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PROPERTIES OF SIMPLE MODEL PROBLEMS FOR REACTING GAS FLOWS

J.K. TEGNÉR *

Abstract. The compressible Navier-Stokes equations for reacting gases are extremely complex. Simpler models have been considered, and for these completely non-physical propagation speeds have been observed. These model problems are stiff, meaning that several different scales are present in the solution. Numerical solution of non-reacting flows almost always involves addition of extra dissipation. It will be shown that this action will render a totally wrong propagation speed for a simple model equation of reacting flows. This problem will be accentuated by increasing stiffness of the problem. Existence and uniqueness of a solution to this model equation is proved. The dependence of the propagation speed on the viscosity and a term governing the stiffness (comparable to the reaction rate for a more complete model) is investigated. A remedy for the wrong propagation speed for this simple model equation is proposed such that the speed is correct though the front is smeared out.

AMS(MOS) subject classification. 65P05,76L05,80A32

Key Words. model problems, traveling waves, artificial viscosity, unphysical propagation speed, "artificial reaction rate"

1. Introduction. The equations of gas dynamics can nowadays be solved numerically by more or less standard methods. New difficulties arise when nonequilibrium gas thermodynamics has to be considered, i.e. chemical reactions are included in the model. This is the case for combustion processes as well as for certain regions of the flow field in hypersonic aerodynamics. Taking many reaction steps into account obviously give a very complex model, and numerical methods are necessary for studying such systems. But even greatly simplified systems are difficult to solve, and much effort has been devoted to understand how numerical methods should be designed to give correct overall pictures of the processes. These model problems can be made more or less complex by applying different levels of idealization of the chemistry.

In the equilibrium thermodynamic case, all numerical methods in current use employ the technique of adding extra dissipation to get a well behaved numerical solution without the use of excessively small computational cells when the problem contains scales which differ by several orders of magnitude. This is usually referred to as artificial viscosity. The result of this action is to smear out all steep gradients in the solution field, but if the extra terms are constructed in a particular fashion, the overall picture is still correct, even in the presence of strong shocks as is the case for hypersonic flows.

When the same procedure is uncritically applied to problems where the chemical reactions are important it will make the reaction waves move with incorrect velocities.

The compressible form of the continuity, Navier-Stokes and energy equations are the standard mathematical model for high temperature flows. When augmented by the models for species concentrations the equations for a reacting mixture can be immensely complex because of the many steps involved in the chemical reactions. For a complete discussion of these equations cf. [1, 2].

The numerical solution of these equations present difficulties not present in the case of non-reacting flows. One of the main difficulties is a result of the widely varying time scales of the different mechanisms in the problem. As an analogy we offer the problems inherent in simulating mechanical systems where some members are very stiff compared to others, and which are usually called "stiff" problems.

In the combustion problem, typically, the time scales of the chemical reactions are orders of magnitude faster than the time scales of the fluid dynamics.

If one tries to advance the solution with time steps appropriate for the slower scales the result might be violent numerical instability. On the other hand, it is uneconomical to use steps to resolve the fastest

* Center for Computational Mathematics and Mechanics, Royal Institute of Technology, S–100 44 Stockholm, Sweden. Partially supported by U.S. Navy contract ONR N00014-90-J-1382 and NSF contract DMS87-19952

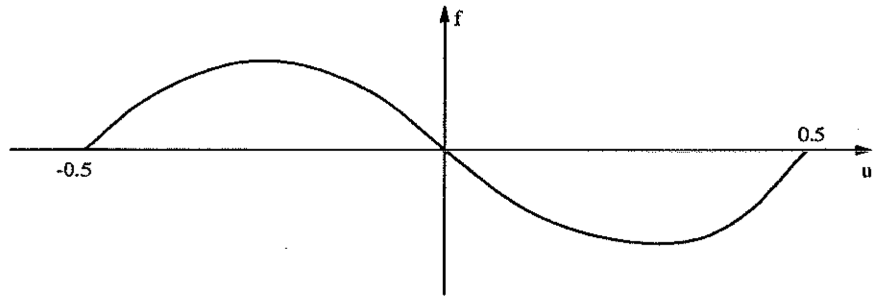

FIG. 1. *The form of $f(u)$*

scale. It is therefore valuable to develop methods which are stable and give an overall correct picture even though they do not capture the fine details governed by the fastest scale. For instance, the steep gradients may be smeared out but it is necessary that fronts - shocks and flame fronts - move at the correct speed and without excessive overshoots.

A future goal is to construct robust numerical methods for this class of problems. This has to be done in several steps. Here, we will study a simplified problem consisting of only one equation, essentially Burger's equation

$$u_t + g(u)u_x = -\mu f(u) + \epsilon u_{xx} \tag{1}$$

with a source term $\mu f(u)$ mimicking the chemistry. The stiffness of the problem can be varied by changing the factor μ in the source term. We will use the following requirements on $f(u)$ in order to prove the existence of a unique, smooth traveling wave solution of Eq.(1).

- $f(u) > 0$ for $-\frac{1}{2} < u < 0$
- $f(u) < 0$ for $0 < u < \frac{1}{2}$
- $f(-\frac{1}{2}) = f(0) = f(\frac{1}{2}) = 0$
- $f'(-\frac{1}{2}) > 0,\ f'(0) < 0,\ f'(\frac{1}{2}) > 0.$

I.e. we require $f(u)$ to have the form depicted in Fig.(1). In addition, the analysis when $\epsilon\mu \to 0$ is simplified if $g(u)$ fulfills

- $g(u)$ is a strictly monotone function.
- $g(u)$ and $f(u)$ are such that $lim_{u\to 0} \frac{f(u)}{g(u)-g(0)} \neq 0$ and bounded.

As boundary conditions we use $u = -\frac{1}{2}$ in front of "the reaction zone" and $u = \frac{1}{2}$ behind, or vice versa.

By making the traveling wave ansatz $u(x,t) = \psi(y),\ y = x + st$, where s is the speed of the wave, Eq.(1) can be written as

$$s\psi' + g(\psi)\psi' = -\mu f(\psi) + \epsilon\psi'' \tag{2}$$

with boundary conditions

$$\begin{cases} \lim_{y\to\infty} \psi(y) = \frac{1}{2} \\ \lim_{y\to-\infty} \psi(y) = -\frac{1}{2} \end{cases} \tag{3}$$

or

$$\begin{cases} \lim_{y\to\infty} \psi(y) = -\frac{1}{2} \\ \lim_{y\to-\infty} \psi(y) = \frac{1}{2} \end{cases} \tag{4}$$

This model is simple enough to allow some of its properties to be extracted by analytic means, and the numerics can then be tested against theory. For example, it is possible to obtain traveling wave solutions which look physically reasonable but which move with an incorrect propagation speed if artificial viscosity is used. Here it is noted that this erroneous behavior becomes more and more pronounced when the stiffness of the problem is increased. It also turns out that for this simplified problem the wave speed depends on the viscosity ϵ and the reaction rate here μ in the combination $\epsilon\mu$.

