varphi_m^{k-1}, \psi_j^k \rangle = 2(-1)^m \alpha_{2j-m+1} \tag{3.16b}$$

and $\alpha_s \neq 0$ only for $0 \leq s \leq S = 2r - 1$, we can replace (2.8c) by

$$\begin{cases} DO\ k = 1, L \\ DO\ j = 1, N_k \\ \gamma_j^k = -2 \sum_{s=0}^{2r-1} (-1)^s \alpha_s \bar{f}_{2j+1-s}^{k-1}. \end{cases} \tag{3.17}$$

The decoding part is obtained from (3.15a) and (3.15c)

$$(3.18a) \qquad \bar{f}_j^{k-1} = \sum_{m=1}^{N_k} \bar{f}_m^k \langle \varphi_m^k, \varphi_j^{k-1} \rangle + \sum_{m=1}^{N_k} \gamma_m^k \langle \psi_m^k, \varphi_j^{k-1} \rangle.$$

Using (3.16b) and

$$(3.18b) \qquad \langle \varphi_m^k, \varphi_j^{k-1} \rangle = \alpha_{j-2m}$$

we can replace (2.9) by

$$(3.19a) \qquad c^{MR} = \{\bar{f}^L, (\gamma^L, \ldots, \gamma^1)\}$$

$$(3.19b) \qquad \begin{cases} DO \ k = L, 1 \\ DO \ j = 1, N_k \\ \bar{f}_{2j-1}^{k-1} = \sum_{s=0}^{r-1} \alpha_{2s+1} \bar{f}_{j-1-s}^k - 2 \sum_{s=0}^{r-1} \alpha_{2s} \gamma_{j-1+s}^k \\ \bar{f}_{2j}^{k-1} = \sum_{s=0}^{r-1} \alpha_{2s} \bar{f}_{j-2}^k + 2 \sum_{s=0}^{r-1} \alpha_{2s+1} \gamma_{j+2}^k. \end{cases}$$

Our main criticism about the compactly supported wavelets is that it leaves very little room to fit the compression algorithm to the particular nature of the data. Once the decision is made to use orthonormal multi-resolution basis (3.2) and to use projection as a reconstruction technique, the only free parameter left is the order of accuracy r. Our goal in data compression is to find a multi-resolution representation (2.12b), (3.19a) in which γ_j^k is significantly different from zero only when there is a new scale of f and not because of inadequacy of the approximation scheme. Therefore it is important to allow for adaptive approximation methods.

In this paper we consider the $\varphi(x)$ to be given and leave the choice of reconstruction subject only to the conservation requirement (2.1b). In the following section we study the simplest choice of taking φ to be the Dirac-δ; this leads us to interpolatory multi-resolution analysis. In section 5 we shall study multi-resolution analysis of cell-averages which corresponds to the box function (1.8b) and in section 6 we outline the general case.

4. Interpolatory Multi-Resolution Analysis.

In this section we take $\varphi = \delta(x)$ (1.8a) for which $\alpha_0 = 1$; this choice represents multi-resolution analysis by interplation techniques: (1.4a) becomes

$$\bar{f}_j^k = \langle f, \frac{1}{h_k}\delta\left(\frac{x - x_j^k}{h_k}\right)\rangle = f(x_j) \tag{4.1a}$$

and (1.5a), the dilation relation,

$$\bar{f}_j^{k+1} = \bar{f}_{2j}^k. \tag{4.1b}$$

This means that we start with the point-values of f on the finest grid, and a lower level of resolution $(k+1)$ is obtained by eliminating the values of f on the k-th grid which have odd indices. Thus the sense of different levels of resolution here is achieved by sampling $f(x)$ with different frequencies.

The conservation property (2.1b) in this case is

$$\bar{f}_j^k = \langle R(\cdot; \bar{f}^k), \frac{1}{h_k}\delta\left(\frac{x - x_j^k}{h_k}\right)\rangle = R(x_j^k; \bar{f}^k). \tag{4.1c}$$

This means that $R(x; \bar{f}^k)$ interpolates $\bar{f}_j^k$ on the k-th grid. In order to stress these points we shall use f^k instead of $\bar{f}^k$ and $I_k(x; f^k)$ instead of $R(x; \bar{f}_k)$, i.e.

$$R(x; \bar{f}_k) = I_k(x; f^k) \tag{4.2a}$$

$$I_k(x_j^k; f^k) = f_j^k \tag{4.2b}$$

Note that the interpolation technique need not be the same for all levels k, and therefore we index it with a subscript k.

We turn now to consider the data compression algorithm (2.8) - (2.9) that is associated with this interpolation. Since $\alpha_\ell = \delta_{\ell,0}$ in this case, the algorithm simplifies considerably. Given a sequence of numbers $\{c_j\}$, $0 \leq j \leq N_0$, we set

$$f_j^0 = c_j, \quad 0 \leq j \leq N_0 \tag{4.3a}$$

and calculate

$$\text{(4.3b)} \qquad \begin{cases} DO\ k = 1, L \\ f_j^k = f_{2j}^{k-1}, \quad 0 \le j \le N_k \\ \hat{d}_j^k = f_{2j-1}^{k-1} - I_k(x_{2j-1}^{k-1}; f^k), \quad 1 \le j \le N_k. \end{cases}$$

At the end of this stage we have obtained c^{MR}, the interpolating multi-resolution representation of c

$$\text{(4.3c)} \qquad c^{MR} = \{f^L, (\hat{d}^L, \dots, \hat{d}^1)\}.$$

Note that here we use the value of f at $x_0^k = 0$ for all levels. Thus we start with an odd number of elements in c, and for all k

$$\text{(4.4)} \qquad I_k(x_0^k; f^k) = f(0) = c_0, \quad I_k(x_{N_k}^k; f^k) = f(1) = c_{N_0};$$

in the periodic case we assume $c_0 = c_{N_0}$. $\hat{d}_j^k$ in (4.3b) is the error committed in interpolating $f(x)$ from the k-th grid at the location x_{2j-1}^{k-1}, which is the center of the interval $[x_{j-1}^k, x_j^k]$.

For purposes of data compression we apply (2.12) to c^{MR} (4.4a). The decoding part of the algorithm starts therefore by inverting the compressed representation (2.12b) to obtain $\{\tilde{d}^1, \dots, \tilde{d}^L\}$. Then we set

$$\text{(4.5a)} \qquad \tilde{f}^L = f^L$$

and calculate

$$\text{(4.5b)} \qquad \begin{cases} DO\ k = L, 1 \\ \tilde{f}_{2j}^{k-1} = \tilde{f}_j^k, \quad 0 \le j \le N_k \\ \tilde{f}_{2j-1}^{k-1} = I_k(x_{2j-1}^{k-1}; \tilde{f}^k) + \tilde{d}_j^k, \quad 1 \le j \le N_k. \end{cases}$$

The multi-resolution representation c^{MR} (4.3c) corresponds to the interpolatory scale-decomposition (2.3)

$$\text{(4.6a)} \qquad I_0(x; f^0) = I_L(x; f^L) + \sum_{k=1}^{L} Q_k(x; f)$$

$$\text{(4.6b)} \qquad Q_k(x; f) = I_{k-1}(x; f^{k-1}) - I_k(x; f^k),$$

which by virtue of (2.4) satisfies

$$Q_k(x_j^m; f) = 0 \quad \text{for} \quad 0 \le j \le N_m, \quad m \ge k, \tag{4.6c}$$

and for $m = k-1$

$$\begin{cases} Q_k(x_{2j}^{k-1}; f) = 0, & 0 \le j \le N_k \\ Q_k(x_{2j-1}^{k-1}; f) = \hat{d}_j^k = f_{2j-1}^{k-1} - I_k(x_{2j-1}^{k-1}; f^k). \end{cases} \tag{4.6d}$$

Note that up to this point we have not assumed linearity of the interpolation procedure and therefore the strategy of interpolation may depend on the nature of the local data. This enables us to use adaptive procedures such as ENO interpolation [2], [3].

