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SINGULAR PERTURBATION ANALYSIS OF BOUNDARY-VALUE PROBLEMS FOR DIFFERENTIAL-DIFFERENCE EQUATIONS VI. SMALL SHIFTS WITH RAPID OSCILLATIONS*

Dedicated to the Memory of our Friend, Hubertus J. Weinitschke

CHARLES G. LANGE† and ROBERT M. MIURA‡

Abstract. This paper continues our study of boundary-value problems for singularly perturbed linear second-order differential-difference equations with small shifts. This study was initiated in the companion paper,"Singular perturbation analysis of boundary-value problems for differential-difference equations. V. Small shifts with layer behavior," this journal, this issue, pp. . Here we extend that study to problems which have solutions that exhibit rapid oscillations. We find restrictions on the sizes of the shifts in terms of the small parameter such that, in general, one cannot replace the shifted terms with truncated Taylor series. In particular, it is shown that even when the shifts are small relative to the width of an oscillation they can affect the solution to leading order. We conclude that oscillatory solutions are more sensitive to small delays than are layer solutions. We show that a suitably modified version of the standard WKB method can be used to obtain leading-order oscillatory solutions of these differential-difference equations. These preliminary studies of differential-difference equations with small shifts provide techniques for treating expected first-exit time problems associated with the membrane potential of neurons for generation of action potentials.

Key words. differential-difference equations, singular perturbations, boundary-value problems, small shifts, rapid oscillations, modified WKB method, exponential polynomials, first-exit times, action potentials

AMS(MOS) subject classifications. 34K10, 34K25, 30C15, 92C20

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1. Introduction. This paper continues our studies of boundary-value problems (BVPs) for singularly perturbed differential-difference equations (DDEs) using singular perturbation analyses coupled with numerical computations ([3]-[6], [8]). In the companion paper [7], we initiated an investigation of BVPs on $0 \leq x \leq 1$ for DDEs with small shifts for which the solutions exhibited layer behavior. Here we extend that study to BVPs for DDEs with small shifts for which the solutions exhibit rapid oscillations. Problems with small shifts arise in determination of the expected first-exit times of the membrane potential through threshold for generation of action potentials in models of neurons [7], [9]-[11]. These preliminary studies develop some of the tools for treating such problems.

In this paper, we address two separate questions. The first concerns the size of the shifts which affect the solution to leading order, and the second concerns whether the oscillatory solutions can be obtained by simple Taylor series expansions of the shifted terms or require the use of a WKB method. For rapidly oscillating problems, the shift affects the solution to leading order even when the shift is small compared to the scale of the oscillations. Specifically, solutions with oscillations of width $O(\varepsilon)$ are affected even when the shift is $O(\varepsilon^2)$. This suggests that small shifts also could play an important role for partial differential equations with rapidly oscillating solutions. On the other hand, for layer problems, the magnitude of the shift must be at least as large as the layer thickness before the shift affects the layer solution to leading order, cf. the companion paper [7]. We conclude that while small shifts affect both layer and oscillatory problems, the effects are more pronounced for oscillatory problems.

The BVPs for the DDEs to be studied in this paper are stated briefly in Section 2. In

Section 3, we analyze both the question of when the size of the shifts affect the solutions to leading order and which dependencies of the shifts on the small parameter, ε, require an analysis where the shifted terms cannot be expanded in Taylor series. There are BVPs for DDEs exhibiting solutions with rapid oscillations all across the interval for shifts which are sufficiently small. Also, there are BVPs with solutions in which oscillations were previously confined to layer regions when the shifts are sufficiently small but where the oscillations can extend into the outer region when the shifts are increased. In this case the layer solution method developed in [7] fails to give the correct inner and outer solutions. In Section 4, these oscillatory solutions which cannot be analyzed by simple Taylor series of the shifted terms are treated using a WKB method [2] which accounts for the small shifts.

2. Statements of the problems. In this paper we study the same classes of model problems investigated in [7] except in parameter ranges where the solutions exhibit oscillatory behavior rather than layer behavior. We refer the reader to [7] for complete details of the BVPs and simply state the DDEs and associated side conditions to establish the notation.

When the shifts are zero, the solutions of the BVPs for the corresponding ODEs exhibit layer behavior, oscillatory behavior, or some combination of these depending on the coefficients. The investigations here will examine questions on the effects of the nonzero shifts on oscillatory behavior and construct leading-order oscillatory solutions using a WKB method. One class of BVPs is given by

$$\varepsilon y''(x;\varepsilon) + a(x)y'(x-\delta(\varepsilon);\varepsilon) + b(x)y(x;\varepsilon) = f(x), \tag{2.1}$$

on $0 < x < 1$, $0 < \varepsilon \ll 1$, and $0 \leq \delta(\varepsilon) \ll 1$, subject to the interval and boundary conditions

$$y(x;\varepsilon) = \phi(x) \quad \text{on} \quad -\delta(\varepsilon) \le x \le 0, \quad y(1;\varepsilon) = \gamma, \tag{2.2}$$

respectively.

Again, it will be convenient to assume $\phi(x) \equiv 1$ and $f(x) \equiv 0$. For sufficiently small $\delta(\varepsilon)$, the first-order derivative term is important near $x = 0$ and/or $x = 1$ (depending on the sign of $a(x)$) and the solutions to this BVP exhibit layer behavior, cf. Figures 1a, 1b, and 3 in [7]. However, as is shown here in Figures 1a and 1b (and in Figures 1c and 1d in [7]), when $\delta(\varepsilon)$ is sufficiently large, the layer structure is destroyed and the solution is dominated by oscillations. These oscillatory solutions are analyzed in Section 4 using a WKB method.

The second class of BVPs is given by DDEs of mixed type

$$\varepsilon^2 y''(x;\varepsilon) + \alpha(x)y(x-\delta(\varepsilon);\varepsilon) + \omega(x)y(x;\varepsilon) + \beta(x)y(x+\eta(\varepsilon);\varepsilon) = f(x), \tag{2.3}$$

on $0 < x < 1$, $0 < \varepsilon \ll 1$, $0 \le \delta(\varepsilon) \ll 1$, and $0 \le \eta(\varepsilon) \ll 1$, subject to the interval conditions

$$\begin{aligned} y(x;\varepsilon) &= \phi(x) \quad \text{on} \quad -\delta(\varepsilon) \le x \le 0, \\ y(x;\varepsilon) &= \psi(x) \quad \text{on} \quad 1 \le x \le 1+\eta(\varepsilon). \end{aligned} \tag{2.4}$$

Again, for convenience, it is assumed that $\phi(x) \equiv 1$ and $\psi(x) \equiv$ constant. Note that there are no first-order derivative terms in (2.3).

If the shifts, $\delta(\varepsilon)$ and $\eta(\varepsilon)$, are both zero and $\alpha(x)+\omega(x)+\beta(x) > 0$ on $0 < x < 1$, then the solution of the corresponding ODE exhibits rapid oscillations all across the interval. For $\delta(\varepsilon)$ and $\eta(\varepsilon)$ sufficiently small, these oscillatory solutions can be analyzed by simply taking Taylor series expansions of the shifted terms. This is carried out in Section 3, cf. Figures 2a and 2b. For $\delta(\varepsilon)$ and/or $\eta(\varepsilon)$, sufficiently large, however, a simple Taylor series expansion of the shifted terms is not valid. Instead these oscillatory solutions can be analyzed by using a WKB method and results are given in Section 4.

If the shifts, $\delta(\varepsilon)$ and $\eta(\varepsilon)$, are both zero and $\alpha(x) + \omega(x) + \beta(x) < 0$ on $0 < x < 1$, then the solution of the corresponding ODE has layers at both the left and right ends, cf. Figure 5a in [7]. However, for $\delta(\varepsilon)$ and/or $\eta(\varepsilon)$, sufficiently large, one or both layers can be destroyed by rapid oscillations. These oscillatory solutions can be analyzed using a WKB method and results for this case are given by the same formulas in Section 4 as for the previous case.

