Nonoscillatory High Order Accurate Self-Similar Maximum Principle Satisfying Shock Capturing Schemes I

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Abstract

This is the first paper in a series in which we construct and analyze a class of nonoscillatory high order accurate self-similar local maximum principle satisfying shock capturing schemes for solving multidimensional systems of conservation laws. In this paper we present a scheme which is of 3rd order of accuracy in the sense of flux approximation, using scalar one-dimensional initial value problems as a model. For this model, we make the schemes satisfy a local maximum principle and a nonoscillatory property. The method uses a simple centered stencil with quadratic reconstruction followed by two modifications, imposed as needed. The first enforces a local maximum principle, the second guarantees that no new extrema develop. The schemes are self-similar in the sense that the numerical flux does not depend explicitly on the grid size, i.e., there are no grid size dependent limits involving free parameters as in, e.g., [12],[13],[14],[15]. Combining the nonoscillatory property and the local maximum principle we achieve TVB (Total Variation Boundedness). Hence we obtain convergence of a subsequence of the numerical solutions as the step size approaches zero. Numerical results are encouraging. Extensions to systems and/or higher dimensions will appear in future papers, as well as extensions to higher orders of accuracy.
1 Introduction

We consider numerical solutions of the hyperbolic conservation law:

\[ u_t + f(u)_x = 0 \]
\[ u(x,0) = u_0(x), \]  
\[ \text{(1.1)} \]

where \( u_0(x) \) is assumed to be a bounded variation function. The main difficulty in solving (1.1) is that the solution may contain discontinuities even if the initial condition is smooth. Among the successful numerical schemes for solving (1.1) we mention the modern nonoscillatory conservative schemes such as TVD (Total Variation Diminishing), UNO (Uniformly High Order Nonoscillatory), ENO (Essentially Nonoscillatory) and TVB (Total Variation Boundedness) schemes (see e.g. [1],[2], [3] and the references listed therein). These schemes are usually total variation stable for one dimension scalar nonlinear problems and are formally higher than first order accurate, hence they can capture sharp shocks without introducing oscillations. These schemes are very successful in numerical experiments. Recently a SNO (Strictly Nonoscillatory) scheme was introduced by Tong in [11], which is of arbitrarily high order of accuracy. We would also like to mention the TVD scheme introduced by Sanders (see [10]). His scheme is TVD in the sense of reconstruction and 3rd order of accuracy except for a degeneracy to second order at isolated extrema. His method involves advancing in time the cell-average and at least one additional quantity, while ours uses only the cell-average. There are, however, similarities in the way in which we enforce a local maximum principle as described below.

The schemes we introduce here are 3rd order accurate, conservative, local maximum principle satisfying, nonoscillatory, and hence TVB schemes. The 3rd order of accuracy is achieved with the usual degeneracy to second order at certain isolated extrema. The TVB property follows from a local maximum principle and a nonoscillatory property. Extensions to multi-dimensional systems are straightforward and will be performed in the future.

In this paper we use a uniform grid (for simplicity only). Extending the schemes to unstructured grids is not difficult. We define a partition \( \{I_j \times [t_n, t_{n+1}]\}_j^n \) of \( R \times R^+ \), where \( I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \) is the \( j \)-th cell, \( x_j = j \cdot h \), \( t_n = n \cdot \tau \), \( h = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \), \( \tau = t_{n+1} - t_n \), and \( \lambda = \tau / h \).

Now we briefly outline the construction of the schemes.
Following the framework of the original ENO schemes constructed in [2], our scheme is of the form

\[ \bar{u}^{n+1} = A \cdot E(\tau) \cdot R(x, \bar{u}^n), \]  

(1.3)

where \( \bar{u}^n = \{ \bar{u}^n_j \} = \{ \frac{1}{h} \int_{I_j} u(x, t_n) \, dx \} \) are cell-averages of the solution \( u(x, t_n) \) at time \( t = t_n \). \( R(x, \bar{u}^n) \) is a reconstruction procedure used to produce a high order accurate global approximation to \( u(x, t_n) \) from its given cell-averages \( \bar{u}^n \). Here we also denote \( R(x, \bar{u}^n) \) to be the global approximation to \( u(x, t_n) \) and consider \( R(x, \bar{u}^n) = \{ R_j(x, \bar{u}^n) \} \) to be a piecewise polynomial and \( R_j(x, \bar{u}^n) \) to be a polynomial in j-th cell \( I_j \). \( E(\tau) \) is the evolution operator of the PDE, and \( A \) is the cell-averaging operator, see [2]. We represent the schemes as following: For time \( t = t_n \), we follow a reconstruction procedure \( R(x, \bar{u}^n) \) to reconstruct the solution to obtain the \( R(x, \bar{u}^n) \) from the given cell-averages \( \bar{u}^n \) at time \( t = t_n \), and perform the evolution operator \( E(\tau) \) of the PDE on the \( R(x, \bar{u}^n) \) to obtain an approximation solution at time \( t = t_{n+1} \) which is the true solution of following hyperbolic conservation law

\[ \begin{align*}
        u_t + f(u)_x &= 0 \\
        u(x, 0) &= R(x, \bar{u}^n),
    \end{align*} \]

(1.4)

at time \( t = \tau \), and then compute the sliding averages of the approximation solution to obtain the cell-averages \( \bar{u}^{n+1} \) at time \( t = t_{n+1} \).

To explain how we get the local maximum principle, the nonoscillatory property, and the resulting TVB property, we rewrite (1.3) as

\[ R(x, \bar{u}^{n+1}) = R(x, A \cdot E(\tau) \cdot R(x, \bar{u}^n)). \]

(1.5)

From the reconstructed solution \( R(x, \bar{u}^n) \) at time \( t = t_n \), we perform the evolution operator \( E(\tau) \) to obtain the solution at time \( t = t_{n+1} \), and the averaging operator \( A \) to obtain the cell-averages \( \bar{u}^{n+1} \) of the solution, and then use the reconstruction procedure to obtain the reconstructed solution \( R(x, \bar{u}^{n+1}) \) at time \( t = t_{n+1} \).

According to (1.5), we define the TVB (Total Variation Bounded) property of the schemes to be

\[ TV(R(x, \bar{u}^{n+1})) \leq C, \]

(1.6)

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i.e. we measure the variation of the reconstructed piecewise polynomial function. This was done earlier by Sanders in [10]. Here the constant C is independent of the step size $h$ and the time level $n$.

The strategy to achieve the TVB property of the scheme is thus simply: we construct the reconstruction solution to satisfy a local maximum principle and a nonoscillatory property.

Our local maximum principle means that for each cell, the reconstructed piecewise polynomial $R(x, \bar{u}^{n+1})$ satisfies, $\forall x \in I_j$,

$$m_j \leq R_j(x, \bar{u}^{n+1}) \leq M_j,$$  

where $m_j = \min_{x \in I_{j-1} \cup I_j \cup I_{j+1}} R(x, \bar{u}^n)$ and $M_j = \max_{x \in I_{j-1} \cup I_j \cup I_{j+1}} R(x, \bar{u}^n)$. This is reasonable under following CFL condition

$$\frac{\bar{c}}{h} \sup_u |f'(u)| \leq 1,$$

where the supreme is evaluated over $[\inf_x(u_0(x)), \sup_x(u_0(x))]$. This condition is optimal.

Our nonoscillatory property means that the number of extrema of the reconstructed piecewise polynomial $R(x, \bar{u}^{n+1})$ is equal to the number of extrema of the $\bar{u}^{n+1}$. Here and below we denote the number of extrema by $N(\cdot)$. Thus the nonoscillatory property means

$$N(R(x, \bar{u}^{n+1})) = N(\bar{u}^{n+1}).$$  

Because of the local maximum principle, for $\forall x, n$,

$$\inf_x(u_0(x)) \leq R(x, \bar{u}^{n+1}) \leq \sup_x(u_0(x)).$$

Because of the nonoscillatory property,

$$N(R(x, \bar{u}^{n+1})) = N(\bar{u}^{n+1}) \leq N(R(x, \bar{u}^n)) = N(\bar{u}^n) \leq N(R(x, \bar{u}^0)) = N(\bar{u}^0) \leq N(u_0(x)).$$

Here $N(\bar{u}^{n+1}) = N(A \cdot E(\tau) \cdot R(x, \bar{u}^n)) \leq N(E(\tau) \cdot R(x, \bar{u}^n)) \leq N(R(x, \bar{u}^n))$, because the $A$ and $E(\tau)$ are both nonoscillatory operators, see [3].
Thus
\[ TV(R(x, \bar{u}^{n+1})) \leq N(R(x, \bar{u}^{n+1})) \cdot (\sup_x (u_0(x)) - \inf_x (u_0(x))) \]
\[ \leq N(u_0(x)) \cdot (\sup_x (u_0(x)) - \inf_x (u_0(x))) \]
\[ = C, \]
where the constant C is independent of \( h \) and \( n \). Hence the TVB property (1.6) follows as soon as we achieve the local maximum principle (1.7) and the nonoscillatory property (1.8) in the reconstruction procedure. The TVB property implies convergence of a subsequence in \( L_1 \). Convergence for nonlinear problems will follow if we obtain a single entropy condition in the convex case (see [4]), and, more generally, if all limit solutions satisfy Kruzkov's entropy condition (see [16]).

Also we construct \( R(x, \bar{u}^{n+1}) \) so that it approximates \( u(x, t_{n+1}) \) up to 3rd order of accuracy in regions in which \( u(x, t_{n+1}) \) is smooth, and has conservation form \( \frac{1}{h} \int_{i_j} R(x, \bar{u}^{n+1}) \, dx = \bar{u}_j^{n+1} \) in each cell.

Hence the main idea of this paper is to achieve 3rd order of accuracy, conservation form, the local maximum principle, the nonoscillatory property, and hence the TVB property by achieving the first four of them in the reconstruction procedure.