In the following we consider data-independent interpolation for which $I_k(\cdot; f)$ is a linear functional of f. In this case we can associate a multi-resolution basis of functions to the representation c^{MR} (4.3c), which is somewhat analogous to that of the wavelets (3.14c). To do so we define

$$\bar{\varphi}_j^k(x) = I_k(x; e_j^k), \quad 0 \le j \le N_k \tag{4.7a}$$

where e_j^k denotes the unit vector of the k-th grid

$$(e_j^k)_i = \delta_{j,i}; \tag{4.7b}$$

clearly

$$\bar{\varphi}_j^k(x_i^k) = \delta_{j,i}. \tag{4.7c}$$

Let $\bar{V}_k$ denote the linear span of $\{\bar{\varphi}_j^k\}$, $0 \le j \le N_k$, and let $\bar{P}_k$ be the interpolatory projection into $\bar{V}_k$

$$(\bar{P}_k g)(x) = \sum_{j=0}^{N_k} g(x_j^k)\bar{\varphi}_j^k(x). \tag{4.8}$$

Clearly for all k

$$I_k(x; f^k) = (\bar{P}_k f)(x) \tag{4.9}$$

and

$$\bar{P}_k I_k = I_k. \tag{4.10}$$

From (4.6c) we get

$$\bar{P}_m Q_k \equiv 0 \quad \text{for} \quad m \geq k \tag{4.11}$$

and from (4.6d)

$$(\bar{P}_{k-1} Q_k)(x) = \sum_{j=1}^{N_k} \hat{d}_j^k \cdot \bar{\varphi}_{2j-1}^{k-1}(x) \equiv \sum_{j=1}^{N_k} \hat{d}_j^k \cdot \hat{\psi}_j^k(x). \tag{4.12a}$$

Using the notation

$$\hat{\psi}_j^k(x) = \bar{\varphi}_{2j-1}^{k-1}(x) \tag{4.12b}$$

$$\bar{Q}_k = \bar{P}_{k-1} Q_k \tag{4.12c}$$

we define

$$W(x; f) = \sum_{k=1}^{L} \bar{Q}_k(x; f) = \sum_{k=1}^{L} \sum_{j=1}^{N_k} \hat{d}_j^k \cdot \hat{\psi}_j^k(x). \tag{4.12d}$$

Theorem 4.1. *If the interpolation scheme satisfies*

$$\bar{P}_{k-1} I_k = I_k \tag{4.13a}$$

then

$$I_0(x; f) = I_L(x; f) + W(x; f). \tag{4.13b}$$

Proof. Because of our assumption (4.13a) and (4.10)

$$\begin{aligned} \bar{Q}_k = \bar{P}_{k-1} Q_k = \bar{P}_{k-1}(I_{k-1} - I_k) = \bar{P}_{k-1} I_{k-1} - \bar{P}_{k-1} I_k \\ = I_{k-1} - I_k. \end{aligned}$$

Therefore

$$W_L = \sum_{k=1}^{L} \bar{Q}_k = \sum_{k=1}^{L} (I_{k-1} - I_k) = I_0 - I_L$$

which proves (4.13b).

Let us return now to condition (4.13a). Because of linearity

$$\bar{P}_{k-1} I_k = \bar{P}_{k-1} \sum_{j=0}^{N_k} f_j \quad \bar{\varphi}_j^k(x) = \sum_{j=0}^{N_k} f_j (P_{k-1} \bar{\varphi}_j^k)(x).$$

Hence (4.13a) is equivalent to the requirement

$$\bar{P}_{k-1} \bar{\varphi}_j^k \equiv \bar{\varphi}_j^k, \tag{4.14a}$$

in other words, $\bar{\varphi}_j^k$ is in $\bar{V}_{k-1}$, i.e. can be expressed as a linear combination of $\{\bar{\varphi}_i^{k-1}\}$, $0 \le i \le N_{k-1}$. When there is a "mother function" $\bar{\varphi}$ such that

$$\bar{\varphi}_j^k = \bar{\varphi}\left(\frac{x - x_j^k}{h_k}\right) \tag{4.14b}$$

(4.14a) implies that $\bar{\varphi}(x)$ must also satisfy a dilation equation.

We see from (4.13b) that $W(x; f)$ is just a rearrangement of terms in $I_0 - I_L$. While $I_0(x; f)$ is represented by the basis $\mathcal{B}$

$$\mathcal{B} = \{\bar{\varphi}_i^0(x)\}_{i=0}^{N_0} \tag{4.15a}$$

with coefficients $\{f(x_j^0)\}_{j=1}^{N_0}$, $W(x; f)$ is represented by the multi-resolution basis $\mathcal{B}^{MR}$

$$\mathcal{B}^{MR} = \{\{\hat{\psi}_j^k\}_{j=1}^{N_k}\}_{k=1}^{L} = \{\{\bar{\varphi}_{2j-1}^{k-1}\}_{j=1}^{N_k}\}_{k=1}^{L}, \tag{4.15b}$$

with coefficients $\hat{d}_j^k$ (4.12a), (4.6d), which are the local interpolation error by I_k at x_{2j-1}^{k-1}. We note that the dimension of the multi-resolution basis is also N_0, thus

$$\dim(\mathcal{B}) = \dim(\mathcal{B}^{MR}). \tag{4.15c}$$

Given a function $f(x)$ on a fixed grid, we can now reduce the dimensionality of its representation in an intelligent way by dropping terms from the RHS of (4.13b) for which $\hat{d}_j^k$ is small in absolute value.

Another point of view is that of local refinement. In this context we start with the coarsest grid of N_L intervals for which the interpolation $I_L(x; f)$ still makes sense, and keep refining the grid by halving its intervals until we get an acceptable approximation to $f(x)$. Hence in order to get a uniform approximation to $f(x)$, we can monitor the coefficients $\hat{d}_j^k$ (4.6d) and refine *locally* only when they are not sufficiently small in absolute value.

As an example let us consider the simplest case of piecewise-linear interpolation, where for all k we take

$$(4.16a) \quad I_k(x; f) = f(x_{j-1}^k) + [f(x_j^k) - f(x_{j-1}^k)] \cdot (x - x_{j-1}^k)/h_k \quad \text{for} \quad x_{j-1}^k \le x \le x_j^k.$$

In this case

$$(4.16b) \qquad \bar{\varphi}_j^k(x) = I_k(x; e_j^k) = \bar{\varphi}\left(\frac{x - x_j^k}{h_k}\right)$$

where $\bar{\varphi}(x)$ is the hat function (1.8c), i.e.

$$(4.16c) \qquad \bar{\varphi}(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Since the hat function satisfies a dilation equation it follows from (4.14) that Theorem 4.1 applies.

In the following we present the multi-resolution version in a grid-refinement mode, i.e. starting from the coarsest grid up. Let us assume that $f(x)$ is to be approximated on a grid with $N = 2^{n_0}$ intervals.

The standard representation in the basis $\mathcal{B}$ is

$$(4.17) \qquad I_0(x; f) = \sum_{j=0}^{2^{n_0}} f(2^{-n_0} \cdot j) \cdot \bar{\varphi}(2^{n_0} x - j),$$

while the representation in the multi-resolution basis $\mathcal{B}^{MR}$ is

$$(4.18a) \qquad W(x;f) = \sum_{k=1}^{n_0} \sum_{j=1}^{2^{k-1}} \beta_j^k(f) \cdot \bar{\varphi}(2^k x - 2j + 1),$$

$$(4.18b) \qquad \beta_j^k(f) = f(2^{-k}(2j-1)) - \frac{1}{2}[f(2^{-k+1}(j-1)) + f(2^{-k+1} \cdot j)].$$

We refer the reader to Figure 1 for a graphical demonstration of the case $n_0 = 3$.

As is customary in numerical analysis we have assume that $I_k(x;f)$ (and consequently $\bar{\varphi}_j^k(x)$) satisfy the given "boundary conditions". When $f(0)$ and $f(1)$ are specified, then $I_L(x;f)$ in (4.13b) assumes these values and therefore

$$(4.19) \qquad W(0;f) = W(1;f) = 0.$$

It is interesting to note that condition (4.13a) is satisfied also by spectral collocation methods: Let $\{\eta_n(x)\}$, $1 \le n < \infty$, be an infinite sequence of linearly independent functions ,and let

$$(4.20a) \qquad I_k(x;f) = \sum_{n=1}^{N_k} a_n^k \eta_n(x), \quad a_n^k = a_n^k(f)$$

where $\{a_n^k\}_{n=1}^{N_k}$ are uniquely determined by the N_k linear equations

$$(4.20b) \qquad I_k(x_j^k;f) = \sum_{n=1}^{N_k} a_n^k \eta_n(x_j^k) = f_j^k, \quad 1 \le j \le N_k.$$

It follows from the uniqueness of the solution for the coefficients in (4.20), that the solution $\{a_n^{k-1}\}_{n=1}^{N_{k-1}}$ to

(4.21a)

$$I_{k-1}(x_j^{k-1};I_k) = \sum_{n=1}^{N_{k-1}} a_n^{k-1} \eta_n(x_j^{k-1}) = I_k(x_j^{k-1};f), \quad 1 \le j \le N_{k-1} = 2N_k$$

is

$$(4.21b) \qquad a_n^{k-1} = \begin{cases} a_n^k & \text{for} \quad 1 \le n \le N_k \\ 0, & \text{for} \quad N_{k+1} \le n \le N_{k-1} \end{cases};$$

this implies

$$I_{k-1}(x; I_k) \equiv I_k, \tag{4.21c}$$

which is equivalent to (4.13a) and (4.14a).

Spectral collocation methods are inherently global. The standard way to reduce the dimensionality of spectral approximations is to eliminate components $\eta_n(x)$ for which $a_n^k(f)$ is small in absolute value. Unfortunately the size of $a_n(f)$ depends on the global behavior of f, and its elimination affects the approxiamtion everywhere. Rewriting the spectral approximation in its multi-resolution basis (4.15b) enables one to reduce the *dimensionality* of the representaton by neglecting terms $\hat{\psi}_j^k(x)$ for which $\hat{d}_j^k$, the *local* approximation error, is small in absolute value. Note that $\hat{\psi}_j^k(x)$ decays away from x_j^k; consequently the error introduced by dropping it from the expansion is restricted to a neighborhood of x_j^k.

