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ABSTRACT. For the low Mach number laminar channel flows, we prove that the fast component (sound wave) and slow component of the solution do not interact up to an $O(1)$ time interval even under the presence of boundary layers to both parts of the solution. We then consider solving for the slow component of the solution with the Leap-Frog/Crank-Nicholson method. An initialization process is incorporated into the algorithm to suppress the sound wave in the initial data.

1. INTRODUCTION

We consider low Mach number ($\epsilon \ll 1$) channel flows described by the simplified Navier-Stokes equation

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \mathbf{F}, \\ \epsilon^2 \{p_t + (\mathbf{u} \cdot \nabla) p\} + \nabla \cdot \mathbf{u} = g, \end{cases} \tag{1}$$

in the domain $0 \le x, y \le 1$, with no-slip boundary condition in one space dimension:

$$\mathbf{u}(x,y,t) = 0, \qquad x = 0, 1, \tag{2}$$

and compatible initial conditions. All data and solutions are assumed to be 1-periodic in y and C^∞-smooth.

The most important feature of this set of equations is that the solution involves sharply different time scales. For laminar flows there are three time scales involving in the solutions. They are acoustic time scale, convective time scale, and viscous time scale. The acoustic time scale and convective time scale, also called fast time scale and slow time scale, are the scales of our interests. In ocean acoustic, attention is given to the propagation of sound wave, the fast component. While in oceanography and metrology, the slow component are usually what people quest for, as the slow component carry most of the energy, and the fast component appears as the perturbations to the slow one.

From the physical point of view, it is believed that the two different scales in the solution are separable at least for a while, in other words, the interaction between

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I am indebted to Professor Kreiss for his helps in the formation of this paper

the two scales is negligible for a certain period of time. This belief has been somewhat justified by Kreiss et al in [6], where for the same equation (1) with periodicity in all space dimensions they showed that the slow and fast components of solution do not interact for an $O(1)$ time interval(the convective time scale) through an asymptotic expansion. The bounded derivative principle was also shown valid to suppress the sound wave.

In this paper we want to carry the research of Kreiss et al for one step further. We consider equations (1) with no-slip boundary conditions in one space dimension. For this problem, one of the most interesting issue remains to be the pattern of the interaction between the fast and slow components. Contrary to the belief that the interaction could happen during the acoustic time scale in the boundary layers and propagate into the interior, we prove in this paper that, similar to the periodic case, the fast and slow waves will not interact during the convective time scale even under the presence of boundary layers in both boundaries $x = 0, 1$. This result suggests that numerically we can resolve the two scales separately, as otherwise the time step may be severely restricted.

The numerical solution of the low Mach number Navier-Stokes equation is another focus of this paper. However, we will limit ourselves in the numerical solutions of the slow components only, and leave the solution of the fast component to a subsequent paper. According to the bounded derivative principle, fast wave will be of order $O(\epsilon^{2p})$ if up to p^{th} time derivatives of the initial data are bounded independently of ϵ[6]. Thus, instead of performing an asymptotic expansion, we choose to discretize the complete equation (1), and solve for the slow solution with such prepared initial data. Because of the embedded sound wave equation in (1), which is highly unsymmetric, we adapt the Leap-Frog/Crank-Nicholson (LF/CN) method for the discretization. The advantage of the LF/CN method is that it is unconditionally stable for the sound wave equation. Thus, no matter how small is the Mach number, the fast component will not be amplified once it is suppressed in the initial conditions. In addition, the LF/CN method being used, which is 2^{nd} order accurate in time, achieves the highest accuracy in time among unconditionally stable multistep methods. Numerical results of the LF/CN methods are also demonstrated.

We show the existence of the unique solution in §2.1. Then in §2.2 sound wave is added to the slow solution via perturbations, and an initialization process is introduced to sort out the slow solution. §2.3 is devoted to the discussions sound wave. In §2.4 we estimate the remainders which contains the interaction between the slow and fast waves.

In §3.1 the numerical discretization of the primitive equation (1) with the LF/CN is detailed. In §3.2 we demonstrate the stability property of the LF/CN method. The numerical implementation of initialization of initial data is presented in §3.3. And finally in §3.4 we present some numerical results.

Equation (1) with inflow boundary condition at $x = 0$ and outflow boundary condition at $x = 1$ will be topic for the near future.

2. Interactions between slow and fast components

2.1. Existence of a slow solution. From the physical point of view it is apparent that a solution without sound wave can be obtained through the evolution starting from the stagnated state. This idea has been used by Kreiss et al to obtain the slow manifold for ODE system with two different time scales[9]. In this section, we will show the existence of a slow solution of the low Mach number Navier-Stokes equation with such idea, for which we need

Assumption 1. *The source functions F and g can be smoothly extended to $-1 \leq t \leq 0$.*

With this week assumption, we can actually construct the slow solution by an asymptotic expansion. Let $\phi(t) \in C^\infty$ be a monotone cut-off function with the property

$$\phi(t) = \begin{cases} 0, & \text{for } t \leq -3/4 \ , \\ 1, & \text{for } t \geq -1/4, \end{cases}$$

we consider the problem

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \phi \mathbf{F}, \\ \epsilon^2 \{p_t + (\mathbf{u} \cdot \nabla) p\} + \nabla \cdot \mathbf{u} = \phi g, \end{cases} \tag{3}$$

starting from $t = -1$ with stagnated initial conditions and no-slip boundary condition

$$\begin{aligned} &\mathbf{u}(x, y, -1) = \mathbf{0}, \quad p(x, y, -1) = 0, \\ &\mathbf{u}(h, y, t) = 0, \quad h = 0, 1. \end{aligned} \tag{4}$$