As noted in [7], the method of steps for solving BVPs for DDEs with small shifts on a unit interval is impractical. When the solutions consist of rapid oscillations, we obtained the numerical solutions of the BVPs by using the iterative numerical scheme and/or the finite difference scheme outlined in Appendix 1 of [7]. In some cases, the iterative scheme using COLSYS [1] did not converge and we had to rely solely on the finite difference solutions.

3. Rapid oscillations. As pointed out in Section 3.1 of [7], the layer region at $x = 0$ is no longer clearly defined if the shift, τ in the layer variables, becomes too large. In particular, if any complex root of the exponential polynomial associated with the Laplace transform solution comes to within $O(\varepsilon)$ of the imaginary axis, then oscillations which are generated in the layer can extend into the outer region, cf. Figure 1c. In this case the layer analysis developed in Section 3 of [7], is no longer completely valid and modifications accounting for the oscillations in the outer region must be devised. The appropriate tool for this analysis is the WKB method [2] with modifications to account for the shift terms. This analysis is carried out in Section 4.

If $\alpha(x) + \omega(x) + \beta(x) > 0$ in (2.3) with small shifts, then the solutions exhibit rapid oscillations all across the interval. In this section, the effects of nonzero shifts on these oscillations will be examined. For this discussion, we will concentrate on equation (2.3). Simple Taylor expansions of the shift terms through second-order derivatives in (2.3) yield the ODE setting $f(x) \equiv 0$)

$$(3.1) \qquad \left\{\varepsilon^2 + \frac{1}{2}\left[\alpha(x)\delta^2(\varepsilon) + \beta(x)\eta^2(\varepsilon)\right]\right\} z''(x;\varepsilon) - \left[\alpha(x)\delta(\varepsilon) - \beta(x)\eta(\varepsilon)\right] z'(x;\varepsilon)$$

$$+ \left[\alpha(x) + \omega(x) + \beta(x)\right] z(x;\varepsilon) = 0,$$

where z replaces y. Higher-order derivative terms occur as $\delta^n z^{(n)}$ and $\eta^n z^{(n)}$, each multiplied by an $O(1)$ coefficient. If the shifts are sufficiently small, then they can be ignored to leading order, namely if $\delta(\varepsilon)$ and $\eta(\varepsilon)$ are both $o(\varepsilon^2)$. However, if they cannot be ignored, how should the BVP be analyzed to obtain a leading-order approximate solution.

In order to see the effects of increasing the shift, consider the sequence of graphs in Figure 2a,b,c for the DDE (2.3) with $\beta(x) \equiv 0$ (i.e., no right shift term), $\phi(x) = 1$, $\psi(x) = 0$, $\gamma = 0$, $\alpha(x) = -1/2$, $\omega(x) = 1$, $f(x) = 0$, and $\delta(\varepsilon) = \varepsilon^2$, $\varepsilon^{3/2}$, and ε with $\varepsilon = 0.01$. Decreasing the power of ε in δ for fixed ε increases the shift without changing the coefficient of y''. For $\alpha(x) < 0$, the effect of the shift is to decrease the amplitude of the oscillation as x increases and the decrease is sharper with increasing shift values, cf. (3.1). For $\delta(\varepsilon) = \varepsilon$, the decrease in amplitude in Figure 2c is so sharp that only a few "visible" oscillations survive (actually there are oscillations all across the interval but with exponentially small amplitude).

When $\alpha(x) > 0$ and $\beta(x) \equiv 0$, then as seen from (3.1), the oscillations grow in amplitude. This is illustrated in Figure 2d for a DDE with only a left shift for $\alpha(x) = 3$, $\omega(x) = 1$, $\beta(x) = 0$, $f(x) = 0$, $\phi(x) = 1$, $\gamma = 0$ (i.e., $\psi(x) = 0$ since $\beta(x) = 0$), and $\delta(\varepsilon) = \varepsilon^2$ with $\varepsilon = 0.01$. The importance of the coefficient of z' in (3.1) is more clearly illustrated by comparing Figures 3a and 3b for the solution of the mixed equation with $\phi(x) = 1$, $\psi(x) = 0$, $\alpha(x) = -1$ and -2, respectively, $\omega(x) = 4$, $\beta(x) = -2$ and -1, respectively, $f(x) = 0$, $\delta(\varepsilon) = \varepsilon^2$ and $2\varepsilon^2$, respectively, and $\eta(\varepsilon) = 2\varepsilon^2$ and ε^2, respectively, with $\varepsilon = 0.01$. The coefficient of z in (3.1) equals $-3\varepsilon^2$ and $3\varepsilon^2$ leading to amplitude increase and decrease, respectively.

The question which remains to be answered is for what dependencies of δ and η on ε does $z(x;\varepsilon)$ as a solution of (3.1) provide a leading-order approximation of the solution $y(x;\varepsilon)$. Although the general question is not settled here, an answer is provided for the special case of constant coefficients and $\delta(\varepsilon) = \tau\varepsilon^p$, $\eta(\varepsilon) = \mu\varepsilon^p$. For this case it is convenient to rewrite (3.1) as

$$\varepsilon^2(1 + P\varepsilon^{2(p-1)})z''(x;\varepsilon) - 2\varepsilon^p Qz'(x;\varepsilon) + Rz(x;\varepsilon) = 0, \tag{3.2}$$

where

$$P \equiv \frac{1}{2}(\alpha\tau^2 + \beta\mu^2), \quad Q \equiv \frac{1}{2}(\alpha\tau - \beta\mu), \; R \equiv \alpha + \omega + \beta. \tag{3.3}$$

Then the exact solution is

$$z(x;\varepsilon) = e^{qx}[c_1 \cos(rx) + c_2 \sin(rx)] \tag{3.4}$$

where c_1 and c_2 are constants of integration and

$$q \equiv \frac{\varepsilon^{p-2}Q}{1+P\varepsilon^{2(p-1)}}, \quad r \equiv \frac{\sqrt{(1+P\varepsilon^{2(p-1)})R-\varepsilon^{2(p-1)}Q^2}}{\varepsilon(1+P\varepsilon^{2(p-1)})}. \tag{3.5}$$

For all $p \geq 1$, $q = O(\varepsilon^{p-2})$ and $r = O(\varepsilon^{-1})$; therefore $\delta^n z^{(n)}$ and $\eta^n z^{(n)}$ are both $O(\varepsilon^{n(p-1)})$ if c_1 and c_2 are $O(1)$. In particular, for $p = 1$, all higher-order derivative terms which have been omitted from (3.2) upon expansion of the shift terms are of the same order of magnitude as those which have been kept. From Figures 2a and 2b, on which are plotted both the numerical solution and the function $z(x;\varepsilon)$ given by (3.4), the agreement between the results is excellent. However, the numerical solution in Figure 2c is not well approximated by $z(x;\varepsilon)$. On the other hand, the numerical solution in Figure 2c is well approximated by the WKB analysis leading to the formulas (4.20) and (4.21).

In summary for the constant coefficient cases, if $\delta(\varepsilon)$ and $\eta(\varepsilon)$ are $o(\varepsilon^2)$ then the reduced equation with $\delta = 0 = \eta$ yields a leading-order solution. If δ and η are $O(\varepsilon^2)$, then the effects of the shift terms cannot be ignored and the approximate equation (3.1) provides a leading-order solution up to δ and η both $o(\varepsilon)$. For δ and η of $O(\varepsilon)$, however, all derivative terms in the expansion of the shifted terms become equally important and (3.1) no longer provides a leading-order solution. Instead, (2.3) must be used without expanding the shifted terms. In Section 4, we develop the WKB solution method to obtain the leading-order oscillatory solutions in this case.

