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1 Introduction

Imagine three-space to be filled with an elastic material and let $\eta(t) : \mathbf{R}^3 \to \mathbf{R}^3$ be the map which takes each material point from its position at time zero to its position at time t. Then the curve of maps, $t \mapsto \eta(t)$, will completely describe the motion of the material.

We shall assume that the material is incompressible, which in mathematical terms means that $\eta(t)$ must satisfy the equation:

$$J(\eta(t)) \equiv 1 \tag{1}$$

where J means the Jacobian determinant. We shall further assume that the material is "neo-Hookean" or that when deformed by a map $\eta : \mathbf{R}^3 \to \mathbf{R}^3$ it acquires potential energy:

$$V(\eta) = \frac{1}{2}\int_{\mathbf{R}^3} (\partial_i \eta^j \partial_i \eta^j - 3) \qquad \text{(repeated indices summed from 1 to 3)} \tag{2}$$

From (1) - (2) one can derive the equations of motion which $\eta(t)$ must satisfy. (see [2]). They are:

$$\ddot{\eta}(t) = \Delta\eta(t) + (\nabla p(t)) \circ \eta(t) \tag{3}$$

where "$\cdot$" means time derivative, "Δ" means Laplacian, "∇" means gradient, and $p(t)$ is a function on $\mathbf{R}^3$ which is chosen so that (1) will hold. (As we shall see below.)

It is known [2] that given initial data $\eta(0)$ and $\dot{\eta}(0)$, there is a positive T and a unique curve $\eta(t)$ $(0 \leq t < T)$ satisfying (1) and (3). In this announcement, we shall show that if $\eta(0)$ is sufficiently near the identity map and if $\dot{\eta}(0)$ is small, then one can take $T = \infty$. Thus if the initial deformations are small, the equations have a unique solution for all time.

To provide context to this theorem we now discuss some related results, each of whose proofs contains ideas that we will use for our proof below. First we consider the case of compressible elastic media. If the potential energy is as in (2), the resulting equation is the (linear) wave equation, so of course solutions persist for all time. However if $V(\eta)$ is not quadratic, the resulting equation is not linear, so it is possible that solutions will develop singularities. F. John [5] [6] analyzes this situation in considerable detail for homogeneous isotropic materials. He finds that for this case the situation is essentially the same as for the simpler case of a single quasi-linear wave equation, which has been studied over many years by Klainerman, John, et al. (see [7] for a survey). Thus we proceed with a brief discussion of quasi-linear wave equations (some of which we shall need for the proof of our theorem).

Let $f = f(x^0, x^1, x^2, x^3)$ (identifying t with x^0) and let f' and f'' denote respectively its first and second order partial derivatives. Let f satisfy

$$\Box f = F(f', f'') \tag{4}$$

where $\Box = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2$, F is linear in f'', and $F(f', f'') = O(|f'|(|f'| + |f''|))$ near zero. Then John and Klainerman prove the following: [8]

Theorem 1. *Let $f^0(x)$ and $f^1(x)$ be smooth functions on $\mathbf{R}^3$ with compact support. Then there exist positive constants $A, B,$ and ε_0 such that if $\varepsilon < \varepsilon_0$, then equation (4) with initial data $f(0,x) = \varepsilon f^0(x)$ and $\partial_t f(0,x) = \varepsilon f'(x)$ has a unique solution for $0 \leq t < T$ with $T \geq Ae^{B/\varepsilon}$.*

Thus a quasi-linear wave equation has solutions for a very long time if the data are small. Moreover as John points out, ([7] p. 27) this result is sharp. That is letting

$$F(f', f'') = \left(\frac{-2\partial_t f}{1 - 2\partial_t f}\right) \Delta f$$

and choosing f^1 with support in the unit ball, we find that (4) does not have a smooth solution on $[0, T)$ if:

$$\begin{aligned} T &> 2e^{1/L} \text{ where} \\ L &= \frac{3}{64\pi} \int_{\mathbf{R}^3} f^1(x) - (f^1(x))^2 dx \end{aligned}$$

Theorem 1 is proven by looking at various weighted spatial L^2 norms of f and its derivatives and showing that they obey certain differential inequalities. In particular one shows that if $N(t)^2$ is a certain sum of terms of type:

$$\int_{\mathbf{R}^3} |(x^\alpha)^l (\partial_\beta)^k f(t,x)|^2 dx \quad \alpha, \beta = 0,1,2,3, \quad \ell \leq k \tag{5}$$

with t identified with x^0, and $x = (x^1, x^2, x^3)$, then one gets an inequality of type:

$$\frac{d}{dt} N(t) \leq const.(1+t)^{-1} N(t)^2 \tag{6}$$

using this and the fact that $\int_0^T (1+t)^{-1} dt = \log(1+T)$ one gets the necessary estimates to prove Theorem 1.