The first order approximation $\mathbf{U}, P$ to $\mathbf{u}, p$ satisfies the following incompressible equation

$$\begin{cases} \mathbf{U}_t + (\mathbf{U} \cdot \nabla)\mathbf{U} + \nabla P = \nu \Delta \mathbf{U} + \phi \mathbf{F}, \\ \nabla \cdot \mathbf{U} = \phi g, \end{cases} \qquad t \geq -1, \tag{5}$$

and initial and boundary conditions

$$\begin{aligned} &\mathbf{U}(x, y, -1) = \mathbf{0}, \\ &\mathbf{U}(h, y, t) = \mathbf{0}, \quad h = 0, 1. \end{aligned} \tag{6}$$

The pressure is determined as a function of $\mathbf{U}$:

$$\begin{aligned} &\Delta P = (\phi g)_t + \nu \Delta (\phi g) + \nabla \cdot (\phi \mathbf{F} - (\mathbf{U} \cdot \nabla)\mathbf{U}), \\ &(1, P(x, y, t)) = 0. \end{aligned} \tag{7}$$

Note that for the solvability of (5,6) we must assume

$$(1, g(\cdot, t)) = \int_0^1 \int_0^1 g(x, y, t) dx dy = 0, \quad t \geq -1. \tag{8}$$

The solution of (5,6) exists for $-1 \leq t \leq \infty$, and it is unique up to a time-dependent function $\bar{P}(t)$. We fix it by requiring

$$\begin{aligned} &(1, P_t + \bar{P}_t + (\mathbf{U} \cdot \nabla) P) = 0, \qquad t \geq -1, \\ &\bar{P}(0) = 0. \end{aligned}$$

If we introduce new variables $\mathbf{u}_1, p_1$, such that

$$\mathbf{u}_1 = \mathbf{u} - \mathbf{U}, \quad p_1 = p - (P + \bar{P}(t)),$$

then we get, for $\mathbf{u}_1, p_1$,

$$\begin{cases} \mathbf{u}_{1t} + (\mathbf{U}\cdot\nabla)\mathbf{u}_1 + (\mathbf{u}_1\cdot\nabla)\mathbf{U} + (\mathbf{u}_1\cdot\nabla)\mathbf{u}_1 + \nabla p_1 = \nu\Delta\mathbf{u}_1, \\ \epsilon^2\{p_{1t} + (\mathbf{U}\cdot\nabla)p_1 + (\mathbf{u}_1\cdot\nabla)P + (\mathbf{u}_1\cdot\nabla)p_1\} + \nabla\cdot\mathbf{u}_1 = \epsilon^2 g_1, \\ \mathbf{u}_1 = 0, \quad p_1 = 0, \qquad t = -1 \\ \mathbf{u}_1(h,y,t) = 0, \quad h = 0,1 \end{cases} \tag{9}$$

with

$$g_1 = -\{P_t + \bar{P}_t + (\mathbf{U}\cdot\nabla)P\}.$$

Problem (3) is now reduced to (9), and the source terms are reduced to $O(\epsilon^2)$. We write the solution of (9) as

$$\mathbf{u}_1 = \epsilon^2\mathbf{U}_1 + \mathbf{u}_2, \quad p_1 = \epsilon^2(P_1 + \bar{P}_1(t)) + p_2,$$

where $\bar{P}_1$ is not yet specified, and $\mathbf{U}_1, P_1$ are the solutions of the linearized incompressible problem

$$\begin{cases} \mathbf{U}_{1t} + (\mathbf{U}\cdot\nabla)\mathbf{U}_1 + (\mathbf{U}_1\cdot\nabla)\mathbf{U} + \nabla P_1 = \nu\Delta\mathbf{U}_1, \\ \nabla\cdot\mathbf{U}_1 = g_1, \qquad (1, P_1(\cdot,t)) = 0, \\ \mathbf{U}_1(h,y,t) = 0, \qquad h = 0,1, \end{cases} \tag{10}$$

with initial condition

$$\mathbf{U}_1 = 0, \quad t = -1.$$

Denote

$$\mathbf{U}^{(1)} = \mathbf{U} + \epsilon^2\mathbf{U}_1, \quad P^{(1)} = P + \epsilon^2 P_1,$$

then the equations for $\mathbf{u}_2, p_2$ are

$$\begin{cases} \mathbf{u}_{2t} + (\mathbf{U}^{(1)}\cdot\nabla)\mathbf{u}_2 + (\mathbf{u}_2\cdot\nabla)\mathbf{U}^{(1)} + (\mathbf{u}_2\cdot\nabla)\mathbf{u}_2 + \nabla p_2 = \nu\Delta\mathbf{u}_2 + \epsilon^4\mathbf{F}_2, \\ \epsilon^2\left\{p_{2t} + (\mathbf{U}^{(1)}\cdot\nabla)p_2 + (\mathbf{u}_2\cdot\nabla)P^{(1)} + (\mathbf{u}_2\cdot\nabla)p_2\right\} + \nabla\cdot\mathbf{u}_2 = \epsilon^4 g_2, \\ \mathbf{u}_2 = 0, \quad p_2 = 0, \qquad t = -1, \\ \mathbf{u}_2(h,y,t) = 0, \quad h = 0,1, \end{cases}$$

where

$$\begin{aligned} \mathbf{F}_2 &= -(\mathbf{U}_1\cdot\nabla)\mathbf{U}_1 \\ g_2 &= -\{P_{1t} + \bar{P}_{1t}(t) + (\mathbf{U}\cdot\nabla)P_1 + (\mathbf{U}_1\cdot\nabla)P\} - \epsilon^2(\mathbf{U}_1\cdot\nabla)P_1. \end{aligned}$$