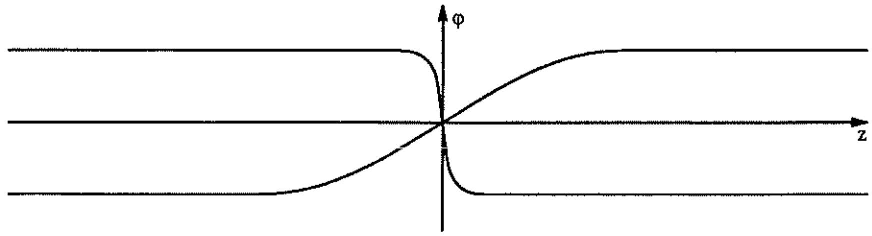

FIG. 2. *Solutions for* $\epsilon\mu \ll 1$

In [3], the problem

$$u_t + u_x = -\mu u(u - 0.5)(u + 0.5) \tag{5}$$

is studied. This equation is solved with different numerical methods and it is shown that the evolving traveling wave gets an unphysical speed when the problem is stiff, i.e. μ is large. This non-physical behavior is explained to be the result of the smearing of the discontinuity.

In [4], the slightly more complex model

$$u_t + \left(\frac{1}{2}u^2 - q_0 Z\right)_x = \beta u_{xx} \tag{6}$$

$$Z_x = K\Phi(u)Z \tag{7}$$

with $K\Phi(u) = K_0 u^\alpha e^{-A/u}$ is considered. Here u is a lumped variable with some features of pressure or temperature, Z is the mass fraction of unburnt gas, q_0 is the heat release and $\beta \geq 0$ is a lumped diffusion coefficient. It is shown that the same non-physical behavior occurs when this set of equations is solved with a specific numerical method provided $K\Delta x$ is sufficiently large or q_0 is large enough for a fixed mesh size (Δx is the increment in the one-space dimension).

In this paper we will prove the existence of traveling wave solutions of Eq.(1) with both sets of boundary conditions, i.e. $u = -\frac{1}{2}$ in front of the reaction zone and $u = \frac{1}{2}$ behind, or vice versa. This differs from the case when the source term $\mu f(u)$ is absent, were only one of the boundary conditions can result in a traveling wave. The same is true when the diffusive term ϵu_{xx} is absent.

For $\epsilon\mu \gg 1$ both sets of boundary conditions will give solutions where the speed of the traveling wave solution is proportional to $\sqrt{\epsilon\mu}$. This knowledge can be used as a guideline when constructing numerical methods for this kind of problem. In realistic cases the viscosity ϵ is very small and one uses an artificial viscosity ϵ_a in order to get a well behaved solution. If now the product $\epsilon_a\mu$ is large one cannot hope to get a good approximate solution of the problem at hand. The remedy, for this model equation, is to use an "artificial reaction rate", μ_a, such that $\epsilon_a\mu_a = \epsilon\mu$. By this procedure the reaction front is smeared out but the speed of the wave is still correct. For $\epsilon\mu \gg 1$ the two solutions have reaction fronts with a thickness of $O(\sqrt{\epsilon/\mu})$.

For $\epsilon\mu \ll 1$ it holds that the speed of the traveling wave solution converges to a finite value for both boundary conditions (3) and (4). That is, if the artificial viscosity ϵ_a is such that the product $\epsilon_a\mu$ is still small the character of the numerical solution is only marginally changed by the increased viscosity. If $g'(\varphi) > 0$ the solution satisfying the boundary conditions (4) has a reaction front of thickness $O(\epsilon)$. For the solution satisfying the boundary conditions (3) the thickness of the reaction front is of $O(1/\mu)$. Since $\epsilon \ll 1/\mu$ it follows that the two solutions above will have the qualitative picture given in Fig.(2). If $g'(\varphi) < 0$ the situation is reversed and the boundary conditions (3) yields the thickness $O(\epsilon)$ and (4) gives the thickness $O(1/\mu)$.

In section 2 we prove existence and uniqueness of two solutions of Eq.(2) satisfying the boundary conditions (3) or (4).

The propagation speed of the traveling wave is analyzed as $\epsilon\mu \to 0$ and $\epsilon\mu \to \infty$ with both sets of boundary conditions in section 3.

Finally in section 4, the theoretical results in section 3 are verified. We use specific $f(u)$ and $g(u)$ and discretize Eq.(1) by a finite difference scheme in space and use a Runge-Kutta method to integrate the solution in time.

2. Existence and Uniqueness. By making the scaling

$$y = \epsilon \hat{y} \text{ and } \hat{f}(u) = \epsilon \mu f(u) \tag{8}$$

Eq.(2) can be written

$$s\psi' + g(\psi)\psi' = -f(\psi) + \psi'' \tag{9}$$

The boundary conditions are

$$\begin{cases} \lim_{y\to\infty} \psi(y) = \frac{1}{2} \\ \lim_{y\to-\infty} \psi(y) = -\frac{1}{2} \end{cases} \tag{10}$$

or

$$\begin{cases} \lim_{y\to\infty} \psi(y) = -\frac{1}{2} \\ \lim_{y\to-\infty} \psi(y) = \frac{1}{2} \end{cases} \tag{11}$$

Here we have neglected the hats over $\hat{f}(\psi)$ and $\hat{y}$.

The position of the wave is fixed by the requirement $\psi(0) = 0$.

Eq.(9) can be written as a system of first order equations by introducing $w = \psi'$

$$\begin{pmatrix} \psi \\ w \end{pmatrix}' = \begin{pmatrix} w \\ (s + g(\psi))w + f(\psi)) \end{pmatrix} \tag{12}$$

Under the assumption that $\psi(y_1) \neq \psi(y_2)$ for $y_1 \neq y_2$ we get

$$\frac{dw}{d\psi} = s + g(\psi) + \frac{f(\psi)}{w}, \; w(-0.5) = w(0.5) = 0 \tag{13}$$

where w now is considered to be a function of ψ.

A smooth solution of Eq.(13) for which $w(\psi) > 0, \; \psi \in (-0.5, 0.5)$ [1]generates a smooth solution of Eq.(9) satisfying the boundary conditions (10). Similarly, a smooth solution for which $w(\psi) < 0, \; \psi \in (-0.5, 0.5)$ generates a smooth solution of Eq.(9) satisfying the boundary conditions (11).

This follows since the properties of $w(\psi)$ above and

$$\frac{d\psi}{dy} = w(\psi) \text{ and } \psi(0) = 0 \tag{14}$$

defines two unique, monotone, smooth solutions $\psi(y)$.

2.1. Existence of a Monotonically Increasing Solution. THEOREM 2.1. *There exists a unique s and a unique, smooth solution of Eq.(13) satisfying $w(\psi) > 0, \; \psi \in (-0.5, 0.5)$.*

To prove theorem 2.1 we will proceed in the following way.

We will look for solutions of Eq.(13) in the two separate intervals $-\frac{1}{2} \leq \psi \leq 0$ and $0 \leq \psi \leq \frac{1}{2}$ emerging from the points $\psi = -\frac{1}{2}, \; w = 0$ and $\psi = \frac{1}{2}, \; w = 0$ respectively. Denote these solutions by $w^-(\psi; s)$ and $w^+(\psi; s)$, s is considered as a parameter.