Finally we remark that under most circumstances

$$\hat{I}_0(x; f) = I_L(x; f) + W(x; f) \tag{4.22}$$

is a meaningful approximation to $f(x)$ in $[0, 1]$ even when $\hat{I}_0(x; f) \neq I_0(x; f)$. After all, what matters is the quality of approximation which is obtained after deleting as many components in W as possible; hence the usefulness of (4.22) should be judged by its performance in this regard.

5. Multi-Resolution Analysis of Cell-Averages.

In this section we consider discrete multi-resolution analysis of cell-averages which is obtained by taking $\varphi(x)$ in (1.4a) to be the box function (1.8b), i.e.

$$\varphi(x) = \begin{cases} 1 & -1 \leq x < 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5.1a}$$

This function has the dilation equation

$$\varphi(x) = \varphi(2x) + \varphi(2x - 1) \Leftrightarrow \alpha_{-1} = \alpha_0 = \frac{1}{2}. \tag{5.1b}$$

Thus the multi-resolution analysis

$$\left\{ \{\bar{f}_j^k\}_{j=1}^{N_k} \right\}_{k=0}^{L} \tag{5.2a}$$

is given by

$$(5.2b) \qquad \bar{f}_j^k = \langle f, \frac{1}{h_k}\varphi\left(\frac{x}{h_k} - j\right)\rangle \equiv \langle f, \varphi_j^k \rangle = \frac{1}{h_k}\int_{x_{j-1}^k}^{x_j^k} f(y)dy$$

and the associated dilation relation (1.5a) is

$$(5.2c) \qquad \bar{f}_j^{k+1} = \frac{1}{2}(\bar{f}_{2j-1}^k + \bar{f}_{2j}^k).$$

We refer to $\bar{f}_j^k$ as the cell-average of $f(x)$ in the j-th cell of the k-th grid. It is convenient to introduce the cell-averaging operator $A(I)$

$$(5.3a) \qquad A(I)f = \frac{1}{|I|}\int_I f dx$$

and to denote

$$(5.3b) \qquad \bar{f}_j^k = A(I_j^k)f, \quad I_j^k = [x_{j-1}^k, x_j^k].$$

(Obviously a "cell" in 1D is just an interval).

Given $\bar{f}^k$, the cell-averages of f on the k-th grid we denote by $R(x; \bar{f}^k)$ a reconstruction procedure which satisfies (2.1), i.e.

$$(5.4a) \qquad R(x; \bar{f}^k) = f(x) + O((h_k)^r),$$

$$(5.4b) \qquad A(I_j^k)R(\cdot; \bar{f}^k) = \bar{f}_j^k.$$

Is there more information in the cell-averages of $f(x)$ than there is in its point-values? To answer this question let us observe that knowing the cell-averages of f is equivalent to knowing the point-values $F(x_j^k)$ of its primitive function

$$(5.5a) \qquad F(x) = \int_0^x f(y)dy :$$

Given $\{F(x_j^k)\}_{j=1}^{N_k}$ we obtain the cell-averages by

$$(5.5b) \qquad \bar{f}_j^k = [F(x_j^k) = F(x_{j-1}^k)]/h_k;$$

note that $F(0) = 0$. Conversely, from given cell-averages $\{\bar{f}_j^k\}_{j=1}^{N_k}$, we get the point-values of the primitive function by

$$(5.5c) \qquad F(x_j^k) = \sum_{i=1} (h_k \bar{f}_i^k), \quad 1 \leq j \leq N_k, \quad F(x_j^0) = F(0) = 0.$$

(see Remark 5.1).

This observation immediately suggests the following reconstruction technique: Interpolate the point-values of the primitive-function by any interpolation technique $I_k(x; F^k)$ and define

$$(5.6) \qquad R_k(x; \bar{f}^k) = \frac{d}{dx} I_k(x; F^k)$$

(this procedure was called "reconstruction via primitive function" in [3]). It is easy to see that (5.6a) satisfies the conservation requirement (5.4b):

$$\begin{aligned} A(I_j^k) R_k(\cdot; \bar{f}^k) &= \frac{1}{h_k} \int_{x_{j-1}^k}^{x_j^k} \frac{d}{dx} I_k(x; F^k) dx = \frac{1}{h_k} [I_k(x_j^k; F^k) - I_k(x_{j-1}^k; F^k)] \\ &= \frac{1}{h_k} [F(x_j^k) - F(x_{j-1}^k)] = \bar{f}_j^k. \end{aligned}$$

Typically if I_k is an interpolation method with formal order of accuracy $r + 1$

$$(5.7a) \qquad I_k(x; F^k) = F(x) + O((h_k)^{r+1} \| F^{(r+1)} \|)$$

then

$$\begin{aligned} (5.7b) \qquad R_k(x; \bar{f}^k) = \frac{d}{dx} I_k(x; F^k) &= \frac{d}{dx} F(x) + O((h_k)^r \| F^{(r+1)} \|) \\ &= f(x) + O((h_k)^r \| f^{(r)} \|). \end{aligned}$$

Assume now that $f(x)$ has $(p-1)$ continuous derivatives and that $f^{(p)}(x)$ is discontinuous but bounded. It is clear from relations (5.7) that the maximal accuracy that can be achieved from either point-values or cell-averages is $O(h^p \| f^{(p)} \|)$: Using cell-averages we gain one order of smoothness in the primitive function (5.5a) but we lose it in the differentiation (5.6). Consequently there is no advantage in using cell-averages rather than point-values of $f(x)$ for continuous data.

There is a significant advantage however in using cell-averages rather than point-values of f when $f(x)$ is discontinuous in a finite number of points ([5]). To see that let us assume that $f(x)$ is discontinuous at $x_d \in (x_{j-1}^k, x_j^k)$ and that in $[a, x_d) \cup (x_d, b]$, $0 \le a < x_d < b \le 1$, f has $(p-1)$ continuous derivatives while $f^{(p)}$ is discontinuous but bounded, $p \ge 1$. Let I^L and I^R denote interpolation of either $f(x)$ or $F(x)$ at grid points in $[a, x_d)$ and $(x_d, b]$, respectively. We note that $F(x)$ is continuous in $[a, b]$, but has a discontinuous derivative at x_d. Consequently, if $F(x)$ is properly resolved on the k-th grid $I^L(x; F^k)$ and $I^R(x; F^k)$ will intersect at some point $\tilde{x}_d \in I_j^k$. Using interpolation with $r \ge p$ in (5.7) we get that this point is a good approximation to the location of the discontinuity within the cell I_j^k, i.e.

$$\tilde{x}_d - x_d = O((h_k)^p \|f^{(p)}\|). \tag{5.8a}$$

On the other hand, having knowledge of point-values $\{f(x_i^k)\}$ in $[a, b]$, there is nothing much we can say about the location of the discontinuity within the cell I_j^k.

We describe now how to apply the subcell-resolution technique of [5] in order to get an $O(h^p)$ approximation $\tilde{F}_{2j-1}^{k-1}$ to $F(x_{2j-1}^{k-1})$

$$\tilde{F}_{2j-1}^{k-1} = F(x_{2j-1}^{k-1}) + O((h_k)^{p+1} \|f^{(p)}\|); \tag{5.8b}$$

recall that x_{2j-1}^{k-1} is the center of I_j^k. Let

$$D(x) = I^R(x; F) - I^L(x; F). \tag{5.9a}$$

Since $D(\tilde{x}_d) = 0$ we assume that

$$D(x_{j-1}^k) \cdot D(x_j^k) < 0. \tag{5.9b}$$

$\tilde{F}_{2j-1}^{k-1}$ is now computed as follows

$$\tilde{F}_{2j-1}^{k-1} = \begin{cases} I^L(x_{2j-1}^{k-1}; F^k) & \text{if} \quad D(x_{2j-1}^{k-1}) \cdot D(x_j^k) \le 0 \\ I^R(x_{2j-1}^{k-1}; F^k) & \text{otherwise.} \end{cases} \tag{5.9c}$$

It is easy to see that if $f(x)$ is a piecewise-polynomial function

$$f(x) = \begin{cases} P_L(x) & a \le x < x_d \\ P_R(x) & x_d < x \le b \end{cases} \tag{5.10a}$$

with

$$\deg(P_L) \le p-1, \quad \deg(P_R) \le p-1, \tag{5.10b}$$

then

$$\tilde{F}^{k-1}_{2j-1} = F(x^{k-1}_{2j-1}), \tag{5.10c}$$

i.e. the procedure (5.9) is exact (provided that $f(x)$ is discontinuous at x_d); statement (5.8b) follows from this observation (see Remark 5.2).

The "reconstruction via primitive function" (5.6) is probably the most convenient way to approximate the function from its cell-averages but there are also other useful techniques. We refer the reader to [3], [4] where a "reconstruction via deconvolution" is described, and to [4], [6] where we present a "reconstruction via collocation" approach which is very general and applies even to unstructured grids in multi-dimensions.