We choose $\bar{P}_1(t)$ such that

$$\begin{aligned} &(1, g_2(\cdot,t)) = 0, \\ &\bar{P}(\cdot,0) = 0. \end{aligned}$$

The source terms have been reduced to $O(\epsilon^4)$. This process can be continued so as to generate the desired asymptotic expansion for (3). We thus can formulate

Theorem 1. *let $\boldsymbol{U}_j, P_j + \bar{P}_j(t), j = 1, \ldots, l$, be defined recursively as the solution of the linear incompressible problem*

$$
(11)\quad \begin{cases} \boldsymbol{U}_{jt} + (\boldsymbol{U}^{(j-1)} \cdot \nabla) \boldsymbol{U}_j + (\boldsymbol{U}_j \cdot \nabla) \boldsymbol{U}^{(j-1)} + \nabla P_j = \nu \Delta \boldsymbol{U}_j + \epsilon^{2j-4} \boldsymbol{F}_j, \\ \nabla \cdot \boldsymbol{U}_j = g_j, \quad (1, P_j(\cdot, t)) = 0, \\ \boldsymbol{U}_j(h, y, t) = 0, \quad h = 0, 1, \end{cases}
$$

with initial condition

$$
\boldsymbol{U}_j = \boldsymbol{0}, \quad t = -1.
$$

Here we define

$$
\begin{aligned}
&\boldsymbol{U}^{(j-1)} = \boldsymbol{U} + \sum_{i=1}^{j-1} \epsilon^{2i} \boldsymbol{U}_i, \quad P^{(j-1)} = P + \bar{P} + \sum_{i=1}^{j-1} \epsilon^{2i}(P_i + \bar{P}_i), \\
&\boldsymbol{F}_1 = 0, \qquad \boldsymbol{F}_j = -(\boldsymbol{U}_{j-1} \cdot \nabla) \boldsymbol{U}_{j-1}, \qquad 2 \le j \le l, \\
&g_j = -\{P_{j-1,t} + \bar{P}_{j-1,t} + (\boldsymbol{U}^{(j-1)} \cdot \nabla) P_{j-1} + (\boldsymbol{U}_{j-1} \cdot \nabla) P^{(j-1)}\} \\
&\qquad - \epsilon^{2j-2}(\boldsymbol{U}_{j-1} \cdot \nabla) P_{j-1},
\end{aligned}
$$

and $\bar{P}_{j-1}(t)$ is chosen such that

$$
\begin{aligned}
&(1, g_j(\cdot, t)) = 0, \qquad -1 \le t \le T, \\
&\bar{P}_{j-1}(0) = 0.
\end{aligned}
$$

All derivatives of the functions $\boldsymbol{U}_j, P_j + \bar{P}_j(t)$ are bounded independently of ϵ. The error terms u_{l+1}, p_{l+1} are defined by

$$
\begin{aligned}
u &= \boldsymbol{U}^{(l)} + u_{l+1}, \\
p &= P^{(l)} + p_{l+1}.
\end{aligned}
$$

They satisfy

$$
(12)\quad \begin{cases} u_{l+1,t} + (\boldsymbol{U}^{(l)} \cdot \nabla) u_{l+1} + (u_{l+1} \cdot \nabla) \boldsymbol{U}^{(l)} + (u_{l+1} \cdot \nabla) u_{l+1} \\ \qquad + \nabla p_{l+1} = \nu \Delta u_{l+1} + \epsilon^{4l} \boldsymbol{F}_{l+1}, \\ \epsilon^2 \{p_{l+1,t} + (\boldsymbol{U}^{(l)} \cdot \nabla) p_{l+1} + (u_{l+1} \cdot \nabla) P^{(l)} + (u_{l+1} \cdot \nabla) p_{l+1}\} \\ \qquad + \nabla \cdot u_{l+1} = \epsilon^{2l+2} g_{l+1}, \\ u_{l+1} = \boldsymbol{0}, \quad p_{l+1} = 0, \qquad t = -1, \\ u_{l+1}(h, y, t) = 0, \quad h = 0, 1. \end{cases}
$$

It is apparently that for sufficiently large l, the solution of (12) is negligible. Hence, we obtain the solution of (3) via an asymptotic expansion, and the solution contains only the slow time scale.

For most application there is $\epsilon^2 \ll \nu \ll 1$. According to the boundary layer theory the tangential component of the solution $\mathbf{U}$ of (5) has boundary layers at $x = 0, 1$. More specifically, without dispute we can make an

Assumption 2. *There exists $T > 0$ such that when $0 \le t \le T$ all first order partial derivatives of the solution of (5) are bounded independently of ν except V_x, which behaves like*

$$|V_x| = \begin{cases} O(\frac{1}{\sqrt{\nu}}), & 0 \le x \le C\nu, \\ O(1), & C\nu \le x \le 1, \end{cases} \tag{13}$$

where C is a constant independent of ν.

From both mathematical and physical point of view, we believe that the solutions of the linearized equations (11) are also boundary layer type solutions. We choose not to go through the complicated justification but only to characterize the solution in the following

Assumption 3. *under Assumption 2 all first order partial derivatives of $U^{(j)}$ and $P^{(j)}$ are bounded independently of ν for $0 \le t \le T$ except $V_x^{(j)}$, which behaves like*

$$|V_x^{(j)}| = \begin{cases} O(1), & C_j\nu \le x \le 1 - C_j\nu \\ O(\frac{1}{\sqrt{\nu}}), & others \end{cases} \quad j = 1, 2, \ldots, l, \tag{14}$$

where $C_j, j = 1, 2, \ldots, l$ are constants independent of ν.

