UCLA

COMPUTATIONAL AND APPLIED MATHEMATICS

Nonoscillatory High Order Accurate Self-Similar Maximum Principle Satisfying Shock Capturing Schemes I

Xu-Dong Liu
Stanley Osher

June 1992

CAM Report 92-27

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA. 90024-1555

Nonoscillatory High Order Accurate Self-Similar Maximum Principle Satisfying Shock Capturing Schemes I

Xu-Dong Liu *   Stanley Osher †

*Department of Mathematics, UCLA, Los Angeles, California, 90024. E-mail: xliu@math.ucla.edu. Research supported by NSF grant DMS88-11863.

†Department of Mathematics, UCLA, Los Angeles, California, 90024. Research supported by NSF grant DMS91-03104 and ARO grant DAAL03-91-G0162

Running head: Nonoscillatory Shock Capturing

Proofs sent to:

Xu-Dong Liu

Department of Mathematics, UCLA
Los Angeles, CA 90024

AMS(MUS) subject classification: Primary 65M10; Secondary 65M05

Keywords: Hyperbolic Conservation Laws, high order, TVB, Nonoscillatory.

Abstract

This is the first paper in a series in which we construct and analyze a class of nonoscillatory high order accurate self-similar local maximum principle satisfying shock capturing schemes for solving multidimensional systems of conservation laws. In this paper we present a scheme which is of 3rd order of accuracy in the sense of flux approximation, using scalar one-dimensional initial value problems as a model. For this model, we make the schemes satisfy a local maximum principle and a nonoscillatory property. The method uses a simple centered stencil with quadratic reconstruction followed by two modifications, imposed as needed. The first enforces a local maximum principle, the second guarantees that no new extrema develop. The schemes are self-similar in the sense that the numerical flux does not depend explicitly on the grid size, i.e., there are no grid size dependent limits involving free parameters as in, e.g., [12],[13],[14],[15]. Combining the nonoscillatory property and the local maximum principle we achieve TVB (Total Variation Boundedness). Hence we obtain convergence of a subsequence of the numerical solutions as the step size approaches zero. Numerical results are encouraging. Extensions to systems and/or higher dimensions will appear in future papers, as well as extensions to higher orders of accuracy.

1 Introduction

We consider numerical solutions of the hyperbolic conservation law:

$$\begin{aligned} &u_t + f(u)_x = 0 \\ &u(x,0) = u_0(x), \end{aligned} \tag{1.1}$$

where $u_0(x)$ is assumed to be a bounded variation function. The main difficulty in solving (1.1) is that the solution may contain discontinuities even if the initial condition is smooth. Among the successful numerical schemes for solving (1.1) we mention the modern nonoscillatory conservative schemes such as TVD (Total Variation Diminishing), UNO (Uniformly High Order Nonoscillatory), ENO (Essentially Nonoscillatory) and TVB (Total Variation Boundedness) schemes (see e.g. [1],[2], [3] and the references listed therein). These schemes are usually total variation stable for one dimension scalar nonlinear problems and are formally higher than first order accurate, hence they can capture sharp shocks without introducing oscillations. These schemes are very successful in numerical experiments. Recently a SNO (Strictly Nonoscillatory) scheme was introduced by Tong in [11], which is of arbitrarily high order of accuracy. We would also like to mention the TVD scheme introduced by Sanders (see [10]). His scheme is TVD in the sense of reconstruction and 3rd order of accuracy except for a degeneracy to second order at isolated extrema. His method involves advancing in time the cell-average and at least one additional quantity, while ours uses only the cell-average. There are, however, similarities in the way in which we enforce a local maximum principle as described below.

The schemes we introduce here are 3rd order accurate, conservative, local maximum principle satisfying, nonoscillatory, and hence TVB schemes. The 3rd order of accuracy is achieved with the usual degeneracy to second order at certain isolated extrema. The TVB property follows from a local maximum principle and a nonoscillatory property. Extensions to multidimensional systems are straightforward and will be performed in the future.

In this paper we use a uniform grid (for simplicity only). Extending the schemes to unstructured grids is not difficult. We define a partition $\{I_j \times [t_n, t_{n+1}]\}_j^n$ of $R \times R^+$, where $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ is the j-th cell, $x_j = j \cdot h$, $t_n = n \cdot \tau$, $h = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $\tau = t_{n+1} - t_n$, and $\lambda = \tau / h$.

Now we briefly outline the construction of the schemes.

Following the framework of the original ENO schemes constructed in [2], our scheme is of the form

$$\bar{u}^{n+1} = A \cdot E(\tau) \cdot R(x, \bar{u}^n), \tag{1.3}$$

where $\bar{u}^n = \{\bar{u}_j^n\} = \{\frac{1}{h}\int\limits_{I_j} u(x,t_n)\,dx\}$ are cell-averages of the solution $u(x,t_n)$ at time $t = t_n$. $R(x,\bar{u}^n)$ is a reconstruction procedure used to produce a high order accurate global approximation to $u(x,t_n)$ from its given cell-averages $\bar{u}^n$. Here we also denote $R(x,\bar{u}^n)$ to be the global approximation to $u(x,t_n)$ and consider $R(x,\bar{u}^n) = \{R_j(x,\bar{u}^n)\}$ to be a piecewise polynomial and $R_j(x,\bar{u}^n)$ to be a polynomial in j-th cell I_j. $E(\tau)$ is the evolution operator of the PDE, and A is the cell-averaging operator, see [2]. We represent the schemes as following: For time $t = t_n$, we follow a reconstruction procedure $R(x,\bar{u}^n)$ to reconstruct the solution to obtain the $R(x,\bar{u}^n)$ from the given cell-averages $\bar{u}^n$ at time $t = t_n$, and perform the evolution operator $E(\tau)$ of the PDE on the $R(x,\bar{u}^n)$ to obtain an approximation solution at time $t = t_{n+1}$ which is the true solution of following hyperbolic conservation law

$$\begin{aligned} &u_t + f(u)_x = 0 \\ &u(x,0) = R(x,\bar{u}^n), \end{aligned} \tag{1.4}$$

at time $t = \tau$, and then compute the sliding averages of the approximation solution to obtain the cell-averages $\bar{u}^{n+1}$ at time $t = t_{n+1}$.

To explain how we get the local maximum principle, the nonoscillatory property, and the resulting TVB property, we rewrite (1.3) as

$$R(x,\bar{u}^{n+1}) = R(x, A \cdot E(\tau) \cdot R(x,\bar{u}^n)). \tag{1.5}$$

From the reconstructed solution $R(x,\bar{u}^n)$ at time $t = t_n$, we perform the evolution operator $E(\tau)$ to obtain the solution at time $t = t_{n+1}$, and the averaging operator A to obtain the cell-averages $\bar{u}^{n+1}$ of the solution, and then use the reconstruction procedure to obtain the reconstructed solution $R(x,\bar{u}^{n+1})$ at time $t = t_{n+1}$.

According to (1.5), we define the TVB (Total Variation Bounded) property of the schemes to be

$$TV(R(x,\bar{u}^{n+1})) \leq C, \tag{1.6}$$

i.e. we measure the variation of the reconstructed piecewise polynomial function. This was done earlier by Sanders in [10]. Here the constant C is independent of the step size h and the time level n.

The strategy to achieve the TVB property of the scheme is thus simply: we construct the reconstruction solution to satisfy a local maximum principle and a nonoscillatory property.

Our local maximum principle means that for each cell, the reconstructed piecewise polynomial $R(x, \bar{u}^{n+1})$ satisfies, $\forall x \in I_j$,

$$m_j \leq R_j(x, \bar{u}^{n+1}) \leq M_j, \tag{1.7}$$

where $m_j = \min_{x \in I_{j-1} \cup I_j \cup I_{j+1}} R(x, \bar{u}^n)$ and $M_j = \max_{x \in I_{j-1} \cup I_j \cup I_{j+1}} R(x, \bar{u}^n)$. This is reasonable under following CFL condition

$$\tfrac{\tau}{h} \sup_u \mid f'(u) \mid \leq 1,$$

where the supreme is evaluated over $[\inf_x(u_0(x)), \sup_x(u_0(x))]$. This condition is optimal.

Our nonoscillatory property means that the number of extrema of the reconstructed piecewise polynomial $R(x, \bar{u}^{n+1})$ is equal to the number of extrema of the $\bar{u}^{n+1}$. Here and below we denote the number of extrema by $N(\cdot)$. Thus the nonoscillatory property means

$$N(R(x, \bar{u}^{n+1})) = N(\bar{u}^{n+1}). \tag{1.8}$$

Because of the local maximum principle, for $\forall x, n$,

$$\inf_x(u_0(x)) \leq R(x, \bar{u}^{n+1}) \leq \sup_x(u_0(x)).$$

Because of the nonoscillatory property,

$$\begin{aligned} N(R(x, \bar{u}^{n+1})) = &\ N(\bar{u}^{n+1}) \\ \leq &\ N(R(x, \bar{u}^n)) = N(\bar{u}^n) \\ \leq &\ N(R(x, \bar{u}^0)) = N(\bar{u}^0) \\ \leq &\ N(u_0(x)). \end{aligned}$$

Here $N(\bar{u}^{n+1}) = N(A \cdot E(\tau) \cdot R(x, \bar{u}^n)) \leq N(E(\tau) \cdot R(x, \bar{u}^n)) \leq N(R(x, \bar{u}^n))$, because the A and $E(\tau)$ are both nonoscillatory operators, see [3].

Thus

$$\begin{aligned} TV(R(x,\bar{u}^{n+1})) \leq & \; N(R(x,\bar{u}^{n+1})) \cdot (\sup_x(u_0(x)) - \inf_x(u_0(x))) \\ \leq & \; N(u_0(x)) \cdot (\sup_x(u_0(x)) - \inf_x(u_0(x))) \\ = & \; C, \end{aligned}$$

where the constant C is independent of h and n. Hence the TVB property (1.6) follows as soon as we achieve the local maximum principle (1.7) and the nonoscillatory property (1.8) in the reconstruction procedure. The TVB property implies convergence of a subsequence in L_1. Convergence for nonlinear problems will follow if we obtain a single entropy condition in the convex case (see [4]), and, more generally, if all limit solutions satisfy Kruzkov's entropy condition (see [16]).

Also we construct $R(x,\bar{u}^{n+1})$ so that it approximates $u(x,t_{n+1})$ up to 3rd order of accuracy in regions in which $u(x,t_{n+1})$ is smooth, and has conservation form $\frac{1}{h}\int_{I_j} R(x,\bar{u}^{n+1})\,dx = \bar{u}_j^{n+1}$ in each cell.

Hence the main idea of this paper is to achieve 3rd order of accuracy, conservation form, the local maximum principle, the nonoscillatory property, and hence the TVB property by achieving the first four of them in the reconstruction procedure.

2 The Reconstruction Procedure

In this section we present the reconstruction procedure to obtain $R(x,\bar{u}^{n+1})$ from the given cell-averages $\bar{u}^{n+1}$. We also know $\{m_j\}$ and $\{M_j\}$. Here $R(x,\bar{u}^{n+1}) = \{R_j(x,\bar{u}^{n+1})\}$ is a piecewise quadratic polynomial approximating the weak solution $u(x,t_{n+1})$ and $R_j(x,\bar{u}^{n+1})$ is quadratic polynomial defined on the j-th cell. According to the previous section, $R(x,\bar{u}^{n+1})$ should satisfy following four properties.

(p_1) Each $R_j(x,\bar{u}^{n+1})$ has 3rd order of accuracy i.e.

$$R_j(x,\bar{u}^{n+1}) = u(x,t_{n+1}) + O(h^3),$$

in regions in which $u(x,t_{n+1})$ is smooth.