We turn now to examine the scale decomposition (2.3) for cell-averages, i.e.

$$R_0(x;\bar{f}^0) = R_L(x;\bar{f}^L) + \sum_{k=1}^{L} Q_k(x;f) \tag{5.11a}$$

with

$$Q_k(x;f) = R_{k-1}(x;\bar{f}^{k-1}) - R_k(x;\bar{f}^k). \tag{5.11b}$$

Note that we have indexed the reconstruction with a subscript k in order to allow for different reconstruction techniques for different levels of resolution. Relations (2.4) become

$$A(I^m_j)\cdot Q_k(\cdot;f) = 0 \quad \text{for} \quad m \ge k \tag{5.12a}$$

$$d^{k-1}_j = A(I^{k-1}_j)\cdot Q_k(\cdot;f) = \bar{f}^{k-1}_j - A(I^{k-1}_j)\cdot R_k(\cdot;\bar{f}_k) \tag{5.12b}$$

Since cell-averages satisfy a dilation relation, knowledge of $\{\bar{f}^k_j\}_{j=1}^{N_k}$ implies knowledge of the cell-averages on all coarser grids. Statement (5.12a) shows that since the reconstruction is conservative, knowledge of $R_k(x;\bar{f}^m)$ implies knowledge of $\{\bar{f}^m_j\}_{j=1}^{N_m}$ and consequently of $R_m(x;\bar{f}^m)$ for $k \le m \le L$. $\bar{d}^{k-1}_j$ in (5.12b) measures how well can the cell-average $\bar{f}^{k-1}_j$ of the finer grid be predicted from knowledge of the cell-averages of the k-th grid. Interpreted differently, this can be taken to say that $f(x)$ is already resolved (in the sense of cell-averages) on the k-th grid, except where $\bar{d}_j^{k-1}$ is unacceptably large. In [6] we show that this point of view provides a natural setting for adaptive mesh refinement methodology for solutions of initial-boundary value problems of hyperbolic type. In the following we outline the basic ideas of [6].

We consider the initial-boundary-value problem for a one-dimensional conservation law

$$(5.13)\qquad \begin{cases} u_t + g(u)_x = 0, & 0 \le x \le 1, \quad t > 0 \\ u(x,0) = u_0(x), & 0 \le x \le 1, \end{cases}$$

with appropriate boundary conditions at $x = 0$, $x = 1$. The problem is discretized on the grid (1.3) by taking cell-averages of the solution $u(x,t)$ over the intervals $\{I_j^k\}$. Let v_j^n denote an approximation to the cell-averages of the solution on the 0-th grid (the finest) at time $t_n = n\tau$

$$(5.14a)\qquad v_j^n \approx A(I_j^0)u(\cdot;t_n), \quad 1 \le j \le N_0.$$

The numerical approximation v^n is evolved in time by a Godunov-type scheme

$$(5.14b)\qquad v_j^{n+1} = v_j^n - \lambda(\bar{g}_j^0 - \bar{g}_{j-1}^0), \quad \lambda = \tau/h_0, \quad 1 \le j \le N_0.$$

where the numerical flux $\bar{g}_j$ is given by

$$(5.14c)\qquad \bar{g}_j^0 = \frac{1}{\tau}\int_0^\tau g(E(t)\cdot R_0(\cdot;v^n)\Big|_{x_j^0})dt;$$

here $E(t)$ is the evolution operator of (5.13) (including boundary conditions) which is basically propagation along characteristic curves; $R_0(x;v^n)$ is the reconstruction (5.4) applied to v^n.

Given v^n on the finest grid we proceed to form its multi-resolution analysis (5.2a) $\{\{v_j^{n,k}\}_{j=1}^{N_k}\}_{k=1}^{L}$ by (5.2c), i.e.

$$(5.15a)\qquad v_j^{n,k} = \frac{1}{2}(v_{2j}^{n,k-1} + v_{2j-1}^{n,k-1}), \quad 1 \le j \le N_k, \quad 1 \le k \le L,$$

and define

$$(5.15b)\qquad \bar{g}_j^k = \frac{1}{\tau}\int_0^\tau g(E(t)\cdot R_k(\cdot;v^{n,k})\Big|_{x_j^0})dt, \quad 0 \le j \le N_0, \quad 1 \le k \le L.$$

Here R_k is the reconstruction of the numerical approximation to $u(x, t_n)$ from the k-th grid; note that these values are defined on the finest grid. In analogy to (5.11) let us define

$$\bar{g}_j^0 = \bar{g}_j^L + \sum_{k=1}^{L} \Delta_k \bar{g}_j, \quad \Delta_k g = \bar{g}_j^{k-1} - \bar{g}_j^k, \quad 1 \le j \le N_0. \tag{5.15c}$$

To simplify our presentation let us consider now the constant coefficient case

$$g(u) = au, \quad a = \text{constant}, \tag{5.16a}$$

where $E(t)$ is just propagation with the constant speed a. Thus

$$E(t) R_k(\cdot; v^{n,k})\Big|_{x_j^0} = R_k(x_j^0 - at; v^{n,k}) \tag{5.16b}$$

and

$$\begin{aligned} \Delta_k \bar{g}_j &= \frac{a}{\tau} \int_{x_j^0 - a\tau}^{x_j^0} [R_{k-1}(x; v^{n,k-1}) - R_k(x; v^{n,k}) dx \\ &= \frac{a}{\tau} \int_{x_j^0 - a\tau}^{x_j^0} Q_k(x; v^n) dx \end{aligned} \tag{5.16c}$$

where $Q_k(x; v^n)$ is (5.11b).

Our task is to obtain an acceptable approximation to $\bar{g}_j^0$ with minimal computational effort. Analyzing the scale coefficients (5.12b)

$$d_j^{n,k-1} = v_j^{n,k-1} - A(I_j^{k-1}) \cdot R_k(\cdot; v^{n,k}) \tag{5.16d}$$

we can estimate the size of $|\Delta^k \bar{g}_j|$ and thus make an intelligent decision on the coarsest level of resolution that will yield an acceptable approximation to $\bar{g}_j^0$. We refer the reader to [6] where we show that this analysis extends also to the nonlinear case and suggests an efficient algorithmic implementation of these ideas.

We turn now to consider the data compression algorithm that it is associated with the multi-resolution analysis of cell-averages (5.2). Since $\alpha_0 = \alpha_{-1} = \frac{1}{2}$ in this case, the algorithm (2.8) - (2.9) simplifies considerably. Given a sequence of numbers $\{c_j\}_{j=1}^{N_0}$ we set

$$\bar{f}_j^0 = c_j, \quad 1 \le j \le N_0 \tag{5.17a}$$

and calculate

(5.17b)
$$\begin{cases} DO \quad k = 1, L \\ \bar{f}_j^k = \frac{1}{2}(\bar{f}_{2j-1}^{k-1} + \bar{f}_{2j}^{k-1}), \quad 1 \le j \le N_k, \\ \hat{d}_j^k = \bar{f}_{2j-1}^{k-1} - A(I_{2j-1}^{k-1}) \cdot R(\cdot; \bar{f}^k), \quad 1 \le j \le N_k. \end{cases}$$

At the end of this stage we obtain the multi-resolution representation of c in the sense of cell-averages (2.9a) i.e.

(5.17c)
$$c^{MR} = \{\bar{f}^L, (\hat{d}^L, \ldots, \hat{d}^1)\}.$$

For purposes of data compression we apply procedure (2.12) to c^{MR}. The decoding part of the algorithm starts therefore by inverting the compressed representation (2.12b) to obtain $\{\tilde{d}^1, \ldots, \tilde{d}^L\}$. Then we set

(5.18a)
$$\tilde{f}^L = \bar{f}^L$$

and calculate

(5.18b)
$$\begin{cases} DO \quad k = L, 1 \\ DO \quad j = 1, N_k \\ \tilde{f}_{2j-1}^{k-1} = A(I_{2j-1}^{k-1}) \cdot R_k(\cdot; \tilde{f}^k) + \tilde{d}_j^k \\ \tilde{f}_{2j}^{k-1} = 2\tilde{f}_j^k - \tilde{f}_{2j-1}^{k-1}. \end{cases}$$

Note that the last relation in (5.18b) is equivalent to defining

$$\tilde{f}_{2j}^{k-1} = A(I_{2j}^{k-1}) \cdot R(\cdot; \tilde{f}^k) - \tilde{d}_j^k.$$

because then, due to conservation

$$\tilde{f}_{2j}^{k-1} + \tilde{f}_{2j-1}^{k-1} = [A(I_{2j}^{k-1}) + A(I_{2j-1}^{k-1})] \cdot R(\cdot; \tilde{f}^k) = 2 \cdot A(I_j^k) \cdot R(\cdot; \tilde{f}^k) = 2 \cdot \tilde{f}_j^k.$$

How does one judge the suitability of a particular choice of a data compression algorithm? Following Daubechies in [1] we suggest to do so by judging the suitability of the function-space for which

(5.19)
$$c^{MR} = \{\bar{f}^L, (0, 0, \ldots, 0)\}$$

(provided that $\bar{f}^L$ is meaningful), i.e. the function-space for which the algorithm achieves absolute compression. Choosing cell-averages and ENO reconstruction with subcell resolution [5] we can obtain absolute comrpession of piecewise-polynomials with polynomial degree p which is smaller than the order of accuracy r of the reconstruction (5.10). Therefore this would be a suitable choice for compression of discontinuous signals (see Remark 5.2).