2 The Reconstruction Procedure

In this section we present the reconstruction procedure to obtain \( R(x, \bar{u}^{n+1}) \) from the given cell-averages \( \bar{u}^{n+1} \). We also know \( \{m_j\} \) and \( \{M_j\} \). Here \( R(x, \bar{u}^{n+1}) = \{R_j(x, \bar{u}^{n+1})\} \) is a piecewise quadratic polynomial approximating the weak solution \( u(x, t_{n+1}) \) and \( R_j(x, \bar{u}^{n+1}) \) is quadratic polynomial defined on the j-th cell. According to the previous section, \( R(x, \bar{u}^{n+1}) \) should satisfy following four properties.

\( (p_1) \) Each \( R_j(x, \bar{u}^{n+1}) \) has 3rd order of accuracy i.e.
\[ R_j(x, \bar{u}^{n+1}) = u(x, t_{n+1}) + O(h^3), \]
in regions in which \( u(x, t_{n+1}) \) is smooth.

\( (p_2) \) Each \( R_j(x, \bar{u}^{n+1}) \) has conservation form, i.e.
\[ \frac{1}{h} \int_{i_j} R_j(x, \bar{u}^{n+1}) \, dx = \bar{u}_j^{n+1}. \]
Each $R_j(x, \bar{u}^{n+1})$ satisfies a local maximum principle i.e. $\forall x \in I_j$, $m_j \leq R_j(x, \bar{u}^{n+1}) \leq M_j$,

where $m_j = \min_{x \in I_{j-1} \cup I_j \cup I_{j+1}} R(x, \bar{u}^n)$ and $M_j = \max_{x \in I_{j-1} \cup I_j \cup I_{j+1}} R(x, \bar{u}^n)$

$p_4$ $R(x, \bar{u}^{n+1})$ satisfies a nonoscillatory property i.e.

\[ N(R(x, \bar{u}^{n+1})) = N(\bar{u}^{n+1}). \]

In the following subsections we design the reconstruction procedure to be two subprocesses. In the first subprocess, by interpolating, we simply get a preliminary reconstructed quadratic polynomial in each cell $I_j$, $P_j(x, \bar{u}^{n+1})$, and thus a preliminary reconstructed piecewise quadratic polynomial $P(x, \bar{u}^{n+1}) = \{P_j(x, \bar{u}^{n+1})\}$ satisfying properties $(p_1)$ and $(p_2)$. In the second subprocess we introduce a modifying operator and apply it twice to each piece $P_j(x, \bar{u}^{n+1})$ of $P(x, \bar{u}^{n+1})$ to obtain the desired $R(x, \bar{u}^{n+1}) = \{R_j(x, \bar{u}^{n+1})\}$ satisfying properties $(p_1)$-$p_4$.

For simplicity, here and below we denote $\bar{u} = \{\bar{u}_j\} = \bar{u}^{n+1} = \{\bar{u}_j^{n+1}\}$, $P(x) = \{P_j(x)\} = P(x, \bar{u}^{n+1}) = \{P_j(x, \bar{u}^{n+1})\}$ and $R(x) = \{R_j(x)\} = R(x, \bar{u}^{n+1}) = \{R_j(x, \bar{u}^{n+1})\}$.

### 2.1 Preliminary Reconstruction Subprocedure

We start with some definitions and observations.

**Definition 1**: We call the j-th cell $I_j$ nondecreasing if $\bar{u}_{j-1} \leq \bar{u}_j \leq \bar{u}_{j+1}$, nonincreasing if $\bar{u}_{j-1} \geq \bar{u}_j \geq \bar{u}_{j+1}$, maximum if $\bar{u}_{j-1} < \bar{u}_j > \bar{u}_{j+1}$, and minimum if $\bar{u}_{j-1} > \bar{u}_j < \bar{u}_{j+1}$. We call $I_j$ monotone if it is nondecreasing or nonincreasing, and extrema if it is maximum or minimum.

**Definition 2**: We say a polynomial $p_j(x)$, which is defined on $I_j$, has the same shape as the cell-averages $\bar{u}$, if the nonconstant $p_j(x)$ satisfies

(i) The polynomial $p_j(x)$ has a maximum in $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ if $I_j$ is a maximum cell and $p_j(x)$ has a minimum in $I_j$ if $I_j$ is a minimum cell.

(ii) $p_j(x)$ is nondecreasing on $I_j$ if $I_j$ is a nondecreasing cell; $p_j(x)$ is nonincreasing on $I_j$ if $I_j$ is a nonincreasing cell.

If $p_j(x)$ is a constant, we again say that it has the same shape as the cell-averages $\bar{u}$.

We say a piecewise polynomial $p(x) = \{p_j(x)\}$ has the same shape as the cell-averages $\bar{u}$ if all of the $p_j(x)$ have this property.
**Definition 3:** We call a polynomial \( p_j(x) \) defined on \( I_j \) a **proper reconstructed polynomial** if

(i) \( \frac{1}{h} \int_{I_j} p_j(x) \, dx = \bar{u}_j \),

(ii) \( \forall x \in I_j, \ p_j(x) - u(x, t_{n+1}) = O(h^3) \) in regions in which \( u(x, t_{n+1}) \) is smooth,

(iii) \( p_j(x) \) has the same shape as the cell-averages \( \bar{u} \).

We also call a piecewise polynomial \( p(x) = \{ p_j(x) \} \) a **proper reconstructed piecewise polynomial** if all of \( p_j(x) \) are proper reconstructed polynomials.

**Definition 4:** Suppose a proper reconstructed piecewise polynomial \( p(x) \) has an extrema at \( x^* \). We say \( p(x) \) has a **false extrema** at \( x^* \), if there is a neighborhood of \( x^* \) so that \( p(x) \) is monotone on both sides of \( x^* \) and of the same kind i.e both nondecreasing or both nonincreasing.

Of course a false extrema of a proper reconstructed piecewise polynomial \( p(x) \) may only occur at an interface between two cells.

**Observation 1:**

a) For a proper reconstructed piecewise polynomial \( p(x) \), we observe that there may be four type of false extrema at an interface, say \( x_{j+\frac{1}{2}} \):

- **false extrema** type (i) \( \bar{u}_j > \bar{u}_{j+1} \) and \( p_j(x_{j+\frac{1}{2}}) < p_{j+1}(x_{j+\frac{1}{2}}) \),
- **false extrema** type (ii) \( \bar{u}_j = \bar{u}_{j+1} \) and \( p_{j+1}(x_{j+\frac{1}{2}}) \geq \bar{u}_j \geq p_{j}(x_{j+\frac{1}{2}}) \) and at least one of these inequalities is strict.
- **false extrema** type (iii) \( \bar{u}_j < \bar{u}_{j+1} \) and \( p_j(x_{j+\frac{1}{2}}) > p_{j+1}(x_{j+\frac{1}{2}}) \),
- **false extrema** type (iv) \( \bar{u}_j = \bar{u}_{j+1} \) and \( p_{j+1}(x_{j+\frac{1}{2}}) \leq \bar{u}_j \leq p_{j}(x_{j+\frac{1}{2}}) \) and at least one of these inequalities is strict.

b) For a proper reconstructed piecewise polynomial \( p(x) \), if \( p(x) \) at any interface \( x_{j+\frac{1}{2}} \), satisfies

(i) if \( \bar{u}_j > \bar{u}_{j+1} \), then \( p_j(x_{j+\frac{1}{2}}) \geq p_{j+1}(x_{j+\frac{1}{2}}) \),

(ii) if \( \bar{u}_j < \bar{u}_{j+1} \), then \( p_j(x_{j+\frac{1}{2}}) \leq p_{j+1}(x_{j+\frac{1}{2}}) \),

(iii) if \( \bar{u}_j = \bar{u}_{j+1} \), then \( p_j(x_{j+\frac{1}{2}}) - \bar{u}_j \cdot (p_{j+1}(x_{j+\frac{1}{2}}) - \bar{u}_j) > 0 \) or \( p_j(x_{j+\frac{1}{2}}) = p_{j+1}(x_{j+\frac{1}{2}}) = \bar{u}_j \),

then \( p(x) \) has no false extrema.

**Observation 2:**

If a proper reconstructed piecewise polynomial \( p(x) \) has no false extrema, then it is a nonoscillatory reconstruction i.e. \( N(p(x)) = N(\bar{u}) \).
We construct the preliminary piecewise polynomial as follows: For each cell $I_j$, we use the centered stencil to interpolate the cell averages $\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}$ and obtain $P_j(x)$, i.e., we require
\[
\frac{1}{h} \int_{I_i} P_j(x) \, dx = \bar{u}_i, \quad i = j - 1, j, j + 1.
\]
We denote $P(x) = \{P_j(x)\}$ and we have

**Lemma 1:**

$P(x)$ is a proper reconstructed piecewise polynomial.

**Proof:** Obviously this $P(x)$ satisfies properties $(p_1)$ and $(p_2)$. For each quadratic polynomial $P_j(x)$, $P_j'(x_{j-\frac{1}{2}}) = (\bar{u}_{j+1} - \bar{u}_j)/h$ and $P_j'(x_{j+\frac{1}{2}}) = (\bar{u}_{j+1} - \bar{u}_j)/h^2$. Hence $P_j(x)$ has the same shape as the cell-averages $\bar{u}$. #

However $P(x)$ might not satisfy properties $(p_3)$ and $(p_4)$, see Figure 1.

**Figure 1**

We observe that some $P_j(x)$ may have overshoot which means $\max_{x \in I_j} P_j(x) - M_j > 0$ and/or undershoot which means $\min_{x \in I_j} P_j(x) - m_j < 0$. We denote the magnitudes of overshoot and undershoot as $\delta_j^+ = \max_{x \in I_j} (P_j(x) - M_j, 0) \geq 0$ and $\delta_j^- = \max_{x \in I_j} (m_j - P_j(x), 0) \geq 0$. Note that, for each $P_j(x)$, $\max(\delta_j^+, \delta_j^-) = O(h^3)$ in regions in which $u(x, t_{n+1})$ is smooth. We also observe that there may also be some false extrema of $P(x) = \{P_j(x)\}$. Note that at each interface $x_{j+\frac{1}{2}}$, $|P_j(x_{j+\frac{1}{2}}) - P_{j+1}(x_{j+\frac{1}{2}})| = O(h^3)$ in regions in which $u(x, t_{n+1})$ is smooth.
Lemma 2:
In each cell and at each interface $I_j$ and $x_{j+\frac{1}{2}}$,

$$\max(\delta_j^+, \delta_j^-) = O(h^3),$$

$$|P_j(x_{j+\frac{1}{2}}) - P_{j+1}(x_{j+\frac{1}{2}})| = O(h^3),$$

in regions in which $u(x, t_{n+1})$ is smooth.