On the other hand, the sharpness of Theorem 1, or the necessary development of singularities is shown by looking at spherical means of functions depending on f and its derivatives. Using these one constructs a function $W(t)$ which obeys:

$$\frac{d}{dt} W(t) \geq const.(1+t)^{-1} W(t)^2 \qquad \text{(the reverse of (6))} \tag{7}$$

and from this gets an upper bound on the time of existence of a smooth solution.

John shows that for compressible materials, the situation is the same as for solutions of (4). He uses the displacement $u(t,x) = \eta(t)(x) - x$ and notes that it satisfies an equation:

$$\partial_t^2 u = c_1 \Delta u + c_2 \nabla \; div(u) + F(u', u'') \tag{8}$$

where F is like the corresponding function in (4) and c_1 and c_2 are positive constants. For (8) one can get estimates like those of (4) thus find that given initial data of compact support, solutions exist as in Theorem 1. [5] Here too

John shows that the result is sharp. [6] Taking spherically symmetric data one gets a solution of the form:

$$u(t,x) = \varphi(t,|x|)x \tag{9}$$

where $\varphi(t,|x|)$ will satisfies a quasi-linear wave equation. Certain combinations of second derivatives of φ obey an inequality like (7) and thus $\varphi(t,|x|)$ remains smooth for only a finite time.

For incompressible elastic materials all results that we know of are for a finite time which depends on the data. [2] proves such a result for materials filling $\mathbf{R}^n$, and [3] and [4] show a similar result for initial-boundary value problems. Furthermore, we do have examples where singularities develop. ([3], section 7)

Another relevant case is that of incompressible fluids (the case $V(\eta) \equiv 0$) for which there is a much larger literature. First there is the old result of Wolibner [10]. It uses estimates based on conservation of vorticity to get existence of solutions for all time for fluids in $\mathbf{R}^2$. The same principle is used by Kato [9] and Yudovič [11] who prove existence for all time for fluids in two-dimensional domains with boundary. This analysis is further extended by Beals, Kato and Majda [1] who show that existence of solutions for all time is guaranteed in $\mathbf{R}^3$ provided the vorticity of the flow obeys a certain bound.

The proof of our result uses a combination of all of the above techniques. We first consider the equation in terms of the displacement. Applying the curl (or exterior derivative) to that, we get a quantity somewhat analogous to the vorticity of a fluid, which satisfies an equation similar to (4). Using norms like $N(t)$ above we get an inequality stronger than (6), and from that we get existence of solutions for all time.

We would like to generalize our proof so that it covers incompressible materials which are not neo-Hookean, but have not been able to do so as yet.

2 Definitions and the Statement of the Theorem

We shall be concerned with functions and vector fields on $\mathbf{R}^3$, and we shall use the exterior derivative, "d", on either of these. Thus for a function p, dp is the same as ∇p and for a vector field u, du is the same as the curl of u.

Also we will denote by δ the formal adjoint of d so that δu is the negative of the divergence of u.

Furthermore if L is any differential operator and $\eta : \mathbf{R}^3 \to \mathbf{R}^3$ is a diffeomorphism, we shall define L_η by:

$$L_\eta f = (L(f \circ \eta^{-1})) \circ \eta. \tag{10}$$

If $\eta(t)$ is a curve of diffeomorphisms, a direct calculation yields:

$$\frac{d}{dt}(L_{\eta(t)} f) = [v \cdot \nabla, L]_{\eta(t)} f \tag{11}$$

where [,] denotes the commutator and v is the vector-field defined by

$$\dot{\eta}(t)(x) = v(\eta(t)(x)). \tag{12}$$

Finally for functions on space-time we define the operators:

$$\begin{aligned}
\Gamma_i &= \partial_i \quad i = 0 \cdots 3 \\
\Gamma_{3+j} &= x^k \partial_i - x^i \partial_k \text{ where } \{ijk\} \quad \text{is an even permutation of } \{123\} \\
\Gamma_{6+k} &= t\partial_k + x^k \partial_0 \quad k = 1 \cdots 3 \\
\Gamma_{10} &= t\partial_0 + \sum_{i=1}^{3} x^i \partial_i
\end{aligned}$$