(p_2) Each $R_j(x,\bar{u}^{n+1})$ has conservation form, i.e.

$$\frac{1}{h}\int_{I_j} R_j(x,\bar{u}^{n+1})\,dx = \bar{u}_j^{n+1}.$$

(p_3) Each $R_j(x, \bar{u}^{n+1})$ satisfies a local maximum principle i.e. $\forall x \in I_j$,

$$m_j \leq R_j(x, \bar{u}^{n+1}) \leq M_j,$$

where $m_j = \min_{x \in I_{j-1} \cup I_j \cup I_{j+1}} R(x, \bar{u}^n)$ and $M_j = \max_{x \in I_{j-1} \cup I_j \cup I_{j+1}} R(x, \bar{u}^n)$

(p_4) $R(x, \bar{u}^{n+1})$ satisfies a nonoscillatory property i.e.

$$N(R(x, \bar{u}^{n+1})) = N(\bar{u}^{n+1}).$$

In the following subsections we design the reconstruction procedure to be two subprocedures. In the first subprocedure, by interpolating, we simply get a preliminary reconstructed quadratic polynomial in each cell I_j $P_j(x, \bar{u}^{n+1})$, and thus a preliminary reconstructed piecewise quadratic polynomial $P(x, \bar{u}^{n+1}) = \{P_j(x, \bar{u}^{n+1})\}$ satisfying properties (p_1) and (p_2). In the second subprocedure we introduce a modifying operator and apply it twice to each piece $P_j(x, \bar{u}^{n+1})$ of $P(x, \bar{u}^{n+1})$ to obtain the desired $R(x, \bar{u}^{n+1}) = \{R_j(x, \bar{u}^{n+1})\}$ satisfying properties (p_1)-(p_4).

For simplicity, here and below we denote $\bar{u} = \{\bar{u}_j\} = \bar{u}^{n+1} = \{\bar{u}_j^{n+1}\}$, $P(x) = \{P_j(x)\} = P(x, \bar{u}^{n+1}) = \{P_j(x, \bar{u}^{n+1})\}$ and $R(x) = \{R_j(x)\} = R(x, \bar{u}^{n+1}) = \{R_j(x, \bar{u}^{n+1})\}$.

2.1 Preliminary Reconstruction Subprocedure

We start with some definitions and observations.

Definition 1: We call the j-th cell I_j nondecreasing if $\bar{u}_{j-1} \leq \bar{u}_j \leq \bar{u}_{j+1}$, nonincreasing if $\bar{u}_{j-1} \geq \bar{u}_j \geq \bar{u}_{j+1}$, maximum if $\bar{u}_{j-1} < \bar{u}_j > \bar{u}_{j+1}$, and minimum if $\bar{u}_{j-1} > \bar{u}_j < \bar{u}_{j+1}$. We call I_j monotone if it is nondecreasing or nonincreasing, and extrema if it is maximum or minimum.

Definition 2: We say a polynomial $p_j(x)$, which is defined on I_j, has the **same shape** as the cell-averages $\bar{u}$, if the nonconstant $p_j(x)$ satisfies

(i) The polynomial $p_j(x)$ has a maximum in $I_j^o = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ if I_j is a maximum cell and $p_j(x)$ has a minimum in I_j^o if I_j is a minimum cell.

(ii) $p_j(x)$ is nondecreasing on I_j if I_j is a nondecreasing cell; $p_j(x)$ is nonincreasing on I_j if I_j is a nonincreasing cell.

If $p_j(x)$ is a constant, we again say that it has the **same shape** as the cell-averages $\bar{u}$.

We say a piecewise polynomial $p(x) = \{p_j(x)\}$ has the **same shape** as the cell-averages $\bar{u}$ if all of the $p_j(x)$ have this property.

Definition 3: We call a polynomial $p_j(x)$ defined on I_j a **proper reconstructed polynomial** if

(i) $\frac{1}{h}\int\limits_{I_j} p_j(x)\,dx = \bar{u}_j$,

(ii) $\forall x \in I_j$, $p_j(x) - u(x,t_{n+1}) = O(h^3)$ in regions in which $u(x,t_{n+1})$ is smooth,

(iii) $p_j(x)$ has the **same shape** as the cell-averages $\bar{u}$.

We also call a piecewise polynomial $p(x) = \{p_j(x)\}$ a **proper reconstructed piecewise polynomial** if all of $p_j(x)$ are **proper reconstructed polynomials.**

Definition 4: Suppose a **proper reconstructed piecewise polynomial** $p(x)$ has a extrema at x^*. We say $p(x)$ has a **false extrema** at x^*, if there is a neighborhood of x^* so that $p(x)$ is monotone on both sides of x^* and of the same kind i.e both nondecreasing or both nonincreasing.

Of course a **false extrema** of a **proper reconstructed piecewise polynomial** $p(x)$ may only occur at an interface between two cells.

Observation 1:

a) For a **proper reconstructed piecewise polynomial** $p(x)$, we observe that there may be four type of **false extrema** at an interface, say $x_{j+\frac{1}{2}}$:

false extrema type (i) $\bar{u}_j > \bar{u}_{j+1}$ and $p_j(x_{j+\frac{1}{2}}) < p_{j+1}(x_{j+\frac{1}{2}})$,

false extrema type (ii) $\bar{u}_j = \bar{u}_{j+1}$ and $p_{j+1}(x_{j+\frac{1}{2}}) \geq \bar{u}_j \geq p_j(x_{j+\frac{1}{2}})$ and at least one of these inequalities is strict.

false extrema type (iii) $\bar{u}_j < \bar{u}_{j+1}$ and $p_j(x_{j+\frac{1}{2}}) > p_{j+1}(x_{j+\frac{1}{2}})$,

false extrema type (iv) $\bar{u}_j = \bar{u}_{j+1}$ and $p_{j+1}(x_{j+\frac{1}{2}}) \leq \bar{u}_j \leq p_j(x_{j+\frac{1}{2}})$ and at least one of these inequalities is strict.

b) For a **proper reconstructed piecewise polynomial** $p(x)$, if $p(x)$ at any interface $x_{j+\frac{1}{2}}$, satisfies

(i) if $\bar{u}_j > \bar{u}_{j+1}$, then $p_j(x_{j+\frac{1}{2}}) \geq p_{j+1}(x_{j+\frac{1}{2}})$,

(ii) if $\bar{u}_j < \bar{u}_{j+1}$, then $p_j(x_{j+\frac{1}{2}}) \leq p_{j+1}(x_{j+\frac{1}{2}})$,

(iii) if $\bar{u}_j = \bar{u}_{j+1}$, then $(p_j(x_{j+\frac{1}{2}})-\bar{u}_j)\cdot(p_{j+1}(x_{j+\frac{1}{2}})-\bar{u}_j) > 0$ or $p_j(x_{j+\frac{1}{2}}) = p_{j+1}(x_{j+\frac{1}{2}}) = \bar{u}_j$,

then $p(x)$ has no **false extrema.**

Observation 2:

If a **proper reconstructed piecewise polynomial** $p(x)$ has no **false extrema**, then it is a nonoscillatory reconstruction i.e. $N(p(x)) = N(\bar{u})$.

We construct the preliminary piecewise polynomial as follows: For each cell I_j, we use the centered stencil to interpolate the cell averages $\bar{u}_{j-1}, \bar{u}_j, \bar{u}_{j+1}$ and obtain $P_j(x)$, i.e., we require

$$\frac{1}{h}\int_{I_i} P_j(x)\,dx = \bar{u}_i, \qquad i = j-1, j, j+1.$$

We denote $P(x) = \{P_j(x)\}$ and we have

Lemma 1:

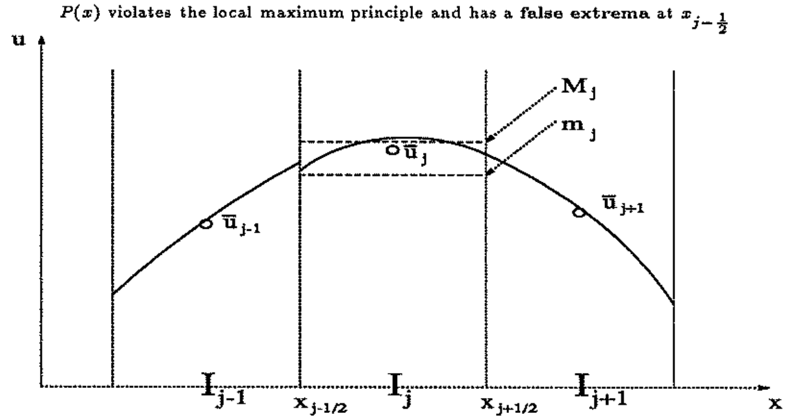
$P(x)$ is a **proper reconstructed piecewise polynomial.**

Proof: Obviously this $P(x)$ satisfies properties (p_1) and (p_2). For each quadratic polynomial $P_j(x)$, $P_j'(x_{j-\frac{1}{2}}) = (\bar{u}_j - \bar{u}_{j-1})/h$ and $P_j'(x_{j+\frac{1}{2}}) = (\bar{u}_{j+1} - \bar{u}_j)/h$. $P_j'(x_{j-\frac{1}{2}}) \cdot P_j'(x_{j+\frac{1}{2}}) = (\bar{u}_j - \bar{u}_{j-1}) \cdot (\bar{u}_{j+1} - \bar{u}_j)/h^2$, hence $P_j(x)$ has the **same shape** as the cell-averages $\bar{u}$. #

However $P(x)$ might not satisfy properties (p_3) and (p_4), see Figure 1.

Figure 1

$P(x)$ violates the local maximum principle and has a false extrema at $x_{j-\frac{1}{2}}$



We observe that some $P_j(x)$ may have overshoot which means $\max_{x\in I_j} P_j(x) - M_j > 0$ and/or undershoot which means $\min_{x\in I_j} P_j(x) - m_j < 0$. We denote the magnitudes of overshoot and undershoot as $\delta_j^+ = max(\max_{x\in I_j} P_j(x) - M_j, 0) \geq 0$ and $\delta_j^- = max(m_j - \min_{x\in I_j} P_j(x), 0) \geq 0$. Note that, for each $P_j(x)$, $max(\delta_j^+, \delta_j^-) = O(h^3)$ in regions in which $u(x, t_{n+1})$ is smooth. We also observe that there may also be some **false extrema** of $P(x) = \{P_j(x)\}$. Note that at each interface $x_{j+\frac{1}{2}}$, $\mid P_j(x_{j+\frac{1}{2}}) - P_{j+1}(x_{j+\frac{1}{2}}) \mid = O(h^3)$ in regions in which $u(x, t_{n+1})$ is smooth.

Lemma 2:
In each cell and at each interface I_j and $x_{j+\frac{1}{2}}$,

$$max(\delta_j^+, \delta_j^-) = O(h^3),$$
$$\mid P_j(x_{j+\frac{1}{2}}) - P_{j+1}(x_{j+\frac{1}{2}}) \mid = O(h^3),$$

in regions in which $u(x, t_{n+1})$ is smooth.

In the following subsection, we will modify $P(x)$ to achieve the local maximum principle in each cell, and to remove all **false extrema** at each interface, hence achieving the nonoscillatory property, while keeping it a **proper reconstructed piecewise polynomial.**

2.2 Advanced Subprocedure

In this subsection we describe the 2nd subprocedure used on the reconstruction. We first introduce an operator which can be used twice in each cell to modify each piece $P_j(x)$ of $P(x)$ to obtain the corresponding piece $R_j(x)$ of $R(x)$ satisfying our desired properties (p_1)-(p_4).

Here and below, we denote the modifying operator by, for each cell I_j,

$$r_j(x) = \mathbf{L}[p_j(x), lb_j, ub_j],$$

where $r_j(x)$ is a quadratic polynomial defined on I_j, $p_j(x)$ is a **proper reconstructed polynomial** defined on I_j, lb_j and ub_j are local lower and upper bounds for the polynomial $r_j(x)$ on I_j. We denote $\delta_j^+ = max(\max\limits_{x \in I_j} p_j(x) - ub_j, 0)$ as magnitude of overshoot of $p_j(x)$ against ub_j, and $\delta_j^- = max(lb_j - \min\limits_{x \in I_j} p_j(x), 0)$ as magnitude of undershoot of $p_j(x)$ against lb_j. We require that lb_j and ub_j satisfy

$$\delta_j^+ = O(h^3), \tag{2.2a}$$

$$\delta_j^- = O(h^3), \tag{2.2b}$$

in regions in which $u(x, t_{n+1})$ is smooth.