These two assumptions will be used later on for the discussions of the interaction between the slow and fast waves.

2.2. Sound wave added as perturbation. In last section we have shown that the sound wave will never be generated if the solution evolves from the stagnated state. However, it is more interesting to investigate the case when the sound wave does present in the solution of the low Mach number flows, as we want to understand the interaction between the sound wave and the slow solution. For this purpose, we simulate a physical process so as to add sound wave to the slow solution at a specific moment, say, $t = 0$, during the evolution of the slow solution, which amounts to considering problem (1) with initial conditions

$$\begin{aligned} \mathbf{u}(x, y, 0) &= \mathbf{U}^{(l)}(x, y, 0) + \mathbf{u}^0(x, y), \\ p(x, y, 0) &= P^{(l)}(x, y, 0) + p^0(x, y), \end{aligned}$$

where $\mathbf{U}^{(l)}(x, y, 0), P^{(l)}(x, y, 0)$ are the solutions of (3,4) at $t = 0$, and $\mathbf{u}^0, p^0$ are any functions satisfying

$$\begin{aligned} &\nabla \times \mathbf{u}^0 = 0, \qquad (1, u^0) = (1, v^0) = 0, \\ &\mathbf{u}^0(h, y) = 0, \quad h = 0, 1, \\ &(1, p^0) = 0. \end{aligned}$$

Note that, characteristically, the velocity of the sound wave is vorticity free.

For problem (1) with the above initial condition at $t = 0$, the asymptotic expansion for the slow solution is obtained through a slightly modified asymptotic expansion which removes all vorticity from the initial condition. For the first order approximation, we consider (5) with initial data solved from

$$
\begin{aligned}
&\nabla \cdot \mathbf{U}(\cdot,0) = g(\cdot,0), \quad \nabla \times \mathbf{U}(\cdot,0) = \nabla \times \mathbf{U}^{(l)}(\cdot,0), \\
&(1, U(\cdot,0)) = \left(1, U^{(l)}(\cdot,0)\right), \quad (1, V(\cdot,0)) = \left(1, V^{(l)}(\cdot,0)\right).
\end{aligned} \tag{15}
$$

The first order approximation of the pressure is determined from (7), subjected to

$$
\begin{aligned}
&\left(1, p(\cdot,0) - P(\cdot,0) - \bar{P}(\cdot,0)\right) = 0, \\
&\left(1, P_t + \bar{P}_t + (\mathbf{U}\cdot\nabla)P\right) = 0.
\end{aligned} \tag{16}
$$

It is not hard to see that $\mathbf{U}, P$ is precisely $\mathbf{U}^{(0)}, P^{(0)}$. For higher order terms of the aymptotic expansion, we solve for $\mathbf{U}_l, P_l$ from (11) with initial conditions obtained from

$$
\begin{aligned}
&\nabla \cdot \mathbf{U}(\cdot,0) = g_j(\cdot,0), \quad \nabla \times \mathbf{U}(\cdot,0) = 0, \quad \mathbf{U}(h,y,0) = 0, \quad h = 0,1, \\
&(1, P_j(\cdot,t)) = 0, \\
&(1, g_j(\cdot,t)) = 0, \\
&\bar{P}_j(0) = 0.
\end{aligned} \tag{17}
$$

Through comparison we conclude that $\mathbf{U}_l, P_l$ is exactly what we have got from the evolution from the stagnated state. Thus, the slow solution remain unchanged after the perturbation by the sound wave. Introduce

$$
\mathbf{u}_{l+1} = \mathbf{u} - \mathbf{U}^{(l)}, \quad p_{l+1} = p - P^{(l)},
$$

then the equations for $\mathbf{u}_{l+1}, p_{l+1}$ are

$$
\begin{cases}
\mathbf{u}_{l+1,t} + (\mathbf{U}^{(l)}\cdot\nabla)\mathbf{u}_{l+1} + (\mathbf{u}_{l+1}\cdot\nabla)\mathbf{U}^{(l)} + (\mathbf{u}_{l+1}\cdot\nabla)\mathbf{u}_{l+1} \\
\qquad + \nabla p_{l+1} = \nu\Delta\mathbf{u}_{l+1} + \epsilon^{4l}\mathbf{F}_{l+1}, \\
\epsilon^2\left\{p_{l+1,t} + (\mathbf{U}^{(l)}\cdot\nabla)p_{l+1} + (\mathbf{u}_{l+1}\cdot\nabla)P^{(l)} + (\mathbf{u}_{l+1}\cdot\nabla)p_{l+1}\right\} \\
\qquad + \nabla\cdot\mathbf{u}_{l+1} = \epsilon^{2l+2}g_{l+1}, \\
\mathbf{u}_{l+1} = \mathbf{u}^0, p_{l+1} = p^0, \quad t = 0, \\
\mathbf{u}_{l+1}(h,y,t) = 0, \quad h = 0,1.
\end{cases} \tag{18}
$$

For relatively large l, the forcing terms in above equation can be ignored. Note that the resulted initial conditions are vorticity free. We will see that the underlying equation of (18) is a typical sound wave equation. The behavior sound wave and the interaction between the sound wave and the slow solution will be discussed in next section.