4. WKB analysis. The WKB method [2] is ideally suited to obtain leading-order oscillatory solutions to the model problems (2.1) - (2.4). In Section 3 we showed that for (2.3), the Taylor expanded equation (3.1) was valid provided δ and η were $o(\varepsilon)$. However, there is a complication in obtaining the WKB solution through 0(1) in those cases. The

WKB ansatz, motivated by the classical ansatz for ODEs, is that the solution can be expressed as the superposition of functions

$$y(x;\varepsilon) = A(x;\varepsilon)\exp\left[\frac{S(x;\varepsilon)}{\varepsilon}\right], \tag{4.1}$$

where $A(x;\varepsilon)$ and $S(x;\varepsilon)$ are smooth functions of x and ε. Substitute (4.1) into (3.2) with $p = 1 + 1/n$; then a self consistent solution requires $S(x;\varepsilon)$ to be a power series in $\varepsilon^{1/n}$. Therefore at least $n+1$ terms in S are required to obtain y to $0(1)$. This complication makes the WKB solution through $0(1)$ more difficult to obtain in spite of being able to expand the shifted terms in (2.3). Because of this complication, we focus on the case of $p = 1$, i.e., where both δ and η are of $0(\varepsilon)$. To illustrate the ideas, the BVP for the DDE given by (2.1) and (2.2) is analyzed in detail. Also results for (2.3) and (2.4) are presented.

For convenience, set $\phi(x) \equiv 1$ and assume $\delta(\varepsilon) = \tau\varepsilon$ with $\tau = O(1)$ as $\varepsilon \to 0$ and $0 < \varepsilon << 1$. Then $S(x;\varepsilon)$ depends only on x. For the zero shift case, i.e., $\delta(\varepsilon) \equiv 0$, $S'(x) = -a(x)$ or $S'(x) = 0$, thus

$$y(x;\varepsilon) \sim k\exp\left[-\int_0^x \frac{b(t)}{a(t)}\,dt\right] + \frac{c_1}{a(x)}\exp\left[\int_0^x \left(\frac{b(t)}{a(t)} - \frac{a(t)}{\varepsilon}\right)dt\right], \tag{4.2}$$

where k and c_1 are arbitrary constants. The first term on the right corresponds to the outer solution. Note that the lower limits in the integrals need not be zero.

For $\delta(\varepsilon) > 0$, substituting (4.1) into (2.1) gives

$$\begin{aligned} &\frac{1}{\varepsilon}S'^2(x)A(x) + 2S'(x)A'(x) + S''(x)A(x) + \varepsilon A''(x) \\ &+ a(x)\left[\frac{S'(x-\delta)A(x-\delta)}{\varepsilon} + A'(x-\delta)\right]\exp\left[\frac{S(x-\delta) - S(x)}{\varepsilon}\right] + b(x)A(x) = 0, \end{aligned} \tag{4.3}$$

where the dependencies on ε have been suppressed. Expand out the shifted terms as follows:

$$S'(x-\delta)A(x-\delta) = S'(x)A(x) - \tau\varepsilon\left[S''(x)A(x) + S'(x)A(x)\right] + O(\varepsilon^2),$$

$$A'(x-\delta) = A'(x) + O(\varepsilon), \tag{4.4}$$

$$\frac{S(x-\delta)-S(x)}{\varepsilon} = -\tau S'(x) + \frac{1}{2}\tau^2\varepsilon S''(x) + O(\varepsilon^2).$$

Define $T(x) \equiv S'(x)$ and assume

$$A(x;\varepsilon) \sim \sum_{j=0}^{\infty} A_j(x)\varepsilon^j, \tag{4.5}$$

then the leading-order equation in ε requires either

$$T(x) = 0 \tag{4.6}$$

or

$$T(x) + a(x)\epsilon^{-\tau T(x)} = 0, \tag{4.7}$$

where (4.6) corresponds to the outer solution.

The phase equation (4.7) can be rewritten as

$$z(x) + \epsilon^{-\nu(x)z(x)} = 0, \tag{4.8}$$

where $z(x) \equiv T(x)/a(x)$ and $\nu(x) \equiv \tau a(x)$. The phase, $S(x)$, then is determined in the following way: 1) specify x and compute $\nu(x)$, 2) determine the nth root of (4.8), say $z_n(x)$, as described in [7], Appendix 2.1, so that changing x corresponds to tracing out the nth root curve in the complex plane, cf. Figure 2b in [7], and 3) determine $T_n(x)$ from $z_n(x)$ and $S_n(x)$ from

$$S_n(x) = \int_0^x a(t)z_n(t)dt, \tag{4.9}$$

where it is convenient to choose $S_n(0) = 0$. As noted earlier, the general solution is a superposition of WKB solutions (4.1) and requires the computation of S_n for a sequence of n values.

It remains to determine the leading-order amplitude $A_0(x)$ corresponding to each $S(x)$. The next higher-order equation in ε is obtained from (4.3) and is given by

$$\text{(4.10)} \quad \{T(x)[1+\tau T(x)]\} A_0'(x) + \left\{\left[1+\tau T(x) - \frac{1}{2}\tau^2 T^2(x)\right] T'(x) + b(x)\right\} A_0(x) = 0,$$

where the approximation $\exp\left[\frac{1}{2}\tau^2 \varepsilon T'(x)\right] \sim 1 + \frac{1}{2}\tau^2 \varepsilon T'(x)$ has been used. Solving for $A_0(x)$ yields

$$\text{(4.11)} \qquad A_0(x) = \frac{1}{T(x)\sqrt{1+\tau\, T(x)}} \exp\left[\frac{1}{2}\tau T(x) - \int_0^x \frac{b(t)\; dt}{T(x)\,[1+\tau T(x)]}\right].$$

Thus the leading-order WKB solution is given by

$$\text{(4.12)} \qquad y(x;\varepsilon) \;\sim\; k \exp\left[-\int_0^x \frac{b(t)}{a(t)}\; dt\right] + \left\{\sum_{n=1}^{\infty} \frac{c_n}{T_n(x)\sqrt{1+\tau\, T_n(x)}}\right.$$

$$\left. \times \exp\left[\frac{S_n(x)}{\varepsilon} + \frac{1}{2}\tau\; T_n(x) - \int_0^x \frac{b(t)dt}{T_n(x)\,[1+\tau\; T_n(x)]}\right] + \text{complex conjugate}\right\},$$

where k and the c_n are arbitrary complex constants. If T_n (or equivalently $z_n(x)$) is real, then the c_n in (4.12) must be divided by 2 to avoid counting the same root twice (cf. discussion of equation (3.18) in [7]). The first term on the right side with coefficient k corresponds to the outer solution. Obviously when $\tau = 0$, i.e., when $\delta(\varepsilon) \equiv 0$, this formula reduces to the standard form in (4.2).

In order to determine the unknown constants k and c_n, the WKB solution is matched with the layer solution near $x = 0$. If Re $z_n(x) < 0$ for all n with $z_1(x)$ sufficiently close to the imaginary axis over some range of x near $x = 0$, then oscillations could be generated in the layer and could continue to oscillate in the "outer region." For this case, the outer solution is not simply an exponential as given by (3.1) in [7], but rather is given by the WKB solution. In this case the leading-order layer equation is given by the first line of (3.14) in [7], hence the Laplace transform is given by

$$\tilde{Y}_0(s) = \frac{1}{s} + \frac{\tilde{y}_0'(0)}{s(s+e^{-\tau s})}. \tag{4.13}$$

The constant term, which corresponds to the residue at $s=0$, matches the constant term in the WKB solution as $x \to 0$, thus

$$\tilde{y}_0'(0) + 1 = k. \tag{4.14}$$

Hence the layer solution has the form

$$\tilde{y}_0(\tilde{x}) = k + \sum_{n=1}^{\infty} \left[\frac{k-1}{s_n(1+\tau s_n)} e^{s_n \tilde{x}} + \frac{k-1}{\bar{s}_n(1+\tau \bar{s}_n)} e^{\bar{s}_n \tilde{x}} \right] \tag{4.15}$$

and matching with the WKB solution for $x \to 0$ and $\tilde{x} \to \infty$ requires

$$c_n = \frac{k-1}{\sqrt{1+\tau s_n}} e^{-\frac{1}{2}\tau s_n}, \tag{4.16}$$

where for convenience, it has been assumed that $a(0) = 1$ so that $T_n(0) = s_n$ (cf. (4.8)).