We require that the modifying operator satisfy following five properties.

(q_1) The function $r_j(x)$ again satisfies the conservation requirement. That is,

$$\frac{1}{h} \int\limits_{I_j} r_j(x)\, dx = \frac{1}{h} \int\limits_{I_j} p_j(x)\, dx = \bar{u}_j.$$

(q_2) The function $r_j(x)$ is as accurate as $p_j(x)$. That is

$$| r_j(x) - p_j(x) | \leq C_2 max(\delta_j^+, \delta_j^-),$$

where C_2 is a constant which only depends on the degree of $p_j(x)$ (in this paper the degree is two).

(q_3) The function $r_j(x)$ is bounded by lb_j and ub_j in I_j i.e. $\forall x \in I_j$,

$$lb_j \leq r_j(x) \leq ub_j.$$

(q_4) The function $r_j(x)$ has the **same shape** as the function $p_j(x)$, in fact, we have

$$r_j'(x) = \epsilon_j p_j'(x),$$

where $0 \leq \epsilon_j < 1$ is a constant.

(q_5) The function $r_j(x)$ is uniformly closer to the cell-average $\bar{u}_j$ than $p_j(x)$, in fact: $\forall x \in I_j$,

$$(r_j(x) - \bar{u}_j) = \epsilon_j (p_j(x) - \bar{u}_j),$$

where $0 \leq \epsilon_j < 1$ is a constant.

Lemma 3:

As long as lb_j and ub_j satisfy (2.2a,b), $r_j(x)$ is a **proper reconstructed polynomial.**

Proof: This follows easily from properties (q_1),(q_2) and (q_4), because $p_j(x)$ is a **proper reconstructed polynomial.** #

Theorem 1 (Modifying Operator)

At the j-th cell, we denote $u_{max} = \max\limits_{x \in I_j} p_j(x)$, $u_{min} = \min\limits_{x \in I_j} p_j(x)$.

(1) If $u_{max} > ub_j$ and $u_{min} \geq lb_j$ (overshoot), we define

$$r_j(x) = \mathbf{L}[p_j(x), lb_j, ub_j] = \epsilon^{(1)} p_j(x) - \epsilon^{(1)} u_{max} + ub_j, \tag{2.3a}$$

where $1 > \epsilon^{(1)} = (ub_j - \bar{u}_j)/(u_{max} - \bar{u}_j) \geq 0$.

(2) If $u_{max} \leq ub_j$ and $u_{min} < lb_j$ (undershoot), we define

$$r_j(x) = \mathbf{L}[p_j(x), lb_j, ub_j] = \epsilon^{(2)} p_j(x) - \epsilon^{(2)} u_{min} + lb_j, \tag{2.3b}$$

where $1 > \epsilon^{(2)} = (lb_j - \bar{u}_j)/(u_{min} - \bar{u}_j) \geq 0$.

(3) If $u_{max} > ub_j$ and $u_{min} < lb_j$ (both overshoot and undershoot),

if $\epsilon^{(1)} < \epsilon^{(2)}$, we define

$$r_j(x) = \mathbf{L}[p_j(x), lb_j, ub_j] = \epsilon^{(1)} p_j(x) - \epsilon^{(1)} u_{max} + ub_j, \tag{2.3c}$$

if $\epsilon^{(1)} \geq \epsilon^{(2)}$, we define

$$r_j(x) = \mathbf{L}[p_j(x), lb_j, ub_j] = \epsilon^{(2)} p_j(x) - \epsilon^{(2)} u_{min} + lb_j. \tag{2.3d}$$

(4) If $u_{max} \leq ub_j$ and $u_{min} \geq lb_j$ (neither overshoot nor undershoot), then

$$r_j(x) = p_j(x). \tag{2.3e}$$

Then $\mathbf{L}[\cdots]$ has all of the five properties (q_1)-(q_5) .

Proof:

For the first statement (1), we denote the difference between $p_j(x)$ and $r_j(x)$ by $d_j(x)$, and have

$$\begin{aligned} d_j(x) = & \; p_j(x) - r_j(x) \\ = & \; (u_{max} - ub_j) \tfrac{(p_j(x) - \bar{u}_j)}{(u_{max} - \bar{u}_j)}. \end{aligned}$$

Because $\int\limits_{I_j} d_j(x)\, dx = 0$, we obtain the conservation property

(q_1) i.e.

$$\tfrac{1}{h} \int\limits_{I_j} r_j(x)\, dx = \tfrac{1}{h} \int\limits_{I_j} p_j(x)\, dx = \bar{u}_j.$$

To obtain property (q_2), we need to use a lemma which we shall prove in the Appendix:

$$\max_{x \in I_j} \mid \tfrac{p_j(x) - \bar{u}_j}{u_{max} - \bar{u}_j} \mid \leq 3.$$

We obtain from this:

$$\mid r_j(x) - p_j(x) \mid \leq 3 max(\delta_j^+, \delta_j^-).$$

We have thus obtained property (q_2).

Because of (2.3a)

$$r_j'(x) = \epsilon^{(1)} p_j'(x). \tag{2.4}$$

Hence we obtain property (q_4).

It is easy to see

$$r_j(x) - \bar{u}_j = \epsilon^{(1)} (p_j(x) - \bar{u}_j), \tag{2.5}$$

where $0 \le \epsilon^{(1)} = (ub_j - \bar{u}_j)/(u_{max} - \bar{u}_j) < 1$ is a constant. Hence we obtain property (q_5).

Denoting $p_j(x_{max}) = u_{max}$ and $p_j(x_{min}) = u_{min}$, where $x_{max}, x_{min} \in I_j$, we know from (2.4) that x_{max} is a point in I_j at which both $r_j(x)$ and $p_j(x)$ achieve their maximum, and x_{min} is a point in I_j at which both $r_j(x)$ and $p_j(x)$ achieve their minimum. Hence

$$\max_{x \in I_j} r_j(x) = r_j(x_{max}) = ub_j,$$
$$\min_{x \in I_j} r_j(x) = r_j(x_{min}) = \bar{u}_j + \epsilon^{(1)}(p_j(x_{min}) - \bar{u}_j) \ge u_{min} \ge lb_j.$$

We obtain property (q_3) i.e.

$$lb_j \le r_j(x) \le ub_j.$$

Hence $r_j(x)$ satisfies all five properties (q_1)-(q_5).

The proof of the second statement (2) is similar.

The proof of the third statement (3) follows easily from (2.5), which guarantees $lb_j \le r_j(x) \le ub_j$. #

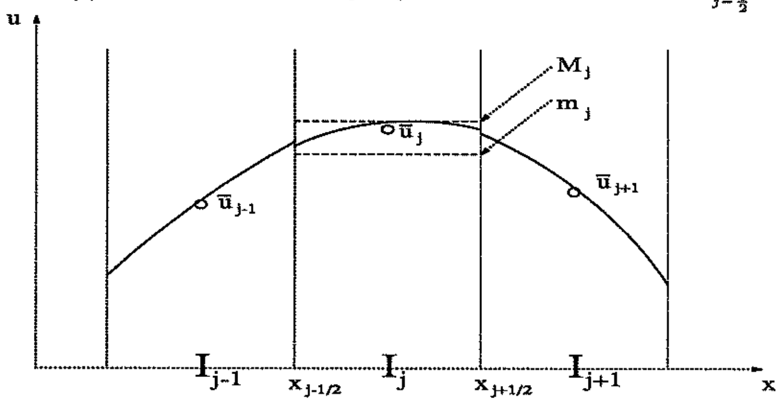
In the following, we outline how to perform the operation in each cell twice to obtain $R(x)$ from $P(x)$ with those five properties (q_1)-(q_5).

First we perform the operation to achieve the local maximum principle (p_3). In each cell I_j, we define the local lower and upper bounds as $lb_j = m_j$ and $ub_j = M_j$, and m_j and M_j are defined as before, and it is easy to see that m_j and M_j satisfy (2.2a,b). We perform the operation on the preliminary **proper reconstructed polynomial** $P_j(x)$ to obtain a quadratic polynomial $P_j^{mp}(x)$.

Remark 2.1: In each cell I_j, because m_j and M_j satisfy (2.2a,b), from Lemma 3, it follows that $P_j^{mp}(x)$ is a **proper reconstructed polynomial.** In addition to these, $P_j^{mp}(x)$ satisfies the local maximum principle, because of property (q_3). By property (q_5) $P_j^{mp}(x)$ is uniformly closer to $\bar{u}_j$ than $P_j(x)$. Therefore $P^{mp}(x) = \{P_j^{mp}(x)\}$ satisfies properties (p_1)-(p_3). See Figure 2.

Figure 2

$P^{mp}(x)$ satisfies the local maximum principle and still has false extrema at $x_{j-\frac{1}{2}}$



Lemma 4:

(i) $P^{mp}(x)$ is a **proper reconstructed piecewise polynomial.**

(ii) $P^{mp}(x)$ satisfies the local maximum principle.

Lemma 5:

At each interface $x_{j+\frac{1}{2}}$,

$$| P_j^{mp}(x_{j+\frac{1}{2}}) - P_{j+1}^{mp}(x_{j+\frac{1}{2}}) |= O(h^3),$$

in regions in which $u(x, t_{n+1})$ is smooth.

Next we perform the operation to enforce the nonoscillatory property (p_4). The only thing left to do is to remove all **false extrema** of $P^{mp}(x) = \{P_j^{mp}(x)\}$. In each monotone cell I_j, we define the local lower and upper bounds $\tilde{lb}_j$ and $\tilde{ub}_j$, which should satisfy (2.2a,b) and which we shall compute later. We perform the operation in each monotone cell on $P_j^{mp}(x)$ and we set $R_j(x) \equiv P_j^{mp}(x)$ in each extrema cell to obtain the final **proper reconstructed polynomial** $R_j(x)$.

Remark 2.2: If $\tilde{lb}_j$ and $\tilde{ub}_j$ satisfy (2.2a,b), from Lemma 3, $R_j(x)$ is a **proper reconstructed polynomial.** In addition to these, $R_j(x)$ satisfies the local maximum principle, because $P_j^{mp}(x)$ satisfies it and $R_j(x)$ is uniformly closer to $\bar{u}_j$ than $P_j^{mp}(x)$ in monotone cells and $R_j(x) = P_j^{mp}(x)$ in extrema cells.

Lemma 6:

If at each monotone cell I_j, $\tilde{lb}_j$ and $\tilde{ub}_j$ satisfy (2.2a,b), then

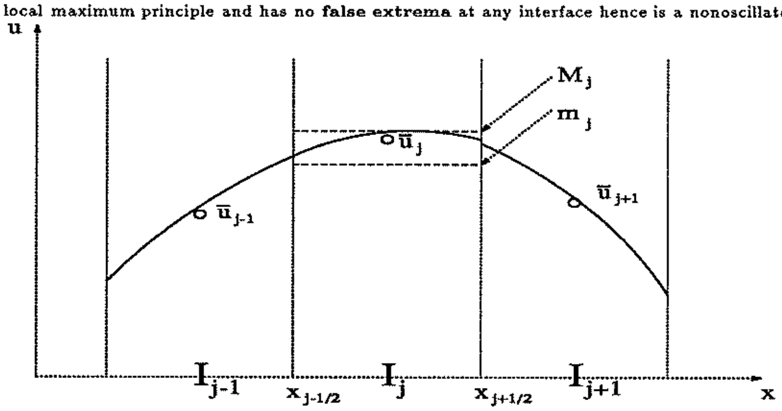
(i) $R(x)$ is a **proper reconstructed piecewise polynomial.**

(ii) $R(x)$ satisfies the local maximum principle.

Hence if we can prove that $R(x) = \{R_j(x)\}$ has no **false extrema**, we will have shown that $R(x)$ is a nonoscillatory reconstruction i.e. $N(R(x)) = N(\bar{u})$. Then $R(x)$ will satisfy all four properties (p_1)-(p_4) which are the 3rd order of accuracy, conservation, local maximum principle satisfying and nonoscillatory properties.