2.3. Sound wave as fast solution. We will show that equation (18) is primarily a sound wave equation. We make a change of time variable

$$
\begin{aligned}
\tau = t/\epsilon, \quad &\hat{\mathbf{u}}(x,y,\tau) = \mathbf{u}_{l+1}(x,y,\epsilon\tau), \\
&\hat{q}(x,y,\tau) = \epsilon p_{l+1}(x,y,\epsilon\tau),
\end{aligned}
$$

and denote for convenience

$$\hat{\mathbf{U}} := \mathbf{U}^{(l)}, \qquad \hat{P} := P^{(l)}.$$

Then (18) is read

$$\begin{cases} \hat{\mathbf{u}}_\tau + \epsilon\{(\hat{\mathbf{U}}\cdot\nabla)\hat{\mathbf{u}} + (\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{U}} + (\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}}\} + \nabla\hat{q} = \epsilon\nu\Delta\hat{\mathbf{u}}, \\ \hat{q}_\tau + \epsilon\{(\hat{\mathbf{U}}\cdot\nabla)\hat{q} + \epsilon(\hat{\mathbf{u}}\cdot\nabla)\hat{P} + (\hat{\mathbf{u}}\cdot\nabla)\hat{q}\} + \nabla\cdot\hat{\mathbf{u}} = 0, \\ \hat{\mathbf{u}} = \mathbf{u}_{l+1,0}, \hat{q} = q_{l+1,0}, \quad \text{at } \tau = 0, \\ \hat{\mathbf{u}}(h, y, \tau) = 0, \quad h = 0, 1. \end{cases} \tag{19}$$

Under the assumption that no confusion is caused we hereafter drop sign ^.

Now we we will try to figure out the leading order terms of (19). We assume at the moment that the solutions away from the boundaries are smooth, hence the convection terms are of order $O(\epsilon)$, and can be dropped in the interior. Near the boundaries $x = 0, 1$, we have the following list of magnitude for each term, where $\delta = \sqrt{\epsilon\nu}$,

$$\begin{array}{llllllllll} u_\tau & +\epsilon\{Uu_x & +Vu_y & +uU_x & +vU_y & +uu_x & +vu_y\} & +q_x & =\epsilon\nu(u_{xx} & +u_{yy}), \\ \delta & +\epsilon\{\delta\cdot 1 & +1\cdot\delta & +\delta\cdot 1 & +1\cdot\delta & +\delta\cdot 1 & +1\cdot\delta\} & +\delta & =\epsilon\nu(\frac{1}{\delta} & +1) \\ v_\tau & +\epsilon\{Uv_x & +Vv_y & +uV_x & +vV_y & +uv_x & +vv_y\} & +q_y & =\epsilon\nu(v_{xx} & +v_{yy}), \\ 1 & +\epsilon\{\delta\cdot\frac{1}{\delta} & +1\cdot 1 & +\delta\frac{1}{\sqrt{\nu}} & +1\cdot 1 & +\delta\cdot\frac{1}{\delta} & +1\cdot 1\} & +1 & =\epsilon\nu(\frac{1}{\delta^2} & +1), \end{array},$$

$$\begin{array}{llllllllll} q_\tau & +\epsilon\{Uq_x & +Vq_y & +\epsilon(uP_x & +vP_y) & +uq_x & +vq_y\} & +u_x & +v_y & =0, \\ 1 & +\epsilon\{1\cdot\delta & +\delta\cdot 1 & +\epsilon(\delta\cdot 1 & +1\cdot 1) & +\delta\cdot\delta & +1\cdot 1\} & +1 & +1 & =0. \end{array}$$

It can be seen that all convection terms are of higher order throughout the domain, thus we ignore them and arrive at a mixed parabolic-hyperbolic equations

$$\begin{cases} \bar{\mathbf{u}}_\tau + \nabla\bar{q} = \epsilon\nu\bar{\mathbf{u}}_{xx}, \\ \bar{q}_\tau + \nabla\cdot\bar{\mathbf{u}} = 0, \\ \bar{\mathbf{u}} = \mathbf{u}_{l+1,0}, \bar{q} = q_{l+1,0} \quad \tau = 0, \\ \bar{\mathbf{u}}(h, y, \tau) = 0, \qquad h = 0, 1. \end{cases} \tag{20}$$

The solutions of the above equation are trigonometric functions with boundary layers. We will characterize these features one by one. We decompose the solutions into two parts:

$$\bar{\mathbf{u}} = \mathbf{u}^i + \mathbf{u}^o,$$
$$\bar{q} = q^i + q^o,$$

where $\mathbf{u}^o, p^o$ satisfy the inviscous equation

$$\begin{cases} \mathbf{u}^o_\tau + \nabla q^o = 0, \\ q^o_\tau + \nabla\cdot\mathbf{u}^o = 0, \\ \mathbf{u}^o = \mathbf{u}_{l+1,0}, q^o = q_{l+1,0}, \quad \tau = 0, \\ u^o(h, y, \tau) = 0, \quad h = 0, 1, \end{cases} \tag{21}$$

while $\mathbf{u}^i, q^i$ satisfy

$$\left\{\begin{array}{l} \mathbf{u}^i_\tau + \nabla q^i = \epsilon\nu\mathbf{u}^i_{xx} + \epsilon\nu\mathbf{u}^o_{xx}, \\ q^i_\tau + \nabla\cdot\mathbf{u}^i = 0, \\ \mathbf{u}^i = 0, q^i = 0, \quad \tau = 0, \\ u^i(h,y,\tau) = 0, v^i(h,y,\tau) = -v^o(h,y,\tau), \quad h = 0,1. \end{array}\right. \tag{22}$$