We will show that a unique $w^-(\psi; s) > 0, \; \psi \in (-0.5, 0), \; w^-(0; s) \geq 0$ exists for every s. Correspondingly, a unique $w^+(\psi; s) > 0, \; \psi \in (0, 0.5), \; w^+(0; s) \geq 0$ exists for every s.

It will be shown that $w^-(\psi; s)$ and $w^+(\psi; s)$ can be matched together for a unique s, i.e. there is exactly one $s = s^*$ for which $w^-(0; s^*) = w^+(0; s^*) > 0$.

It will also be shown that this solution is smooth.

The lemmas below are needed in the proof of theorem 2.1.

LEMMA 2.2. *The point $(-0.5, 0)$ is a saddle point in the (ψ, w)-plane and there exists a unique solution of Eq.(12) leaving the point with increasing ψ and w^- as y increases. Further, this solution is continuous in both y and the parameter s.*

[1] by $\psi \in (a, b]$ we denote $a < \psi \leq b$

Proof: Linearize Eq.(12) around the critical point $(\psi, w) = (-0.5, 0)$, i.e. put $\psi = -0.5 + x_1$ and $w = x_2$ and neglect the non-linear terms

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ where } A = \begin{pmatrix} 0 & 1 \\ f'(-0.5) & s + g(-0.5) \end{pmatrix} \tag{15}$$

Where by $f'(-0.5)$ we denote $\frac{df(-0.5)}{d\psi}$. The eigenvalues of the system matrix A above are

$$\lambda_{1,2} = \frac{s + g(-0.5)}{2} \pm \sqrt{(\frac{s + g(-0.5)}{2})^2 + f'(-0.5)} \tag{16}$$

and since $f'(-0.5) > 0$ the eigenvalues are of different sign, $\lambda_2 < 0 < \lambda_1$. Hence $(-0.5, 0)$ is a saddle point.

The eigenvectors corresponding to the different eigenvalues are

$$\frac{1}{\sqrt{1+\lambda_1^2}} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \text{ and } \frac{1}{\sqrt{1+\lambda_2^2}} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \tag{17}$$

The eigenvector corresponding to the positive eigenvalue specifies the direction for the solution which leaves the saddle point as y increases. The only possible solution is the one which leaves the critical point with increasing values on ψ and w (the orbit which leaves the saddle point with decreasing values on ψ and w can never reach the critical point $(0.5, 0)$).

For existence, uniqueness and continuity see [5].

•

LEMMA 2.3. *If $w^-(\psi; s)$ leaves the saddle point as a monotonic increasing function of ψ then $0 < w^-(\psi; s) \leq C$, $\psi \in (-0.5, 0)$. It also holds that $0 \leq w^-(0; s) \leq C$. Here C is a constant which depends on $f(\psi)$, $g(\psi)$, $\psi \in [-0.5, 0]$ and s.*

Proof: For $w > 0$ all solution curves are such that ψ increases with increasing y. From Eq.(12) it follows that in $-0.5 < \psi < 0$ all solution curves to this system increases in w if w is sufficiently small. I.e. the orbit leaving the saddle point $(-0.5, 0)$ with increasing ψ and w cannot cross the axis $w = 0$ in $-0.5 < \psi < 0$.

By doing the same linearization as in the proof of lemma 2.2, but this time around the critical point $(0, 0)$, we obtain the eigenvalues

$$\lambda_{1,2} = \frac{s + g(0)}{2} \pm \sqrt{(\frac{s + g(0)}{2})^2 + f'(0)} \tag{18}$$

and the corresponding eigenvectors

$$\frac{1}{\sqrt{1+\lambda_1^2}} \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \text{ and } \frac{1}{\sqrt{1+\lambda_2^2}} \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \tag{19}$$

For varying s we have the following possibilities, [6]

- For $s \geq -g(0) + 2\sqrt{-f'(0)}$ both eigenvalues are positive. Hence all orbits are directed away from the origin in a small neighborhood of the critical point $(0, 0)$ as y increases.
- For $-g(0) - 2\sqrt{-f'(0)} < s < -g(0) + 2\sqrt{-f'(0)}$ the origin is either a center or a spiral point.
- For $s \leq -g(0) - 2\sqrt{-f'(0)}$ both eigenvalues are negative and the origin is an improper node with limiting directions specified by the eigenvectors.

In all of the cases above $w^-(0; s) \geq 0$, equality is only possible in the last case. If $w^-(0; s) = 0$ there is no possibility to find a solution connecting $(-0.5, 0)$ with $(0.5, 0)$.

It remains to prove that $w^-(\psi; s) \leq C$. If $w \geq 1$ Eq.(13) yields

$$\frac{dw}{d\psi} \leq C_0 \tag{20}$$

Here C_0 is a constant which depends on f, g and s. From this it follows that $w(\psi; s) \leq 1 + C_0(\psi + 0.5)$. We conclude that $C = 1 + \frac{C_0}{2}$.

•

LEMMA 2.4. *If $w^-(0; s) > 0$ then $w_s =: \frac{dw^-}{ds} > 0$ for every fixed $\psi \in (-0.5, 0]$.*

Proof: Eq.(13) gives us

$$\frac{dw_s}{d\psi} = 1 - \frac{f(\psi)}{w^2} w_s \tag{21}$$

Therefore $\frac{dw_s}{d\psi} > 0$ in $-\frac{1}{2} < \psi < 0$ for sufficiently small w_s. It is obvious that $w_s(-\frac{1}{2}) = 0$ and lemma 2.2 yields $\frac{dw_s}{d\psi}(-\frac{1}{2}) > 0$. This follows since $\frac{dw(-0.5)}{d\psi} = \lambda_2$ and $\frac{d\lambda_2}{ds} > 0$. It then follows that $w_s(\psi) > 0, \ -\frac{1}{2} < \psi \leq 0$.

•

LEMMA 2.5. *There exists a $s = s^- \leq -g(0) - 2\sqrt{-f'(0)}$ such that*

$$w(0; s) = \begin{cases} 0 & \text{for } s \leq s^- \\ > 0 & \text{for } s > s^- \\ +\infty & \text{for } s \to \infty \end{cases} \tag{22}$$

Further, $w^-(0; s)$ varies continuously with s.

Proof: We have

$$\frac{dw^-}{d\psi} = s + g(\psi) + \frac{f(\psi)}{w^-} \tag{23}$$

From lemma 2.3 we know that $w^-(\psi; s) \geq 0, \ -\frac{1}{2} \leq \psi \leq 0$ and since $f(\psi) \geq 0, \ -\frac{1}{2} \leq \psi \leq 0$ this leads to

$$\frac{dw^-}{d\psi} > s + g(\psi) \tag{24}$$

in the same interval. Thus, $w^-(0; s) > \frac{s + g_{min}}{2}$ as $s \to \infty$.