Also for any multi-index $\alpha = (\alpha_0 \cdots \alpha_{10})$ we define the higher order operator:

$$\Gamma^\alpha = \Gamma_0^{\alpha_0} \Gamma_1^{\alpha_1} \cdots \Gamma_{10}^{\alpha_{10}}. \tag{13}$$

Γ_0 to Γ_3 of course commute with the wave operator $\Box$, and since Γ_4 to Γ_9 are infinitessimal generators of the Lorentz group, they commute with $\Box$ also. However Γ_{10} generates similarity transformations and for it we have the identity:

$$[\Gamma_0, \Box] = -2\Box. \tag{14}$$

Using the Γ_p we define certain L^2 norms on functions of space time which depend on time and are stronger than the Sobolev norms. We write:

$$\|f\|^2_{LS,s} = \sum_{|\alpha| \leq s} \int_{\mathbf{R}^3} |\Gamma^\alpha f(t,x)|^2 dx \tag{15}$$

(LS stands for Lorentz-similarity). We shall also need supremum norms for functions and their derivatives and for this we write:

$$\begin{aligned} \|f\|_{C^k} &= \sum_{|\alpha|\leq k} \sup_{x\in\mathbf{R}^3} |D^\alpha f(x)| \\ \text{where} \qquad D^\alpha &= \partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}. \end{aligned}$$

Having introduced the necessary notation, we can now discuss our basic equation (3). We introduce $q(t)(x) = p(t)(\eta(t)(x))$ and then write the equation as:

$$\ddot{\eta} = \Delta\eta + d_\eta q. \tag{16}$$

We would like to show that q is determined by η and its derivatives, and to do so we first take a time derivative of equation (1). This gives:

$$\delta_\eta(\dot{\eta}) = 0. \tag{17}$$

Then applying a second time derivative and using (11) gives:

$$\delta_\eta(\ddot{\eta}) - [v\cdot\nabla,\delta]_\eta\dot{\eta} = 0. \tag{18}$$

where v is defined as in (12). Also a direct calculation yields:

$$[v\cdot\nabla,\delta]_\eta\dot{\eta} = tr((D_\eta\dot{\eta})^2) \text{ where "tr" means trace of a matrix.} \tag{19}$$

Now applying δ_η to (16) and using (18) and (19) we find:

$$tr((D_\eta\dot{\eta})^2) = \delta_\eta\Delta\eta - \Delta_\eta q. \tag{20}$$

Thus given $\eta(t)$ and $\dot{\eta}(t)$ we get $q(t)$ as the solution of (20) and it follows that the right side of (16) is a function of η and $\dot{\eta}$. Our main theorem is:

Theorem 2: *Given η_0 and η_1 smooth vector fields on $\mathbf{R}^3$ such that $\eta_0 - Id$ and $\dot{\eta}_1$ are rapidly decreasing; there exists a maximal $T > 0$ and a unique curve $\eta(t)$ $(0 \leq t < T)$ which satisfies* (1) and (16) (q defined as in (20)) such that $\eta(0) = \eta_0$ and $\dot{\eta}(0) = \eta_1$. Furthermore there exists $\varepsilon > 0$ such that if $\|\eta_0 - Id\|_{LS,5}$ and $\|\eta_1\|_{LS,4}$ are less than ε, then $T = \infty$.

3 Outline of Proof

We let $u(t,x) = \eta(t)(x) - x$ be the displacement as defined in §1. Then (16) is equivalent to:

$$\Box u = d_\eta q. \tag{21}$$

Also since $\eta(t)(x) = x + u(t,x)$, we find that the equation (1) is equivalent to:

$$\delta(u) = Q(Du) + det(Du) \tag{22}$$

where

$$Q(Du) = \sum_{i=1}^{3} det \begin{pmatrix} \partial_i u^i & \partial_i u^{i+1} \\ \partial_{i+1} u^i & \partial_{i+1} u^{i+1} \end{pmatrix} \text{ (indices taken mod 3)} \tag{23}$$

and det means determinant. Thus (21),(22) is equivalent to (1), (3) and from [2] we have the following short time existence theorem.