Let us discuss (21) first. We define the vorticity and dilatation of $\mathbf{u}^o$ by

$$\xi^o = \nabla\times\mathbf{u}^o, \qquad s^o = \nabla\cdot\mathbf{u}^o, \tag{23}$$

then apparently

$$\xi^o = 0, \quad s^o_{\tau\tau} = \Delta s^o, \quad q^o_{\tau\tau} = \Delta q^o,$$

i,e, the divergence and pressure satisfy the wave equations. Denote the initial data of s^o, q^o by

$$\begin{array}{ll} s^o = s^o_0, \quad s^o_\tau = s^o_{\tau,0}, \quad (1, s^o_0) = (1, s^o_{\tau,0}) = 0, & \\ q^o = q^o_0, \quad q^o_\tau = q^o_{\tau,0}, \quad (1, q^o_0) = (1, q^o_{\tau,0}) = 0, & \end{array} \quad t = 0. \tag{24}$$

Then by Fourier expansions we obtain

$$\begin{array}{l} s^o(x,y,\tau) = \displaystyle\sum_{\mathbf{k},k_2\neq 0} \hat{s}^o(\mathbf{k},\tau)\cos(k_1\pi x)e^{i2\pi k_2 y}, \\ \xi^o(x,y,\tau) = \displaystyle\sum_{\mathbf{k},k_2\neq 0} \hat{\xi}^o(\mathbf{k},\tau)\sin(k_1\pi x)e^{i2\pi k_2 y}, \end{array} \qquad \mathbf{k} = (k_1,k_2)\in Z^2, \tag{25}$$

with

$$\begin{array}{l} \hat{s}^o(\mathbf{k},\tau) = \hat{s}^o_0(\mathbf{k})\cos 2\pi k\tau + \dfrac{1}{2\pi k}\hat{s}^o_{\tau,0}(\mathbf{k})\sin 2\pi k\tau, \\ \hat{q}^o(\mathbf{k},\tau) = \hat{q}^o_0(\mathbf{k})\cos 2\pi k\tau + \dfrac{1}{2\pi k}\hat{q}^o_{\tau,0}(\mathbf{k})\sin 2\pi k\tau, \end{array} \qquad k^2 = \frac{1}{4}k_1^2 + k_2^2.$$

From (24), we get $\mathbf{u}^o$ of the form

$$\begin{array}{l} u^o(x,y,\tau) = \displaystyle\sum_{\mathbf{k},k_2\neq 0} \hat{u}^o(k,\tau)\sin(k_1\pi x)e^{i2\pi k_2 y}, \\ v^o(x,y,\tau) = \displaystyle\sum_{\mathbf{k},k_2\neq 0} \hat{v}^o(k,\tau)\cos(k_1\pi x)e^{i2\pi k_2 y}, \end{array} \qquad \mathbf{k} = (k_1,k_2)\in Z^2. \tag{26}$$

Obviously we have

Lemma 1. *There are constant C_k independent of ϵ, ν such that*

$$\max_{0\le\tau\le\frac{T}{\epsilon}} \{||u^o(\cdot,\tau)||_{H^k} + ||q^o(\cdot,\tau)||_{H^k}\} \le C_k\{||u_{l+1,0}||_{H^k} + ||q_{l+1,0}||_{H^k}\}.$$

Given $\mathbf{u}^o, q^o$, we can solve for $\mathbf{u}^i, q^i$ from (22) by Laplace transform, which turns the equations into ode's[5]. In applications, very often the perturbation of sound wave is of order $O(\epsilon)$, we thus assume

$$\mathbf{u}_{l+1,0} = O(\epsilon), \quad q_{l+1,0} = O(\epsilon). \tag{27}$$

Then, from lemma 1 we have

$$\mathbf{u}^o(x,y,\tau) = O(\epsilon), \quad q^o(x,y,\tau) = O(\epsilon). \tag{28}$$

There is no difficulty to show

Lemma 2. *If the initial conditions of (20) satisfy (27), then $\mathbf{u}^i, q^i$ are the following boundary layer type functions.*

$$\begin{aligned}
u^i(x,y,\tau) &= O\left(\epsilon(e^{-x/\sqrt{\epsilon\nu}} + e^{(x-1)/\sqrt{\epsilon\nu}})\right), \\
u^i_x(x,y,\tau) &= O\left(\epsilon(e^{-x/\sqrt{\epsilon\nu}} + e^{(x-1)/\sqrt{\epsilon\nu}})\right), \\
v^i(x,y,\tau) &= O\left(\epsilon(e^{-x/\sqrt{\epsilon\nu}} + e^{(x-1)/\sqrt{\epsilon\nu}})\right), \\
v^i_x(x,y,\tau) &= O\left(\sqrt{\frac{\epsilon}{\nu}}(e^{-x/\sqrt{\epsilon\nu}} + e^{(x-1)/\sqrt{\epsilon\nu}})\right), \\
q^i(x,y,\tau) &= O\left(\epsilon(e^{-x/\sqrt{\epsilon\nu}} + e^{(x-1)/\sqrt{\epsilon\nu}})\right), \\
q^i_x(x,y,\tau) &= O\left(\epsilon(e^{-x/\sqrt{\epsilon\nu}} + e^{(x-1)/\sqrt{\epsilon\nu}})\right).
\end{aligned} \tag{29}$$

Therefore, with initial conditions (27) the leading terms of the sound wave satisfy

$$\bar{\mathbf{u}} = O(\epsilon), \quad \bar{q} = O(\epsilon). \tag{30}$$

This result will be used later for measuring the interaction between the slow wave and the sound wave. This is done by estimating the remainders, for which we need to highlight another property of the fast solutions.

Lemma 3. *Let $\phi = \phi(x,y,\tau)$ denote $\bar{u}, \bar{q}$ or any of their space derivatives, if the initial data satisfy $(1,\bar{u}_0) = (1,\bar{v}_0) = (1,\bar{q}_0) = 0$ then*

$$\int_0^T \phi(x,y,\frac{t}{\epsilon})dt = O(\epsilon^2),$$

i.e, the integral is bounded independent of x, y.