For negative s assume that

$$-\frac{f_{max} + C_1}{s} \leq w^-(\psi; s) \tag{25}$$

where C_1 is a positive constant. This gives us

$$\frac{dw^-}{d\psi} \leq \frac{C_1}{f_{max} + C_1} s + g(\psi) \tag{26}$$

Therefore, for $s < -g_{max} \frac{f_{max} + C_1}{C_1}$ we have that $\frac{dw^-}{d\psi} < 0$ and hence that $w^-(\psi; s) \leq -\frac{f_{max} + C_1}{s}$. This is also valid for $\psi = 0$, i.e. $0 \leq w^-(0; s) \leq -\frac{f_{max} + C_1}{s}$ as $s \to -\infty$.

We know from above that by making s sufficiently negative we can get the solution arbitrarily close to the origin. For $s \leq -g(0) - 2\sqrt{-f'(0)}$ the origin is an attractor. Hence there exists an $s^- \leq -g(0) - 2\sqrt{-f'(0)}$ such that the solution tends to the origin as $y \to \infty$ for $s \leq s^-$.

For continuity in the parameter s see [6].

•

LEMMA 2.6. *For $s = -g(0)$ we have $w^-(0; s) > 0$ i.e. $w^-(\psi) > 0, \ -\frac{1}{2} < \psi \leq 0$.*

Proof: This follows since with this choice on s the origin is either a center or a spiral point, [6].

•

By making the substitution

$$\tilde{y} = -y,\ \tilde{\psi} = -\psi,\ \tilde{s} = -s,\ \tilde{g}(\tilde{\psi}) = -g(-\psi) \text{ and } \tilde{f}(\tilde{\psi}) = -f(-\psi) \tag{27}$$

Eq.(12) transforms into

$$\frac{d}{d\tilde{y}}\begin{pmatrix} \tilde{\psi} \\ w \end{pmatrix} = \begin{pmatrix} w \\ (\tilde{s} + \tilde{g}(\tilde{\psi}))w + \tilde{f}(\tilde{\psi})) \end{pmatrix} \tag{28}$$

where $\tilde{f}(\tilde{\psi})$ and $\tilde{g}(\tilde{\psi})$ satisfies the requirements given in section 1.

For these new variables the proofs of the previous lemmas are valid. After transforming back to the original variables the lemmas below follow.

LEMMA 2.7. *The point* $(0.5, 0)$ *is a saddle point in the* ψ, w*-plane and there exists a unique solution of Eq.(12) leaving the point with decreasing* ψ *and increasing* w^+ *as* y *decreases. Further, this solution is continuous in both* y *and the parameter* s.

LEMMA 2.8. *If* $w^+(\psi; s)$ *leaves the saddle point with increasing* w^+ *and decreasing* ψ. *It then holds that* $0 < w^+(\psi; s) \leq C,\ \psi \in (0, 0.5)$. *It also holds that* $0 \leq w^+(0; s) \leq C$. *Here* C *is a constant which depends on* $f(\psi),\ g(\psi),\ \psi \in [0, 0.5]$ *and* s.

LEMMA 2.9. *If* $w^+(0; s) > 0$ *then* $w_+ =: \frac{dw^+}{ds} < 0$ *for every fixed* $\psi \in [0, 0.5)$.

LEMMA 2.10. *There exists a* $s = s^+ \geq -g(0) + 2\sqrt{-f'(0)}$ *such that*

$$w(0; s) = \begin{cases} 0 & \textit{for } s \geq s^+ \\ > 0 & \textit{for } s < s^+ \\ +\infty & \textit{for } s \to -\infty \end{cases} \tag{29}$$

Further, $w^+(0; s)$ *varies continuously with* s.

LEMMA 2.11. *For* $s = -g(0)$ *we have* $w^+(0; s) > 0$ *i.e.* $w^+(\psi; s) > 0,\ 0 \leq \psi < \frac{1}{2}$.

Proof of Theorem 2.1:

Lemma 2.3 and lemma 2.8 shows that $w^-(\psi; s)\ w^+(\psi; s)$ will reach the axis $\psi = 0$ with finite values.

Suppose $w^-(0; -g(0)) < w^+(0; -g(0))$. The case $w^-(0; -g(0)) > w^+(0; -g(0))$ is treated in the same way. From lemma 2.6 and lemma 2.11 it follows that $0 < w^-(0; -g(0)) < w^+(0; -g(0))$. By increasing s we will get $w^-(0; s)$ to increase and $w^+(0; s)$ to decrease, this follows from lemma 2.4 and lemma 2.9.

Lemma 2.5 and lemma 2.10 gives that there exists a $s = s^*$ such that $-g(0) < s^* < s^+$ for which it holds $w^-(0; -g(0)) < w^-(0; s^*) = w^+(0; s^*) < w^+(0; -g(0))$.

The uniqueness of s^* is guaranteed by lemma 2.4 and lemma 2.9.

The fact that $w(\psi) > 0,\ -\frac{1}{2} < \psi < \frac{1}{2}$ follows from lemma 2.3 and lemma 2.8 together with $w(0; s^*) > 0$.

The existence, continuity and uniqueness of $w(\psi)$ follows from lemma 2.2 and lemma 2.7.

We have found a continuous solution of Eq.(13). From the Stable and Unstable Manifold Theorem, [5], we know that this solution is as smooth as the righthand side of the equation in a neighborhood of the saddle points $(-0.5, 0)$ and $(0.5, 0)$. In the rest of the interval, where those neighborhoods are excluded, $w > 0$ and $\frac{dw}{d\varphi}$ is bounded and continuous. Continued derivation of Eq.(13) yields that $w(\varphi)$ is smooth.

•

2.2. Existence of a Monotonically Decreasing Solution. Now we investigate the boundary conditions from (11).

In the same way as in section 2.1 we can show

THEOREM 2.12. *There exists a unique* s *and a unique, smooth solution of Eq.(13) satisfying* $w(\psi) < 0,\ \psi \in (-0.5, 0.5)$.

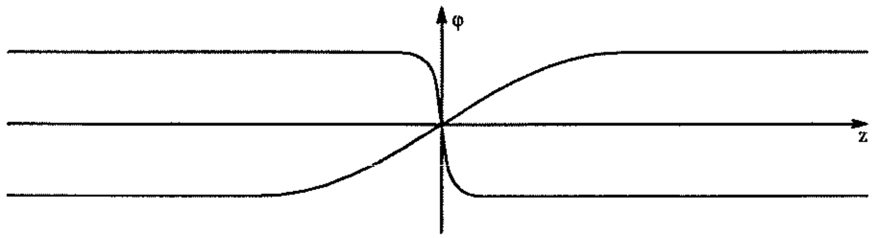

FIG. 3. *Solutions for* $\epsilon\mu \ll 1$

3. Properties of the Solution. In this section we will investigate the propagation speed of the traveling wave solution for small and large values of the product $\epsilon\mu$ with both sets of boundary conditions.