Theorem 3: *Given $u^0 \in H^4(\mathbf{R}^3, \mathbf{R}^3)$ and $u^1 \in H^3(\mathbf{R}^3, \mathbf{R}^3)$ such that $\delta(u^0) = Q(Du^0) + J(u^0)$ and $\delta(u^1) = 0$, there exists $T > 0$ and a unique solution of $u(t,x)$ (21)-(22) for $0 \leq t < T$ such that $u(0,x) = u^0(x)$ $\partial_t u(0,x) = u^1(x)$. Furthermore if $u^0 \in H^{s+1}(\mathbf{R}^3, \mathbf{R}^3)$ and $u^1 \in H^s(\mathbf{R}^3, \mathbf{R}^3)$ then $u \in \cap_{k=0}^{s} C^k([0,T),\ H^{s+1-k}(\mathbf{R}^3, \mathbf{R}^3))$.*

For our estimates we will need the $\| \ \|_{LS}$ norms, which are examples of the norms $N(t)$ of §1. Hence we need to know that $u(t,x)$ goes to zero rapidly as $|x| \to \infty$. Thus we introduce $H^{s,\ell}$ norms:

$$\|f\|^2_{s,\ell} = \int_{\mathbf{R}^3} |D^s f(x)|^2 (1 + |x|^2)^\ell dx. \tag{24}$$

A slight modification of the proof of [2] shows that in Theorem 3, H^s can be replaced by $H^{s,\ell}$ for any positive integer ℓ. Thus norms of the form $N(t)$ will be finite for solutions $u(t,x)$.

To prove the last statement of Theorem 2 it suffices to show that u remains bounded as t gets large. To do this we will derive an equivalent equation for du and show that solutions of that equation must remain bounded.

To derive an equation for du we first apply d_η to (21) getting:

$$d_\eta \Box u = 0. \tag{25}$$

However, by direct calculation we find that for any $v : \mathbf{R}^3 \to \mathbf{R}^3$,

$$d_\eta v = A((D\eta)^{-1} Dv) \tag{26}$$

where $(D\eta)^{-1}$ denotes the matrix inverse of $D\eta$ and A applied to a matrix means the matrix minus its transpose. Thus (25) can be written:

$$A((D\eta)^{-1}\Box Du) = 0. \tag{27}$$

Since $D\eta = Id + Du$ (and therefore $(D\eta)^{-1} - Id = -Du(D\eta)^{-1}$) we can rewrite (27) as:

$$A(\Box Du) = A(Du(D\eta)^{-1}\Box Du)$$

or

$$\Box du = A(Du(D\eta)^{-1}\Box Du) \tag{28}$$

Using LS norms we will derive estimates for (28) which show that it has solutions for all time.

To analyze (28) would like to eliminate the highest order terms on the right side – namely $\Box Du$, and to do this must first express Du as a function of du. Letting $\omega = du$ and using (22), we find that u satisfies the elliptic system:

$$du = \omega, \quad \delta u = Q(Du) + det(Du). \tag{29}$$

Proposition 4. *There exists a ball B about zero in $H^3(\mathbf{R}^3, \mathbf{R}^3)$ such that given $\omega \in H^2(\mathbf{R}^3, \mathbf{R}^3)$ there is at most one u in B which satisfies (29).*

Proof. Use the fact that u is determined by du and plus the fact that $Q(Du)$ and $det(DU)$ are $O(|Du|^2)$.

Proposition 4 defines a map $G(\omega) = Du$ where u is the solution of (29). Thus (28) can be written:

$$\Box du = A(Du(D\eta)^{-1}\Box G(du)). \tag{30}$$

We proceed to analyze $\Box G(du)$, our first step being to compute:

$$\partial_\alpha G(du), \quad \alpha = 0, \ldots, 3$$

Applying ∂_α to (29) we get:

$$d\partial_\alpha u = \partial_\alpha \omega \qquad \delta\partial_\alpha u = Q'(Du)D\partial_\alpha u + det'(Du)D\partial_\alpha u \tag{31}$$

where Q' and det' are the derivatives of Q and det. Thus $\partial_\alpha G(\omega) = D\partial_\alpha u$ where $\partial_\alpha u$ is the solution of (31). We write $D\partial_\alpha u = G'(\omega)\partial_\alpha \omega$.

Applying ∂_α again we find that $\partial_\alpha^2 u$ satisfies:

$$\begin{aligned} d\partial_\alpha^2 u &= \partial_\alpha^2 \omega \\ \delta\partial_\alpha^2 u &= Q'(Du)D\partial_\alpha^2 u + det'(Du)D\partial_\alpha^2 u \\ &\quad + Q''(Du)(D\partial_\alpha u, D\partial_\alpha u) + det''(Du)(D\partial_\alpha u, D\partial_\alpha u) \end{aligned} \tag{32}$$

where Q'' and det'' are second derivatives of Q and det. Thus we can write $\partial_\alpha^2 u = c_\alpha + b_\alpha$ where c_α solves:

$$dc_\alpha = \partial_\alpha^2 \omega \quad \delta c_\alpha = Q'(Du)Dc_\alpha + det'(Du)Dc_\alpha \tag{33}$$

and b_α solves:

$$\begin{aligned} db_\alpha &= 0 \\ \delta b_\alpha &= Q'(Du)Db_\alpha + det'(Du)Db_\alpha \\ &\quad + Q''(Du)(D\partial_\alpha u, D\partial_\alpha u) + det''(Du)(D\partial_\alpha u, D\partial_\alpha u). \end{aligned} \tag{34}$$