Finally we consider the case of data-independent (i.e. linear) reconstruction procedures and describe their equivalent representation in a multi-resolution basis of functions. To gain some insight let us first consider "reconstruction via primitive function" (5.6) where the interpolation method is data-independent; thus Q_k in (5.11b) is

$$Q_k(x;f) = \frac{d}{dx}[I_{k-1}(x;F^{k-1}) - I_k(x;F^k)]. \tag{5.20a}$$

Using Theorem 4.1 we get that

$$Q_k(x;f) = \sum_{j=1}^{N_k} \hat{d}_j^k(F)\bar{\psi}_j^k(x) \tag{5.20b}$$

where we define

$$\hat{d}_j^k(F) = [F_{2j-1}^{k-1} - I_k(x_{2j-1}^{k-1};F^k)]/h_{k-1} \tag{5.20c}$$

$$\bar{\psi}_j^k(x) = h_{k-1}\cdot\frac{d}{dx}I_{k-1}(x;e_{2j-1}^{k-1}). \tag{5.20d}$$

The reason for the above scaling is that the interpolation error of the primitive function is $O((h_k)^{r+1})$ while

$$\frac{d}{dx}I_{k-1}(x;e_{2j-1}^{k-1}) = O\left(\frac{1}{h_{k-1}}\right).$$

For example let us consider the case of piecewise linear interpolation (4.16) for the primitive function. Here

$$\bar{\psi}(x) = \begin{cases} 1 & x_{2j-2}^{k-1} < x \le x_{2j-1}^{k-1} \\ -1 & x_{2j-1}^{k-1} < x \le x_{2j}^{k-1} \\ 0 & \text{otherwise} \end{cases} \tag{5.21a}$$

and

$$\hat{d}_j^k(F) = -\frac{1}{2h_{k-1}}(F_{2j-2}^{k-1} - 2F_{2j-1}^{k-1} + F_{2j}^{k-1}) = \frac{1}{2}(\bar{f}_{2j}^{k-1} - \bar{f}_{2j-1}^{k-1}) = \hat{d}_j^k \tag{5.21b}$$

where $\hat{d}_j^k$ is the reconstruction error (5.12b) for piecewise-constant reconstruction. We observe that this is exactly the Haar basis, which is the compactly supported orthonormal wavelets for $r = 1$.

Returning now to general reconstruction via primitive function, we see from (2.10d) that

$$A(I_i^{k-1}) \cdot \bar{\psi}_j^k = \int_{x_{i-1}^{k-1}}^{x_i^{k-1}} \frac{d}{dx} I_{k-1}(x; e_{2j-1}^{k-1})dx = (e_{2j-1}^{k-1} - e_{2j}^{k-1})_i, \tag{5.22a}$$

which implies that

$$\bar{\psi}_j^k(x) = R_{k-1}(x; e_{2j-1}^{k-1} - e_{2j}^{k-1}). \tag{5.22b}$$

Applying $A(I_j^{k-1})$ to Q_k in (5.20b) and comparing to (5.12b) we see that

$$\hat{d}_j^k = \hat{d}_j^k(F). \tag{5.22c}$$

Thus we have shown

$$R_0(x; \bar{f}^0) = R_L(x; \bar{f}^L) + \sum_{k=1}^{L} \sum_{j=1}^{N_k} \hat{d}_j^k \cdot R_{k-1}(x; e_{2j-1}^{k-1} - e_{ij}^{k-1}). \tag{5.23}$$

In the next section we describe multi-resolution bases in the general case.

Remark 5.1. In the periodic case it is convenient to work with functions that have zero average in $[0, 1]$ so that $F(x)$ is also periodic and $F(1) = 0$ in (5.5). Therefore it is helpful to define

$$\hat{f}(x) = f(x) - K, \quad K = \int_0^1 f(x)dx, \tag{5.24a}$$

or on the discrete level

$$\hat{c}_j = c_j - K, \quad K = \frac{1}{N_0} \sum_{j=1}^{N_0} c_j. \tag{5.24b}$$

Remark 5.2. If we know that $f(x)$ has $q-1$ continuous derivatives and a discontinuity of the q-th derivative in x_d, $x_{j-1}^k < x_d < x_j^k$ we can extend the subcell resolution technique of (5.8) - (5.10) to this case as follows: $\frac{d^q}{dx^q} F(x)$ has a discontinuous first derivative at x_d. If it is sufficiently resolved on the grid, we expect $\frac{d^q}{dx^q} I_L(x;F)$ and $\frac{d^q}{dx^q} I_k(x;F)$ to intersect at $\tilde{x}_d$ in I_j^k,

(5.25a) $$\tilde{x}_d - x_d = O(h^{p-q}).$$

It follows therefore that if we replace $D(x)$ in (5.9) by

(5.25b) $$D(x) = \frac{d^q}{dx^q} I^R(x;F) - \frac{d^q}{dx^q} I^L(x;F),$$

we get a subcell-resolution technique which is exact for the corresponding piecewise-polynomial problem (5.10); this implies (5.8b).

Remark 5.3. Extrapolating the analysis of the information contents in cell-averages vs. point-values, we get that weighted-averages with respect to the hat-function (1.8c) contain information that will enable us to obtain subcell resolution of δ-distributions; this may be useful for compression of digital images and propagation of singularities.

6. Multi-Resolution Analysis of Weighted-Averages.

In this section we revisit the framework outlined in section 2 and describe its functional structure. We consider now general $\varphi(x)$ which satisfies:

(i) Dilation relation (1.6)

(6.1a) $$\varphi(x) = 2\sum_{\ell} \alpha_\ell \varphi(2x - \ell),$$

(ii) Even diagonal dominance (2.5a)

(6.1b) $$|\alpha_0| > \sum_{\ell \neq 0} |\alpha_{2\ell}|,$$

and assume that we are given $\{\bar{f}_j^0\}_{j=1}^{N_0}$, weighted averages of $f(x)$ with respect to $\varphi(x)$, i.e.

(6.2a) $$\bar{f}_j^0 = \langle f, \frac{1}{h_0}\varphi\left(\frac{x}{h_0} - j\right)\rangle$$

and define

$$\bar{f}_j^k = \langle f, \varphi_j^k \rangle, \quad \varphi_j^k = \frac{1}{h_k}\varphi\left(\frac{x}{h_k} - j\right). \tag{6.2b}$$

The set

$$\{\{\bar{f}_j^k\}_{j=1}^{N_k}\}_{k=1}^{L} \tag{6.2c}$$

is a discrete multi-resolution analysis of f in the sense that knowledge of $\bar{f}^k$ implies knowledge of $\bar{f}^{k+1}$ via relation (1.5). On the functional level we define

$$V_k = \text{ linear span } \{\varphi_j^k\}_{j=1}^{N_k}, \tag{6.3a}$$

which forms a multi-resolution analysis in Mallat's sense [7], i.e.

$$V_k \supset V_{k+1}. \tag{6.3b}$$

Our prediction tool in climbing up from coarse to finer grid is a reconstruction $R(x; \bar{f}^k)$ (2.1) which is a conservative r-th order approximation to $f(x)$. Let e_j^k be the unit vector (4.7b) and denote

$$\bar{\varphi}_j^k(x) = R(x; e_j^k), \tag{6.4a}$$

$$\bar{V}^k = \text{linear span } \{\bar{\varphi}_j^k\}_{j=1}^{N_k}. \tag{6.4b}$$

We observe that the conservation property implies that $\{\varphi_j^k\}_{j=1}^{N_k}$ and $\{\bar{\varphi}_j^k\}_{j=1}^{N_k}$ are bi-orthonormal systems,

$$\langle \varphi_i^k, \bar{\varphi}_j^k \rangle = \langle \varphi_i^k, R(x; e_j^k) \rangle = \delta_{j,i}. \tag{6.4c}$$

Furthermore, $Q_k(x; f)$, the k-th scale component of f (2.3b)

$$Q_k(x; f) = R_{k-1}(x; \bar{f}^{k-1}) - R_k(x; \bar{f}^k) \tag{6.5a}$$

satisfies

$$\langle \varphi_j^m, Q_k(\cdot; f) \rangle = 0 \Longleftrightarrow Q_k \perp V_m \ , \quad m \geq k \tag{6.5b}$$

$$d_j^{k-1} = \langle \varphi_j^{k-1}, Q_k(\cdot; f) \rangle = \bar{f}_j^{k-1} - \langle \varphi_j^{k-1}, R_k(\cdot; \bar{f}^k) \rangle \ , \quad 1 \leq j \leq N_{k-1}. \tag{6.5c}$$

Here $\{d_j^{k-1}\}_{j=1}^{N_{k-1}}$, the coefficients of the k-th scale of $f(x)$, are the local approximation error in predicting $\bar{f}_j^{k-1}$ from the k-th grid. These quantities provide the information needed for a data-compression algorithm; however only half of these quantities are independent. In section 2 we suggested to store the N_k values of d_j^{k-1} with odd indices, i.e.

$$\hat{d}_j^k = d_{2j-1}^{k-1}, \quad 1 \le j \le N_k, \tag{6.6a}$$

and to use the redundancy relation (2.4c) to set up a system of N_k linear equation for $\{d_{2j}^{k-1}\}_{j=1}^{N_k}$ with a RHS which depends on the known $\{\hat{d}_j^k\}$. We showed that condition (6.1b) implies that this system is diagonally dominant and thus solvable for any reasonable boundary conditions. (6.1b) is a constraint on the choice of $\varphi(x)$ which is satisfied anyway by the examples considered in this paper, and in fact can be taken to be a definition of suitable $\varphi(x)$. We note however that (6.1b) is just a *sufficient* condition for this strategy. We denote the result of this procedure in a matrix form by (2.5d)

$$d^{k-1} = \mathbf{D} \cdot \hat{d}^k, \tag{6.6b}$$

where $\mathbf{D}$ is a rectangular $N_{k-1} \times N_k$ matrix.