In the following subsection, we will modify $P(x)$ to achieve the local maximum principle in each cell, and to remove all false extrema at each interface, hence achieving the nonoscillatory property, while keeping it a proper reconstructed piecewise polynomial.

2.2 Advanced Subprocedure

In this subsection we describe the 2nd subprocedure used on the reconstruction. We first introduce an operator which can be used twice in each cell to modify each piece $P_j(x)$ of $P(x)$ to obtain the corresponding piece $R_j(x)$ of $R(x)$ satisfying our desired properties $(p_1)-(p_4)$.

Here and below, we denote the modifying operator by, for each cell $I_j$,

$$r_j(x) = L[p_j(x), lb_j, ub_j],$$

where $r_j(x)$ is a quadratic polynomial defined on $I_j$, $p_j(x)$ is a proper reconstructed polynomial defined on $I_j$, $lb_j$ and $ub_j$ are local lower and upper bounds for the polynomial $r_j(x)$ on $I_j$. We denote $\delta_j^+ = \max(\max_{x \in I_j} P_j(x) - ub_j, 0)$ as magnitude of overshoot of $p_j(x)$ against $ub_j$, and $\delta_j^- = \max(lb_j - \min_{x \in I_j} p_j(x), 0)$ as magnitude of undershoot of $p_j(x)$ against $lb_j$. We require that $lb_j$ and $ub_j$ satisfy

$$\delta_j^+ = O(h^3),$$  \hspace{1cm} (2.2a)

$$\delta_j^- = O(h^3),$$  \hspace{1cm} (2.2b)

in regions in which $u(x, t_{n+1})$ is smooth.

We require that the modifying operator satisfy following five properties.

$(q_1)$ The function $r_j(x)$ again satisfies the conservation requirement. That is,

$$\frac{1}{h} \int_{I_j} r_j(x) \, dx = \frac{1}{h} \int_{I_j} p_j(x) \, dx = \bar{u}_j.$$
(q2) The function $r_j(x)$ is as accurate as $p_j(x)$. That is

$$|r_j(x) - p_j(x)| \leq C_2 \max(\delta_j^+, \delta_j^-),$$

where $C_2$ is a constant which only depends on the degree of $p_j(x)$ (in this paper the degree is two).

(q3) The function $r_j(x)$ is bounded by $lb_j$ and $ub_j$ in $I_j$ i.e. $\forall x \in I_j$,

$$lb_j \leq r_j(x) \leq ub_j.$$

(q4) The function $r_j(x)$ has the same shape as the function $p_j(x)$, in fact, we have

$$r_j'(x) = \epsilon_j p_j'(x),$$

where $0 \leq \epsilon_j < 1$ is a constant.

(q5) The function $r_j(x)$ is uniformly closer to the cell-average $\bar{u}_j$ than $p_j(x)$, in fact: $\forall x \in I_j$,

$$(r_j(x) - \bar{u}_j) = \epsilon_j (p_j(x) - \bar{u}_j),$$

where $0 \leq \epsilon_j < 1$ is a constant.

Lemma 3:

As long as $lb_j$ and $ub_j$ satisfy (2.2a,b), $r_j(x)$ is a proper reconstructed polynomial.

Proof: This follows easily from properties (q1),(q2) and (q4), because $p_j(x)$ is a proper reconstructed polynomial. #

Theorem 1 (Modifying Operator)

At the j-th cell, we denote $u_{max} = \max_{x \in I_j} p_j(x)$, $u_{min} = \min_{x \in I_j} p_j(x)$.

1. If $u_{max} > ub_j$ and $u_{min} \geq lb_j$ (overshoot), we define

$$r_j(x) = L[p_j(x), lb_j, ub_j] = \epsilon^{(1)} p_j(x) - \epsilon^{(1)} u_{max} + ub_j,$$ (2.3a)

where $1 > \epsilon^{(1)} = (ub_j - \bar{u}_j)/(u_{max} - \bar{u}_j) \geq 0$.

2. If $u_{max} \leq ub_j$ and $u_{min} < lb_j$ (undershoot), we define

$$r_j(x) = L[p_j(x), lb_j, ub_j] = \epsilon^{(2)} p_j(x) - \epsilon^{(2)} u_{min} + lb_j,$$ (2.3b)

where $1 > \epsilon^{(2)} = (lb_j - \bar{u}_j)/(u_{min} - \bar{u}_j) \geq 0$.

3. If $u_{max} > ub_j$ and $u_{min} < lb_j$ (both overshoot and undershoot),
if $\epsilon^{(1)} < \epsilon^{(2)}$, we define
\[ r_j(x) = L[p_j(x), lb_j, ub_j] = \epsilon^{(1)}p_j(x) - \epsilon^{(1)}u_{\text{max}} + ub_j, \quad (2.3c) \]
if $\epsilon^{(1)} \geq \epsilon^{(2)}$, we define
\[ r_j(x) = L[p_j(x), lb_j, ub_j] = \epsilon^{(2)}p_j(x) - \epsilon^{(3)}u_{\text{min}} + lb_j. \quad (2.3d) \]

(4) If $u_{\text{max}} \leq ub_j$ and $u_{\text{min}} \geq lb_j$ (neither overshoot nor undershoot), then
\[ r_j(x) = p_j(x). \quad (2.3e) \]

Then $L[\cdot \cdot \cdot]$ has all of the five properties $(q_1)$-$(q_5)$.

**Proof:**
For the first statement (1), we denote the difference between $p_j(x)$ and $r_j(x)$ by $d_j(x)$, and have
\[ d_j(x) = p_j(x) - r_j(x) = (u_{\text{max}} - ub_j) \frac{p_j(x) - ub_j}{u_{\text{max}} - ub_j}. \]

Because $\int_{l_j}^{} d_j(x) \, dx = 0$, we obtain the conservation property
(q1) i.e.
\[ \frac{1}{h} \int_{l_j}^{} r_j(x) \, dx = \frac{1}{h} \int_{l_j}^{} p_j(x) \, dx = \bar{u}_j. \]

To obtain property $(q_2)$, we need to use a lemma which we shall prove in the Appendix:
\[ \max_{x \in l_j} | \frac{p_j(x) - \bar{u}_j}{u_{\text{max}} - \bar{u}_j} | \leq 3. \]

We obtain from this:
\[ | r_j(x) - p_j(x) | \leq 3max(\delta_j^+, \delta_j^-). \]

We have thus obtained property $(q_2)$.

Because of (2.3a)
\[ r'_j(x) = \epsilon^{(1)}p'_j(x). \quad (2.4) \]

Hence we obtain property $(q_4)$.

It is easy to see
\[ r_j(x) - \bar{u}_j = \epsilon^{(1)}(p_j(x) - \bar{u}_j), \quad (2.5) \]

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where $0 \leq \epsilon^{(1)} = (u_b - \bar{u}_j)/(u_{\max} - \bar{u}_j) < 1$ is a constant. Hence we obtain property $(q_5)$. 

Denoting $p_j(x_{\max}) = u_{\max}$ and $p_j(x_{\min}) = u_{\min}$, where $x_{\max}, x_{\min} \in I_j$, we know from (2.4) that $x_{\max}$ is a point in $I_j$ at which both $r_j(x)$ and $p_j(x)$ achieve their maximum, and $x_{\min}$ is a point in $I_j$ at which both $r_j(x)$ and $p_j(x)$ achieve their minimum. Hence

$$
\max_{x \in I_j} r_j(x) = r_j(x_{\max}) = u_b_j,
$$

$$
\min_{x \in I_j} r_j(x) = r_j(x_{\min}) = \bar{u}_j + \epsilon^{(1)}(p_j(x_{\min}) - \bar{u}_j) \geq u_{\min} \geq l_b_j.
$$

We obtain property $(q_3)$ i.e.

$$
l_b_j \leq r_j(x) \leq u_b_j.
$$

Hence $r_j(x)$ satisfies all five properties $(q_1)$-$(q_5)$.

The proof of the second statement (2) is similar.

The proof of the third statement (3) follows easily from (2.5), which guarantees $l_b_j \leq r_j(x) \leq u_b_j$. #

In the following, we outline how to perform the operation in each cell twice to obtain $R(x)$ from $P(x)$ with those five properties $(q_1)$-$(q_5)$.

First we perform the operation to achieve the local maximum principle $(p_3)$. In each cell $I_j$, we define the local lower and upper bounds as $l_b_j = m_j$ and $u_b_j = M_j$, and $m_j$ and $M_j$ are defined as before, and it is easy to see that $m_j$ and $M_j$ satisfy (2.2a,b). We perform the operation on the preliminary proper reconstructed polynomial $P_j(x)$ to obtain a quadratic polynomial $P_j^{mp}(x)$.

**Remark 2.1:** In each cell $I_j$, because $m_j$ and $M_j$ satisfy (2.2a,b), from Lemma 3, it follows that $P_j^{mp}(x)$ is a proper reconstructed polynomial. In addition to these, $P_j^{mp}(x)$ satisfies the local maximum principle, because of property $(q_3)$. By property $(q_5)$ $P_j^{mp}(x)$ is uniformly closer to $\bar{u}_j$ than $P_j(x)$. Therefore $P^{mp}(x) = \{P_j^{mp}(x)\}$ satisfies properties $(p_1)$-$(p_3)$. See Figure 2.
Lemma 4:
(i) $P_{mp}(x)$ is a proper reconstructed piecewise polynomial.
(ii) $P_{mp}(x)$ satisfies the local maximum principle.

Lemma 5:
At each interface $x_{j\pm\frac{1}{2}}$, 

$$|P_{j}^{mp}(x_{j\pm\frac{1}{2}}) - P_{j+1}^{mp}(x_{j\pm\frac{1}{2}})| = O(h^3),$$

in regions in which $u(x,t_{n+1})$ is smooth.