Figure 3

$R(x)$ satisfies the local maximum principle and has no false extrema at any interface hence is a nonoscillatory reconstruction



The following is the **Advanced Subprocedure**:

STEP 1: Achieve the local maximum principle:

(i) In each cell I_j, set $lb_j = m_j$ and $ub_j = M_j$,

(ii) do j=1,2,3,$\cdots$

$$P_j^{mp}(x) = \mathbf{L}[P_j(x), m_j, M_j],$$

end do.

Here we perform the operation from first cell to the last cell in sequential pattern.

STEP 2: In each monotone cell I_j, define the proper local bounds $\tilde{lb}_j$ and $\tilde{ub}_j$, and modify $P_j^{mp}(x)$ to obtain the nonoscillatory reconstruction $R_j(x)$ by an odd-even pattern. In each extrema cell, set $R_j(x) = P_j^{mp}(x)$. That is

First we modify $P_j^{mp}(x)$ in the odd numbered cells:

do j=1,3,5,$\cdots$

if $(\bar{u}_{j-1} \le \bar{u}_j \le \bar{u}_{j+1})$ then

$$\tilde{ub}_j = max(\tfrac{1}{2}(\bar{u}_j + \bar{u}_{j+1}), P_{j+1}^{mp}(x_{j+\frac{1}{2}}))$$

$$\tilde{lb}_j = min(\tfrac{1}{2}(\bar{u}_j + \bar{u}_{j-1}), P_{j-1}^{mp}(x_{j-\frac{1}{2}}))$$

$$R_j(x) = \mathbf{L}[P_j^{mp}(x), \tilde{lb}_j, \tilde{ub}_j]$$

else if $(\bar{u}_{j-1} \ge \bar{u}_j \ge \bar{u}_{j+1})$ then

$\tilde{ub}_j = max(\frac{1}{2}(\bar{u}_j + \bar{u}_{j-1}), P^{mp}_{j-1}(x_{j-\frac{1}{2}}))$
$\tilde{lb}_j = min(\frac{1}{2}(\bar{u}_j + \bar{u}_{j+1}), P^{mp}_{j+1}(x_{j+\frac{1}{2}}))$
$R_j(x) = \mathbf{L}[P^{mp}_j(x), \tilde{lb}_j, \tilde{ub}_j]$
else
$R_j(x) = P^{mp}_j(x)$
end if.
end do

Next after we have obtained $R_j(x)$ in the odd numbered cells, we modify $P^{mp}_j(x)$ in the even numbered cells:

do j=2,4,6,$\cdots$
if $(\bar{u}_{j-1} \le \bar{u}_j \le \bar{u}_{j+1})$ then
$\tilde{ub}_j = max(\frac{1}{2}(\bar{u}_j + \bar{u}_{j+1}), R_{j+1}(x_{j+\frac{1}{2}}))$
$\tilde{lb}_j = min(\frac{1}{2}(\bar{u}_j + \bar{u}_{j-1}), R_{j-1}(x_{j-\frac{1}{2}}))$
$R_j(x) = \mathbf{L}[P^{mp}_j(x), \tilde{lb}_j, \tilde{ub}_j]$
else if $(\bar{u}_{j-1} \ge \bar{u}_j \ge \bar{u}_{j+1})$ then
$\tilde{ub}_j = max(\frac{1}{2}(\bar{u}_j + \bar{u}_{j-1}), R_{j-1}(x_{j-\frac{1}{2}}))$
$\tilde{lb}_j = min(\frac{1}{2}(\bar{u}_j + \bar{u}_{j+1}), R_{j+1}(x_{j+\frac{1}{2}}))$
$R_j(x) = \mathbf{L}[P^{mp}_j(x), \tilde{lb}_j, \tilde{ub}_j]$
else
$R_j(x) = P^{mp}_j(x)$
end if.
end do

Remark 2.3: Among monotone cells, we modify each $P^{mp}_j(x)$ on the odd numbered cells first, and then modify each $P^{mp}_j(x)$ on the even numbered cells. The reason is that this odd-even pattern may save some unnecessary work which can not be avoided by modifying the $P^{mp}_j(x)$ in sequence. However no matter which pattern we choose, the error propagation is very local, because $P^{mp}(x) = \{P^{mp}_j(x)\}$ has the **same shape** as the cell-averages $\bar{u}$ and because of property (q_5) of the modifying operator.

In the following Theorem, we prove that in each monotone cell I_j, $\tilde{lb}_j$ and $\tilde{ub}_j$ satisfy (2.2a,b), hence $R(x) = \{R_j(x)\}$ is a **proper reconstructed piecewise polynomial**; $R(x)$ has no **false extrema**, hence $R(x)$ is a nonoscillatory reconstruction, which means

$$N(R(x)) = N(\bar{u}).$$

Here $R(x)$ will satisfy all four properties (p_1)-(p_4).

Theorem 2

Our scheme using two reconstruction subprocedures is a 3rd order accurate, conservative, local maximum principle satisfying, nonoscillatory, and hence a TVB scheme under the following CFL condition

$$\frac{\tau}{h}\sup_u \mid f'(u) \mid \leq 1,$$

where the supremum is evaluated over $[\inf_x(u_0(x)), \sup_x(u_0(x))]$. The CFL condition is optimal.

Proof

From the **Remark 2.1 and 2.2**, we only need to prove that, in each monotone cell I_j, $\tilde{lb}_j$ and $\tilde{ub}_j$ satisfy (2.2a,b), and that the $R(x)$ has no **false extrema**.

We recall that the **proper reconstructed piecewise polynomial** $P^{mp}(x)$ may have four types of **false extrema** at some interfaces $x_{j+\frac{1}{2}}$ i.e.

false extrema type (i) $\bar{u}_j > \bar{u}_{j+1}$ and $P_j^{mp}(x_{j+\frac{1}{2}}) < P_{j+1}^{mp}(x_{j+\frac{1}{2}})$,

false extrema type (ii) $\bar{u}_j = \bar{u}_{j+1}$ and $P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \geq \bar{u}_j \geq P_j^{mp}(x_{j+\frac{1}{2}})$ (at least one of the inequalities is strict).

false extrema type (iii) $\bar{u}_j < \bar{u}_{j+1}$ and $P_j^{mp}(x_{j+\frac{1}{2}}) > P_{j+1}^{mp}(x_{j+\frac{1}{2}})$,

false extrema type (iv) $\bar{u}_j = \bar{u}_{j+1}$ and $P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \leq \bar{u}_j \leq P_j^{mp}(x_{j+\frac{1}{2}})$ (at least one of the inequalities is strict).

In the following, we shall show that there is no **false extrema** of $R(x)$ at any interface $x_{j+\frac{1}{2}}$:

$$\begin{array}{llll} \text{if} & \bar{u}_j > \bar{u}_{j+1}, & \text{then} & R_j(x_{j+\frac{1}{2}}) \geq R_{j+1}(x_{j+\frac{1}{2}}), \\ \text{if} & \bar{u}_j < \bar{u}_{j+1}, & \text{then} & R_j(x_{j+\frac{1}{2}}) \leq R_{j+1}(x_{j+\frac{1}{2}}), \\ \text{if} & \bar{u}_j = \bar{u}_{j+1}, & \text{then} & (R_j(x_{j+\frac{1}{2}}) - \bar{u}_j)\cdot(R_{j+1}(x_{j+\frac{1}{2}}) - \bar{u}_j) > 0 \\ & & & \text{or } R_j(x_{j+\frac{1}{2}}) = R_{j+1}(x_{j+\frac{1}{2}}) = \bar{u}_j. \end{array} \tag{2.6}$$

We shall also show that in each monotone cell I_j, $\tilde{lb}_j$ and $\tilde{ub}_j$ satisfy (2.2a,b).

Here if $P^{mp}(x)$ satisfies (2.6) at some interfaces, $R(x)$ will also satisfy (2.6) at same interfaces, because $R_j(x)$ is uniformly closer to $\bar{u}_j$ than $P_j^{mp}(x)$ in the monotone cells and $R_j(x) = P_j^{mp}(x)$ in the extrema cells.

Now if $P^{mp}(x)$ does not satisfy (2.6) at some interfaces, we prove that $R(x)$ still does satisfy (2.6) at same interfaces.

Suppose there is a type (i) **false extrema** of $P^{mp}(x)$ at an interface, say $x_{j+\frac{1}{2}}$: $\bar{u}_j > \bar{u}_{j+1}$, $P_j^{mp}(x_{j+\frac{1}{2}}) < P_{j+1}^{mp}(x_{j+\frac{1}{2}})$. W.L.O.G. we assume j is odd.

Case 1. If we choose, in the advanced subprocedure,

$$\tilde{lb}_j = min(\tfrac{1}{2}\bar{u}_j + \tfrac{1}{2}\bar{u}_{j+1}, P_{j+1}^{mp}(x_{j+\frac{1}{2}})),$$
$$\tilde{ub}_{j+1} = max(\tfrac{1}{2}\bar{u}_j + \tfrac{1}{2}\bar{u}_{j+1}, R_j(x_{j+\frac{1}{2}})),$$

which means $\bar{u}_{j-1} \geq \bar{u}_j > \bar{u}_{j+1}$ and $\bar{u}_j > \bar{u}_{j+1} \geq \bar{u}_{j+2}$.

If $\tilde{lb}_j = \frac{1}{2}\bar{u}_j + \frac{1}{2}\bar{u}_{j+1}$, then $R_j(x_{j+\frac{1}{2}}) \geq \tilde{lb}_j$, $\tilde{ub}_{j+1} = R_j(x_{j+\frac{1}{2}})$, and $R_{j+1}(x_{j+\frac{1}{2}}) \leq R_j(x_{j+\frac{1}{2}})$; If $\tilde{lb}_j = P_{j+1}^{mp}(x_{j+\frac{1}{2}})$ then $R_j(x_{j+\frac{1}{2}}) \geq P_{j+1}^{mp}(x_{j+\frac{1}{2}})$, thus $R_j(x_{j+\frac{1}{2}}) \geq R_{j+1}(x_{j+\frac{1}{2}})$ because $P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \geq R_{j+1}(x_{j+\frac{1}{2}})$. Hence (2.6) is satisfied.

If $\tilde{lb}_j > \min_{x\in I_j} P_j^{mp}(x)$, then $0 < \tilde{lb}_j - \min_{x\in I_j} P_j^{mp}(x) = \tilde{lb}_j - P_j^{mp}(x_{j+\frac{1}{2}}) \leq P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - P_j^{mp}(x_{j+\frac{1}{2}}) = O(h^3)$. And if $\tilde{ub}_{j+1} < \max_{x\in I_{j+1}} P_{j+1}^{mp}(x)$, then $0 < \max_{x\in I_{j+1}} P_{j+1}^{mp}(x) - \tilde{ub}_{j+1} = P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - \tilde{ub}_{j+1} \leq P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - P_j^{mp}(x_{j+\frac{1}{2}}) = O(h^3)$. The $\tilde{lb}_j$ and $\tilde{ub}_{j+1}$ satisfy (2.2a,b).

Case 2: If we choose, in the advanced subprocedure,

$$\tilde{lb}_j = min(\tfrac{1}{2}\bar{u}_j + \tfrac{1}{2}\bar{u}_{j+1}, P_{j+1}^{mp}(x_{j+\frac{1}{2}})),$$
$$R_{j+1}(x) = P_{j+1}^{mp}(x),$$

which means $\bar{u}_{j-1} \geq \bar{u}_j > \bar{u}_{j+1}$ and $\bar{u}_j > \bar{u}_{j+1} < \bar{u}_{j+2}$.