We will investigate if Eq.(2) can be simplified for certain values of the parameters ϵ and μ. Therefore, let $\varphi(z) = \psi(y)$ where $z = \frac{y}{\alpha}$ and α is a scaling factor. This leads to

$$\frac{1}{\alpha}(s + g(\varphi))\varphi' = -\mu f(\varphi) + \frac{\epsilon}{\alpha^2}\varphi'' \tag{30}$$

The boundary conditions are

$$\begin{cases} \lim_{z\to\infty} \varphi(z) = \frac{1}{2} \\ \lim_{z\to-\infty} \varphi(z) = -\frac{1}{2} \end{cases} \tag{31}$$

or

$$\begin{cases} \lim_{z\to\infty} \varphi(z) = -\frac{1}{2} \\ \lim_{z\to-\infty} \varphi(z) = \frac{1}{2} \end{cases} \tag{32}$$

In section 2 we proved the existence of solutions of Eq.(30) with either (31) or (32) as boundary conditions. Our plan here is to investigate the behavior of these two solutions for $\epsilon\mu \to 0$ and $\epsilon\mu \to \infty$.

If we assume that $g'(\varphi) > 0$ and that ϵ, μ are given so that $\epsilon\mu \ll 1$ we have

1. For $\alpha = \mu^{-1}$ the solution of Eq.(30) satisfying the boundary conditions (31) has derivatives of O(1). For this solution

$$\lim_{\epsilon\mu\to 0} s = -g(0) \tag{33}$$

2. For $\alpha = \epsilon$ the solution of Eq.(30) satisfying the boundary conditions (32) has derivatives of O(1). For this solution

$$\lim_{\epsilon\mu\to 0} s = -\int_{-0.5}^{0.5} g(\varphi)\, d\varphi \tag{34}$$

If $\epsilon\mu \ll 1$ and if the two solutions above are considered on the same scale we realize that the solution in (2) above will be much steeper than the one in (1). This follows since $\epsilon\mu \ll 1 \Leftrightarrow \epsilon \ll 1/\mu$. See Fig.(3) for the case $g'(\varphi) > 0$.

If $g'(\varphi) < 0$ the situation is reversed. I.e. the solution of Eq.(30) satisfying the boundary conditions (31) have derivatives of O(1) if an $\alpha = \epsilon$ is used. Likewise, the solution satisfying the boundary conditions (32) have derivatives of O(1) if an $\alpha = 1/\mu$ is used.

We conclude that for $\epsilon\mu \ll 1$ one of the two solutions of Eq.(30) have derivatives of O(1) if $\alpha = 1/\mu$ is used. This solution is considered in section 3.1. The solution satisfying the other set of boundary conditions is considered in section 3.2 when an $\alpha = \epsilon$ is used.

In section 3.3 we will investigate the nature of the two solutions of Eq.(30) for large values of the product $\epsilon\mu$. Independent of the sign of $g'(\varphi)$ we show that for given ϵ, μ with $\epsilon\mu \gg 1$ we have

1. The two solutions of Eq.(30) satisfying the boundary conditions (31) or (32) have bounded derivatives when $\alpha = \sqrt{\frac{\epsilon}{\mu}}$ is used. The important difference here is that for this case, the speed of the traveling wave solutions is proportional to $\sqrt{\epsilon\mu}$.

3.1. The Scaling $\alpha = 1/\mu$. In this section we will consider the solution of Eq.(30) which has derivatives of O(1) if $\alpha = 1/\mu$ is used for $\epsilon\mu \ll 1$. If $g'(\varphi) > 0$ this solution satisfies the boundary conditions (31). If $g'(\varphi) < 0$ this solution satisfies the other set of boundary conditions.

With the above choice of α we have

(35) $$(s + g(\varphi))\varphi' = -f(\varphi) + \delta\varphi'' \text{ where } \delta = \epsilon\mu$$

In the same way as for Eq.(13) this can, by introducing $w = \varphi'$, be written

(36) $$\frac{dw}{d\varphi} = \frac{1}{\delta}\left(s + g(\varphi) + \frac{f(\varphi)}{w}\right),\ w(-0.5) = w(0.5) = 0$$

From section 2 we know that Eq.(36) has two solutions. We shall use the notation so that $w^{(1)}(\varphi) > 0$ and $w^{(2)}(\varphi) < 0$ for $\varphi \in (-0.5, 0.5)$. Below we will show that one of these solutions, together with its derivative will stay bounded as $\epsilon\mu \to 0$.

Let

(37) $$w_0(\varphi) = \frac{f(\varphi)}{g(0) - g(\varphi)} \text{ and } s_0^* = -g(0)$$

Hence the ansatz $w = w_0 + \delta p$ and $s = s_0^* + \delta s_1$ in Eq.(36) gives

(38) $$p' = \frac{1}{\delta}\left(-\frac{f(\varphi)}{w_0(\varphi)}\frac{p}{w_0(\varphi) + \delta p} + s_1 - w_0'(\varphi)\right)$$

Let $w_1(\varphi)$ be the solution of

(39) $$-\frac{f(\varphi)}{w_0(\varphi)}\frac{w_1}{w_0(\varphi) + \delta w_1} + s_1^* - w_0'(\varphi) = 0$$

where $s_1^* = w_0'(0)$. This leads to

(40) $$w_1(\varphi) = \frac{s_1^* - w_0'(\varphi)}{g(0) - g(\varphi) - \delta(s_1^* - w_0'(\varphi))}w_0$$

Now make the ansatz $p = w_1 + \delta r$ and $s_1 = s_1^* + \delta s_2$ in Eq.(38)

(41) $$w_1'(\varphi) + \delta r' = \frac{1}{\delta}\left(-\frac{f(\varphi)}{w_0(\varphi)}\frac{w_1(\varphi) + \delta r}{w_0(\varphi) + \delta(w_1(\varphi) + \delta r)} + s_1^* + \delta s_2 - w_0'(\varphi)\right)$$

We have

(42) $$\frac{w_1(\varphi) + \delta r}{w_0(\varphi) + \delta(w_1(\varphi) + \delta r)} = \frac{w_1(\varphi)}{w_0(\varphi) + \delta w_1(\varphi)} + \delta\frac{w_0(\varphi)}{w_0(\varphi) + \delta w_1(\varphi)}\frac{r}{w_0(\varphi) + \delta(w_1(\varphi) + \delta r)}$$

If we use this in Eq.(41) and impose the boundary conditions from Eq.(36) we get

(43) $$r' = \frac{1}{\delta}\left(-\frac{f(\varphi)}{\tilde{w}_1(\varphi)}\frac{r}{\tilde{w}_1(\varphi) + \delta^2 r} + s_2 - w_1'(\varphi)\right),\ r(-0.5) = r(0.5) = 0$$

if we put $\tilde{w}_1(\varphi) = w_0(\varphi) + \delta w_1(\varphi)$

Consider the solutions of Eq.(43) in each of the intervals $\varphi \in [-0.5, 0]$ and $\varphi \in [0, 0.5]$. From section 2 we know that there exists two solutions in each of these intervals. We use the notation for these solutions

(44) $$r_1^-(\varphi; s_2) \text{ and } r_2^-(\varphi; s_2) \text{ for } \varphi \in [-0.5, 0]$$

and

(45) $$r_1^+(\varphi; s_2) \text{ and } r_2^+(\varphi; s_2) \text{ for } \varphi \in [0, 0.5]$$

From section 2 we also know that there exists a unique $s_2 = s_2^{(1)}$ such that r_1^- and r_1^+ can be matched together. Likewise, r_2^- and r_2^+ can be matched together for $s_2 = s_2^{(2)}$. Further,

$$\tilde{w}_1(\varphi) + \delta^2 r_1^-(\varphi; s_2) \geq 0 \text{ and } \tilde{w}_1(\varphi) + \delta^2 r_1^+(\varphi; s_2) \geq 0 \tag{46}$$

and

$$\tilde{w}_1(\varphi) + \delta^2 r_2^-(\varphi; s_2) \leq 0 \text{ and } \tilde{w}_1(\varphi) + \delta^2 r_2^+(\varphi; s_2) \leq 0 \tag{47}$$

These inequalities holds for all values on s_2 and equality is only possible at $\varphi = -0.5$, $\varphi = 0$ or $\varphi = 0.5$.