It is important to observe that the functional structure and the corresponding data compression algorithm (2.8) - (2.9) apply also to data-dependent (nonlinear) reconstruction procedures. It is only at this point that we turn to the data-independent case and define the projection $\bar{P}_k$ into $\bar{V}_k$ by

$$\bar{P}_k f = \sum_{j=1}^{N_k} \langle f, \varphi_j^k \rangle \bar{\varphi}_j^k(x); \tag{6.7a}$$

clearly for all k

$$R_k(x; \bar{f}^k) = \bar{P}_k f, \tag{6.7a}$$

$$\bar{P}_k \cdot R_k = R_k. \tag{6.7c}$$

Theorem 6.1. *If*

$$\bar{P}_{k-1} \cdot R_k = R_k \ , \quad 1 \le k \le L, \tag{6.8}$$

then

$$(6.9a) \qquad R_0(x;\bar{f}^0) = R_L(x;\bar{f}^L) + \sum_{k=1}^{L}\sum_{j=1}^{N_k} \hat{d}_j^k \cdot \bar{\psi}_j^k(x)$$

where $\hat{d}_j^k$ *is* (6.6a) *and*

$$(6.9b) \qquad \bar{\psi}_j^k(x) = R_{k-1}(x;\mathbf{D}\cdot e_j^k) = \sum_{i=1}^{N_{k-1}} (\mathbf{D}\cdot e_j^k)_i \; \bar{\varphi}_i^{k-1}(x), \quad 1 \le j \le N_k.$$

Proof: First we observe that (6.8) together with (6.7) imply

$$\bar{Q}_k = \bar{P}_{k-1}\cdot Q_k = \bar{P}_{k-1}\cdot R_{k-1} - \bar{P}_{k-1}\cdot R_k = R_{k-1} - R_k$$

and therefore

$$R_0(x;\bar{f}^0) = R_L(x;\bar{f}^L) + \sum_{k=1}^{L} \bar{Q}_k(x;f).$$

Using the representation

$$d^{k-1} = \mathbf{D}\cdot\hat{d}^k = \mathbf{D}\cdot(\sum_j \hat{d}_j^k e_j^k) = \sum_j \hat{d}_j^k(\mathbf{D}\cdot e_j^k),$$

(6.5c) and (6.9b), we get

$$\begin{aligned}\bar{Q}_k = \bar{P}_{k-1}\cdot Q_k &= \sum_{j=1}^{N_{k-1}} \langle Q_k, \varphi_i^{k-1}\rangle \bar{\varphi}_i^{k-1} = \sum_{i=1}^{N_{k-1}} d_i^{k-1}\bar{\varphi}_i^{k-1} \\ &= \sum_{i=1}^{N_{k-1}} \left[\sum_{j=1}^{N_k} \hat{d}_j^k(\mathbf{D}\cdot e_j^k)_i\right] \bar{\varphi}_i^{k-1}(x) = \sum_{j=1}^{N_k} \hat{d}_j^k \left[\sum_{i=1}^{N_{k-1}} (\mathbf{D}\cdot e_j^k)_i \bar{\varphi}_i^{k-1}\right] \\ &= \sum_{j=1}^{N_k} \hat{d}_j^k \cdot \bar{\psi}_j^k(x),\end{aligned}$$

which proves the theorem.

Condition (6.8), which can also be formulated by

$$\bar{P}_{k-1} \cdot \bar{V}_k = \bar{V}_k \Longleftrightarrow \bar{P}_{k-1} \cdot \bar{\varphi}_j^k = \bar{\varphi}_j^k, \quad 1 \le j \le N_k, \tag{6.10a}$$

implies of course that

$$\bar{V}_{k-1} \supset \bar{V}_k. \tag{6.10b}$$

Let us denote the complement of $\bar{V}_k$ in $\bar{V}_{k-1}$ by $\bar{W}_k$, i.e.

$$\bar{W}_k = \bar{V}_{k-1} - \bar{V}_k. \tag{6.11a}$$

Clearly $\{\bar{\psi}_j^k\}_{j=1}^{N_k}$ is a basis of $\bar{W}_k$ and $Q_k(\cdot;f)$ (6.5a), considered as an operator $Q_k \cdot f$, is the projection of $f(x)$ into $\bar{W}_k$. It follows from (6.5b) that

$$\bar{W}_k \perp V_m, \quad m \ge k; \tag{6.11b}$$

furthermore

$$\bar{V}_{k-1} = \bar{V}_k \oplus \bar{W}_k, \tag{6.11c}$$

where the direct-sum decomposition

$$\bar{P}_{k-1} \cdot f = \bar{P}_k \cdot f + \bar{Q}_k \cdot f, \quad (Q_k = \bar{Q}_k) \tag{6.11d}$$

corresponds to the relations

$$\bar{P}_{k-1} \cdot f = \sum_{i=1}^{N_{k-1}} \bar{f}_i^{k-1} \bar{\varphi}_i^{k-1} = R_{k-1}(\cdot; \bar{f}^{k-1}), \tag{6.11e}$$

$$\bar{P}_k \cdot f = \sum_{j=1}^{N_k} \bar{f}_j^k \bar{\varphi}_j^k = R_k(\cdot; \bar{f}^k) = \sum_{i=1}^{N_{k-1}} \langle \varphi_i^{k-1}, R_k - (\cdot; \bar{f}^k) \rangle \bar{\varphi}_i^{k-1}, \tag{6.11f}$$

$$\bar{Q}_k \cdot f = \sum_{j=1}^{N_k} \hat{d}_j^k \bar{\psi}_j^k = Q_k(\cdot; f) = \sum_{i=1}^{N_{k-1}} \langle \bar{\varphi}_i^{k-1}, Q_k(\cdot; f) \rangle \bar{\varphi}_i^{k-1} \tag{6.11g}$$

$$= \sum_{i=1}^{N_{k-1}} d_i^{k-1} \bar{\varphi}_i^{k-1}.$$

Theorem 6.1 can be expressed in these terms by

$$\bar{P}_{k-1} \cdot \bar{P}_k = \bar{P}_k \Rightarrow \bar{V}_0 = \bar{V}_L \oplus \bar{W}_L \oplus \cdots \oplus \bar{W}_1. \tag{6.12}$$

Recalling that $\bar{V}_k$ and V_k are bi-orthonormal spaces in the sense of (6.4c), and observing the similarity in relations to those satisfied by the wavelets of section 3, it seems suitable to name $\{\{\bar{\psi}_j^k\}_{j=1}^{N_k}\}_{k=1}^{L}$ "generalized wavelets" or "pseudo-wavelets". If, as is done for wavelets in section 3, we choose the reconstruction $R(x; \bar{f}^k)$ to be the conservative projection of $f(x)$ into V_k, i.e.

$$R(x; \bar{f}^k) = \sum_{j=1}^{N_k} \beta_j^k \varphi_j^k(x) \tag{6.13a}$$

where β_j^k are determined by the conservation relations

$$\bar{f}_i^k = \sum_{j=1}^{N_k} \beta_j^k \langle \varphi_j^k, \varphi_i^k \rangle, \quad 1 \le i \le N_k \tag{6.13b}$$

we get that

$$\beta^k = (\Phi_k)^{-1} \bar{f}^k \tag{6.13c}$$

where Φ_k is the symmetric $N_k \times N_k$ matrix, the elements of which are

$$(\bar{\Phi}_k)_{i,j} = \langle \varphi_i^k, \varphi_j^k \rangle. \tag{6.13d}$$

In this case

$$\bar{\varphi}_j^k(x) = \sum_{i=1}^{N_k} (\Phi_k^{-1} e_j^k)_i \varphi_i^k(x), \tag{6.14a}$$

$$\varphi_i^k(x) = \sum_{j=1}^{N_k} (\Phi_k e_i^k)_j \bar{\varphi}_j^k(x), \tag{6.14b}$$

and consequently

$$\bar{V}_k = V_k. \tag{6.14c}$$

It follows then from (6.11b) that

$$\{\bar{\psi}_j^k\}_{j=1}^{N_k} \perp \{\bar{\varphi}_j^m\}_{j=1}^{N_m}, \quad m \geq k \tag{6.15a}$$

and

$$\langle \bar{\psi}_j^k, \bar{\psi}_{j'}^{k'} \rangle = 0 \text{ for } k \neq k'. \tag{6.15b}$$

This shows that the particular choice of taking the reconstruction R to be the conservative projection onto V_k results in bi-orthogonal wavelets. If we now limit the choice of φ to those functions for which $\{\varphi_j^k\}_{j=1}^{N_k}$ also forms an orthonormal set (i.e. $\Phi_k = I$ in (6.13d)), we get that in addition to (6.15b) wavelets of the same resolution level $\{\psi_j^k\}_{j=1}^{N_k}$ are also orthogonal (Here $\{\psi_j^k\}$ denote a linear combination of $\{\bar{\psi}_j^k\}$ corresponding to the relation (3.13) – see Remark 6.1). If we further restrict the choice of φ to functions of compact support, we get the Daubechies' wavelets of section 3.