Next we perform the operation to enforce the nonoscillatory property ($p_4$). The only thing left to do is to remove all false extrema of $P_{mp}(x) = \{P_{j}^{mp}(x)\}$. In each monotone cell $I_j$, we define the local lower and upper bounds $\tilde{l}b_j$ and $\tilde{u}b_j$, which should satisfy (2.2a,b) and which we shall compute later. We perform the operation in each monotone cell on $P_{j}^{mp}(x)$ and we set $R_j(x) \equiv P_{j}^{mp}(x)$ in each extrema cell to obtain the final proper reconstructed polynomial $R_j(x)$.

Remark 2.2: If $\tilde{l}b_j$ and $\tilde{u}b_j$ satisfy (2.2a,b), from Lemma 3, $R_j(x)$ is a proper reconstructed polynomial. In addition to these, $R_j(x)$ satisfies the local maximum principle, because $P_{j}^{mp}(x)$ satisfies it and $R_j(x)$ is uniformly closer to $u_j$ than $P_{j}^{mp}(x)$ in monotone cells and $R_j(x) = P_{j}^{mp}(x)$ in extrema cells.

Lemma 6:
If at each monotone cell $I_j$, $\tilde{l}b_j$ and $\tilde{u}b_j$ satisfy (2.2a,b), then
(i) $R(x)$ is a proper reconstructed piecewise polynomial.
(ii) $R(x)$ satisfies the local maximum principle.
Hence if we can prove that \( R(x) = \{ R_j(x) \} \) has no false extrema, we will have shown that \( R(x) \) is a nonoscillatory reconstruction i.e. \( N(R(x)) = N(\bar{u}) \). Then \( R(x) \) will satisfy all four properties \((p_1)-(p_4)\) which are the 3rd order of accuracy, conservation, local maximum principle satisfying and nonoscillatory properties.

**Figure 3**

\( R(x) \) satisfies the local maximum principle and has no false extrema at any interface hence is a nonoscillatory reconstruction

The following is the **Advanced Subprocedure**:

**STEP 1**: Achieve the local maximum principle:
(i) In each cell \( I_j \), set \( \bar{l}_j = m_j \) and \( \bar{u}_j = M_j \),
(ii) do \( j=1,2,3,\cdots \)

\[
P_{j}^{\text{mp}}(x) = L[P_j(x), m_j, M_j],
\]

end do.

Here we perform the operation from first cell to the last cell in sequential pattern.

**STEP 2**: In each monotone cell \( I_j \), define the proper local bounds \( \bar{l}_j \) and \( \bar{u}_j \), and modify \( P_{j}^{\text{mp}}(x) \) to obtain the nonoscillatory reconstruction \( R_j(x) \) by an odd-even pattern. In each extrema cell, set \( R_j(x) = P_{j}^{\text{mp}}(x) \). That is

First we modify \( P_{j}^{\text{mp}}(x) \) in the odd numbered cells:

do \( j=1,3,5,\cdots \)

if \( (\bar{u}_{j-1} \leq \bar{u}_j \leq \bar{u}_{j+1}) \) then

\[
\bar{u}_j = \max(\frac{1}{2}(\bar{u}_j + \bar{u}_{j+1}), P_{j+1}^{\text{mp}}(x_{j+\frac{1}{2}}))
\]

\[
\bar{l}_j = \min(\frac{1}{2}(\bar{u}_j + \bar{u}_{j-1}), P_{j-1}^{\text{mp}}(x_{j-\frac{1}{2}}))
\]

\( R_j(x) = L[P_j^{\text{mp}}(x), \bar{l}_j, \bar{u}_j] \)

else if \( (\bar{u}_{j-1} \geq \bar{u}_j \geq \bar{u}_{j+1}) \) then

...
\[
\begin{align*}
\bar{u}_j &= \max\left(\frac{1}{2}(\bar{u}_j + \bar{u}_{j-1}), P_j^{mp}(x_{j-\frac{1}{2}})\right) \\
\bar{b}_j &= \min\left(\frac{1}{2}(\bar{u}_j + \bar{u}_{j+1}), P_j^{mp}(x_{j+\frac{1}{2}})\right) \\
R_j(x) &= \mathbf{L}[P_j^{mp}(x), \bar{b}_j, \bar{u}_j] \\
\text{else} & \\
R_j(x) &= P_j^{mp}(x) \\
\text{end if.}
\end{align*}
\]

Next after we have obtained \(R_j(x)\) in the odd numbered cells, we modify \(P_j^{mp}(x)\) in the even numbered cells:

\[
\text{do } j=2, 4, 6, \ldots
\]

\[
\text{if } (\bar{u}_{j-1} \leq \bar{u}_j \leq \bar{u}_{j+1}) \text{ then}
\]

\[
\begin{align*}
\bar{u}_j &= \max\left(\frac{1}{2}(\bar{u}_j + \bar{u}_{j+1}), R_{j+1}(x_{j+\frac{1}{2}})\right) \\
\bar{b}_j &= \min\left(\frac{1}{2}(\bar{u}_j + \bar{u}_{j-1}), R_{j-1}(x_{j-\frac{1}{2}})\right) \\
R_j(x) &= \mathbf{L}[P_j^{mp}(x), \bar{b}_j, \bar{u}_j] \\
\text{else if } (\bar{u}_{j-1} \geq \bar{u}_j \geq \bar{u}_{j+1}) \text{ then}
\end{align*}
\]

\[
\begin{align*}
\bar{u}_j &= \max\left(\frac{1}{2}(\bar{u}_j + \bar{u}_{j-1}), R_{j-1}(x_{j-\frac{1}{2}})\right) \\
\bar{b}_j &= \min\left(\frac{1}{2}(\bar{u}_j + \bar{u}_{j+1}), R_{j+1}(x_{j+\frac{1}{2}})\right) \\
R_j(x) &= \mathbf{L}[P_j^{mp}(x), \bar{b}_j, \bar{u}_j] \\
\text{else}
\end{align*}
\]

\[
\begin{align*}
R_j(x) &= P_j^{mp}(x) \\
\text{end if.}
\end{align*}
\]

\textbf{Remark 2.3:} Among monotone cells, we modify each \(P_j^{mp}(x)\) on the odd numbered cells first, and then modify each \(P_j^{mp}(x)\) on the even numbered cells. The reason is that this odd-even pattern may save some unnecessary work which can not be avoided by modifying the \(P_j^{mp}(x)\) in sequence. However no matter which pattern we choose, the error propagation is very local, because \(P^{mp}(x) = \{P_j^{mp}(x)\}\) has the same shape as the cell-averages \(\bar{u}\) and because of property \((q_6)\) of the modifying operator.

In the following Theorem, we prove that in each monotone cell \(I_j\), \(\bar{b}_j\) \(\bar{u}_j\) and satisfy (2.2a,b), hence \(R(x) = \{R_j(x)\}\) is a proper reconstructed piecewise polynomial; \(R(x)\) has no false extrema, hence \(R(x)\) is a nonoscillatory reconstruction, which means

\[
N(R(x)) = N(\bar{u}).
\]

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Here \( R(x) \) will satisfy all four properties \((p_1)-(p_4)\).

**Theorem 2**

Our scheme using two reconstruction subprocedures is a 3rd order accurate, conservative, local maximum principle satisfying, nonoscillatory, and hence a TVB scheme under the following CFL condition

\[
\frac{\alpha}{\tau} \sup_u |f'(u)| \leq 1,
\]

where the supremum is evaluated over \([\inf_x(u_0(x)), \sup_x(u_0(x))]\). The CFL condition is optimal.

**Proof**

From the Remark 2.1 and 2.2, we only need to prove that, in each monotone cell \( I_j, \bar{l}b_j \) and \( \bar{u}b_j \) satisfy \((2.2a,b)\), and that the \( R(x) \) has no false extrema.

We recall that the proper reconstructed piecewise polynomial \( P^{mp}(x) \) may have four types of false extrema at some interfaces \( x_{j+\frac{1}{2}} \) i.e.

- **false extrema** type (i) \( \bar{u}_j > \bar{u}_{j+1} \) and \( P_j^{mp}(x_{j+\frac{1}{2}}) < P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \),
- **false extrema** type (ii) \( \bar{u}_j = \bar{u}_{j+1} \) and \( P_j^{mp}(x_{j+\frac{1}{2}}) \geq \bar{u}_j \geq P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \) (at least one of the inequalities is strict),
- **false extrema** type (iii) \( \bar{u}_j < \bar{u}_{j+1} \) and \( P_j^{mp}(x_{j+\frac{1}{2}}) > P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \),
- **false extrema** type (iv) \( \bar{u}_j = \bar{u}_{j+1} \) and \( P_j^{mp}(x_{j+\frac{1}{2}}) \leq \bar{u}_j \leq P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \) (at least one of the inequalities is strict).

In the following, we shall show that there is no false extrema of \( R(x) \) at any interface \( x_{j+\frac{1}{2}} \):

\[
\begin{align*}
&\text{if } \bar{u}_j > \bar{u}_{j+1}, \text{ then } R_j(x_{j+\frac{1}{2}}) \geq R_{j+1}(x_{j+\frac{1}{2}}), \\
&\text{if } \bar{u}_j < \bar{u}_{j+1}, \text{ then } R_j(x_{j+\frac{1}{2}}) \leq R_{j+1}(x_{j+\frac{1}{2}}), \\
&\text{if } \bar{u}_j = \bar{u}_{j+1}, \text{ then } (R_j(x_{j+\frac{1}{2}}) - \bar{u}_j) \cdot (R_{j+1}(x_{j+\frac{1}{2}}) - \bar{u}_j) > 0 \quad (2.6) \\
&\text{or } R_j(x_{j+\frac{1}{2}}) = R_{j+1}(x_{j+\frac{1}{2}}) = \bar{u}_j.
\end{align*}
\]

We shall also show that in each monotone cell \( I_j, \bar{l}b_j \) and \( \bar{u}b_j \) satisfy \((2.2a,b)\).

Here if \( P^{mp}(x) \) satisfies \((2.6)\) at some interfaces, \( R(x) \) will also satisfy \((2.6)\) at same interfaces, because \( R_j(x) \) is uniformly closer to \( \bar{u}_j \) than \( P_j^{mp}(x) \) in the monotone cells and \( R_j(x) = P_j^{mp}(x) \) in the extrema cells.