Note, r_1^- specifies the solution which leaves the saddle point $w = 0, \varphi = -0.5$ with increasing values on z. Also, r_2^+ specifies the solution which leaves the saddle point $w = 0, \varphi = 0.5$ with increasing values on z. Hence, the matching of these solutions will not result in a solution of Eq.(35). The same argument are used to exclude the matching of r_2^- and r_1^+.

For the matched solutions of Eq.(43) we shall use the notation $w_2^{(1)}(\varphi)$ and $w_2^{(2)}(\varphi)$, i.e.

$$w_2^{(1)}(\varphi) = \begin{cases} r_1^-(\varphi; s_2^{(1)}) \text{ if } \varphi \in [-0.5, 0] \\ r_1^+(\varphi; s_2^{(1)}) \text{ if } \varphi \in (0, 0.5] \end{cases} \text{ and } w_2^{(2)}(\varphi) = \begin{cases} r_2^-(\varphi; s_2^{(2)}) \text{ if } \varphi \in [-0.5, 0] \\ r_2^+(\varphi; s_2^{(2)}) \text{ if } \varphi \in (0, 0.5] \end{cases} \tag{48}$$

Finally, we have for the two solutions of Eq.(36)

$$w^{(1)}(\varphi) = w_0(\varphi) + \delta w_1(\varphi) + \delta^2 w_2^{(1)}(\varphi) \geq 0 \tag{49}$$

and

$$w^{(2)}(\varphi) = w_0(\varphi) + \delta w_1(\varphi) + \delta^2 w_2^{(2)}(\varphi) \leq 0 \tag{50}$$

with the corresponding values of s given by

$$s^{(1)} = s_0^* + \delta s_1^* + \delta^2 s_2^{(1)} \text{ and } s^{(2)} = s_0^* + \delta s_1^* + \delta^2 s_2^{(2)} \tag{51}$$

According to the way we have defined them we have $w^{(1)}(\varphi) > 0$ and $w^{(2)}(\varphi) < 0$ for $\varphi \in (-0.5, 0.5)$.

If $g'(\varphi) > 0$ it follows that $w_0(\varphi) > 0$, $\varphi \in (-0.5, 0.5)$. Hence there is no possibility for $\delta^2 w_2^{(2)}(\varphi)$ to remain small. This solution is instead considered in section 3.2 with an $\alpha = \epsilon$.

On the other hand, the perturbations in $w^{(1)}(\varphi)$ remains small with this choice of α and we can prove

THEOREM 3.1. *If $g'(\varphi) > 0$ then*

$$w^{(1)}(\varphi) \to w_0(\varphi) \text{ and } s^{(1)} \to s_0^* \text{ as } \delta \to 0 \tag{52}$$

The following bounds hold

$$\max_{\varphi \in [-0.5, 0.5]} |w_2^{(1)}(\varphi)| \leq K_1 \frac{1}{\delta}, \quad \max_{\varphi \in [-0.5, 0.5]} \left|\frac{d}{d\varphi} w_2^{(1)}(\varphi)\right| \leq K_2 \frac{1}{\delta^2} \text{ and } |s_2^{(1)}| \leq K_3 \tag{53}$$

If $g'(\varphi) < 0$ then

$$w^{(2)}(\varphi) \to w_0(\varphi) \text{ and } s^{(2)} \to s_0^* \text{ as } \delta \to 0 \tag{54}$$

and similar bounds hold for $w^{(2)}(\varphi)$, $\frac{d}{d\varphi} w_2^{(2)}(\varphi)$ and $s_2^{(2)}$.

The following lemmas are needed in the proof of the theorem above

LEMMA 3.2. *Assume that $g'(\varphi) > 0$. Then the following hold if $s_2 > d_1$*

$$0 < r_1^-(\varphi; s_2) \leq \frac{1}{\delta} C(\varphi + 0.5), \ \varphi \in [-0.5, 0) \tag{55}$$

and

$$0 > r_1^+(\varphi; s_2) \geq \frac{1}{\delta} C(\varphi - 0.5), \ \varphi \in (0, 0.5] \tag{56}$$

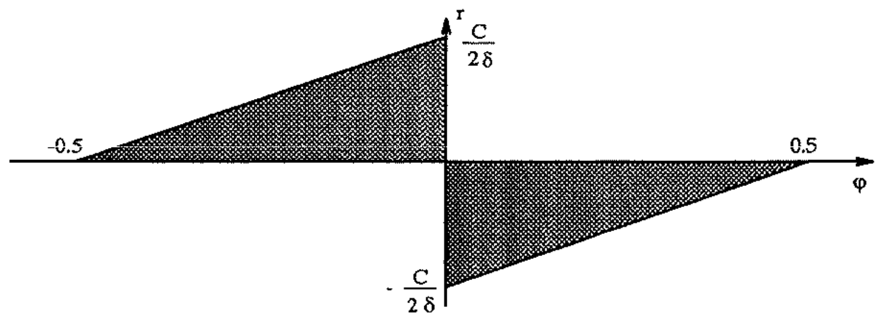

FIG. 4. *Bounds on r_1^- and r_1^+*

Here $C = s_2 - \min_{\varphi\in[-0.5,0.5]} w_1'(\varphi)$ and $d_1 = \max_{\varphi\in[-0.5,0.5]} w_1'(\varphi)$. I.e. if $s_2 > d_1$ then r_1^- and r_1^+ have to stay in the shaded regions in Fig.(4).

Proof of lemma 3.2:

In a neighborhood of $\varphi = -0.5$ we obtain from Eq.(43)

$$\frac{d}{d\varphi} r_1^-(-0.5; s_2) = (s_2 - w_1'(-0.5)) \frac{f'(-0.5)}{(g(0) - g(-0.5))^2} + O(\delta) \tag{57}$$

Hence, for $s_2 > d_1$ and δ sufficiently small it follows that $r_1^-(\varphi; s_2) > 0$ in a neighborhood of $\varphi = -0.5$. Further, Eq.(43) gives that on the axis $r = 0$

$$0 < r'(\varphi; s_2), \ \varphi \in (-0.5, 0] \text{ for } s_2 > d_1 \tag{58}$$

Hence

$$0 < r_1^-(\varphi; s_2), \ \varphi \in (-0.5, 0] \text{ for } s_2 > d_1 \tag{59}$$

for δ sufficiently small. If we us this in Eq.(43) we realize that

$$r'(\varphi; s_2) \leq \frac{1}{\delta} C, \ \varphi \in (-0.5, 0] \text{ for } s_2 > d_1 \tag{60}$$

This follows since our assumption $g'(\varphi) > 0$ yields a $\tilde{w}_1(-\varphi) \geq 0$. Hence

$$\frac{f(\varphi)}{\tilde{w}_1(\varphi)(\tilde{w}_1(\varphi) + \delta^2 r)} \geq 0 \tag{61}$$

leading to estimate above.