Proof: The proof is based on the fact that $\bar{\mathbf{u}}$ and $\bar{q}$ are trigonometric functions in τ. We consider the solutions of (21) first. Let ϕ^o be any of u^o, v^o, q^o and their derivatives. Apparently there is

$$\int_0^T \phi^o(x,y,\frac{t}{\epsilon})dt = \epsilon\int_0^{\frac{T}{\epsilon}} \phi^o(x,y,\tau)d\tau = O(\epsilon^2).$$

Next we will show that the similar property exists for the solutions of the boundary layer equations of (22). Note that because of (28) the forcing term $\epsilon\nu\mathbf{u}^o_{xx}$ is of order $O(\epsilon^2)$, we thus ignore it at the moment. Consider

$$
(31) \qquad \begin{cases} \mathbf{u}^i_\tau + \nabla q^i = \epsilon\nu\mathbf{u}^i_{xx}, \\ q^i_\tau + \nabla\cdot\mathbf{u}^i = 0, \\ \mathbf{u}^i = 0, q^i = 0, \quad \tau = 0, \\ u^i(h,y,\tau) = 0, v^i(h,y,\tau) = -v^o(h,y,\tau), \quad h = 0,1. \end{cases}
$$

Through Fourier transform in y, sine and cosine transform in τ, this set of equations is turned to a set of ODE's, and the solution exists and is unique. The solution of (31) is a trigonometric function in τ, its integration in τ are bounded independently of τ. The solution of the equation with an additional source term $\epsilon\nu\mathbf{u}^o_{xx}$ can be obtained with the Duhammel's principle, and its integration in time interval $[0, T/\epsilon]$ is of order ϵ. So if ϕ^i is any of the $\mathbf{u}^i, q^i$ and their spatial derivatives, then there is

$$
\int_0^{T/\epsilon} \phi^i(x,y,\tau)d\tau = O(\epsilon).
$$

Thus finally we conclude that

$$
\int_0^{T} \phi(x,y,\frac{t}{\epsilon})dt = O(\epsilon^2),
$$

which completes our proof $\square$

2.4. Estimate of the remainders. We decompose the solution of (18) as

$$
\mathbf{u}_{l+1} = \hat{\mathbf{U}} + \bar{\mathbf{u}} + \mathbf{u}^r, \quad p_{l+1} = \hat{P} + \bar{p} + p^r,
$$

where $\mathbf{u}^r, p^r$ is called the reminders, which represent the interaction between the slow wave and the fast wave. If the magnitude of $\mathbf{u}^r, p^r$ is big, then we know the interaction is strong, otherwise it is week. Under assumption (27), we can show that $\mathbf{u}^r, p^r$ are of higher order in ϵ than the fast and slow waves for $O(1)$ time interval, the length of convective time scale. This result will help us in designing numerical algorithms.

Denote

$$
\hat{Q} = \epsilon\hat{P}, \quad \bar{q} = \epsilon\bar{p}, \quad q^r = \epsilon p^r,
$$

and introduce

$$
\begin{aligned} \hat{\mathbf{U}}^{(1)} &= \hat{\mathbf{U}} + \bar{\mathbf{u}}, \\ \hat{Q}^{(1)} &= \hat{Q} + \bar{q}, \end{aligned}
$$

then the remainder terms $\mathbf{u}^r, q^r$ satisfy

$$(32)\qquad \begin{cases} \mathbf{u}_t^r + (\hat{\mathbf{U}} \cdot \nabla)\mathbf{u}^r + (\mathbf{u}^r \cdot \nabla)\hat{\mathbf{U}} + (\mathbf{u}^r \cdot \nabla)\mathbf{u}^r + \dfrac{1}{\epsilon}\nabla q^r \\ \qquad = \nu\Delta\mathbf{u}^r - \{(\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} + (\hat{\mathbf{U}} \cdot \nabla)\bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla)\hat{\mathbf{U}} - \nu\bar{\mathbf{u}}_{yy}\}, \\ q_t^r + (\hat{\mathbf{U}} \cdot \nabla)q^r + (\mathbf{u}^r \cdot \nabla)\hat{Q} + (\mathbf{u}^r \cdot \nabla)q^r + \dfrac{1}{\epsilon}\nabla \cdot \mathbf{u}^r \\ \qquad = -\{(\bar{\mathbf{u}} \cdot \nabla)\bar{q} + (\hat{\mathbf{U}} \cdot \nabla)\bar{q} + (\bar{\mathbf{u}} \cdot \nabla)\hat{Q}\}, \\ \mathbf{u}^r = 0, \quad q^r = 0, \quad \tau = 0, \\ \mathbf{u}^r(h, y, \tau) = 0, \quad h = 0, 1. \end{cases}$$

We now prove

Theorem 2. *Assume the initial data satisfy*

$$u - \hat{U} = u_{l+1} = O(\epsilon), \quad q - \hat{Q} = q_{l+1} = O(\epsilon), \qquad t = 0.$$

Then for $0 \le t \le T$ the solution of (32) satisfies

$$\|u^r\| + \|q^r\| = C_r\epsilon^2,$$

where C_r depends on ν only.