Now if \( P^{mp}(x) \) does not satisfy \((2.6)\) at some interfaces, we prove that \( R(x) \) still does satisfy \((2.6)\) at same interfaces.
Suppose there is a type (i) false extrema of $P_{j+1}(x)$ at an interface, say $x_{j+1}$: $\bar{u}_j > \bar{u}_{j+1}$, $P_{j+1}(x_{j+\frac{1}{2}}) < P_{j+1}(x_{j+\frac{1}{2}})$. W.L.O.G. we assume $j$ is odd.

Case 1. If we choose, in the advanced subprocedure,
\[
\begin{align*}
\bar{u}_j &= \min(\frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}, P_{j+1}(x_{j+\frac{1}{2}})), \\
\bar{u}_j &= \max(\frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}, R_j(x_{j+\frac{1}{2}})),
\end{align*}
\]
which means $\bar{u}_{j-1} > \bar{u}_j > \bar{u}_{j+1} > \bar{u}_{j+2}$.

If $\bar{u}_j = \frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}$, then $R_j(x_{j+\frac{1}{2}}) > \bar{u}_j$, $u_{j+1} = R_j(x_{j+\frac{1}{2}})$, and $R_{j+1}(x_{j+\frac{1}{2}}) \leq R_j(x_{j+\frac{1}{2}})$; If $\bar{u}_j = P_{j+1}(x_{j+\frac{1}{2}})$ then $R_j(x_{j+\frac{1}{2}}) \geq P_{j+1}(x_{j+\frac{1}{2}})$, thus $R_j(x_{j+\frac{1}{2}}) \geq R_{j+1}(x_{j+\frac{1}{2}})$ because $P_{j+1}(x_{j+\frac{1}{2}}) \geq R_{j+1}(x_{j+\frac{1}{2}})$. Hence (2.6) is satisfied.

If $\bar{u}_j > \min_{x \in \bar{E}_{j+1}} P_{j+1}(x)$, then $0 < \bar{u}_j - \min_{x \in \bar{E}_{j+1}} P_{j+1}(x) = l_{j+1} - P_{j+1}(x_{j+\frac{1}{2}}) \leq P_{j+1}(x_{j+\frac{1}{2}}) - P_j(x_{j+\frac{1}{2}}) = O(h^3)$. And if $\bar{u}_j < \max_{x \in \bar{E}_{j+1}} P_{j+1}(x)$, then $0 < \max_{x \in \bar{E}_{j+1}} P_{j+1}(x) - \bar{u}_j \leq P_{j+1}(x_{j+\frac{1}{2}}) - \bar{u}_j = O(h^3)$. The $l_{j+1}$ and $u_{j+1}$ satisfy (2.2a,b).

Case 2: If we choose, in the advanced subprocedure,
\[
\begin{align*}
\tilde{u}_j &= \min(\frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}, P_{j+1}(x_{j+\frac{1}{2}})), \\
R_{j+1}(x) &= P_{j+1}(x),
\end{align*}
\]
which means $\bar{u}_{j-1} > \bar{u}_j > \bar{u}_{j+1} > \bar{u}_{j+2}$.

Here we claim that $l_{j+1} = P_{j+1}(x_{j+\frac{1}{2}})$ (It is easy to see $P_{j+1}(x_{j+\frac{1}{2}}) = \frac{1}{6}(-\bar{u}_{j+2} + 5\bar{u}_{j+1} + 2\bar{u}_j) + P_{j+1}(x_{j+\frac{1}{2}})-\frac{1}{2} (\tilde{u}_j + \bar{u}_{j+1}) \leq \frac{1}{6}(-\bar{u}_{j+2} + 2\bar{u}_{j+1} - \bar{u}_j) < 0$. We know that, if $P_{j+1}(x_{j+\frac{1}{2}}) < \bar{u}_{j+1}$ then $P_{j+1}(x_{j+\frac{1}{2}}) - (\frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}) < 0$; if not $P_{j+1}(x_{j+\frac{1}{2}}) - (\frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}) < P_{j+1}(x_{j+\frac{1}{2}}) - (\frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}) < 0$. Thus $P_{j+1}(x_{j+\frac{1}{2}}) < \frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}$ and $\bar{u}_j = P_{j+1}(x_{j+\frac{1}{2}})$). Hence $R_{j+1}(x_{j+\frac{1}{2}}) = P_{j+1}(x_{j+\frac{1}{2}}) = \bar{u}_j \leq R_j(x_{j+\frac{1}{2}})$, (2.6) is satisfied.

If $l_{j+1} > \max_{x \in \bar{E}_{j+1}} P_{j+1}(x)$, then $0 < l_{j+1} - \min_{x \in \bar{E}_{j+1}} P_{j+1}(x) = \bar{u}_j - P_{j+1}(x_{j+\frac{1}{2}}) = O(h^3)$. Hence the $l_{j+1}$ satisfy (2.2b).

Case 3: If we choose, in the advanced subprocedure,
\[
\begin{align*}
\bar{u}_j &= P_{j+1}(x), \\
\bar{u}_{j+1} &= \max(\frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}, R_j(x_{j+\frac{1}{2}})),
\end{align*}
\]
which means \( u_{j-1} < \bar{u}_j > \bar{u}_{j+1} \) and \( \bar{u}_j > \bar{u}_{j+1} \geq \bar{u}_{j+2} \).

Similarly we have \( ub_j^{\tilde{\gamma}_{j+1}} = R_j(x_{j+\frac{1}{2}}) = P_j^{mp}(x_{j+\frac{1}{2}}) \). Hence \( R_{j+1}(x_{j+\frac{1}{2}}) \leq ub_j^{\tilde{\gamma}_{j+1}} = R_j(x_{j+\frac{1}{2}}) \), (2.6) is satisfied.

If \( ub_j^{\tilde{\gamma}_{j+1}} < \max_{x \in I_{j+1}} P_j^{mp}(x) \), then \( 0 < \max_{x \in I_{j+1}} P_j^{mp}(x) - ub_j^{\tilde{\gamma}_{j+1}} = P_j^{mp}(x_{j+\frac{1}{2}}) - ub_j^{\tilde{\gamma}_{j+1}} \geq P_j^{mp}(x_{j+\frac{1}{2}}) - P_j^{mp}(x_{j+\frac{1}{2}}) = O(h^3) \). Hence \( ub_j^{\tilde{\gamma}_{j+1}} \) satisfy (2.2a).

Case 4: If we choose, in the advanced subprocedure,

\[
R_j(x) = P_j^{mp}(x), \\
R_{j+1}(x) = P_j^{mp}(x),
\]

which means \( \tilde{u}_{j-1} < \bar{u}_j > \tilde{u}_{j+1} \) and \( \bar{u}_j > \tilde{u}_{j+1} < \tilde{u}_{j+2} \).

Similarly we have \( P_j^{mp}(x_{j+\frac{1}{2}}) \geq \frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1} \geq P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \), (2.6) is satisfied.

Next suppose there is type (ii) false extrema of \( P_j^{mp}(x) \) at an interface \( x_{j+\frac{1}{2}}: \bar{u}_j = \tilde{u}_{j+1} \) and \( P_j^{mp}(x_{j+\frac{1}{2}}) \geq \bar{u}_j + \tilde{u}_{j+1} \geq P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \) (two equals do not hold at same time). W.L.O.G. we assume \( j \) is odd.

We have

\[
\begin{align*}
\tilde{b}_j &= \min(\frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}, P_j^{mp}(x_{j+\frac{1}{2}})) \\
&= \min(\bar{u}_j, P_{j+1}^{mp}(x_{j+\frac{1}{2}})) \\
&= \bar{u}_j, \\
ub^{\tilde{\gamma}_{j+1}} &= \max(\frac{1}{2} \bar{u}_j + \frac{1}{2} \bar{u}_{j+1}, R_j(x_{j+\frac{1}{2}})) \\
&= \max(\bar{u}_j, R_j(x_{j+\frac{1}{2}})) \\
&= \bar{u}_j.
\end{align*}
\]

Because \( R_{j+1}(x_{j+\frac{1}{2}}) \leq ub^{\tilde{\gamma}_{j+1}} = \bar{u}_j \) and \( R_{j+1}(x_{j+\frac{1}{2}}) \geq \bar{u}_j \) (\( P_j^{mp}(x_{j+\frac{1}{2}}) \geq \bar{u}_j \)), we have \( R_{j+1}(x_{j+\frac{1}{2}}) = \bar{u}_j \). Because \( R_j(x_{j+\frac{1}{2}}) \geq \tilde{b}_j = \bar{u}_j \) and \( R_j(x_{j+\frac{1}{2}}) \leq \bar{u}_j \) (\( P_j^{mp}(x_{j+\frac{1}{2}}) \leq \bar{u}_j \)), we have \( R_j(x_{j+\frac{1}{2}}) = \bar{u}_j \). Hence \( R_j(x_{j+\frac{1}{2}}) \) and \( R_{j+1}(x_{j+\frac{1}{2}}) \) are both equal to \( \bar{u}_j \), (2.6) is satisfied.

If \( \tilde{b}_j > \min_{x \in I_j} P_j^{mp}(x) \), then \( 0 < \tilde{b}_j - \min_{x \in I_j} P_j^{mp}(x) = \tilde{b}_j - P_j^{mp}(x_{j+\frac{1}{2}}) = \bar{u}_j - P_j^{mp}(x_{j+\frac{1}{2}}) \leq P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - P_j^{mp}(x_{j+\frac{1}{2}}) = O(h^3) \); if \( ub^{\tilde{\gamma}_{j+1}} < \max_{x \in I_{j+1}} P_j^{mp}(x) \), then \( 0 < \max_{x \in I_{j+1}} P_j^{mp}(x) - ub^{\tilde{\gamma}_{j+1}} = P_j^{mp}(x_{j+\frac{1}{2}}) - \bar{u}_j \leq P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - P_j^{mp}(x_{j+\frac{1}{2}}) = O(h^3) \). Hence the \( \tilde{b}_j \) and \( ub^{\tilde{\gamma}_{j+1}} \) satisfy (2.2a,b).