Here we claim that $\tilde{lb}_j = P_{j+1}^{mp}(x_{j+\frac{1}{2}})$ (It is easy to see $P_{j+1}(x_{j+\frac{1}{2}}) = \frac{1}{6}(-\bar{u}_{j+2}+5\bar{u}_{j+1}+2\bar{u}_j)$ and $P_{j+1}(x_{j+\frac{1}{2}})-\frac{1}{2}(\bar{u}_j+\bar{u}_{j+1}) \leq \frac{1}{6}(-\bar{u}_{j+2}+2\bar{u}_{j+1}-\bar{u}_j) < 0$. We know that, if $P_{j+1}^{mp}(x_{j+\frac{1}{2}}) < \bar{u}_{j+1}$ then $P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - (\frac{1}{2}\bar{u}_j + \frac{1}{2}\bar{u}_{j+1}) < 0$; if not $P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - (\frac{1}{2}\bar{u}_j + \frac{1}{2}\bar{u}_{j+1}) < P_{j+1}(x_{j+\frac{1}{2}}) - (\frac{1}{2}\bar{u}_j + \frac{1}{2}\bar{u}_{j+1}) < 0$. Thus $P_{j+1}^{mp}(x_{j+\frac{1}{2}}) < \frac{1}{2}\bar{u}_j + \frac{1}{2}\bar{u}_{j+1}$ and $\tilde{lb}_j = P_{j+1}^{mp}(x_{j+\frac{1}{2}})$). Hence $R_{j+1}(x_{j+\frac{1}{2}}) = P_{j+1}^{mp}(x_{j+\frac{1}{2}}) = \tilde{lb}_j \leq R_j(x_{j+\frac{1}{2}})$, (2.6) is satisfied.

If $\tilde{lb}_j > \min_{x\in I_j} P_j^{mp}(x)$, then $0 < \tilde{lb}_j - \min_{x\in I_j} P_j^{mp}(x) = \tilde{lb}_j - P_j^{mp}(x_{j+\frac{1}{2}}) = P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - P_j^{mp}(x_{j+\frac{1}{2}}) = O(h^3)$. Hence the $\tilde{lb}_j$ satisfy (2.2b).

Case 3: If we choose, in the advanced subprocedure,

$$R_j(x) = P_j^{mp}(x),$$
$$\tilde{ub}_{j+1} = max(\tfrac{1}{2}\bar{u}_j + \tfrac{1}{2}\bar{u}_{j+1}, R_j(x_{j+\frac{1}{2}})),$$

which means $\bar{u}_{j-1} < \bar{u}_j > \bar{u}_{j+1}$ and $\bar{u}_j > \bar{u}_{j+1} \geq \bar{u}_{j+2}$.

Similarly we have $u\tilde{b}_{j+1} = R_j(x_{j+\frac{1}{2}}) = P_j^{mp}(x_{j+\frac{1}{2}})$. Hence $R_{j+1}(x_{j+\frac{1}{2}}) \leq u\tilde{b}_{j+1} = R_j(x_{j+\frac{1}{2}})$, (2.6) is satisfied.

If $u\tilde{b}_{j+1} < \max_{x \in I_{j+1}} P_{j+1}^{mp}(x)$, then $0 < \max_{x \in I_{j+1}} P_{j+1}^{mp}(x) - u\tilde{b}_{j+1} = P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - u\tilde{b}_{j+1} = P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - P_j^{mp}(x_{j+\frac{1}{2}}) = O(h^3)$. Hence $u\tilde{b}_{j+1}$ satisfy (2.2a).

Case 4: If we choose, in the advanced subprocedure,

$$R_j(x) = P_j^{mp}(x),$$
$$R_{j+1}(x) = P_{j+1}^{mp}(x),$$

which means $\bar{u}_{j-1} < \bar{u}_j > \bar{u}_{j+1}$ and $\bar{u}_j > \bar{u}_{j+1} < \bar{u}_{j+2}$.

Similarly we have $P_j^{mp}(x_{j+\frac{1}{2}}) \geq \frac{1}{2}\bar{u}_j + \frac{1}{2}\bar{u}_{j+1} \geq P_{j+1}^{mp}(x_{j+\frac{1}{2}})$, (2.6) is satisfied.

Next suppose there is type (ii) **false extrema** of $P^{mp}(x)$ at an interface $x_{j+\frac{1}{2}}$: $\bar{u}_j = \bar{u}_{j+1}$ and $P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \geq \bar{u}_{j+1} = \bar{u}_j \geq P_j^{mp}(x_{j+\frac{1}{2}})$ (two equals do not hold at same time). W.L.O.G. we assume j is odd.

We have

$$\begin{aligned}
l\tilde{b}_j &= min(\tfrac{1}{2}\bar{u}_j + \tfrac{1}{2}\bar{u}_{j+1}, P_{j+1}^{mp}(x_{j+\frac{1}{2}})) \\
&= min(\bar{u}_j, P_{j+1}^{mp}(x_{j+\frac{1}{2}})) \\
&= \bar{u}_j, \\
u\tilde{b}_{j+1} &= max(\tfrac{1}{2}\bar{u}_j + \tfrac{1}{2}\bar{u}_{j+1}, R_j(x_{j+\frac{1}{2}})) \\
&= max(\bar{u}_j, R_j(x_{j+\frac{1}{2}})) \\
&= \bar{u}_j.
\end{aligned}$$

Because $R_{j+1}(x_{j+\frac{1}{2}}) \leq u\tilde{b}_{j+1} = \bar{u}_j$ and $R_{j+1}(x_{j+\frac{1}{2}}) \geq \bar{u}_j$ ($P_{j+1}^{mp}(x_{j+\frac{1}{2}}) \geq \bar{u}_j$), we have $R_{j+1}(x_{j+\frac{1}{2}}) = \bar{u}_j$; Because $R_j(x_{j+\frac{1}{2}}) \geq l\tilde{b}_j = \bar{u}_j$ and $R_j(x_{j+\frac{1}{2}}) \leq \bar{u}_j$ ($P_j^{mp}(x_{j+\frac{1}{2}}) \leq \bar{u}_j$), we have $R_j(x_{j+\frac{1}{2}}) = \bar{u}_j$. Hence $R_j(x_{j+\frac{1}{2}})$ and $R_{j+1}(x_{j+\frac{1}{2}})$ are both equal to $\bar{u}_j$, (2.6) is satisfied.

If $l\tilde{b}_j > \min_{x \in I_j} P_j^{mp}(x)$, then $0 < l\tilde{b}_j - \min_{x \in I_j} P_j^{mp}(x) = l\tilde{b}_j - P_j^{mp}(x_{j+\frac{1}{2}}) = \bar{u}_j - P_j^{mp}(x_{j+\frac{1}{2}}) \leq P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - P_j^{mp}(x_{j+\frac{1}{2}}) = O(h^3)$; if $u\tilde{b}_{j+1} < \max_{x \in I_{j+1}} P_{j+1}^{mp}(x)$, then $0 < \max_{x \in I_{j+1}} P_{j+1}^{mp}(x) - u\tilde{b}_{j+1} = P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - \bar{u}_j \leq P_{j+1}^{mp}(x_{j+\frac{1}{2}}) - P_j^{mp}(x_{j+\frac{1}{2}}) = O(h^3)$. Hence the $l\tilde{b}_j$ and $u\tilde{b}_{j+1}$ satisfy (2.2a,b).

We can prove (2.6) and (2.2a,b) if there is type (iii) or (iv) **false extrema** of $P^{mp}(x)$ at an interface $x_{j+\frac{1}{2}}$.

Now we can say that after performing this **advanced Subprocedure**, at any interface $x_{j+\frac{1}{2}}$, we have (2.6)

$$\begin{array}{llll} \text{if} & \bar{u}_j > \bar{u}_{j+1}, & \text{then} & R_j(x_{j+\frac{1}{2}}) \geq R_{j+1}(x_{j+\frac{1}{2}}), \\ \text{if} & \bar{u}_j < \bar{u}_{j+1}, & \text{then} & R_j(x_{j+\frac{1}{2}}) \leq R_{j+1}(x_{j+\frac{1}{2}}), \\ \text{if} & \bar{u}_j = \bar{u}_{j+1}, & \text{then} & (R_j(x_{j+\frac{1}{2}}) - \bar{u}_j) \cdot (R_{j+1}(x_{j+\frac{1}{2}}) - \bar{u}_j) > 0 \\ & & & \text{or } R_j(x_{j+\frac{1}{2}}) = R_{j+1}(x_{j+\frac{1}{2}}) = \bar{u}_j. \end{array}$$

There are no **false extrema** of $R(x)$. Therefore $N(R(x)) = N(\bar{u})$. The nonoscillatory property is achieved.

In all the cases, (2.2a,b) are satisfied or $R_j(x) = P_j^{mp}(x)$. Thus for all of $R_j(x)$,

$$R_j(x) - u(x, t_{n+1}) = O(h^3).$$

in regions in which $u(x, t_{n+1})$ is smooth.

Thus **Theorem 2** has been proven. #

3 Simple Implementation of the Schemes

In this section we follow the idea and the analysis in [4] to obtain the explicit form of the schemes. The form (1.5) of the scheme is called the abstract form in which we need to evaluate the exact solution in the small for the IVP(1.1) with the initial data $R(x)$, and then take the cell-average, which results in conservation form:

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda\left(\frac{1}{\tau}\int_0^\tau f(v(x_{j+\frac{1}{2}}, \eta))\, d\eta - \frac{1}{\tau}\int_0^\tau f(v(x_{j-\frac{1}{2}}, \eta))\, d\eta\right), \tag{3.1}$$

where $\lambda = \tau/h$ and $v(x,t)$ is the exact solution of the following equation

$$\begin{array}{l} v_t + f(v)_x = 0 \\ v(x, 0) = R(x). \end{array} \tag{3.2}$$

To evaluate the integral in (3.1)

$$\frac{1}{\tau}\int_0^\tau f(v(x_{j+\frac{1}{2}}, \eta))\, d\eta, \tag{3.3}$$

we try to derive a simple but adequate approximation following [2]. Note that the integrand is a smooth function of t.

The first step is to discretize the integral in (3.3) by using a numerical quadrature such as Gauss or other quadratures with 3rd order of accuracy. We could use 2 point Gauss or 3 point Simpson's quadrature, i.e. in general

$$\frac{1}{\tau}\int_0^\tau f(v(x_{j+\frac{1}{2}},\eta))\,d\eta = \sum_{k=0}^{K} \alpha_k f(v(x_{j+\frac{1}{2}},\beta_k\tau)). \tag{3.4}$$

The second step is to approximate $v(x,t)$ by its Taylor expansion which is obtained by the following local Cauchy-Kowalewski procedure i.e.

$$\begin{aligned} v_t &= -f'v_x \\ v_{xt} &= -[f''v_x^2 + f'v_{xx}] \\ v_{tt} &= -[f''v_tv_x + f'v_{xt}]. \end{aligned} \tag{3.5}$$

Thus in (3.4) for $v(x_{j+\frac{1}{2}},\beta_k\tau)$, if we approximate it from the left cell of the interface $x_{j+\frac{1}{2}}$, we obtain

$$\begin{aligned} v(x_{j+\frac{1}{2}},\beta_k\tau) \approx & \;\; \tilde{v}_j(x_{j+\frac{1}{2}},\beta_k\tau) \\ = & \;\; v(x_{j+\frac{1}{2}},0) + v_t(x_{j+\frac{1}{2}},0)(\beta_k\tau) + \tfrac{1}{2}v_{tt}(x_{j+\frac{1}{2}},0)(\beta_k\tau)^2, \end{aligned} \tag{3.6}$$

where $v(x_{j+\frac{1}{2}},0) = R_j(x_{j+\frac{1}{2}})$, $v_t(x_{j+\frac{1}{2}},0)$ and $v_{tt}(x_{j+\frac{1}{2}},0)$ are obtained through the local Cauchy-Kowalewski procedure (3.5) from $v(x,0) = R_j(x)$, and

$$\tilde{v}_j(x_{j+\frac{1}{2}},\beta_k\tau) = v(x_{j+\frac{1}{2}},\beta_k\tau) + O(h^3).$$