The estimate in the lemma follows from Eq.(59) and Eq.(60)

The second part of the lemma is proved in the same way.

•

LEMMA 3.3. *Assume that $g'(\varphi) > 0$. Then the following estimates hold if $s_2 < d_2$*

$$0 > r_1^-(\varphi; s_2) \geq -\frac{1}{\delta} D(\varphi + 0.5), \ \varphi \in [-0.5, 0) \tag{62}$$

and

$$0 < r_1^+(\varphi; s_2) \leq -\frac{1}{\delta} D(\varphi - 0.5), \ \varphi \in (0, 0.5] \tag{63}$$

Here $D = -s_2 + \max_{\varphi\in[-0.5,0.5]} w_1'(\varphi)$ and $d_2 = \min_{\varphi\in[-0.5,0.5]} w_1'(\varphi)$. I.e. if $s_2 < d_2$ then r_1^- and r_1^+ have to stay in the shaded regions in Fig.(5).

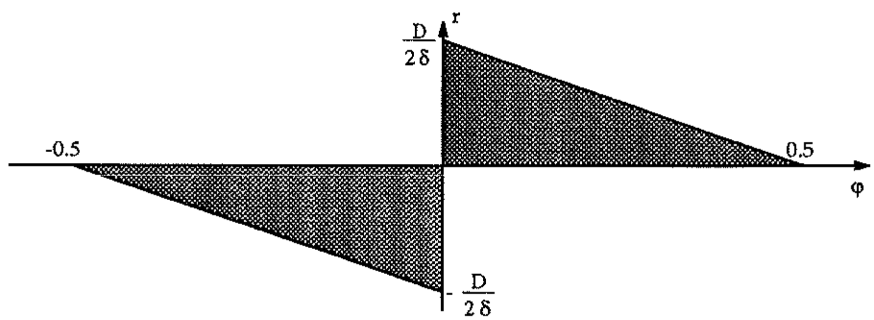

FIG. 5. *Bounds on* r_1^- *and* r_1^+

The proof of lemma 3.3 is similar to the proof of lemma 3.2.

Proof of theorem 3.1:

As noted above, we can find $r_1^-(\varphi; s_2)$ which solves Eq.(43) for $-\frac{1}{2} \leq \varphi \leq 0$ and $r_1^+(\varphi; s_2)$ which solves Eq.(43) $0 \leq \varphi \leq \frac{1}{2}$. It has already been shown that r_1^- and r_1^+ can be matched together for a unique $s_2 = s_2^{(1)}$. From lemma 2.4 it follows that $r_1^-(\varphi; s_2)$ increases continuously with increasing s_2. Lemma 2.9 gives that $r_1^+(\varphi; s_2)$ decreases continuously with increasing s_2. Hence, lemmas 3.2, 3.3 give that for the matched solution it holds that $d_2 \leq s_2^{(1)} \leq d_1$. These two lemmas also give

(64) $$|w_2^{(1)}(\varphi)| \leq \frac{1}{\delta}\hat{K}_1(0.5+\varphi)(0.5-\varphi)$$

where $\hat{K}_1$ is a constant independent of δ. Together with Eq.(40) this can be used in Eq.(43) to get bounds on the derivative, $\frac{d}{d\varphi}w_2^{(1)}(\varphi)$, for δ sufficiently small

(65) $$|\frac{d}{d\varphi}w_2^{(1)}(\varphi)| \leq \frac{1}{\delta^2}\hat{K}_2(0.5+\varphi)(0.5-\varphi)$$

where $\hat{K}_2$ is a constant independent of δ.

I.e. we have found a $s = s^{(1)} = -g(0) + \delta w_0'(0) + \delta^2 s_2^{(1)}$ such that the continuous solution of Eq.(36) can be written

(66) $$w^{(1)}(\varphi) = w_0(\varphi) + \delta w_1(\varphi) + \delta^2 w_2^{(1)}(\varphi)$$

where both $\delta w_2^{(1)}(\varphi)$ and $\delta^2 \frac{d}{d\varphi} w_2^{(1)}(\varphi)$ are bounded independently of δ.

The case when $g'(\varphi) < 0$ is proved in the same way.

•

3.2. The Scaling $\alpha = \epsilon$. In this section we will consider the solution of Eq.(30) which has derivatives of O(1) if $\alpha = \epsilon$ is used for $\epsilon\mu \ll 1$. If $g'(\varphi) > 0$ this solution satisfies the boundary conditions (32). If $g'(\varphi) < 0$ this solution satisfies the other set of boundary conditions.

With the above choice of α we have

(67) $$(s + g(\varphi))\varphi' = -\delta f(\varphi) + \varphi'' \text{ where } \delta = \epsilon\mu$$

For $\delta = 0$ this is a standard model in gas dynamics. It is known that for this case only one of the set of boundary conditions can yield a solution. If $g'(\varphi) > 0$ the admissible boundary conditions are (32), [7].

In the same way as before, we get by introducing $w = \varphi'$

(68) $$\frac{dw}{d\varphi} = s + g(\varphi) + \delta\frac{f(\varphi)}{w(\varphi)},\ w(-0.5) = w(0.5) = 0$$

As in the previous section we realize that there exist two solutions of Eq.(68), $w^{(1)}(\varphi)$ and $w^{(2)}(\varphi)$, such that $w^{(1)}(\varphi) > 0$ and $w^{(2)}(\varphi) < 0$ for $\varphi \in (-0.5, 0.5)$

Below we will show that one of these solutions has derivatives of $O(1)$ as $\epsilon\mu \to 0$. This is the solution which was unbounded when an $\alpha = \frac{1}{\mu}$ was used in section 3.1.

Let

(69) $$s_0^* = -\int_{-0.5}^{0.5} g(\varphi)\, d\varphi$$

and let w_0 be the solution of

(70) $$\frac{dw}{d\varphi} = s_0^* + g(\varphi)$$

Note that if $g'(\varphi) > 0$ this yields a solution $\varphi(z)$ which satisfies the boundary condition (32).