We can prove (2.6) and (2.2a,b) if there is type (iii) or (iv) false extrema of \( P_j^{mp}(x) \) at an interface \( x_{j+\frac{1}{2}} \).
Now we can say that after performing this advanced Subprocedure, at any interface \( x_{j+\frac{1}{2}} \), we have (2.6)

\[
\begin{align*}
\text{if } \bar{u}_j > \bar{u}_{j+1}, \text{ then } & \quad R_j(x_{j+\frac{1}{2}}) \geq R_{j+1}(x_{j+\frac{1}{2}}), \\
\text{if } \bar{u}_j < \bar{u}_{j+1}, \text{ then } & \quad R_j(x_{j+\frac{1}{2}}) \leq R_{j+1}(x_{j+\frac{1}{2}}), \\
\text{if } \bar{u}_j = \bar{u}_{j+1}, \text{ then } & \quad (R_j(x_{j+\frac{1}{2}}) - \bar{u}_j) \cdot (R_{j+1}(x_{j+\frac{1}{2}}) - \bar{u}_j) > 0 \\
& \quad \text{or } R_j(x_{j+\frac{1}{2}}) = R_{j+1}(x_{j+\frac{1}{2}}) = \bar{u}_j.
\end{align*}
\]

There are no false extrema of \( R(x) \). Therefore \( N(R(x)) = N(\bar{u}) \). The nonoscillatory property is achieved.

In all the cases, (2.2a,b) are satisfied or \( R_j(x) = P_j^{mp}(x) \). Thus for all of \( R_j(x) \),

\[ R_j(x) - u(x, t_{n+1}) = O(h^3) \]

in regions in which \( u(x, t_{n+1}) \) is smooth.

Thus Theorem 2 has been proven. 

#

3 Simple Implementation of the Schemes

In this section we follow the idea and the analysis in [4] to obtain the explicit form of the schemes. The form (1.5) of the scheme is called the abstract form in which we need to evaluate the exact solution in the small for the IVP(1.1) with the initial data \( R(x) \), and then take the cell-average, which results in conservation form:

\[
\bar{u}_{j}^{n+1} = \bar{u}_{j}^{n} - \lambda \left( \frac{1}{\tau} \int_{0}^{\tau} f(v(x_{j+\frac{1}{2}}, \eta)) \, d\eta - \frac{1}{\tau} \int_{0}^{\tau} f(v(x_{j-\frac{1}{2}}, \eta)) \, d\eta \right), \tag{3.1}
\]

where \( \lambda = \tau/h \) and \( v(x, t) \) is the exact solution of the following equation

\[
\begin{align*}
v_t + f(v)_x &= 0 \\
v(x, 0) &= R(x).
\end{align*} \tag{3.2}
\]

To evaluate the integral in (3.1)

\[
\frac{1}{\tau} \int_{0}^{\tau} f(v(x_{j+\frac{1}{2}}, \eta)) \, d\eta, \tag{3.3}
\]

we try to derive a simple but adequate approximation following [2]. Note that the integrand is a smooth function of \( t \).
The first step is to discretize the integral in (3.3) by using a numerical quadrature such as Gauss or other quadratures with 3rd order of accuracy. We could use 2 point Gauss or 3 point Simpson’s quadrature, i.e. in general

$$\frac{1}{\tau} \int_{0}^{\tau} f(v(x_{j+\frac{1}{2}}, \eta)) \, d\eta = \sum_{k=0}^{K} \alpha_k f(v(x_{j+\frac{1}{2}}, \beta_k \tau)).$$  \hspace{1cm} (3.4)

The second step is to approximate $v(x, t)$ by its Taylor expansion which is obtained by the following local Cauchy-Kowalewski procedure i.e.

$$v_t = -f' v_x$$
$$v_{xt} = -[f'' v_x^2 + f' v_{xx}]$$
$$v_{tt} = -[f'' v_t v_x + f' v_{xt}].$$  \hspace{1cm} (3.5)

Thus in (3.4) for $v(x_{j+\frac{1}{2}}, \beta_k \tau)$, if we approximate it from the left cell of the interface $x_{j+\frac{1}{2}}$, we obtain

$$v(x_{j+\frac{1}{2}}, \beta_k \tau) \approx \tilde{v}_j(x_{j+\frac{1}{2}}, \beta_k \tau) = v(x_{j+\frac{1}{2}}, 0) + v_t(x_{j+\frac{1}{2}}, 0)(\beta_k \tau) + \frac{1}{2} v_{tt}(x_{j+\frac{1}{2}}, 0)(\beta_k \tau)^2,$$  \hspace{1cm} (3.6)

where $v(x_{j+\frac{1}{2}}, 0) = R_j(x_{j+\frac{1}{2}}), v_t(x_{j+\frac{1}{2}}, 0)$ and $v_{tt}(x_{j+\frac{1}{2}}, 0)$ are obtained through the local Cauchy-Kowalewski procedure (3.5) from $v(x, 0) = R_j(x)$, and

$$\tilde{v}_j(x_{j+\frac{1}{2}}, \beta_k \tau) = v(x_{j+\frac{1}{2}}, \beta_k \tau) + O(h^3).$$

If we approximate it from the right cell of the $x_{j+\frac{1}{2}}$, we obtain

$$v(x_{j+\frac{1}{2}}, \beta_k \tau) \approx \tilde{v}_{j+1}(x_{j+\frac{1}{2}}, \beta_k \tau) = v(x_{j+\frac{1}{2}}, 0) + v_t(x_{j+\frac{1}{2}}, 0)(\beta_k \tau) + \frac{1}{2} v_{tt}(x_{j+\frac{1}{2}}, 0)(\beta_k \tau)^2,$$  \hspace{1cm} (3.7)

where $v(x_{j+\frac{1}{2}}, 0) = R_{j+1}(x_{j+\frac{1}{2}}), v_t(x_{j+\frac{1}{2}}, 0)$ and $v_{tt}(x_{j+\frac{1}{2}}, 0)$ are obtained through the local Cauchy-Kowalewski procedure (3.5) from $v(x, 0) = R_{j+1}(x)$, and

$$\tilde{v}_{j+1}(x_{j+\frac{1}{2}}, \beta_k \tau) = v(x_{j+\frac{1}{2}}, \beta_k \tau) + O(h^3).$$

Thus

$$f(v(x_{j+\frac{1}{2}}, \beta_k \tau)) = f(\tilde{v}_j(x_{j+\frac{1}{2}}, \beta_k \tau)) \quad \text{or} \quad f(\tilde{v}_{j+1}(x_{j+\frac{1}{2}}, \beta_k \tau))$$  \hspace{1cm} (3.8)
The last step in the derivation of the numerical flux is to approximate 
\( f(v(x_{j+\frac{1}{2}}, \beta_k \tau)) \) by

\[
f(v(x_{j+\frac{1}{2}}, \beta_k \tau)) = \tilde{h}(\tilde{v}_j(x_{j+\frac{1}{2}}, \beta_k \tau), \tilde{v}_{j+1}(x_{j+\frac{1}{2}}, \beta_k \tau)),
\]  

(3.9)

where \( \tilde{h}(\cdot, \cdot) \) is any two-point Lipschitz continuous monotone flux which is nondecreasing for the first argument and nonincreasing for the second argument. Some possible choices are

(i) Engquist-Osher

\[
h^{EO}(a, b) = \int_0^b \min(f'(s), 0) \, ds + \int_0^a \min(f'(s), 0) \, ds + f(0);
\]

(3.10)

(ii) Godunov

\[
h^G(a, b) = \begin{cases} 
\min_{a \leq u \leq b} f(u) & \text{if } a \leq b, \\
\max_{a \geq u \geq b} f(u) & \text{if } a > b;
\end{cases}
\]

(3.11)

(iii) Lax-Friedrichs

\[
h^{LF}(a, b) = \frac{1}{2}[f(a) + f(b) - \alpha(b - a)] \quad \alpha = \max |f'(u)|,
\]

(3.12)

where the maximum is taken over the whole region in which \( a, b \) varies, e.g. in \([\inf_{x}(u_0(x)), \sup_{x}(u_0(x))]\), where \( u_0(x) \) is the initial function;

(iv) Local-Lax-Friedrichs

\[
h^{LLF}(a, b) = \frac{1}{2}[f(a) + f(b) - \beta(b - a)] \quad \beta = \max_{\min(a, b) \leq u \leq \max(a, b)} |f'(u)|.
\]

(3.13)

For convex \( f, f'' \geq 0 \), one has \( \beta = \max(|f'(a)|, |f'(b)|) \),

(v) Roe with entropy fix

\[
h^{RF}(a, b) = \begin{cases} 
f(a) & \text{if } f'(u) \geq 0 \text{ for } u \in [\min(a, b), \max(a, b)] \\
f(b) & \text{if } f'(u) \leq 0 \text{ for } u \in [\min(a, b), \max(a, b)] \\
h^{LLF}(a, b) & \text{otherwise.}
\end{cases}
\]

(3.14)

Hence now we obtain the explicit form of the scheme, which is

\[
\tilde{u}_j^{n+1} = \tilde{u}_j^n - \lambda (\tilde{f}_{j+\frac{1}{2}} - \tilde{f}_{j-\frac{1}{2}}),
\]

(3.15)
where

\[ \tilde{f}_{j+\frac{1}{2}} = \sum_{k=0}^{K} \alpha_k \tilde{h}(\tilde{v}_j(x_{j+\frac{1}{2}}, \beta_k \tau), \tilde{v}_{j+1}(x_{j+\frac{1}{2}}, \beta_k \tau)). \]  

(3.16)

The flux \( \tilde{f}_{j+\frac{1}{2}} \) is an adequate approximation to the flux \( \frac{1}{\tau} \int_0^\tau f(v(x_{j+\frac{1}{2}}, \eta)) \, d\eta \); for details see [2].