Thus the WKB solution becomes

$$\begin{aligned} y(x;\varepsilon) \sim k \exp\left[-\int_0^x \frac{b(t)}{a(t)}\,dt \right] + 2(k-1) \sum_{n=1}^{\infty} \operatorname{Re} \left\{ \frac{1}{T_n(x)\sqrt{(1+\tau s_n)(1+\tau T_n)}} \right. \\ \left. \times \exp\left\{ \frac{S_n(x)}{\varepsilon} + \frac{1}{2}\tau[T_n(x) - S_n] - \int_0^x \frac{b(t)dt}{T_n(t)[1+\tau T_n(t)]} \right\} \right\}. \end{aligned} \tag{4.17}$$

Two concrete examples now will be described. (Note that the solutions plotted in Figures 1c and 1d in [7] can be treated similarly.) First, consider the BVP for the DDE with $a(x) = 1$, $b(x) = 1$, $f(x) = 0$, $\phi(x) = 1$, $\gamma = 1$, and $\varepsilon = 0.01$. The solution is shown in Figure 1a for $\delta(\varepsilon) = \tau\epsilon$ with $\tau = 1.5$. In this case the layer analysis described in Section 3.1 in [7] fails because the oscillations in the "layer" extend into the "outer" region. The WKB solution is given by

$$(4.18) \qquad y(x;\varepsilon) \sim ke^{-x} + (k-1)\left\{ \frac{exp\left\{\left[\frac{s_1}{\epsilon} - \frac{1}{s_1(1+\tau s_1)}\right]x\right\}}{s_1(1+\tau s_1) + c.c.} \right\}$$

$$\equiv ke^{-x} + (k-1)[E(x) + c.c.],$$

where in (4.17) only the first term in the sum has been retained and $T_1 = s_1$, $S_1 = s_1 x$. The value of k, and hence $\tilde{y}'(0)$ from (4.14), is obtained by applying the boundary condition at $x = 1$. Thus

$$(4.19) \qquad \tilde{y}'(0) \sim \frac{e-1}{1 + e[E(1) + c.c.]}.$$

From Appendix 2.1 in [7], $s_1 = -0.022 + 1.0334i$ and from (4.19), $\tilde{y}'(0) \sim 1.18$, while the numerical results give $\tilde{y}'(0) = 1.25$.

As a second example, the solution to the BVP for the DDE with $a(x) = \epsilon^{-\nu x}$ with $\nu = 0.5$, $\varepsilon = 0.01$, $b(x) = 1$, $\gamma = 1$, and $\tau = 1.5$ is plotted in Figure 1b. Near $x = 0$, two of the roots of the phase equation (4.7) have small real parts which explains the rapid oscillations and the lack of a boundary layer. However, as x increases, the real parts of all the roots of (4.7) become negative and lie outside an $O(\varepsilon)$ strip along the imaginary axis. (This behavior occurs because $a(x)$ is a decreasing function of x.) Thus the oscillations die out in the middle of the interval and the solution blends into the outer solution having $k = \gamma$. A comparison of the leading-order WKB solution with the numerical solution shows excellent agreement with errors of $O(\varepsilon)$, as expected.

For the BVP, (2.3)-(2.4), the WKB analysis is similar and only the leading-order results are recorded here. The principal differences with the BVP treated above is that there are no first-order derivative terms and possibly two distinct shifts. The WKB ansatz (4.1) is substituted into (2.3) (with $f(x) \equiv 0$) and expansions similar to (4.4) are carried out. In Section 3, we have shown that if δ and η are both $o(\varepsilon)$, then the approximation (3.1) is adequate. However, if δ (and/or η) is $O(\varepsilon)$, then this approximation is no longer valid and the effects of δ (and/or η) are important to leading order. If $\delta = \tau\varepsilon$ and $\eta = \mu\varepsilon$ where τ and μ are constants, then the leading-order equation requires

$$T^2(x) + \alpha(x)e^{-\tau T(x)} + \omega(x) + \beta(x)e^{\mu T(x)} = 0 \tag{4.20}$$

where $T(x) \equiv S'(x)$. This exponential polynomial in T corresponds to $R(T;\tau,\mu)$ studied in Appendix 2.3 in [7]. The leading-order amplitude, $A_0(x)$ where (4.5) is assumed, is obtained from the next order equation in ε and is given by

$$A_0(x) = \frac{1}{\sqrt{T(x) - \frac{1}{2}\tau\alpha(x)e^{-\tau T(x)} + \frac{1}{2}\mu\beta(x)e^{\mu T(x)}}} \times \exp\left\{-\frac{1}{4}\int_0^x \frac{\tau\alpha'(t)e^{-\tau T(t)} - \mu\beta'(t)e^{\mu T(x)}}{T(t) - \frac{\tau\alpha(x)}{2}e^{-\tau T(x)} + \frac{\mu\beta(x)}{2}e^{\mu T(x)}}\,dt\right\}. \tag{4.21}$$

We give two concrete examples of the formulas (4.20) and (4.21). The first example is the BVP (2.3) - (2.4) with $\beta(x) = 0$ (i.e., no right shift term), $\phi(x) = 1$, $\gamma = 0$, $\alpha(x) = -1/2$, $\omega(x) = 1$, $f(x) = 0$, and $\delta(\varepsilon) = \varepsilon$ with $\varepsilon = 0.01$. The solution is plotted in Figure 2c. Since $\alpha + \omega + \beta > 0$, the solution for $\delta = 0$ is oscillatory. In this case, for $\delta = \varepsilon$ the oscillations near $x = 0$ decay rapidly to the outer solution, but this solution cannot be approximated to leading order by the solution (3.4). However, this solution is well approximated by using the roots of (4.20) and the formula (4.21) in (4.1).

The second example is the BVP (2.3) - (2.4) with $\alpha(x) = -2e^{-x}$, $\omega(x) = -1$, $\beta(x) = 0$, $f(x) = 1$, $\phi(x) = 1$, $\gamma = 0$ (i.e., $\psi(x) = 0$ since $\beta(x) = 0$), and $\delta(\varepsilon) = \tau\varepsilon$ with $\tau = 3$. The solution is plotted in Figure 4. The reader is reminded that for $\alpha+\omega+\beta < 0$ and $\tau = 0$, there are only layers and no oscillations. This example demonstrates that small shifts can destroy the boundary layer at $x = 0$ and generate oscillations which extend beyond the layer. Again the WKB analysis yields (4.20) and (4.21), and rapid oscillations are generated because there are two roots of (4.20) with real part of $O(\varepsilon)$. The solution oscillates beyond the layer region, then dies out and blends with the outer solution. However, in this case, note that there remains a layer at $x = 1$ in which there are no oscillations. Using the roots in (4.20) and the formula (4.21) in (4.1) gives a good approximation to the solution in Figure 4.

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FIGURE CAPTIONS

Figure 1. Graphs of numerical solutions of BVP (2.1) - (2.2) for $b(x) = 1$, $f(x) = 0$, $\phi(x) = 1$, $\gamma = 1$, $\varepsilon = 0.01$, and $\delta(\varepsilon) = \tau\varepsilon$ with $\tau = 1.5$. Figure 1a has $a(x) = 1$ and Figure 1b has $a(x) = e^{-\nu x}$ with $\nu = 0.5$.

Figure 2. Graphs of numerical solutions of BVP (2.3) - (2.4) for $\omega(x) = 1$, $\beta(x) = 0$, $f(x) = 0$, $\phi(x) = 1$, $\gamma = 0$ (i.e., $\psi(x) = 0$ since $\beta(x) = 0$), and $\varepsilon = 0.01$. Figures 2a, 2b, and 2c correspond to $\alpha(x) = -1/2$ with the shift $\delta(\varepsilon) = \varepsilon^2$, $\varepsilon^{3/2}$, and ε, respectively. Figure 2d corresponds to $\alpha(x) = 3$ with the shift $\delta(\varepsilon) = \varepsilon^2$. Both numerical solutions (solid curves) and approximate solutions (3.4) (dashed) are plotted on Figures 2a, 2b, and 2d.