In particular $Dc_\alpha = G'(\omega)\partial_\alpha^2 \omega$. Then summing over α we get:

$$\begin{aligned} \Box G(\omega) &= D(c_0 - \sum_{i=1}^{3} c_i) + D(b_0 - \sum_{i=1}^{3} b_i) \\ &\stackrel{def}{=} DC + DB \end{aligned} \tag{35}$$

Here

$$DC = G'(\omega)(\Box du) \tag{36}$$

Also combining (35) with (30) we get:

$$\Box du - A(Du(D\eta)^{-1}DC) = A(Du(D\eta)^{-1}DB) \tag{37}$$

so using (36) we get:

$$\Box du - A(Du(D\eta)^{-1}G'(w)\Box du) = A(Du(D\eta)^{-1}DB). \tag{38}$$

The second term on the left side of (38) can be thought of as a linear operator applied to $\Box du$ and if Du is small it will have small norm. Hence the map $\Phi(\Box du) = \Box du - A(Du(D\eta)^{-1}G'(du)\Box du)$ will be a linear map which is near the identity. Thus Φ will be invertible and we can write:

$$\Box du = \Phi^{-1}(A(Du(D\eta)^{-1}DB). \tag{39}$$

Our theorem can now be proven by getting estimates for the solution of (39).

We note that DB, which is a sum of Db_α, depends on $D\partial_\alpha u$ in a bi-linear fashion. Therefore:

$$Du(D\eta)^{-1}DB = O(|Du||\partial_\alpha Du|^2) \tag{40}$$

and since Φ goes to the identity as $Du \to 0$, we find:

$$\Phi^{-1}(A(Du(D\eta)^{-1}DB) = O(|Du||\partial_\alpha Du|^2) \tag{41}$$

also.

Let E denote the energy associated with the wave equation, i.e.

$$E(g) = \frac{1}{2}\int_{\mathbf{R}^3}(|Dg|^2 + |\partial_t g|^2). \tag{42}$$

Then $\frac{d}{dt}E(g) = \int_{\mathbf{R}^3}(\Box g)(\partial_t g)$. Therefore:

$$\frac{d}{dt}E(du) = \int\langle\Phi^{-1}(A(Du(D\eta)^{-1}DB)), \partial_t du\rangle. \tag{43}$$

Then applying Γ^α to (39) (with $|\alpha| = 4$) and using (42) and the commutation properties of $\Box$ and Γ^α, one gets:

$$\begin{aligned}\frac{d}{dt}E(\Gamma^\alpha du) &= \int\langle\Gamma^\alpha\Phi(A(Du(D\eta)^{-1}DB)), \Gamma^\alpha\partial_t du\rangle \\ &\leq K\|\partial_t du\|_{LS,4}\|\partial_\alpha du\|_{LS,4}\|Du\|^2_{C^3}\end{aligned} \tag{44}$$

for some constant K.

But Klainerman's inequality (see [7], appendix 2) combined with the estimates for $Du = G(du)$ tells us that:

$$\|Du\|^2_{C^3} \leq \frac{K}{1+t)^2}\sum_{|\alpha|\leq 4}E(\Gamma^\alpha du). \tag{45}$$

Let $E_4(du) = \sum_{|\alpha|\leq 4}E(\Gamma^\alpha du)$. Then $E_4(du)$ bounds $\|\partial_\alpha du\|^2_{(L,4)}$, so from (44) and (45) we find that there is a constant K such that:

$$\frac{d}{dt}E_4(du) \leq K(1+t)^{-2}E_4(du)^2. \tag{46}$$

But integrating this inequality we find that if $E_4(du)(0) < \frac{1}{K}$, then:

$$E_4(du)(t) \leq \frac{E_4(du)(0)}{1 - KE_4(du)(0)(1 - \frac{1}{1+t})} \tag{47}$$

for all positive t.

From this it follows that if du (and its derivatives) is small at time zero, then it must remain bounded for all time. Thus the solution of (39) remains bounded, and hence the solution of (21) must remain bounded as well.

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