If we approximate it from the right cell of the $x_{j+\frac{1}{2}}$, we obtain

$$\begin{aligned} v(x_{j+\frac{1}{2}},\beta_k\tau) \approx & \;\; \tilde{v}_{j+1}(x_{j+\frac{1}{2}},\beta_k\tau) \\ = & \;\; v(x_{j+\frac{1}{2}},0) + v_t(x_{j+\frac{1}{2}},0)(\beta_k\tau) + \tfrac{1}{2}v_{tt}(x_{j+\frac{1}{2}},0)(\beta_k\tau)^2, \end{aligned} \tag{3.7}$$

where $v(x_{j+\frac{1}{2}},0) = R_{j+1}(x_{j+\frac{1}{2}})$, $v_t(x_{j+\frac{1}{2}},0)$ and $v_{tt}(x_{j+\frac{1}{2}},0)$ are obtained through the local Cauchy-Kowalewski procedure (3.5) from $v(x,0) = R_{j+1}(x)$, and

$$\tilde{v}_{j+1}(x_{j+\frac{1}{2}},\beta_k\tau) = v(x_{j+\frac{1}{2}},\beta_k\tau) + O(h^3).$$

Thus

$$f(v(x_{j+\frac{1}{2}},\beta_k\tau)) = f(\tilde{v}_j(x_{j+\frac{1}{2}},\beta_k\tau)) \quad \text{or} \quad f(\tilde{v}_{j+1}(x_{j+\frac{1}{2}},\beta_k\tau)) \tag{3.8}$$

The last step in the derivation of the numerical flux is to approximate $f(v(x_{j+\frac{1}{2}}, \beta_k \tau))$ by

$$f(v(x_{j+\frac{1}{2}}, \beta_k \tau)) = \tilde{h}(\tilde{v}_j(x_{j+\frac{1}{2}}, \beta_k \tau), \tilde{v}_{j+1}(x_{j+\frac{1}{2}}, \beta_k \tau)), \tag{3.9}$$

where $\tilde{h}(\cdot,\cdot)$ is any two-point Lipschitz continuous monotone flux which is nondecreasing for the first argument and nonincreasing for the second argument. Some possible choices are

(i) Engquist-Osher

$$h^{EO}(a,b) = \int_0^b min(f'(s),0)\,ds + \int_0^a min(f'(s),0)\,ds + f(0); \tag{3.10}$$

(ii) Godunov

$$h^{G}(a,b) = \begin{cases} min_{a \le u \le b} f(u) & \text{if } a \le b, \\ max_{a \ge u \ge b} f(u) & \text{if } a > b; \end{cases} \tag{3.11}$$

(iii) Lax-Friedrichs

$$h^{LF}(a,b) = \tfrac{1}{2}[f(a) + f(b) - \alpha(b-a)] \qquad \alpha = max \mid f'(u) \mid, \tag{3.12}$$

where the maximum is taken over the whole region in which a, b varies, e.g. in $[\inf_x(u_0(x)), \sup_x(u_0(x))]$, where $u_0(x)$ is the initial function;

(iv) Local-Lax-Friedrichs

$$h^{LLF}(a,b) = \tfrac{1}{2}[f(a) + f(b) - \beta(b-a)] \qquad \beta = \max_{min(a,b) \le u \le max(a,b)} \mid f'(u) \mid . \tag{3.13}$$

For convex f, $f'' \ge 0$, one has $\beta = max(\mid f'(a) \mid, \mid f'(b) \mid)$,

(v) Roe with entropy fix

$$h^{RF}(a,b) = \begin{cases} f(a) & \text{if } f'(u) \ge 0 \text{ for } u \in [min(a,b), max(a,b)] \\ f(b) & \text{if } f'(u) \le 0 \text{ for } u \in [min(a,b), max(a,b)] \\ h^{LLF}(a,b) & \text{otherwise.} \end{cases} \tag{3.14}$$

Hence now we obtain the explicit form of the scheme, which is

$$\bar{u}_j^{n+1} = \bar{u}_j^n - \lambda(\bar{f}_{j+\frac{1}{2}} - \bar{f}_{j-\frac{1}{2}}), \tag{3.15}$$

where

$$\bar{f}_{j+\frac{1}{2}} = \sum_{k=0}^{K} \alpha_k \tilde{h}(\tilde{v}_j(x_{j+\frac{1}{2}}, \beta_k \tau), \tilde{v}_{j+1}(x_{j+\frac{1}{2}}, \beta_k \tau)). \tag{3.16}$$

The flux $\bar{f}_{j+\frac{1}{2}}$ is an adequate approximation to the flux $\frac{1}{\tau}\int_0^{\tau} f(v(x_{j+\frac{1}{2}}, \eta))\, d\eta$; for details see [2].

Here we mention that, for computational reasons, in the reconstruction procedure, we impose a requirement $m_j \le \bar{u}_j^{n+1} \le M_j$ in each cell (This, of course, was implied for the abstract form of the scheme). That is, if $\bar{u}_j^{n+1} > M_j$, we set $\bar{u}_j^{n+1} = M_j$; if $\bar{u}_j^{n+1} < m_j$, we set $\bar{u}_j^{n+1} = m_j$. Because the true solution should satisfy the requirement, we will obtain a better approximation for the cell-average $\bar{u}_j^{n+1}$

We note that for linear equation, the explicit form is equivalent to the abstract scheme if we use any of the Engquist-Osher flux (3.10), the Godunov flux (3.11), or the Roe flux with entropy fix (3.14). All of them are just simple upwind differencing. Thus, in the linear case, the schemes satisfy the maximum principle and are nonoscillatory, and thus are TVB schemes. There is a subsequence of the numerical solutions converging to the weak solution of (1.1) as the step size approaches zero. Of course, the solution to the linear equation is unique, and hence the schemes are convergent, in this simple case.

In the next section we will test the schemes on linear and nonlinear equations with convex and nonconvex fluxes. The numerical experiments show that the schemes are 3rd order accurate for smooth solutions, satisfy the maximum principle and the nonoscillatory property, and converge to the entropy solutions.

4 Numerical Experiments

In this section we use some model problems to numerically test our schemes. We used the Roe flux with entropy fix as numerical flux and 2 point Gauss quadrature in all of our examples.

Example 1. We solve the model equation

$$\begin{array}{ll} u_t + u_x = 0 & -1 \le x \le 1 \\ u(x,0) = u_0(x) & u_0(x) \text{ periodic with period 2.} \end{array} \tag{4.1}$$

Four initial data $u_0(x)$ are used. The first one is $u_0(x) = sin(\pi x)$ and we list the errors at time t=10 in Table 1. The second one is $u_0(x) = sin^4(\pi x)$ and we list the errors at time t=10 in Table 2.

TABLE 1 (τ/h=0.9, t=10)

l	L_1 error	L_1 order	L_∞ error	L_∞ order
20	5.7694407036673D-03		4.5009125264985D-03	
40	6.9744677300436D-04	3.05	5.4686841904228D-04	3.04
80	8.5439845931126D-05	3.03	6.7079922749236D-05	3.03

TABLE 2 (τ/h=0.6, t=10)

l	L_1 error	L_1 order	L_∞ error	L_∞ order
80	1.4690161632608D-02		2.1347851475350D-02	
160	2.0083513922684D-03	2.87	2.8168317475599D-03	2.92
320	2.6268572179140D-04	2.93	3.5474728536100D-04	2.99

Here and below l is the total number of cells and the step size $h = 2/l$ in all examples.

For the first two initial data, we obtain 3rd order of accuracy in the smooth region in both L_1 and L_∞ norms. We note that standard ENO schemes applied to the example with the second initial data experienced an (easily fixed) loss of accuracy, see [17], [18]. No such degeneracy was found with our present method.

The third initial function is

$$u_0(x) = \begin{cases} 1 & -\frac{1}{5} \le x \le \frac{1}{5} \\ 0 & \text{otherwise} \end{cases}$$

and the fourth is

$$u_0(x) = \begin{cases} (1 - (\frac{10}{3}x)^2)^{\frac{1}{2}} & -\frac{3}{10} \le x \le \frac{3}{10} \\ 0 & \text{otherwise.} \end{cases}$$

We see the good resolution of the solutions in Figures 4-5.

Figure 4 ($\tau/h = 0.9$)

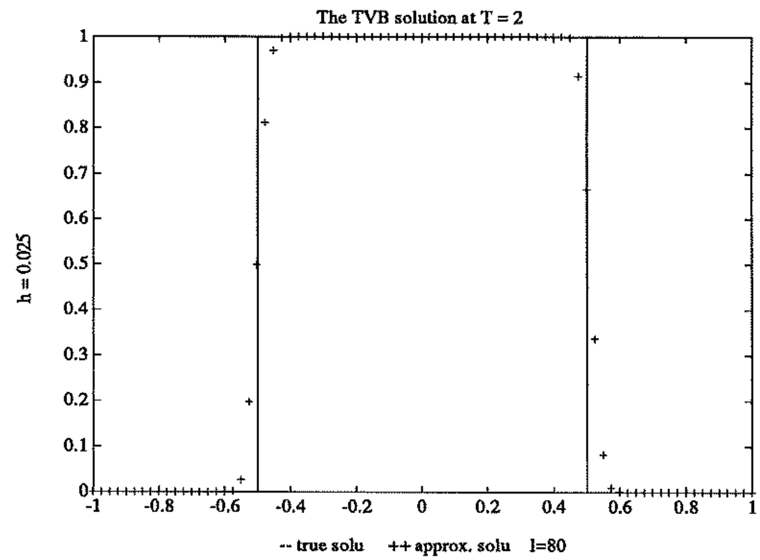
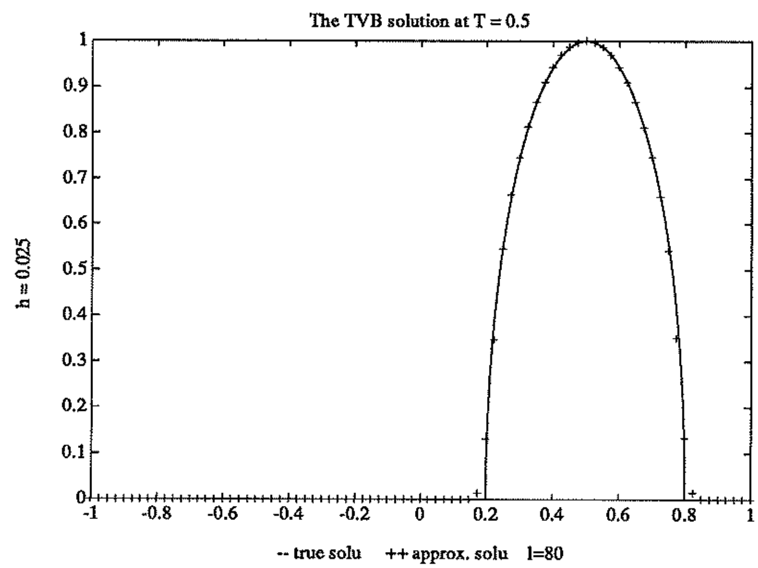


Figure 5 ($\tau/h = 0.9$)



Example 2. We solve Burgers' equation with a periodic boundary condition

$$\begin{array}{ll} u_t + (\frac{1}{2}u^2)_x = 0 & -1 \le x \le 1 \\ u(x,0) = u_0(x) & u_0(x)\text{periodic with period 2.} \end{array} \tag{4.2}$$

For the initial data $u_0(x) = 1 + \frac{1}{2}sin(\pi x)$, the exact solution is smooth up to $t = \frac{2}{\pi}$, then it develops a moving shock which interacts with a rarefaction wave.

At $t = 0.3$ the solution is still smooth. We list the errors in Table 3. Note we have close to 3rd order of accuracy in L_1 and more than 2nd order of accuracy in L_∞.

TABLE 3 (τ/h=0.6, t=0.3)

l	L_1 error	L_1 order	L_∞ error	L_∞ order
160	4.6548730388854D-06		1.3542618619078D-05	
320	7.7529803426540D-07	2.59	2.8349390184257D-06	2.26
640	1.2524533052373D-07	2.63	5.8732024554864D-07	2.27
1280	1.8075141183793D-08	2.79	1.2430677109876D-07	2.24
2560	2.5408037626108D-09	2.83	2.6569131561516D-08	2.23

At $t = \frac{2}{\pi}$ the shock just begins to form, at $t = 1.1$ the interaction between the shock and the rarefaction waves is over, and the solution becomes monotone between shocks. In Figures 6-7 we can see the excellent behaviors of the schemes in both cases. The errors 0.1 away from the shock (i.e. $| x - \text{shock location} | \geq 0.1$) are listed in Table 4 at $t = 1.1$. These errors are of even much smaller magnitude than the ones in the smooth case of Table 3 and show 3rd order of accuracy both in L_1 and L_∞ in the smooth regions 0.1 away from the shock. This shows that error propagation of the scheme is very local.