Figure 3. Graphs of numerical solutions of BVP (2.3) - (2.4) for $\omega(x) = 4$, $f(x) = 0$, $\phi(x) = 1$, $\psi(x) = 0$, and $\varepsilon = 0.01$. Figure 3a corresponds to $\alpha(x) = -1$, $\beta(x) = -2$, $\delta(\varepsilon) = \varepsilon^2$, and $\eta(\varepsilon) = 2\varepsilon^2$. Figure 3b corresponds to $\alpha(x) = -2$, $\beta(x) = -1$, $\delta(\varepsilon) = 2\varepsilon^2$, and $\eta(\varepsilon) = \varepsilon^2$.

Figure 4. Graph of numerical solution of BVP (2.3) - (2.4) for $\alpha(x) = -2e^{-x}$, $\omega(x) = -1$, $\beta(x) = 0$, $f(x) = 1$, $\phi(x) = 1$, $\gamma = 0$ (i.e., $\psi(x) = 0$ since $\beta(x) = 0$), and $\delta(\varepsilon) = \tau\varepsilon$ with $\tau = 3$.

FIGURE 1a

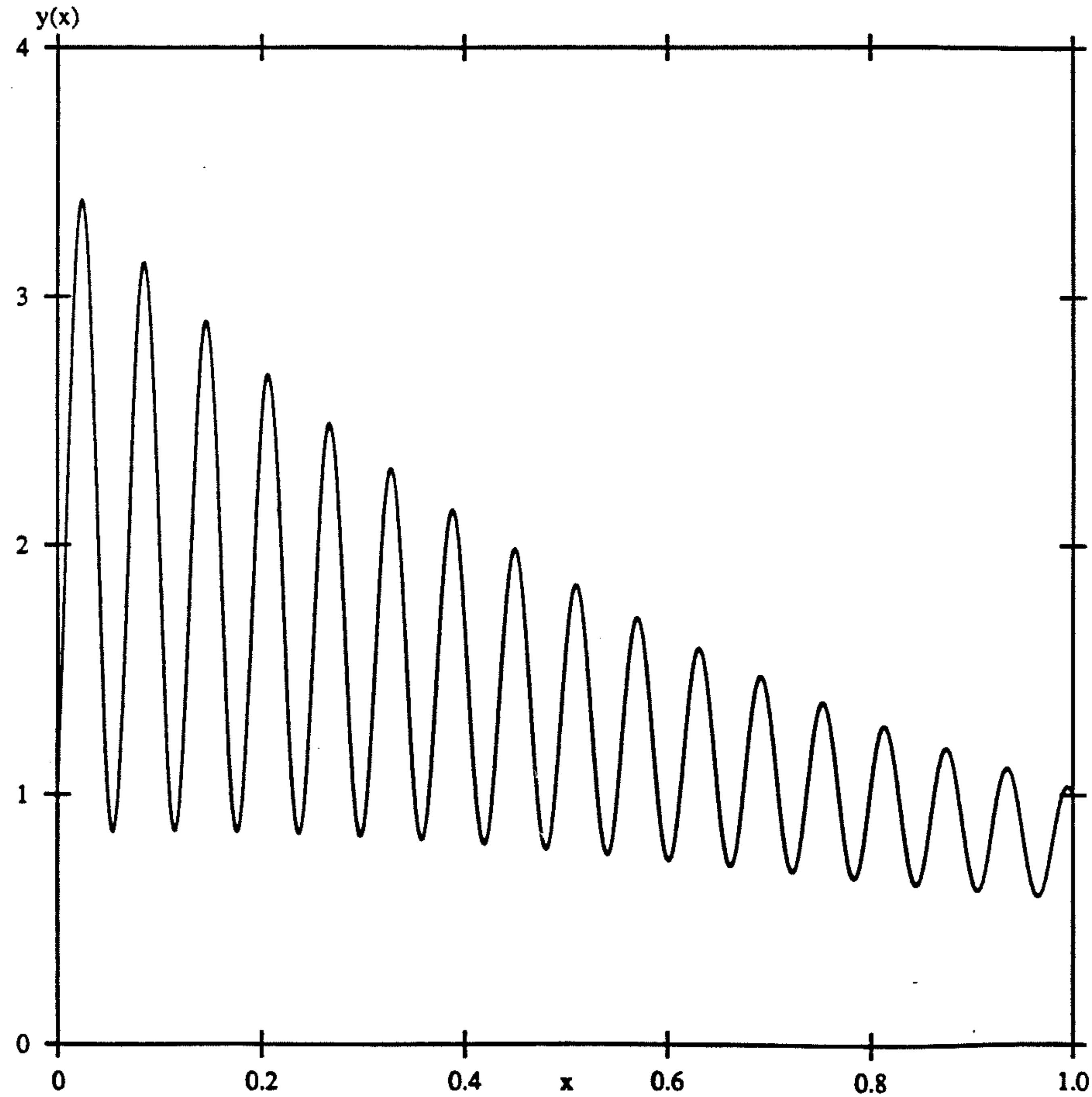

FIGURE 1b

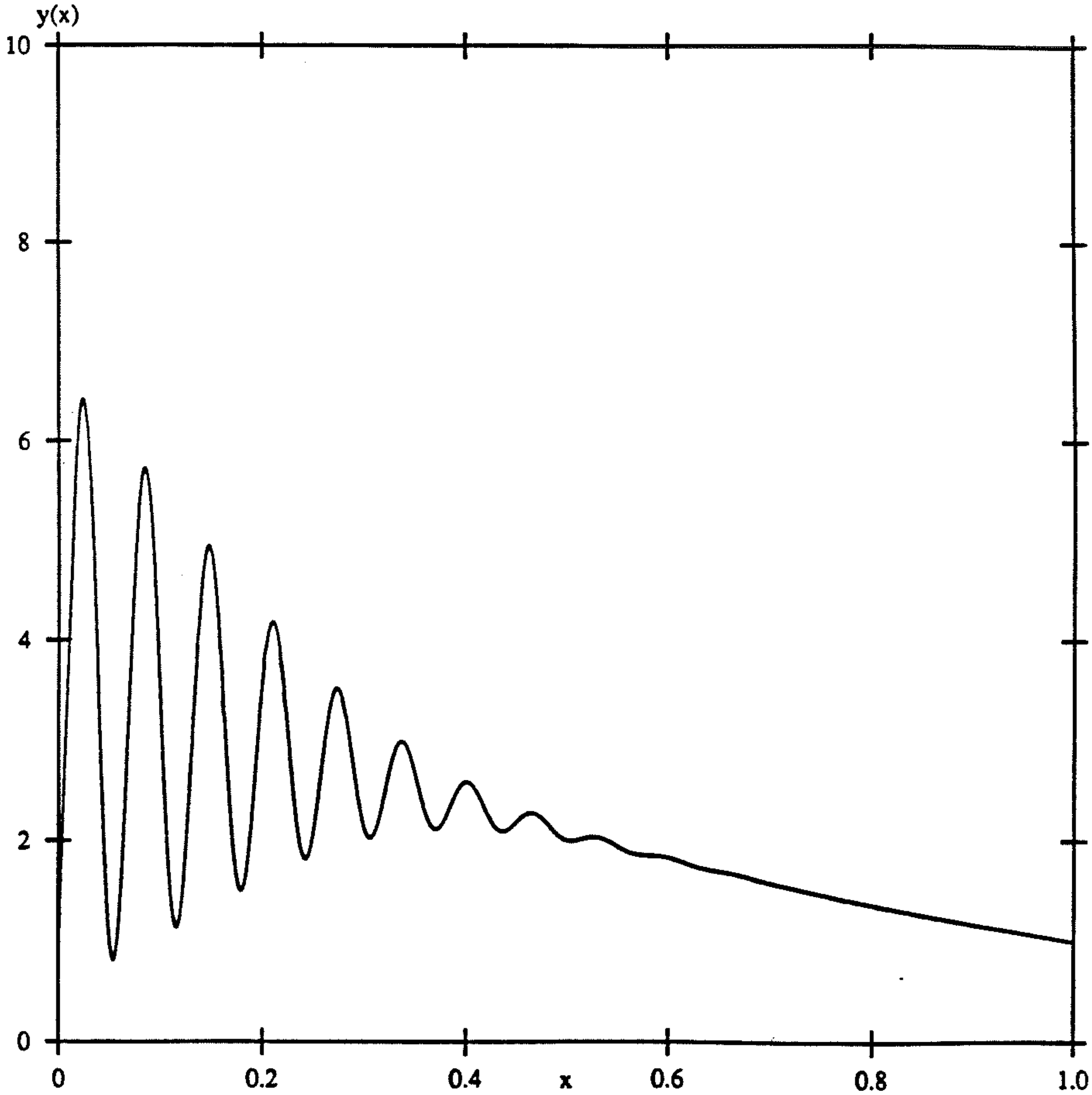

FIGURE 2a

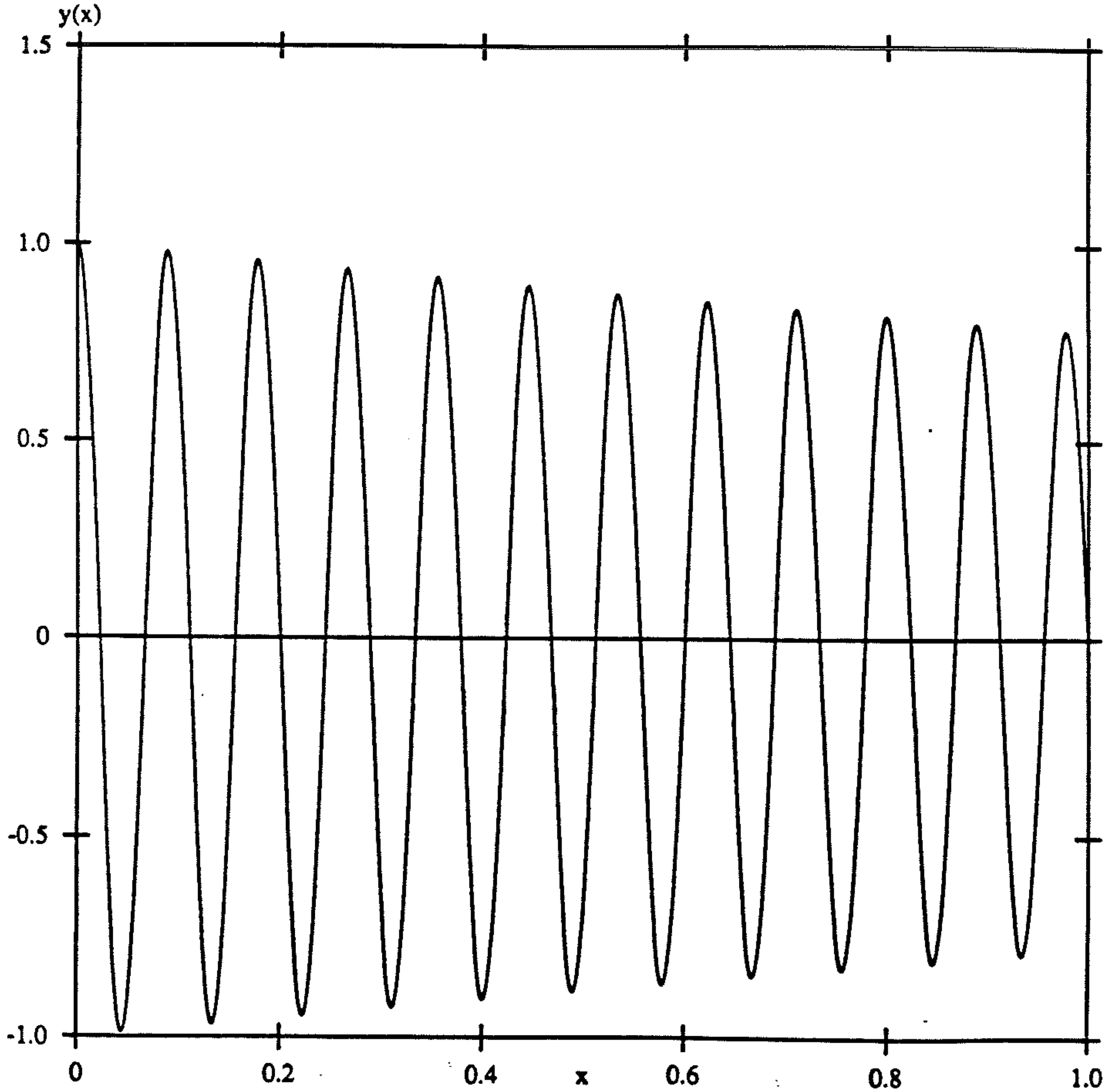