Here we mention that, for computational reasons, in the reconstruction procedure, we impose a requirement \( m_j \leq \tilde{u}_j^{n+1} \leq M_j \) in each cell (This, of course, was implied for the abstract form of the scheme). That is, if \( \tilde{u}_j^{n+1} > M_j \), we set \( \tilde{u}_j^{n+1} = M_j \); if \( \tilde{u}_j^{n+1} < m_j \), we set \( \tilde{u}_j^{n+1} = m_j \). Because the true solution should satisfy the requirement, we will obtain a better approximation for the cell-average \( \tilde{u}_j^{n+1} \).

We note that for linear equation, the explicit form is equivalent to the abstract scheme if we use any of the Engquist-Osher flux (3.10), the Godunov flux (3.11), or the Roe flux with entropy fix (3.14). All of them are just simple upwind differencing. Thus, in the linear case, the schemes satisfy the maximum principle and are nonoscillatory, and thus are TVB schemes. There is a subsequence of the numerical solutions converging to the weak solution of (1.1) as the step size approaches zero. Of course, the solution to the linear equation is unique, and hence the schemes are convergent, in this simple case.

In the next section we will test the schemes on linear and nonlinear equations with convex and nonconvex fluxes. The numerical experiments show that the schemes are 3rd order accurate for smooth solutions, satisfy the maximum principle and the nonoscillatory property, and converge to the entropy solutions.

4 Numerical Experiments

In this section we use some model problems to numerically test our schemes. We used the Roe flux with entropy fix as numerical flux and 2 point Gauss quadrature in all of our examples.

Example 1. We solve the model equation

\[ u_t + u_x = 0 \quad -1 \leq x \leq 1 \]

\[ u(x, 0) = u_0(x) \quad \text{periodic with period 2.} \]  

(4.1)
Four initial data $u_0(x)$ are used. The first one is $u_0(x) = \sin(\pi x)$ and we list the errors at time $t=10$ in Table 1. The second one is $u_0(x) = \sin^4(\pi x)$ and we list the errors at time $t=10$ in Table 2.

**TABLE 1 ( $\tau/h=0.9$, $t=10$ )**

<table>
<thead>
<tr>
<th>$l$</th>
<th>$L_1$ error</th>
<th>$L_1$ order</th>
<th>$L_\infty$ error</th>
<th>$L_\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>5.7694407036673D-03</td>
<td></td>
<td>4.5009125264985D-03</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>6.974677300436D-04</td>
<td>3.05</td>
<td>5.4686841904228D-04</td>
<td>3.04</td>
</tr>
<tr>
<td>80</td>
<td>8.5439845931126D-05</td>
<td>3.03</td>
<td>6.7079922749236D-05</td>
<td>3.03</td>
</tr>
</tbody>
</table>

**TABLE 2 ( $\tau/h=0.6$, $t=10$ )**

<table>
<thead>
<tr>
<th>$l$</th>
<th>$L_1$ error</th>
<th>$L_1$ order</th>
<th>$L_\infty$ error</th>
<th>$L_\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>1.4690161632608D-02</td>
<td></td>
<td>2.1347851475350D-02</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>2.008513922684D-03</td>
<td>2.87</td>
<td>2.8168317475599D-03</td>
<td>2.92</td>
</tr>
<tr>
<td>320</td>
<td>2.628572179140D-04</td>
<td>2.93</td>
<td>3.5474728536100D-04</td>
<td>2.99</td>
</tr>
</tbody>
</table>

Here and below $l$ is the total number of cells and the step size $h = 2/l$ in all examples.

For the first two initial data, we obtain 3rd order of accuracy in the smooth region in both $L_1$ and $L_\infty$ norms. We note that standard ENO schemes applied to the example with the second initial data experienced an (easily fixed) loss of accuracy, see [17], [18]. No such degeneracy was found with our present method.

The third initial function is

$$u_0(x) = \begin{cases} 
1 & \quad \frac{1}{5} \leq x \leq \frac{1}{5} \\
0 & \quad \text{otherwise}
\end{cases}$$

and the fourth is

$$u_0(x) = \begin{cases} 
(1 - (\frac{10}{3}x)^2)^{\frac{1}{2}} & \quad \frac{3}{10} \leq x \leq \frac{3}{10} \\
0 & \quad \text{otherwise}
\end{cases}$$

We see the good resolution of the solutions in Figures 4-5.
Example 2. We solve Burgers’ equation with a periodic boundary condition
\[
\begin{align*}
&u_t + \left(\frac{1}{2}u^2\right)_x = 0, \quad -1 \leq x \leq 1 \\
&u(x, 0) = u_0(x) \quad u_0(x) \text{ periodic with period 2.} \quad (4.2)
\end{align*}
\]
For the initial data \( u_0(x) = 1 + \frac{1}{2} \sin(\pi x) \), the exact solution is smooth up to \( t = \frac{2}{\pi} \), then it develops a moving shock which interacts with a rarefaction wave.

At \( t = 0.3 \) the solution is still smooth. We list the errors in Table 3. Note we have close to 3rd order of accuracy in \( L_1 \) and more than 2nd order of accuracy in \( L_\infty \).

**TABLE 3 ( \( \tau/h=0.6, \ t=0.3 \) )**

<table>
<thead>
<tr>
<th>( l )</th>
<th>( L_1 ) error</th>
<th>( L_1 ) order</th>
<th>( L_\infty ) error</th>
<th>( L_\infty ) order</th>
</tr>
</thead>
<tbody>
<tr>
<td>160</td>
<td>4.6548730388854D-06</td>
<td></td>
<td>1.3542618619078D-05</td>
<td></td>
</tr>
<tr>
<td>320</td>
<td>7.7529803426540D-07</td>
<td>2.59</td>
<td>2.834930184257D-06</td>
<td>2.26</td>
</tr>
<tr>
<td>640</td>
<td>1.2524533052373D-07</td>
<td>2.63</td>
<td>5.8732024554864D-07</td>
<td>2.27</td>
</tr>
<tr>
<td>1280</td>
<td>1.8075141183793D-08</td>
<td>2.79</td>
<td>1.2430677109876D-07</td>
<td>2.24</td>
</tr>
<tr>
<td>2560</td>
<td>2.5408037626108D-09</td>
<td>2.83</td>
<td>2.6569131561516D-08</td>
<td>2.23</td>
</tr>
</tbody>
</table>

At \( t = \frac{2}{\pi} \) the shock just begins to form, at \( t = 1.1 \) the interaction between the shock and the rarefaction waves is over, and the solution becomes monotone between shocks. In Figures 6-7 we can see the excellent behaviors of the schemes in both cases. The errors 0.1 away from the shock (i.e. \( |x - \text{shock location}| \geq 0.1 \) ) are listed in Table 4 at \( t = 1.1 \). These errors are of even much smaller magnitude than the ones in the smooth case of Table 3 and show 3rd order of accuracy both in \( L_1 \) and \( L_\infty \) in the smooth regions 0.1 away from the shock. This shows that error propagation of the scheme is very local.

**TABLE 4 ( \( \tau/h=0.66, \ t=1.1 \) )**

<table>
<thead>
<tr>
<th>( l )</th>
<th>( L_1 ) error</th>
<th>( L_1 ) order</th>
<th>( L_\infty ) error</th>
<th>( L_\infty ) order</th>
</tr>
</thead>
<tbody>
<tr>
<td>160</td>
<td>1.4188176779673D-06</td>
<td></td>
<td>3.8623190183884D-06</td>
<td></td>
</tr>
<tr>
<td>320</td>
<td>1.7619541036046D-07</td>
<td>3.01</td>
<td>4.5400412962415D-07</td>
<td>3.09</td>
</tr>
<tr>
<td>640</td>
<td>2.1941647813206D-08</td>
<td>3.01</td>
<td>5.651208345725D-08</td>
<td>3.01</td>
</tr>
<tr>
<td>1280</td>
<td>2.7394705290687D-09</td>
<td>3.00</td>
<td>7.0478263225482D-09</td>
<td>3.00</td>
</tr>
<tr>
<td>2560</td>
<td>3.4273836711438D-10</td>
<td>3.00</td>
<td>8.8725682356738D-10</td>
<td>2.99</td>
</tr>
</tbody>
</table>
Figure 6 ( $\tau / h = 0.66$ )

The TVB solution at $T = 0.6366$

Figure 7 ( $\tau / h = 0.66$ )

The TVB solution at $T = 1.1$
For the initial data

\[ u_0(x) = \begin{cases} 
2 & -1 \leq x < -0.5 \\
1 & -0.5 \leq x < 0 \\
0 & 0 \leq x \leq 1, 
\end{cases} \]

we see one rarefaction wave, and two shocks waves which interact with each other. Good resolution is observed in Figure 8.

**Figure 8 ( \( \tau/h = 0.5 \) )**

![Graph showing the TVB solution at T = 0.4]

**Example 3.** we use two nonconvex fluxes to test the convergence to the physically correct solutions. The true solutions are obtained from the Lax-Friedrichs scheme on a very fine grid.

The first one is a Riemann problem with the flux

\[ f(u) = \frac{1}{4}(u^2 - 1)(u^2 - 4), \]

and the initial data

\[ u_0(x) = \begin{cases} 
u_l & x < 0 \\
u_r & x > 0. \end{cases} \]

The two cases we test are (i) \( \nu_l = 2, \nu_r = -2 \), Figure 9; (ii) \( \nu_l = -3, \nu_r = 3 \), Figure 10. For more details concluding this problem see [2].
The figure shows two graphs illustrating the TVB solution at different times for two different values of $\tau/h$.

**Figure 9** ($\tau/h = 0.33$)

The TVB solution at $T = 1$.

**Figure 10** ($\tau/h = 0.05$)

The TVB solution at $T = 0.04$.

The second flux is the Buckley-Leverett flux used to model oil recovery [2],

$$f(u) = \frac{4u^2}{4u^2 + (1-u)^2},$$

with initial data $u = 1$ in $[-\frac{1}{2}, 0]$ and $u = 0$ elsewhere.
The result is displayed in Figure 11.

**Figure 11 (\(\tau/h = 0.3\))**

In this example, we observe convergence with excellent resolution to the entropy solutions in both cases.

In all the examples that we have illustrated above, we observe that the schemes are of 3rd order of accuracy with the somewhat less than usual degeneracy to 2nd order at certain isolated extrema, local maximum principle satisfying and nonoscillatory properties as we proved in §2, and convergent, with excellent resolution, to the entropy solutions.