TABLE 4 (τ/h=0.66, t=1.1)

l	L_1 error	L_1 order	L_∞ error	L_∞ order
160	1.4188176779673D-06		3.8623190183884D-06	
320	1.7619541036046D-07	3.01	4.5400412962415D-07	3.09
640	2.1941677813206D-08	3.01	5.6512083457250D-08	3.01
1280	2.7394705290687D-09	3.00	7.0478263225482D-09	3.00
2560	3.4273836711438D-10	3.00	8.8725682356738D-10	2.99

Figure 6 ($\tau/h = 0.66$)

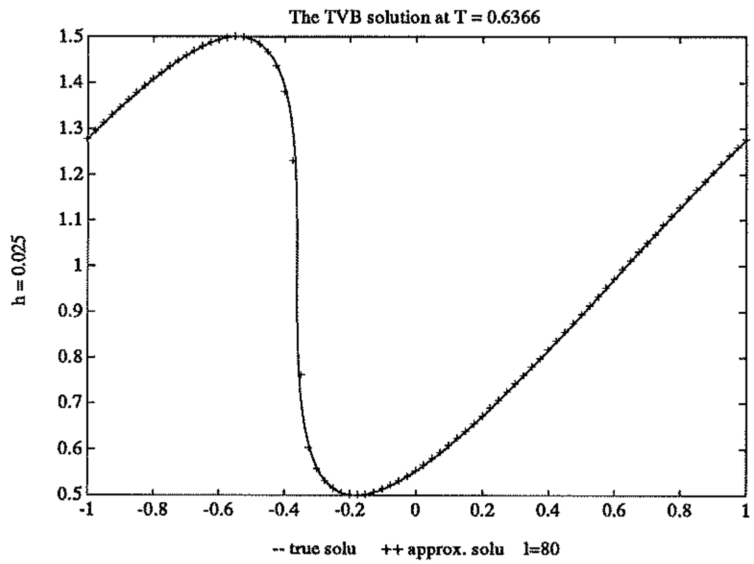
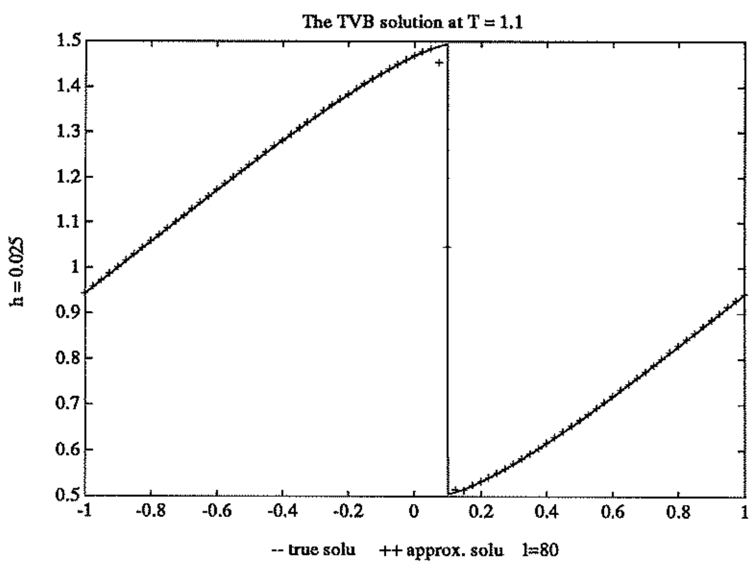


Figure 7 ($\tau/h = 0.66$)

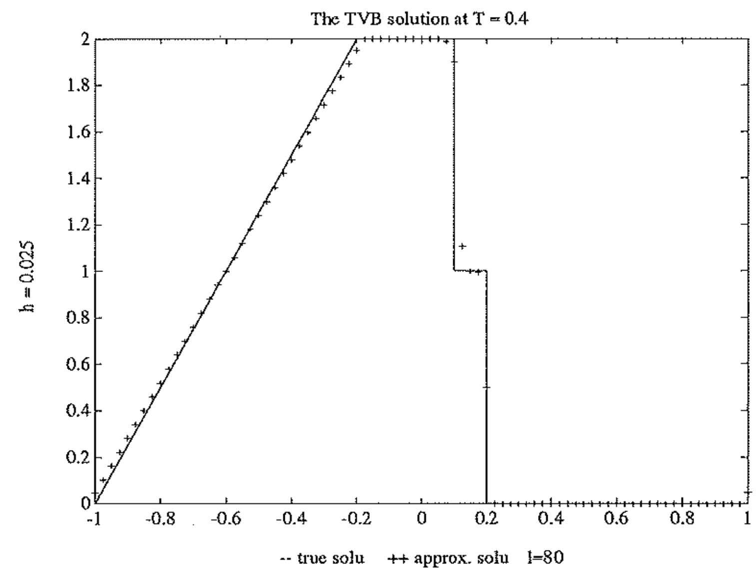


For the initial data

$$u_0(x) = \begin{cases} 2 & -1 \leq x < -0.5 \\ 1 & -0.5 \leq x < 0 \\ 0 & 0 \leq x \leq 1, \end{cases}$$

we see one rarefaction wave, and two shocks waves which interact with each other. Good resolution is observed in Figure 8.

Figure 8 ($\tau/h = 0.5$)



Example 3. we use two nonconvex fluxes to test the convergence to the physically correct solutions. The true solutions are obtained from the Lax-Friedrichs scheme on a very fine grid.

The first one is a Riemann problem with the flux

$$f(u) = \tfrac{1}{4}(u^2 - 1)(u^2 - 4),$$

and the initial data

$$u_0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0. \end{cases}$$

The two cases we test are (i) $u_l = 2$, $u_r = -2$, Figure 9; (ii) $u_l = -3$, $u_r = 3$, Figure 10. For more details concluding this problem see [2]

Figure 9 ($\tau/h = 0.33$)

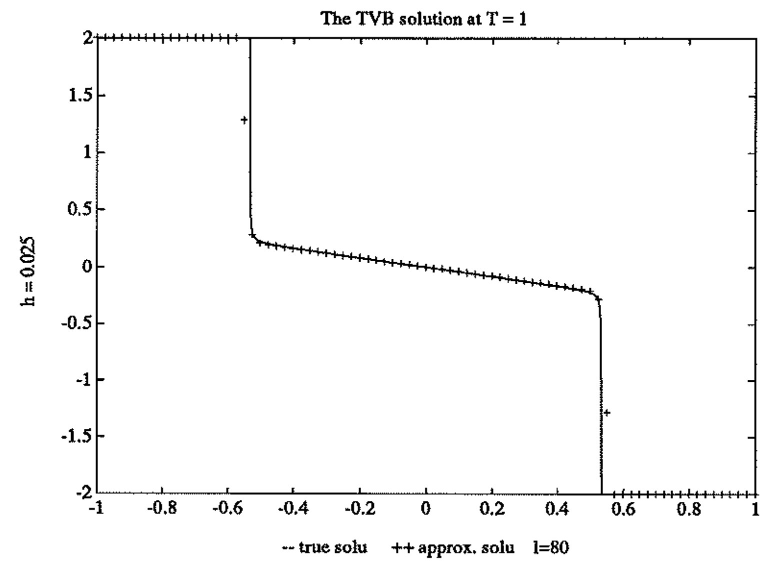
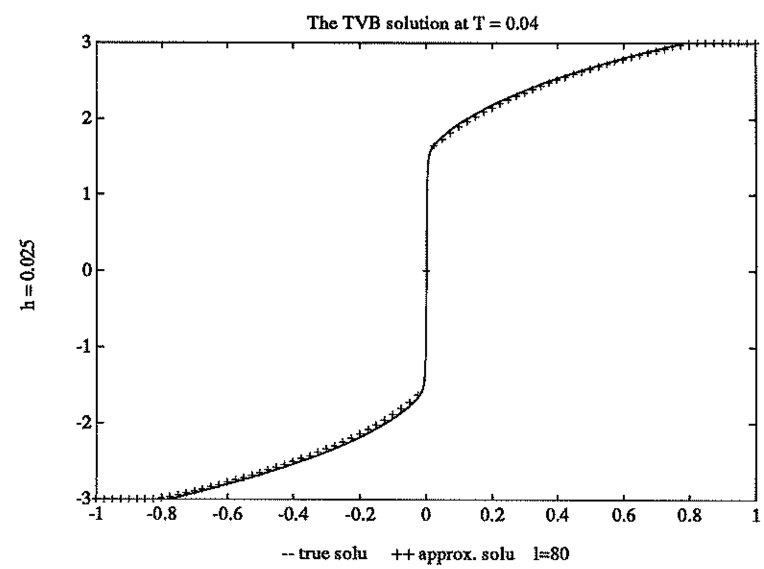


Figure 10 ($\tau/h = 0.05$)



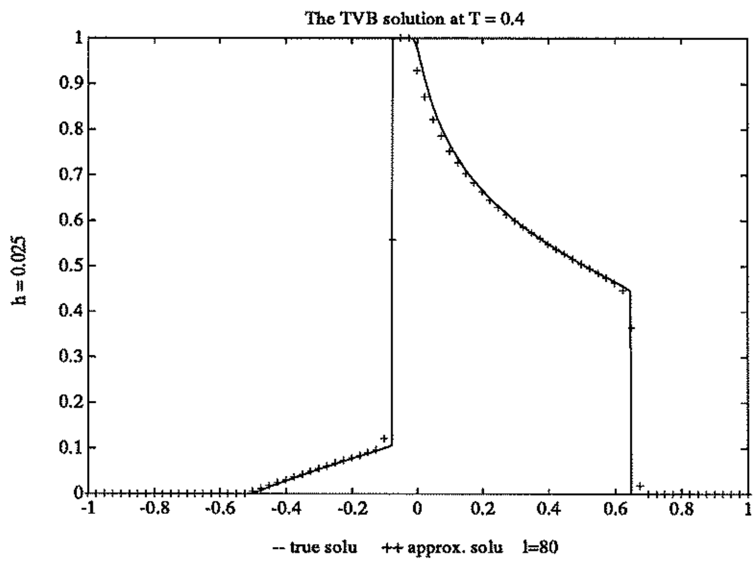
The second flux is the Buckley-Leverett flux used to model oil recovery [2],

$$f(u) = 4u^2/(4u^2 + (1-u)^2),$$

with initial data $u = 1$ in $[-\frac{1}{2}, 0]$ and $u = 0$ elsewhere.

The result is displayed in Figure 11.

Figure 11 ($\tau/h = 0.3$)



In this example, we observe convergence with excellent resolution to the entropy solutions in both cases.

In all the examples that we have illustrated above, we observe that the schemes are of 3rd order of accuracy with the somewhat less than usual degeneracy to 2nd order at certain isolated extrema, local maximum principle satisfying and nonoscillatory properties as we proved in §2, and convergent, with excellent resolution, to the entropy solutions.

5 Summary and Prospects

We now review the general idea of this paper. For the maximum principle, the local bounds $m = \{m_j\}$ and $M = \{M_j\}$ are chosen from data obtained at previous time steps. The true solution of equation $u_t + f(u)_x = 0$ is bounded by these numbers locally.

However for equations with source terms of the form $u_t + f(u)_x = \phi(u,t)$, the solution $u(x, t_{n+1})$ at $t = t_{n+1}$ should also satisfy

$$m_j^{new} \leq u(x, t_{n+1}) \leq M_j^{new} \qquad x \in I_j, \tag{5.1}$$

where we set $m_j^{new} = v_m(\tau)$ and $v_m(t)$ is the solution of the ODE $u_t = \phi(u,t)$

with initial data $v = m_j$, and $M_j^{new} = v_M(\tau)$ and $v_M(t)$ is the solution of the ODE $u_t = \phi(u,t)$ with initial data $v = M_j$ with m_j and M_j defined as before. The condition (5.1) is reasonable to be used in solving equation $u_t + f(u)_x = \phi(u,t)$, because of well known comparison theorems.