FIGURE 2b

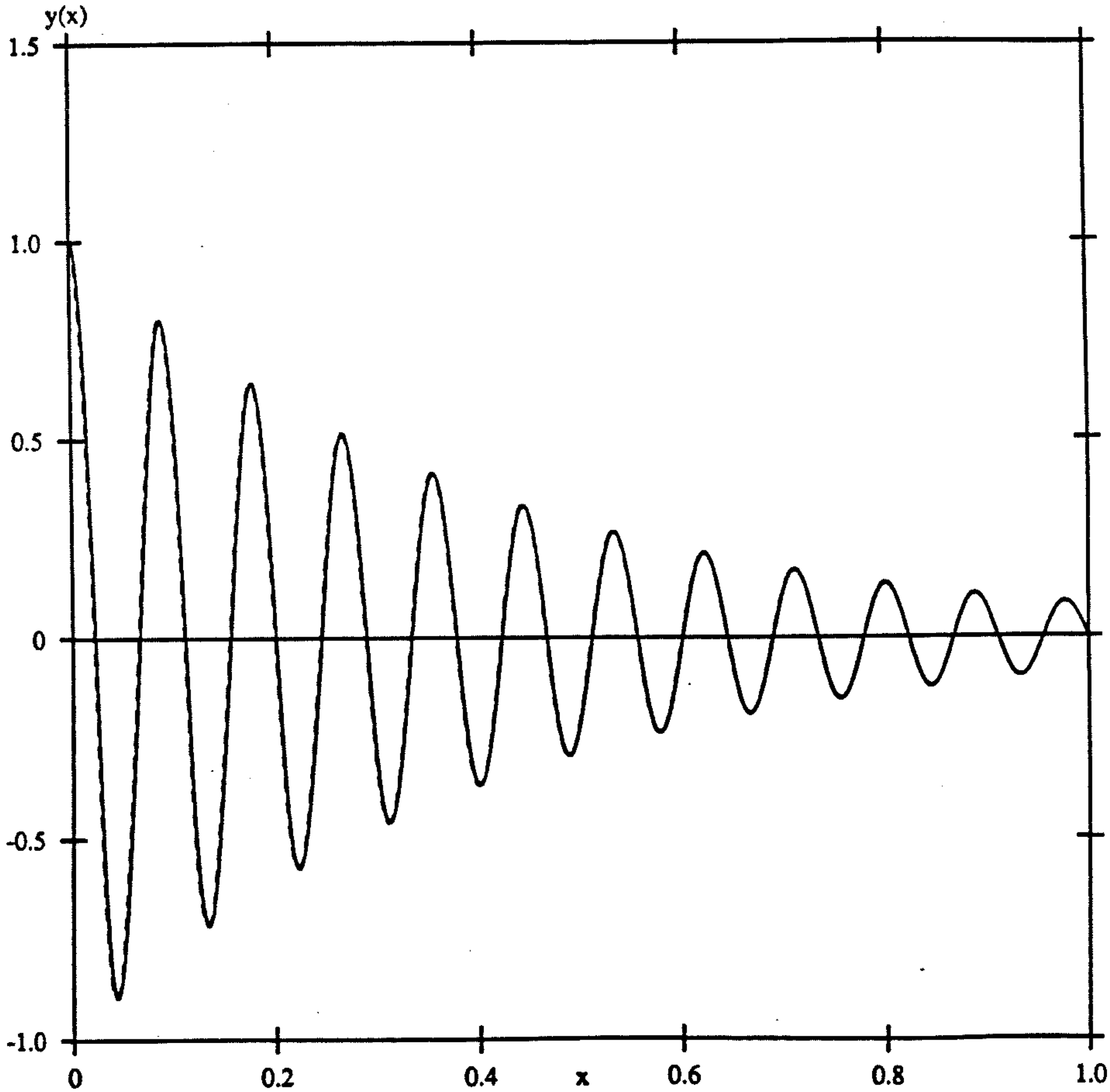

FIGURE 2C

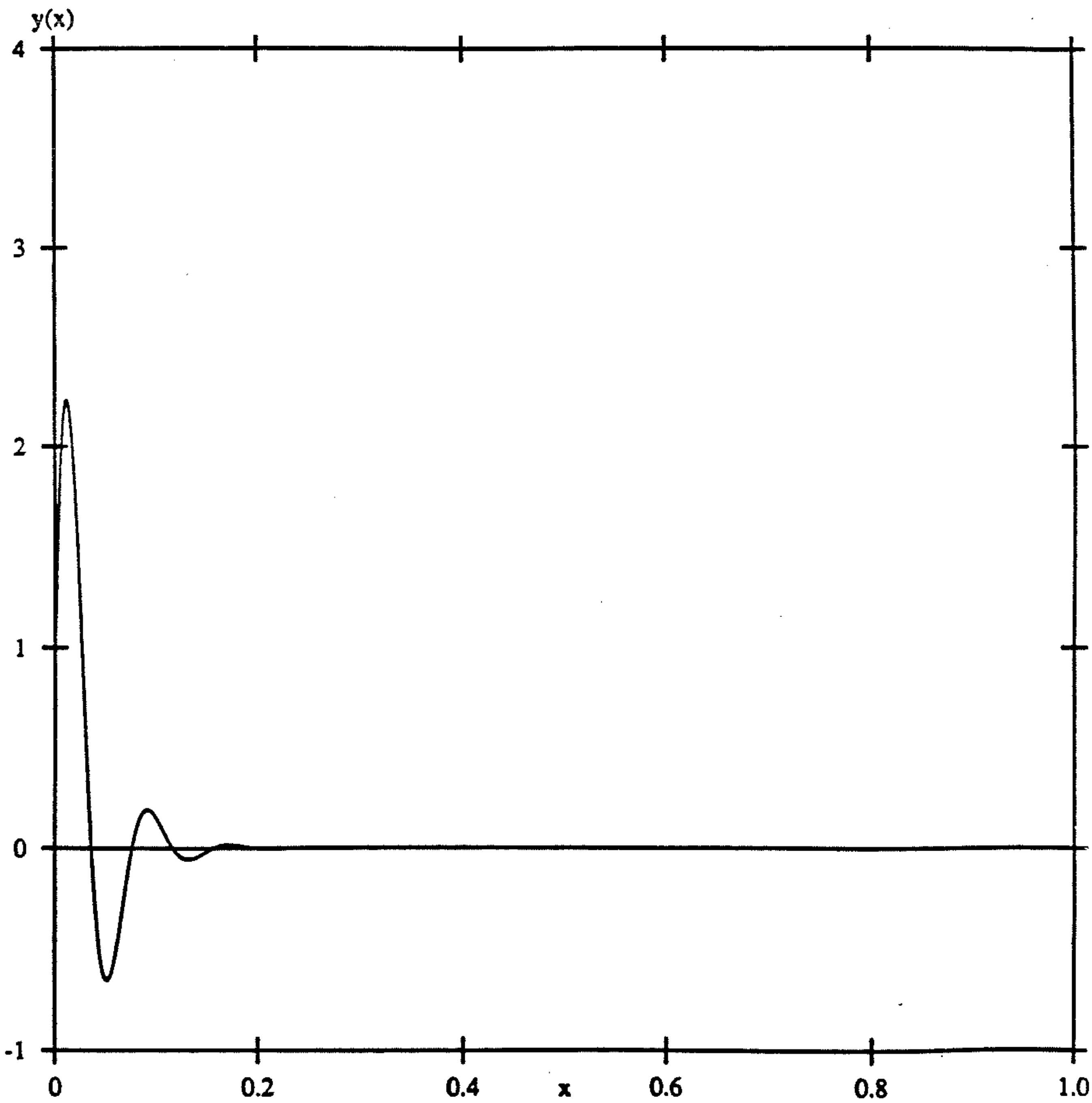

FIGURE 2d

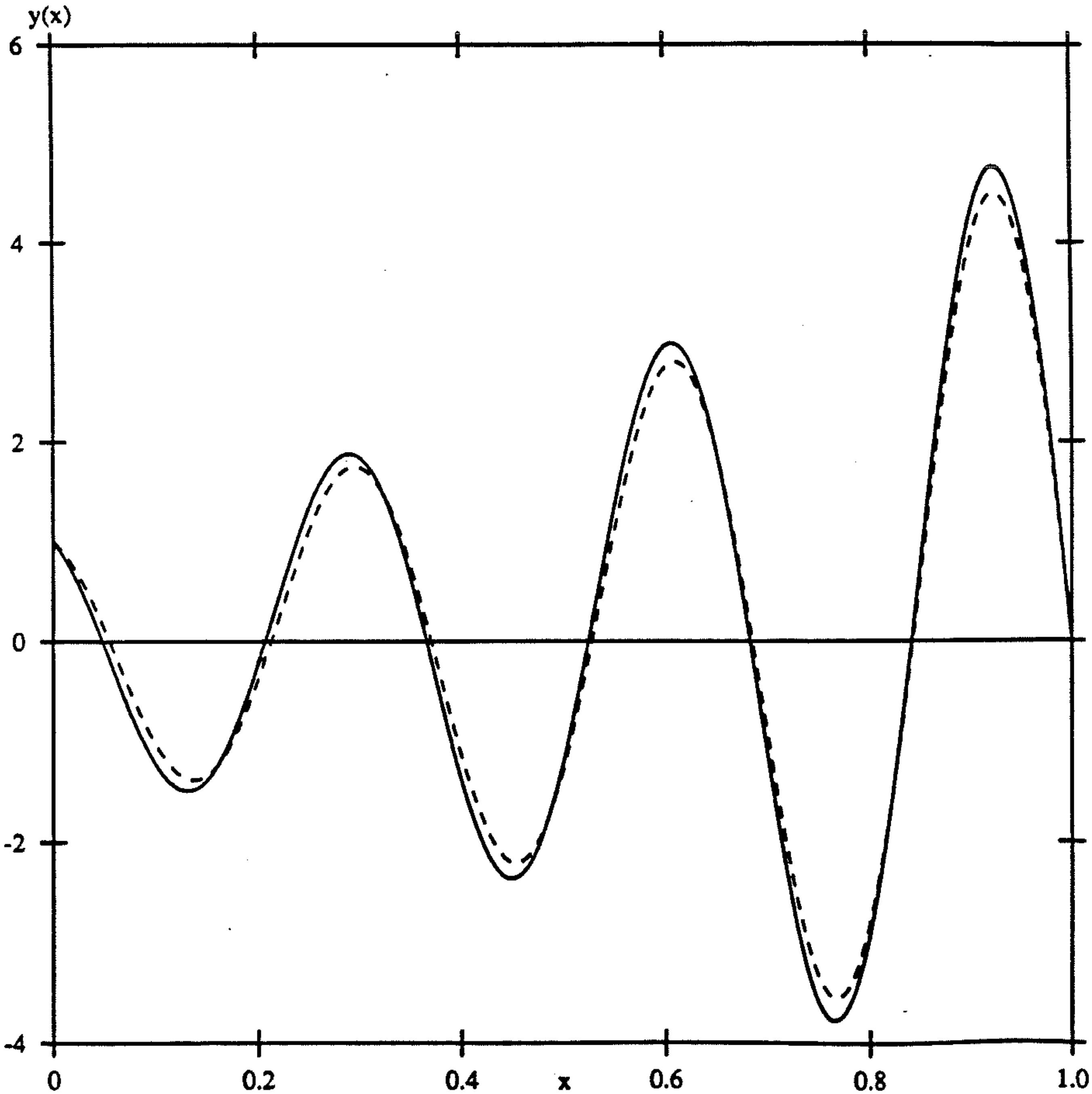

FIGURE 3a

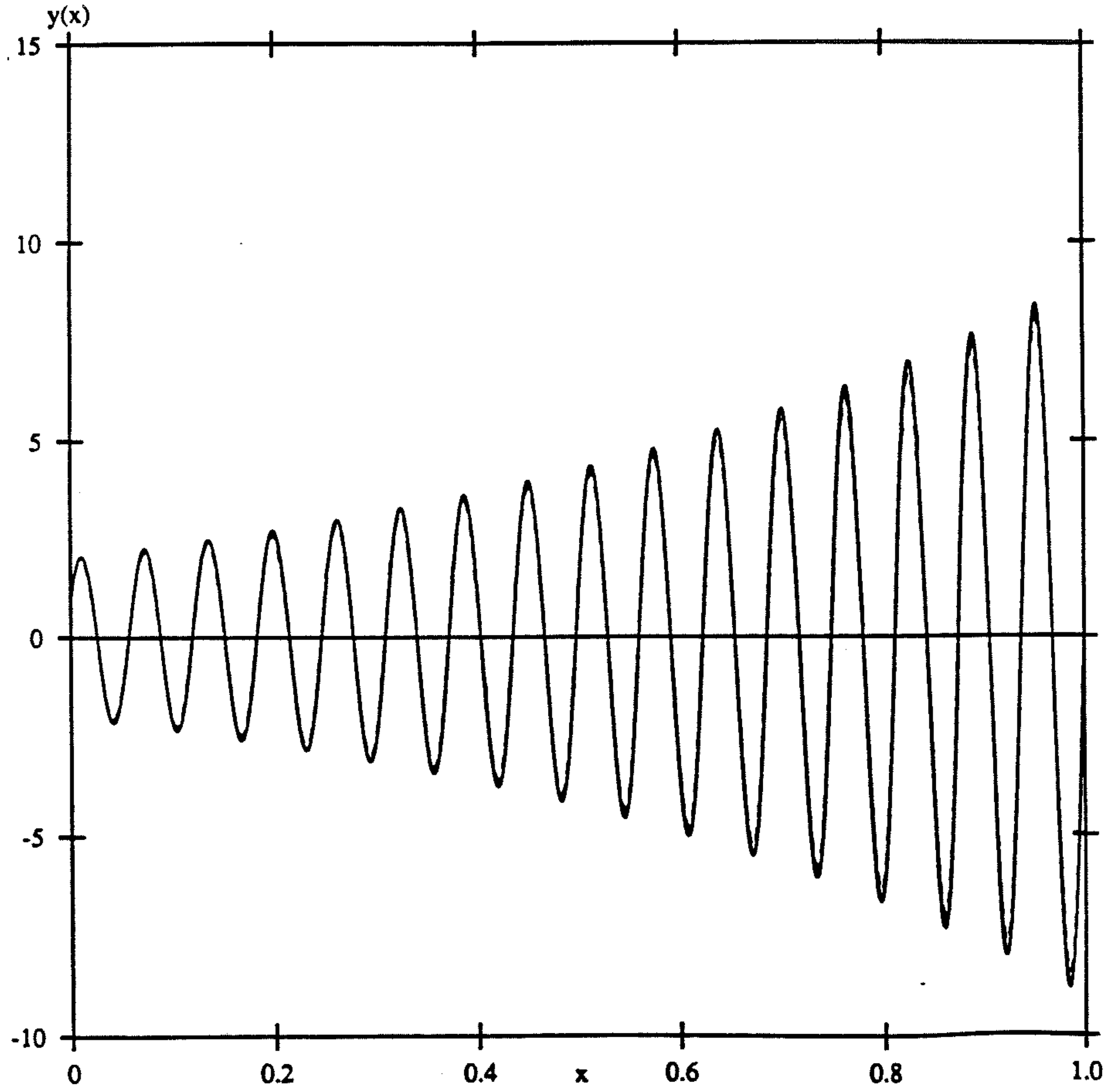

FIGURE 3b

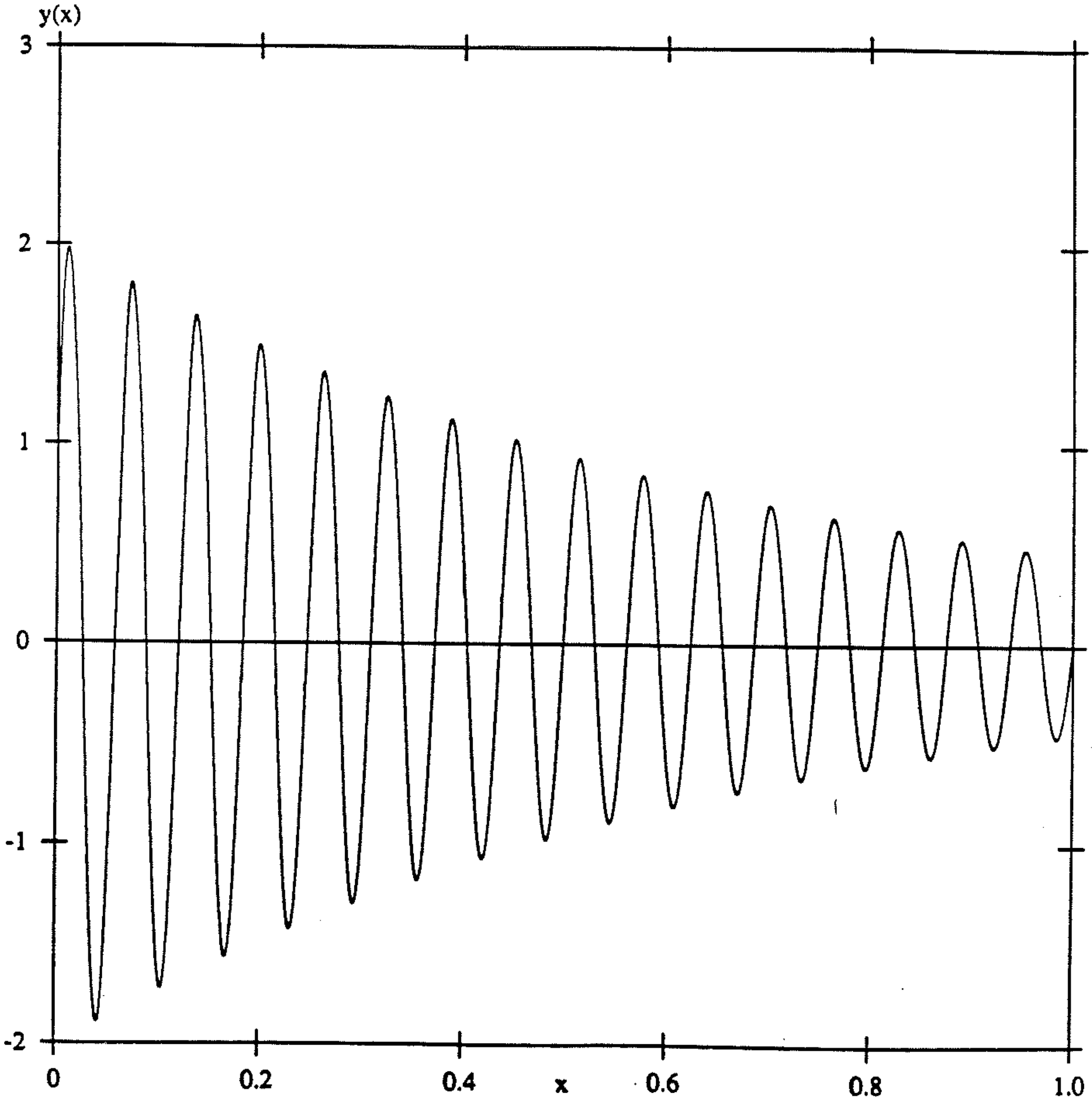

FIGURE 4