Taking the reconstruction $R(x; \bar{f}^k)$ to be a conservative projection onto V_k (6.13) is a natural choice form the point of view of functional analysis, but it is much too restrictive from the point of view of numerical analysis. We have the freedom to choose

$$R(x; \bar{f}^k) = \sum_{j=1}^{N_k} \beta_j^k \mu_j^k(x), \quad \mu_j^k(x) = I_k(x; e_j^k) \tag{6.16a}$$

where I_k is any reasonable interpolation scheme. In this case we get from the conservation requirement (2.1b) that

$$\bar{f}_i^k = \sum_{j=1}^{N_k} \beta_j^k \langle \mu_j^k, \varphi_i^k \rangle, \quad 1 \leq i \leq N_k. \tag{6.16b}$$

If both $\varphi_j^k(x)$ and $\mu_j^k(x)$ are "decent" approximations to Dirac's-δ at x_j^k (which they should in order to be numerically useful), then the $N_k \times N_k$ matrix $\mathbf{B}$

$$(\mathbf{B}_k)_{i,j} = \langle \mu_j^k, \varphi_i^k \rangle, \tag{6.16c}$$

is expected to be diagonally dominant and therfore invertible. We get therefore that $\beta^k = \mathbf{B}_k^{-1} \bar{f}^k$ and

$$R(x; \bar{f}^k) = \sum_{j=1}^{N_k} (\mathbf{B}_k^{-1} \bar{f}^k)_j \mu_j^k(x) = I_k(x; \mathbf{B}_k^{-1} \bar{f}^k). \tag{6.17a}$$

Observe that

$$(6.17\text{b}) \qquad \bar{\varphi}_i^k(x) = \sum_{j=1}^{N_k} (\mathbf{B}_k^{-1} e_i^k)_j \mu_j^k(x) = I_k(x; \mathbf{B}_k^{-1} e_i^k)$$

and that the reconstructed values at the grid-points are given by

$$(6.17\text{c}) \qquad R(x_j^k; \bar{f}^k) = (\mathbf{B}_k^{-1} \bar{f}^k)_j.$$

It is important to notice that the "reconstruction via collocation" described above extends immediately, just by a change of notation, to the multi-dimensional case where $x \in \mathbb{R}^d$ and $\varphi(x)$ is an appropriate averaging function in $\mathbb{R}^d$. All we have to do is to arrange the nodes of the multi-dimensional grid in a one-dimensional array $\{x_i^k\}$, $1 \leq i \leq N_k$ and to take $I_k(x; f)$ to be a multi-dimensional interpolation scheme. With this change of notation (6.16) - (6.17) *is* a reconstruction procedure for the multi-dimensional case. Furthermore, this reconstruction via collocation can be generalized to unstructured grids in $\mathbb{R}^d$ by identifying each element of the grid by an appropriate x_i^k, and taking $\mu_i^k(x)$ to be an appropriate unit interpolation function for this element.

Comparing the choice of general reconstruction via collocation to that of the bi-orthogonal wavelets we see that we lose a bit in functional structure, but gain the possibility of using the well developed machinery (including computer software) of interpolation schemes. Using this arsenal wisely we can hopefully achieve better compression in the representation of digital data and functions.

In the next section we suggest a modified data-compression algorithm which enables us to control the error due to truncation.

7. Error Control

In this section we introduce the truncation operation

$$(7.1) \qquad (\hat{d}_k^{tr})_j = tr(\hat{d}_j^k; \mathcal{E}_k) = \begin{cases} 0 & |\hat{d}_j^k| \leq \mathcal{E}_k \\ \hat{d}_j^k & \text{otherwise} \end{cases}$$

which is to be applied to the multi-resolution representation c^{MR} (2.9a) in order to compress both the digital representation of the discrete input data (2.8a) and the dimensionality of the representation of $f(x)$ in the multi-resolution basis (6.9a). Obviously this strategy gives us direct control over the rate of compression through an appropriate choice of the tolerance-levels $\{\mathcal{E}_k\}_{k=1}^{L}$. However once we use the truncated values (7.1) in the decoding algorithm (2.9b) or the multi-resolution representation of $f(x)$ (6.9a) we get an error which can be estimated by analysis but cannot be directly controlled. This strategy is therefore suitable for applications where we are limited in capacity and we have to settle for whatever quality is possible under this limitation.

There are other applications where quality control is of utmost importance, yet we would like to be as economical as possible with respect to storage and speed of computation. To accomplish this goal we present a modification of the encoding algorithm which keeps track of the cumulative error in a predetermined decoding procedure and truncates accordingly. This enables us to specify the desired level of accuracy in the decompressed signal as well as in the reduced functional representation. As is to be expected (from considerations of the uncertainty principle), we cannot specify compression rate at the same time.

First we describe this nonlinear encoding procedure in the interpolatory case of section 4, where the predetermined decoding procedure is (4.5), i.e.

(i) Set

$$\tilde{f}^L = f^L \tag{7.2a}$$

(ii) Calculate

$$\left\{\begin{array}{l} DO\ k = L, 1 \\ \tilde{f}_0^{k-1} = \tilde{f}_0^L \\ \left\{\begin{array}{l} DO\ j = 1, N_k \\ \tilde{f}_{2j}^{k-1} = \tilde{f}_j^k \\ \tilde{f}_{2j-1}^{k-1} = I_k(x_{2j-1}^{k-1}; \tilde{f}^k) + \tilde{d}_j^k. \end{array}\right. \end{array}\right. \tag{7.2b}$$

Given any tolerance-level $\mathcal{E}$ for accuracy, our task is to come up with a compressed representation

$$\{f^L, (\tilde{d}^L, \dots, \tilde{d}^1)\} \tag{7.3a}$$

such that

$$\|f^0 - \tilde{f}^0\|_\infty = \max_{1\le i\le N_0} |f_i^0 - \tilde{f}_i^0| \le \mathcal{E} \tag{7.3b}$$

for $\tilde{f}^0$ which is obtained by the decoding (7.2). The modified encoding procedure is described algorithmically by the following:

(i) Set

$$\tilde{f}_j^L = f_j^L = f(x_j^L), \quad 0 \le j \le N_L. \tag{7.4a}$$

(ii) Calculate

$$\begin{cases} DO\ k = L, 1 \\ \tilde{f}_0^{k-1} = f_0^L = f(0) \\ \begin{cases} DO\ j = 1, N_k \\ \tilde{f}_{2j}^{k-1} = \tilde{f}_j^k \\ f^{PR} = I_k(x_{2j-1}^{k-1}; \tilde{f}^k) \\ \tilde{d}_j^k = tr(f_{2j-1}^{k-1} - f^{PR}; \mathcal{E}_k) \\ \tilde{f}_{2j-1}^{k-1} = f^{PR} + \tilde{d}_j^k. \end{cases} \end{cases} \tag{7.4b}$$

Observe that unlike (4.3b), the $k - DO$ loop in (7.4b) is done in reverse; here "PR" stands for "predicted".

Let us denote the pointwise error on the k-th grid by E_j^k, i.e.

$$E_j^k = f_j^k - \tilde{f}_j^k. \tag{7.5a}$$

Recalling (4.1b) we get from (7.4b) that

$$|E_{2j}^{k-1}| = |E_j^k| \le \mathcal{E}_{k-1} \tag{7.5b}$$

$$|E_{2j-1}^{k-1}| = |f_{2j-1}^{k-1} - f^{PR} - tr(f_{2j-1}^{k-1} - f^{PR}; \mathcal{E}_k)| \le \mathcal{E}_k; \tag{7.5c}$$

therefore

$$\|E^{k-1}\|_\infty \le \max(\mathcal{E}_k, \mathcal{E}_{k+1}), \quad 1 \le k \le L-1, \ \|E^L\|_\infty = 0, \tag{7.5d}$$

which implies that

(7.5e) $$\|E^{k-1}\|_\infty \le \max(\mathcal{E}_k, \dots, \mathcal{E}_L).$$

We see that the best policy is to choose

(7.5f) $$\mathcal{E}_k = \mathcal{E}, \quad 1 \le k \le L$$

and then (7.3b) follows from (7.5e) for $k = 1$.

Next we describe the modified encoding procedure for the multi-resolution analysis of cell-averages of section 5, where the predetermined decoding procedure is (5.18b), i.e.

(i) Set

(7.6a) $$\tilde{f}^L = \bar{f}^L$$

(ii) Calculate

(7.6b) $$\begin{cases} DO\ k = L, 1 \\ \begin{cases} DO\ j = 1, N_k \\ \tilde{f}^{k-1}_{2j-1} = A(I^{k-1}_{2j-1}) \cdot R_k(\cdot; \tilde{f}^k) + \tilde{d}^k_j \\ \tilde{f}^{k-1}_{2j} = 2\tilde{f}^k_j - \tilde{f}^{k-1}_{2j-1} \ . \end{cases} \end{cases}$$

Given any tolerance level $\mathcal{E}$ for accuracy, our task is to come up with a compressed representation

(7.7a) $$\{\bar{f}^L, (\tilde{d}^L, \dots, \tilde{d}^1)\}$$

so that

(7.7b) $$\|\bar{f}^0 - \tilde{f}^0\| \le \mathcal{E}$$

for $\tilde{f}^0$ obtained by (7.6); for the moment we leave the norm in (7.7b) unspecified.