For the general system case, the condition (5.1) might not be adequate and we need to know a bit more about the structure of the solution.

However the nonoscillatory property (1.8) naturally generalizes in all cases. Hence we can achieve a generalized nonoscillatory property (1.8) in the multidimensional and system cases up to 3rd order of accuracy. This will be done in future work.

For higher order of accuracy, we will consider the reconstructed polynomial with a corresponding higher degree. We have proved that the modifying operator (Theorem 1) works for higher degree polynomials and shall show this proof in future work. Our next main project is to use this operator to modify higher degree reconstructed polynomials and we may well combine it with the nonoscillatory ideas of Tong in [11] to achieve the TVB property with arbitrarily high order of accuracy.

6 Appendix

Lemma 7

$$\max_{x\in I_j} \mid (p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \le 3 \quad (6.1a)$$
$$\min_{x\in I_j} \mid (p_j(x) - \bar{u}_j)/(u_{min} - \bar{u}_j) \mid \le 3 \quad (6.1b)$$

in Theorem 1.

Proof

W.L.O.G. we only prove (6.1a).

Case 1: $p_j(x)$ is a linear function, then $\mid (p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \le 1$ and (6.1a) is satisfied.

Now if $p_j(x)$ is not a linear function, we denote $p_j(x) = a_2(x - x^*)^2 + a_0$, $x_{j-\frac{1}{2}} = x_j - \frac{1}{2}h$, $x_{j+\frac{1}{2}} = x_j + \frac{1}{2}h$, $x_{j+\frac{1}{2}} - x^* = \theta h$, $x_{j-\frac{1}{2}} - x^* = (\theta - 1)h$, and x^* is the extrema point of $p_j(x)$ in the interval $(-\infty, +\infty)$. We have

$$(p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) = (a_2(x - x^*)^2 + a_0 - \bar{u}_j)/(u_{max} - \bar{u}_j).$$

Case 2: $x^* \notin I_j$. Thus $\theta > 1$ or $\theta < 0$. $p_j(x)$ is monotone in I_j. So $u_{max} = p_j(x_{j+\frac{1}{2}})$ or $u_{max} = p_j(x_{j-\frac{1}{2}})$. W.L.O.G. we assume $u_{max} = p_j(x_{j+\frac{1}{2}})$, then $\max\limits_{x\in I_j} \mid (p_j(x)-\bar{u}_j)/(u_{max}-\bar{u}_j) \mid = \max(1, \mid (p_j(x_{j-\frac{1}{2}})-\bar{u}_j)/(u_{max}-\bar{u}_j) \mid)$. Because $\frac{1}{3}a_2(x-x^*)^3|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + a_0 h = \bar{u}_j h$, we have

$$\begin{aligned}
(p_j(x_{j-\frac{1}{2}}) - \bar{u}_j)/(u_{max} - \bar{u}_j) = &\ (a_2(x_{j-\frac{1}{2}} - x^*)^2 + a_0 - \bar{u}_j)/(u_{max} - \bar{u}_j) \\
= &\ (a_2(x_{j-\frac{1}{2}} - x^*)^2 - \tfrac{1}{3}a_2(x-x^*)^3|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}})/ \\
&\ (a_2(x_{j+\frac{1}{2}} - x^*)^2 - \tfrac{1}{3}a_2(x-x^*)^3|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}) \\
= &\ (3(\theta-1)^2 - \theta^3 + (\theta-1)^3)/(3\theta^2 - \theta^3 + (\theta-1)^3) \\
= &\ (-3\theta+2)/(3\theta-1),
\end{aligned}$$

and

$$\max_{\theta\notin[0,1]} \mid (-3\theta+2)/(3\theta-1) \mid = 2,$$

and

$$\mid (p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \leq 2.$$

We obtain

$$\max_{x\in I_j} \mid (p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \leq 2 \quad \text{if} \quad u_{max} = p_j(x_{j+\frac{1}{2}}).$$

By the same argument we also have

$$\max_{x\in I_j} \mid (p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \leq 2 \quad \text{if} \quad u_{max} = p_j(x_{j-\frac{1}{2}}).$$

Case 3: $x^* \in I_j$, $p_j(x^*) = \min\limits_{x\in I_j} p_j(x)$, and $u_{max} = max(p_j(x_{j+\frac{1}{2}}), p_j(x_{j-\frac{1}{2}}))$. W.L.O.G. we assume $p_j(x_{j+\frac{1}{2}}) = u_{max}$, hence $x_{j+\frac{1}{2}} - x^* = \theta h$ and $x_{j-\frac{1}{2}} - x^* = (\theta-1)h$ where $\frac{1}{2} \leq \theta \leq 1$. Because

$$\mid (p_j(x_{j-\frac{1}{2}}) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \leq \mid (p_j(x_{j+\frac{1}{2}}) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid = 1,$$

so $\max\limits_{x\in I_j} \mid (p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid = \max(1, \mid (p_j(x^*) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid)$.

$$\begin{aligned}
(p_j(x^*) - \bar{u}_j)/(u_{max} - \bar{u}_j) = &\ (a_0 - \bar{u}_j)/(a_2(x_{j+\frac{1}{2}} - x^*)^2 + a_0 - \bar{u}_j) \\
= &\ (\tfrac{-1}{3}a_2(x-x^*)^3|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}})/ \\
&\ (a_2(x_{j+\frac{1}{2}} - x^*)^2 h + \tfrac{-1}{3}a_2(x-x^*)^3|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}) \\
= &\ (3\theta^2 - 3\theta + 1)/(-3\theta + 1),
\end{aligned}$$

Hence, for $\frac{1}{2} \leq \theta \leq 1$

$$\mid (p_j(x^*) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \leq \max_{\frac{1}{2} \leq \theta \leq 1} \mid (3\theta^2 - 3\theta + 1)/(-3\theta + 1) \mid = \tfrac{1}{2}.$$

Hence

$$\max_{x \in I_j} \mid (p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \leq 1$$

Case 4: $x^* \in I_j$ and $p_j(x^*) = \max_{x \in I_j} p_j(x)$. $\max_{x \in I_j} \mid (p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid = \max(\mid (p_j(x_{j+\frac{1}{2}}) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid, \mid (p_j(x_{j-\frac{1}{2}}) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid)$. W.L.O.G. we assume $\mid (p_j(x_{j+\frac{1}{2}}) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \geq \mid (p_j(x_{j-\frac{1}{2}}) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid$. Hence $x_{j+\frac{1}{2}} - x^* = \theta h$ and $x_{j-\frac{1}{2}} - x^* = (\theta - 1)h$, where $\frac{1}{2} \leq \theta \leq 1$. Hence

$$\begin{aligned} (p_j(x_{j+\frac{1}{2}}) - \bar{u}_j)/(u_{max} - \bar{u}_j) = & \ (a_2(x_{j+\frac{1}{2}} - x^*)^2 + \tfrac{-1}{3h} a_2 (x - x^*)^3 \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}})/ \\ & \ (\tfrac{-1}{3h} a_2 (x - x^*)^3 \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}}) \\ = & \ (3\theta - 1)/(-3\theta^2 + 3\theta - 1). \end{aligned}$$

Thus $\mid (p_j(x_{j+\frac{1}{2}}) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid = \mid (3\theta - 1)/(-3\theta^2 + 3\theta - 1) \mid \leq 3$ for $\frac{1}{2} \leq \theta \leq 1$. Hence $\max_{x \in I_j} \mid (p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \leq 3$.

In all four cases we obtain (6.1a) i.e.

$$\max_{x \in I_j} \mid (p_j(x) - \bar{u}_j)/(u_{max} - \bar{u}_j) \mid \leq 3.$$

Similarly we obtain (6.1b) i.e.

$$\max_{x \in I_j} \mid (p_j(x) - \bar{u}_j)/(u_{min} - \bar{u}_j) \mid \leq 3.$$

The lemma is proven. #

References

[1] A. Harten, "On a Class of High Resolution Total-Variation-Stable Finite-Difference Schemes," *SIAM J. Numer. Anal.*, Vol.21, No.1, 1984, pp.1-23.

[2] A. Harten, B. Engquist, S. Osher and S. Chakravarthy, "Uniformly High Order Accurate Essentially Non-Oscillatory Schemes III," *J. Comp. Phys.*, Vol. 71, 1987, pp. 231-303; also *ICASE Report* No. 86-22, April 1986.

[3] A. Harten and Stanley Osher, "Uniformly High-Order Accurate Nonoscillatory Schemes I," *SIAM J. Numer. Anal.*, Vol. 24, No. 2, 1987, pp. 279-309.

[4] Stanley Osher, "Convergence of Generalized MUSCL Schemes," *SIAM J. Numer. Anal.*, Vol. 22, No. 5, 1985, pp. 947-961.

[5] Stanley Osher and E. Tadmor, " On the Convergence of Difference Approximations to Scalar Conservation Laws," *Math. Comp.*, Vol. 50, No. 181, 1988, pp.19-51.

[6] Chi-Wang Shu and Stanley Osher, "Efficient Implementation of Essentially Non-oscillatory Shock-Capturing Schemes," *J. Comp. Phys.*, Vol. 77, 1988, pp.439-471.

[7] Chi-Wang Shu and Stanley Osher, "Efficient Implementation of Essentially Non-oscillatory Shock-Capturing Schemes, II" *J. Comp. Phys.*, Vol. 83, 1989, pp.32-78.

[8] B. Engquist and Stanley Osher, "Stable and Entropy Satisfying Approximations for Transonic Flow Calculations," *Math. Comp.*, Vol. 34, No. 149, 1980, pp.45-75.

[9] A. Harten, S. Osher, B. Engquist, and S. Chakravarthy, "Some Results on Uniformly High Order Accurate Essentially Nonoscillatory Schemes," *Appl. Numer. Math.*, Vol. 2, 1986, pp. 347-377.

[10] Richard Sanders, "A Third-Order Accurate Variation Nonexpansive Difference Schemes for Single Nonlinear Conservation Laws," *Math. Comp.*, Vol. 51, No. 184, 1988, pp.535-558.

[11] Boning Tong, "Uniformly High Order Accurate Strictly Non-Oscillatory Numerical Schemes for Hyperbolic Conservation Law," PhD thesis, 1992.

[12] Chi-Wang Shu, "TVB Uniformly High-Order Schemes for Conservation Laws," *Math. Comp.*, Vol. 49, No. 179, 1987 pp.105-121.

[13] Chi-Wang Shu, "TVB Boundary Treatment for Numerical Solutions of Conservation Laws," *Math. Comp.*, Vol. 49, No. 179, 1987 pp.123-134.

[14] Bernardo Cockburn and Chi-Wang Shu, "TVB Runge-Kutta Local Projection Discontinuous Galerkin Finite Element Method for Conservation Laws II: General Framework," *Math. Comp.*, Vol. 52, No. 186, 1989, pp.411-435.

[15] Bernardo Cockburn, San-Yih Lin and Chi-Wang Shu, "TVB Runge-Kutta Local Projection Discontinuous Galerkin Finite Element Method for Conservation Laws III: One-Dimensional Systems," *J. Comp. Phys.*, Vol. 84, 1989, pp.90-113.

[16] S.N.Kruzkov, "First Order Quasilinear Equations in Several Independent Variables", *Math. USSR Sbornik*, Vol. 10, No. 2, 1970, pp.217-243.

[17] A. Rogerson and E. Meiburg, "A Numerical Study of the Convergence Properties of ENO Schemes," *J. Scientific Computing*, Vol.5, No.2, 1990, pp.151-167.

[18] Chi-Wang Shu, "Numerical Experiments on the Accuracy of ENO and Modified ENO Schemes," *J. Scientific Computing*, Vol.5, No.2, 1990, pp.127-149.