5 Summary and Prospects

We now review the general idea of this paper. For the maximum principle, the local bounds \(m = \{m_j\}\) and \(M = \{M_j\}\) are chosen from data obtained at previous time steps. The true solution of equation \(u_t + f(u)_x = 0\) is bounded by these numbers locally.

However for equations with source terms of the form \(u_t + f(u)_x = \phi(u, t)\), the solution \(u(x, t_{n+1})\) at \(t = t_{n+1}\) should also satisfy

\[
m_j^{\text{new}} \leq u(x, t_{n+1}) \leq M_j^{\text{new}} \quad x \in I_j, \tag{5.1}
\]

where we set \(m_j^{\text{new}} = v_m(\tau)\) and \(v_m(t)\) is the solution of the ODE \(u_t = \phi(u, t)\)
with initial data \( v = m_j \), and \( M_j^{\text{new}} = v_M(\tau) \) and \( v_M(t) \) is the solution of the ODE \( u_t = \phi(u, t) \) with initial data \( v = M_j \) with \( m_j \) and \( M_j \) defined as before. The condition (5.1) is reasonable to be used in solving equation \( u_t + f(u)_x = \phi(u, t) \), because of well known comparison theorems.

For the general system case, the condition (5.1) might not be adequate and we need to know a bit more about the structure of the solution.

However the nonoscillatory property (1.8) naturally generalizes in all cases. Hence we can achieve a generalized nonoscillatory property (1.8) in the multidimensional and system cases up to 3rd order of accuracy. This will be done in future work.

For higher order of accuracy, we will consider the reconstructed polynomial with a corresponding higher degree. We have proved that the modifying operator (Theorem 1) works for higher degree polynomials and shall show this proof in future work. Our next main project is to use this operator to modify higher degree reconstructed polynomials and we may well combine it with the nonoscillatory ideas of Tong in [11] to achieve the TVB property with arbitrarily high order of accuracy.

6 Appendix

Lemma 7

\[
\begin{align}
\max_{x \in I_j} \left| \frac{(p_j(x) - \bar{u}_j)}{(u_{\max} - \bar{u}_j)} \right| & \leq 3 \quad (6.1a) \\
\min_{x \in I_j} \left| \frac{(p_j(x) - \bar{u}_j)}{(u_{\min} - \bar{u}_j)} \right| & \leq 3 \quad (6.1b)
\end{align}
\]

in Theorem 1.

Proof
W.L.O.G. we only prove (6.1a).

Case 1: \( p_j(x) \) is a linear function, then \( \left| \frac{(p_j(x) - \bar{u}_j)}{(u_{\max} - \bar{u}_j)} \right| \leq 1 \) and (6.1a) is satisfied.

Now if \( p_j(x) \) is not a linear function, we denote \( p_j(x) = a_2(x - x^*)^2 + a_0 \), \( x_{j-\frac{1}{2}} = x_j - \frac{1}{2}h \), \( x_{j+\frac{1}{2}} = x_j + \frac{1}{2}h \), \( x_{j+1} - x^* = \theta h \), \( x_{j-\frac{1}{2}} - x^* = (\theta - 1)h \), and \( x^* \) is the extrema point of \( p_j(x) \) in the interval \((-\infty, +\infty)\). We have

\[
\frac{(p_j(x) - \bar{u}_j)}{(u_{\max} - \bar{u}_j)} = \frac{(a_2(x - x^*)^2 + a_0 - \bar{u}_j)}{(u_{\max} - \bar{u}_j)}.
\]
Case 2: \( x^* \notin I_j \). Thus \( \theta > 1 \) or \( \theta < 0 \). \( p_j(x) \) is monotone in \( I_j \). So \( u_{\text{max}} = p_j(x_{j+\frac{1}{2}}) \) or \( u_{\text{max}} = p_j(x_{j-\frac{1}{2}}) \). W.L.O.G. we assume \( u_{\text{max}} = p_j(x_{j+\frac{1}{2}}) \), then max

\[
\frac{\left(p_j(x) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j} = \max(1, \frac{\left(p_j(x_{j-\frac{1}{2}}) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j}).
\]

Because \( \frac{1}{3} a_2(x - x^*)^3 \left| x_{j+\frac{1}{2}} \right| + a_0 h = \bar{u}_j h \), we have

\[
\frac{\left(p_j(x_{j-\frac{1}{2}}) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j} = \frac{a_2(x_{j-\frac{1}{2}} - x^*)^2 + a_0 - \bar{u}_j}{u_{\text{max}} - \bar{u}_j} = \frac{a_2(x_{j-\frac{1}{2}} - x^*)^2}{u_{\text{max}} - \bar{u}_j} - \frac{\left(1 - \frac{1}{3} a_2(x - x^*)^3 \left| x_{j+\frac{1}{2}} \right| \right)}{u_{\text{max}} - \bar{u}_j}
\]

\[
= \frac{(3\theta - 1)^2 - \theta^3}{(3\theta^2 - \theta^3 + (\theta - 1)^3)} = \frac{(-3\theta + 2)/(3\theta - 1)}{1},
\]

and

\[
\max_{\theta \in [0,1]} \frac{(-3\theta + 2)/(3\theta - 1)}{1} = 2,
\]

and

\[
\frac{\left(p_j(x) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j} \leq 2.\]

We obtain

\[
\max_{x \in I_j} \frac{\left(p_j(x) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j} \leq 2 \quad \text{if} \quad u_{\text{max}} = p_j(x_{j+\frac{1}{2}}).
\]

By the same argument we also have

\[
\max_{x \in I_j} \frac{\left(p_j(x) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j} \leq 2 \quad \text{if} \quad u_{\text{max}} = p_j(x_{j-\frac{1}{2}}).
\]

Case 3: \( x^* \in I_j \), \( p_j(x^*) = \min_{x \in I_j} p_j(x) \), and \( u_{\text{max}} = \max(p_j(x_{j+\frac{1}{2}}), p_j(x_{j-\frac{1}{2}})) \). W.L.O.G. we assume \( p_j(x_{j+\frac{1}{2}}) = u_{\text{max}} \), hence \( x_{j+\frac{1}{2}} - x^* = \theta h \) and \( x_{j-\frac{1}{2}} - x^* = (\theta - 1) h \) where \( \frac{1}{2} \leq \theta \leq 1 \). Because

\[
\frac{\left(p_j(x_{j-\frac{1}{2}}) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j} \leq \frac{\left(p_j(x_{j+\frac{1}{2}}) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j} \quad \text{and} \quad \max_{x \in I_j} \frac{\left(p_j(x) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j} = \max(1, \frac{\left(p_j(x^*) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j}).
\]

\[
\frac{\left(p_j(x^*) - \bar{u}_j\right)}{u_{\text{max}} - \bar{u}_j} = \frac{a_0 - \bar{u}_j}{a_2(x_{j+\frac{1}{2}} - x^*)^2 + a_0 - \bar{u}_j} = \frac{a_2(x_{j+\frac{1}{2}} - x^*)^2 h + \frac{1}{3} a_2(x - x^*)^3 \left| x_{j+\frac{1}{2}} \right|}{(3\theta^2 - 3\theta + 1)/(-3\theta + 1)},
\]

\[33\]
Hence, for $\frac{1}{2} \leq \theta \leq 1$

$$| (p_j(x^*) - \bar{u}_j)/(u_{\max} - \bar{u}_j) | \leq \max_{\frac{1}{2} \leq \theta \leq 1} | (3\theta^2 - 3\theta + 1)/(-3\theta + 1) | = \frac{1}{2}.$$  

Hence

$$\max_{x \in I_j} | (p_j(x) - \bar{u}_j)/(u_{\max} - \bar{u}_j) | \leq 1$$

Case 4: $x^* \in I_j$ and $p_j(x^*) = \max_{x \in I_j} p_j(x). \max_{x \in I_j} | (p_j(x) - \bar{u}_j)/(u_{\max} - \bar{u}_j) |

= \max_{x \in I_j} (| (p_j(x_{j+\frac{1}{2}}) - \bar{u}_j)/(u_{\max} - \bar{u}_j) |, | (p_j(x_{j-\frac{1}{2}}) - \bar{u}_j)/(u_{\max} - \bar{u}_j) |). W.L.O.G. 

we assume $| (p_j(x_{j+\frac{1}{2}}) - \bar{u}_j)/(u_{\max} - \bar{u}_j) | \geq | (p_j(x_{j-\frac{1}{2}}) - \bar{u}_j)/(u_{\max} - \bar{u}_j) |$. Hence $x_{j+\frac{1}{2}} - x^* = \theta h$ and $x_{j-\frac{1}{2}} - x^* = (\theta - 1)h$, where $\frac{1}{2} \leq \theta \leq 1$. Hence

$$(p_j(x_{j+\frac{1}{2}}) - \bar{u}_j)/(u_{\max} - \bar{u}_j) = (a_2(x_{j+\frac{1}{2}} - x^*)^2 + \frac{1}{\theta h} a_2(x - x^*)^3 | x_{j+\frac{1}{2}}^{x_{j-\frac{1}{2}}})/(x_{j-\frac{1}{2}}^{x_{j+\frac{1}{2}}}) 

= (3\theta - 1)/(-3\theta^2 + 3\theta - 1).$$

Thus $| (p_j(x_{j+\frac{1}{2}}) - \bar{u}_j)/(u_{\max} - \bar{u}_j) | = | (3\theta - 1)/(-3\theta^2 + 3\theta - 1) | \leq 3$ for $\frac{1}{2} \leq \theta \leq 1$. Hence $\max_{x \in I_j} | (p_j(x) - \bar{u}_j)/(u_{\max} - \bar{u}_j) | \leq 3$.

In all four cases we obtain (6.1a) i.e.

$$\max_{x \in I_j} | (p_j(x) - \bar{u}_j)/(u_{\max} - \bar{u}_j) | \leq 3.$$  

Similarly we obtain (6.1b) i.e.

$$\max_{x \in I_j} | (p_j(x) - \bar{u}_j)/(u_{\min} - \bar{u}_j) | \leq 3.$$  

The lemma is proven. #

References