The modified encoding procedure is described algorithmically by the following:

(i) Compute the multi-resolution analysis of the input data by

(7.8a) $$\begin{cases} DO\ k = 1, L \\ \begin{cases} DO\ j = 1, N_k \\ \bar{f}^k_j = \frac{1}{2}(\bar{f}^{k-1}_{2j-1} + \bar{f}^{k-1}_{2j}) \end{cases} \end{cases}$$

(ii) Set

$$\tilde{f}^L = \bar{f}^L \tag{7.8b}$$

(iii) Calculate

$$\left\{\begin{array}{l} DO\ k = L, 1 \\ \left\{\begin{array}{l} DO\ j = 1, N_k \\ \bar{f}^{PR} = A(I_{2j-1}^{k-1}) \cdot R(\cdot; \tilde{f}^k) \\ \tilde{d}_j^k = tr(\bar{f}_{2j-1}^{k-1} - \bar{f}^{PR} - (\bar{f}_j^k - \tilde{f}_j); \mathcal{E}_k) \\ \tilde{f}_{2j-1}^{k-1} = \bar{f}^{PR} + \tilde{d}_j^k \\ \tilde{f}_{2j}^{k-1} = 2\tilde{f}_j^k - \tilde{f}_{2j-1}^{k-1}. \end{array}\right. \end{array}\right. \tag{7.8c}$$

Let us denote the error in the computed cell-averages by

$$\bar{E}_j^k = \bar{f}_j^k - \tilde{f}_j^k \tag{7.9a}$$

and

$$\bar{E}^{PR} = \bar{f}_{2j-1}^{k-1} - \bar{f}^{PR}. \tag{7.9b}$$

With this notation we get from (7.8c) that

$$\bar{E}_{2j-1}^{k-1} = \bar{E}^{PR} - tr(\bar{E}^{PR} - \bar{E}_j^k; \mathcal{E}_k), \tag{7.9c}$$

$$\frac{1}{2}(\bar{E}_{2j-1}^{k-1} + \bar{E}_{2j}^{k-1}) = \bar{E}_j^k. \tag{7.10a}$$

Subtracting (7.9c) from (7.10a) we get

$$\frac{1}{2}(\bar{E}_{2j}^{k-1} - \bar{E}_{2j-1}^{k-1}) = \bar{E}_j^k - \bar{E}^{PR} + tr(\bar{E}^{PR} - \bar{E}_j^k; \mathcal{E}_k). \tag{7.10b}$$

Let us now examine the two possibilities in (7.10):

$$|\bar{E}_j^k - \bar{E}^{PR}| > \mathcal{E}_k \Rightarrow \frac{1}{2}(\bar{E}_{2j-1}^{k-1} + \bar{E}_{2j}^{k-1}) = 0 \Rightarrow \bar{E}_{2j-1}^{k-1} = \bar{E}_{2j}^{k-1} = \bar{E}_j^k, \tag{7.11a}$$

$$|\bar{E}_j^k - \bar{E}^{PR} \le \mathcal{E}_k \Rightarrow \left\{\begin{array}{l} \frac{1}{2}(\bar{E}_{2j}^{k-1} - \bar{E}_{2j-1}^{k-1}) = \bar{E}_j^k - \bar{E}^{PR} \\ \frac{1}{2}(\bar{E}_{2j}^{k-1} + \bar{E}_{2j-1}^{k-1}) = \bar{E}_j^k \end{array}\right. . \tag{7.11b}$$

From (7.11) we get the following inequalities

(7.12a) $$\max(|\bar{E}^{k-1}_{2j-1}|, |\bar{E}^{k-1}_{2j}|) = \frac{1}{2}|\bar{E}^{k-1}_{2j} + \bar{E}^{k-1}_{2j-1}| + \frac{1}{2}|\bar{E}^{k-1}_{2j} - \bar{E}^{k-1}_{2j-1}| \le |\bar{E}^k_j| + \mathcal{E}_k,$$

(7.12b)
$$(|\bar{E}^{k-1}_{2j-1}| + |\bar{E}^{kj-1}_{2j}|) = \max(|\bar{E}^{k-1}_{2j} + \bar{E}^{k-1}_{2j-1}|, |\bar{E}^{k-1}_{2j} - \bar{E}^{k-1}_{2j-1}|) \le 2\max(|\bar{E}^k_j|, \mathcal{E}_k).$$

Recalling that $\bar{E}^L = 0$ we get from (7.12a)

(7.13a) $$\|\bar{E}^{k-1}\|_\infty \le \|\bar{E}^k\|_\infty + \mathcal{E}_k \le \cdots \le \sum_{\ell=k}^{L} \mathcal{E}_\ell;$$

Recalling that $h_{k-1} = \frac{1}{2}h_k$ we get from (7.12b)

(7.13b) $$\begin{aligned}\|\bar{E}^{k-1}\|_{\ell_1} = h_{k-1}\sum_{i=1}^{N_{k-1}} |\bar{E}^{k-1}_i| &= h_{k-1}\sum_{j=1}^{N_k}(|\bar{E}^{k-1}_{2j-1}| + |\bar{E}^{k-1}_{2j}|) \\ &\le h_k \sum_{j=1}^{N_k} \max(|\bar{E}^k_j|, \mathcal{E}_k).\end{aligned}$$

It follows from (7.13) that

(7.14a) $$\|\bar{E}^0\|_\infty \le \sum_{\ell=1}^{L} \mathcal{E}_\ell,$$

and if we choose $\{\mathcal{E}_\ell\}_{\ell=1}^L$ such that

(7.14b) $$\mathcal{E}_k \ge \sum_{m=k+1}^{L} \mathcal{E}_m$$

then the ℓ_1-error is

(7.14c) $$\|\bar{E}^0\|_{\ell_1} \le \mathcal{E}_1.$$

Given $\mathcal{E}$ it makes good sense (see Remark 7.1) to choose the tolerance-levels $\mathcal{E}_k$ to be

(7.15a) $$\mathcal{E}_k = \mathcal{E}/2^k, \qquad 1 \le k \le L,$$

in which case we get in (7.7b)

$$\|\bar{f}^0 - \tilde{f}^0\|_\infty = \|\bar{E}^0\|_\infty \le \mathcal{E}, \tag{7.15b}$$

$$\|\bar{f}^0 - \tilde{f}^0\|_{\ell_1} = \|\bar{E}^0\|_{\ell_1} \le \mathcal{E}/2. \tag{7.15c}$$

We turn now to examine the multi-resolution expansion (6.9a) corresponding to the modified encoding algorithm

$$\tilde{R}(x; f) = R_L(x; \tilde{f}^L) + \sum_{k-1}^{L} \sum_{j=1}^{N_k} \tilde{d}_j^k \bar{\psi}_j^k(x); \tag{7.16a}$$

here we use R generically for both interpolation and reconstruction from cell-averages. When R is a linear procedure which is projective in the sense of (6.8) we get that

$$\sum_{j=1}^{N_k} \tilde{d}_j^k \bar{\psi}_j^k(x) = R_{k-1}(x; \tilde{f}^{k-1}) - R_k(x; \tilde{f}^k),$$

and consequently in (7.16a)

$$\tilde{R}(x; f) = R_0(x; \tilde{f}^0). \tag{7.16b}$$

We observe that although $\tilde{d}_j^k = 0$ wherever the appropriate truncation criterion in (7.4b), (7.8c) is met, the resulting approximation satisfies the specified accuracy requirement, i.e. for interpolation

$$|\tilde{R}(x_j^0; f) - f(x_j^0)| \le \mathcal{E}, \quad 0 \le j \le N_0, \tag{7.17a}$$

and for reconstruction from cell-averages

$$|A(I_j^0) \cdot \tilde{R}(\cdot; f) - \bar{f}_j^0| \le \mathcal{E}, \quad 1 \le j \le N_0, \tag{7.17b}$$

and also

$$\frac{1}{N_0} \sum_{j=1}^{N_0} |A(I_j^0) \cdot \tilde{R}(\cdot; f) - \bar{f}_j^0| \le \mathcal{E}/2. \tag{7.17c}$$

We see that using the coefficients $\{\tilde{d}_j^k\}$ which are obtained from the modified algorithm and dropping $\bar{\psi}_j^k(x)$ for which $\tilde{d}_j^k = 0$, we can get a compressed representation of $f(x)$ which is accurate in the sense of (7.17) to a prescribed tolerance.

Remark 7.1. Given cell-averages $\{\bar{f}_j^0\}_{j=1}^{N_0}$ we could evaluate the point-values of the primitive function $\{F(x_j^0)\}_{j=0}^{N_0}$ by (5.5c) and apply the interpolatory data-compression algorithm (7.2) - (7.4) to these input data. Observe that the uniform tolerance in (7.5) corresponds in this case to the geometric choice (7.15a) for the cell-averages. Also observe that there is no need to prepare the multi-resolution analysis (7.8a) in this case. Hence even if we select to use the algorithm (7.6) - (7.8) for the cell-averages it pays to use reconstruction via primitive function (5.6).

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