The ansatz $w = w_0 + \delta w_1$ and $s = s_0^* + \delta s_1$ in Eq.(68) yields

(71) $$\frac{dw_1}{d\varphi} = s_1 + \frac{f(\varphi)}{w_0(\varphi) + \delta w_1},\ w_1(-0.5) = w_1(0.5) = 0$$

As in section 3.1 we know that there exist $s_1 = s_1^{(1)}$ and $s_1 = s_1^{(2)}$ such that $w_1^{(1)}(\varphi)$ and $w_1^{(2)}(\varphi)$ are solutions of Eq.(71). It is also known that

(72) $$w^{(1)}(\varphi) = w_0(\varphi) + \delta w_1^{(1)}(\varphi) > 0,\ \varphi \in (-0.5, 0.5)$$

and

(73) $$w^{(2)}(\varphi) = w_0(\varphi) + \delta w_1^{(2)}(\varphi) > 0,\ \varphi \in (-0.5, 0.5)$$

We have

THEOREM 3.4. *If $g'(\varphi) > 0$ then*

(74) $$w^{(2)}(\varphi) \to w_0(\varphi) \text{ and } s^{(2)} \to s_0^* \text{ as } \delta \to 0$$

Further, $w^{(2)}(\varphi)$ is smooth independently of δ. If $g'(\varphi) < 0$ then

(75) $$w^{(1)}(\varphi) \to w_0(\varphi) \text{ and } s^{(1)} \to s_0^* \text{ as } \delta \to 0$$

Except for the smoothness this is proved in the same way as theorem 3.1. Smoothness follows in the same way as in the proof of theorem 2.1.

3.3. The Scaling $\alpha = \sqrt{\epsilon/\mu}$. In this section we will consider the two solutions of Eq.(30) which has derivatives of O(1) if $\alpha = \sqrt{\epsilon/\mu}$ is used for $\epsilon\mu \gg 1$.

For this case it will be shown that the speed of the traveling wave, s, is proportional to $\sqrt{\epsilon\mu}$, except for the special case that

(76) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} f(\varphi)\, d\varphi = 0$$

With the above choice of α we have

(77) $$\delta(s + g(\varphi))\varphi' = -f(\varphi) + \varphi''$$

where in this case $\delta = \sqrt{\frac{1}{\epsilon\mu}} > 0$. In the same way as for Eq.(13) we get

(78) $$\frac{dw}{d\varphi} = \delta(s + g(\varphi)) + \frac{f(\varphi)}{w},\ w(-0.5) = w(0.5) = 0$$

4. Numerical Results. In order to verify the theoretical results we use given functions $f(u)$ and $g(u)$. Eq.(30) is analyzed by perturbation analyses. Eq.(1) is solved numerically. These results are then compared with each other. This is only shown for the case when $\epsilon\mu \gg 1$ since this is the case when the character of the solution is greatly dependent of the involved physical parameters, the "viscosity", ϵ and the "reaction rate", μ.

Here we will use

$$f(u) = (1 + \beta(u + \frac{1}{2}))(u^3 - \frac{1}{4}u) \tag{88}$$

and

$$g(u) = u + \frac{1}{2} \tag{89}$$

Consider the scaling $\alpha = \sqrt{\frac{\epsilon}{\mu}}$, i.e. the case which resolves the solution when the product $\epsilon\mu \gg 1$. For this case a perturbation analyses can be performed when $\beta \ll 1$ and $\delta \ll \beta$ (as before, $\delta = 1/\sqrt{\epsilon\mu}$ on this scaling). The result of this analyses shows that

$$\varphi = \varphi_0 + \beta\varphi_1 + O(\beta^2) \text{ and } s + \frac{1}{2} = \hat{s} = \frac{\beta}{\delta}(\hat{s}_0 + \hat{s}_1\beta + O(\beta^2)) \tag{90}$$

Here

$$\varphi_0 = \frac{1 - e^{\hat{z}}}{2(1 + e^{\hat{z}})} \tag{91}$$

where $\hat{z} = z/\sqrt{2}$ and $\hat{s}_0 \approx -0.0707$ satisfies Eq.(30) with boundary conditions from Eq.(32). When the boundary conditions from Eq.(31) are used we get

$$\varphi_0 = \frac{1 - e^{-\hat{z}}}{2(1 + e^{-\hat{z}})} \tag{92}$$

and $\hat{s}_0 \approx 0.0707$. This form of s agrees with theorem 3.5.

In order to solve Eq.(1) numerically an ordinary Runge Kutta time stepping method has been used to advance the solution in time. As initial value we use the first term in the expansion for φ. Thus,

$$u_t + uu_x = -\mu f(u) + \epsilon u_{xx} \tag{93}$$

has been discretized spatially, after a suitable scaling, and then advanced in time. As initial value we have used $u(x, 0) = \varphi_0(x)$. We have performed the calculations for $\beta = 0$ and $\beta \neq 0$ with both sets of boundary conditions. The computations have been performed for $\epsilon\mu \ll 1$ and for $\epsilon\mu \gg 1$.

The case when $\epsilon\mu \ll 1$ gives results which agrees well with the theory. As stated above these results will not be considered further here.

For the case when $\epsilon\mu \gg 1$ and $\alpha = \sqrt{\epsilon/\mu}$ we have two cases depending upon which boundary conditions that are used. We have used a fixed β, $\beta = 0.2$, and varied the product $\epsilon\mu$. With boundary conditions from Eq.(3) we get

$\epsilon\mu$	1000	5000	10000
$\hat{s}$	0.428	0.955	1.350

When boundary conditions from Eq.(4) are used we get

$\epsilon\mu$	1000	5000	10000
$\hat{s}$	-0.426	-0.953	-1.350

These two tables verify the fact from section 3.3 that $\hat{s} \sim \frac{b}{\delta}$ where $\delta = 1/\sqrt{\epsilon\mu}$ and b has different sign depending on which boundary conditions that are used. In the perturbation analyses it was predicted that $\hat{s} = \frac{\beta}{\delta}(\hat{s}_0 + O(\beta))$ where $\hat{s}_0 \approx \pm 0.0707$. For comparison we give the following table

$\epsilon\mu$	1000	5000	10000
$\lvert\hat{s}_0\rvert\frac{\beta}{\delta}$	0.45	1.00	1.41

i.e. we have reasonable agreement between the theory and the numerical calculations despite the fact that β is as big as 0.2.

5. Conclusions. According to our theory, and also verified numerically, the solution of the model equation, Eq.(1), behaves in a spurious way when the product $\epsilon\mu$ increases. In this case the speed of the wave is proportional to $\sqrt{\epsilon\mu}$ and the direction in which the wave travels depends on which boundary boundary conditions that are used. As pointed out in the introduction, it is common practice to add extra dissipation in order to get a well behaved, numerical solution. For Eq.(1) this would mean that ϵ is substituted with $\epsilon_a > \epsilon$. This would render an incorrect speed of the traveling wave solution. A possible remedy for this simplified equation is to use an artificial μ_a such that the product $\epsilon\mu$ is kept constant. From section 3.3 we have that the thickness of the wave is proportional to $\sqrt{\epsilon/\mu}$. Therefore, if one uses $\epsilon_a = \epsilon a$ and $\mu_a = \mu/a$ it follows that the thickness of this "artificial" wave is $\sqrt{\epsilon_a/\mu_a} = a\sqrt{\epsilon/\mu}$ i.e. the wave will be smeared out but it will still have the correct propagation speed. It seems reasonable to believe that problems arise even for a more realistic problem when extra dissipation is added. It is difficult to know what kind of problems would occur for this case and what the remedy should be. In analogy with our simplified problem it might be so that the problem arising from the addition of extra dissipation can be treated with an artificial reaction rate.

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