Proof: We will have to exploit the oscillatory behavior of the source function

$$\mathbf{F}_1 = -(\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} + (\hat{\mathbf{U}} \cdot \nabla)\bar{\mathbf{u}} + (\bar{\mathbf{u}} \cdot \nabla)\hat{\mathbf{U}} - \nu\bar{\mathbf{u}}_{yy}\},$$
$$g_1 = -\{(\bar{\mathbf{u}} \cdot \nabla)\bar{q} + (\hat{\mathbf{U}} \cdot \nabla)\bar{q} + (\bar{\mathbf{u}} \cdot \nabla)\hat{Q}\}.$$

According to (30) we can write

$$\mathbf{F}_1 = \epsilon\tilde{\mathbf{F}}_1(t/\epsilon, t) + O(\epsilon^2), \quad g_1 = \epsilon\tilde{g}_1(t/\epsilon, t) + O(\epsilon^2),$$

where $\tilde{\mathbf{F}}_1$ and $\tilde{g}_1$ are the sum of the products between the fast and slow functions. Now we split $\mathbf{u}^r, p^r$ into

$$\mathbf{u}^r = \mathbf{u}_1 + \mathbf{u}_2, \quad q^2 = q_1 + q_2,$$

where $\mathbf{u}_1, q_1$ satisfy

$$(33)\qquad \begin{cases} \mathbf{u}_{1t} + \nabla q_1 = \nu\Delta\mathbf{u}_1 + \mathbf{F}_1, \\ q_{1t} + \nabla \cdot \mathbf{u}_1 = g_1, \end{cases}$$

with initial conditions

$$\mathbf{u}_1 = 0, \quad q_1 = 0, \quad t = 0.$$

Let $S(t)$ be the solution operator for the homogeneous equations of (33), $\mathbf{w} = (\mathbf{u}_1, q_1)$, and $\mathbf{G} = (\mathbf{F}_1, g_1)$, then the solutions of (33) can be written as

$$\mathbf{w}(t) = \int_0^t S(\xi)\mathbf{G}(\xi)d\xi.$$

By integration by parts we have

$$\mathbf{w}(t) = S(t)\int_0^t \mathbf{G}(\xi)d\xi - \int_0^t S'(\xi)\left(\int_0^\xi \mathbf{G}(\zeta)d\zeta\right)d\xi.$$

By energy method we know that

$$\|S(t)\|_2 \leq C, \quad \|S_t(t)\|_2 \leq C.$$

Because

$$\mathbf{G}(t) = \epsilon \tilde{\mathbf{G}}(t/\epsilon) + O(\epsilon^2),$$

and, by Lemma 3, there is

$$\int_0^t \tilde{\mathbf{G}}(\frac{\xi}{\epsilon}) d\xi = O(\epsilon).$$

We thus obtain the bounds on $\mathbf{w}$:

$$\mathbf{w}(t) = O(\epsilon^2), \qquad 0 \leq t \leq T.$$

To get the bound on $\mathbf{u}_2, q_2$, we introduce

$$\begin{aligned} \hat{\mathbf{U}}^{(2)} &= \hat{\mathbf{U}}^{(1)} + \mathbf{u}_1, \\ \hat{Q}^{(2)} &= \hat{Q}^{(1)} + q_1. \end{aligned}$$

Then equations for $\mathbf{u}_2, q_2$ are

$$\begin{cases} \mathbf{u}_{2t} + (\hat{\mathbf{U}}^{(2)} \cdot \nabla)\mathbf{u}_2 + (\mathbf{u}_2 \cdot \nabla)\hat{\mathbf{U}}^{(2)} + (\mathbf{u}_2 \cdot \nabla)\mathbf{u}_2\} + \nabla q_2 = \nu \Delta \mathbf{u}_2 + \mathbf{F}_2, \\ q_{2t} + (\hat{\mathbf{U}}^{(2)} \cdot \nabla) q_2 + (\mathbf{u}_2 \cdot \nabla)\hat{Q}^{(2)} + (\mathbf{u}_2 \cdot \nabla) q_2\} + \nabla \cdot \mathbf{u}_2 = g_2, \\ \mathbf{u}_2 = 0, \quad q_2 = 0, \quad t = 0, \\ \mathbf{u}_2(h, y, t) = 0, \quad h = 0, 1, \end{cases}$$

where

$$\begin{aligned} \mathbf{F}_2 &= -\{(\mathbf{u}_1 \cdot \nabla)\mathbf{u}_1 + (\hat{\mathbf{U}}^{(1)} \cdot \nabla)\mathbf{u}_1 + (\mathbf{u}_1 \cdot \nabla)\hat{\mathbf{U}}^{(1)}\}, \\ g_2 &= -\{(\mathbf{u}_1 \cdot \nabla) q_1 + (\hat{\mathbf{U}}^{(1)} \cdot \nabla) q_1 + (\mathbf{u}_1 \cdot \nabla)\hat{Q}^{(1)}\}. \end{aligned}$$

From our estimates for $\mathbf{u}_1, q_1$ we have

$$\mathbf{F}_2 = O(\epsilon^2), \quad g_2 = O(\epsilon^2), \qquad 0 \leq t \leq T.$$

By typical energy method (see [5] for instance) we can derive

$$\|\mathbf{u}_2\|_2 + \|q_2\|_2 = O(\epsilon^2).$$

Finally we have

$$\begin{aligned} \mathbf{u}^r &= \mathbf{u}_1 + \mathbf{u}_2 = O(\epsilon^2), \\ q^r &= q_1 + q_2 = O(\epsilon^2), \end{aligned}$$

which completes the proof □

Now we can formulate our main result.

Theorem 3. *Assume the initial data satisfy*

$$\nabla \cdot u_0 = g(\cdot, 0) + O(\epsilon), \quad p_0 = P(\cdot, 0) + \bar{P}(0) + O(1).$$

The compressible problem (1,2) has a unique solution in $0 \leq t \leq T$. It can be written in the form

$$\begin{aligned} u &= U + \bar{u} + O(\epsilon^2), \\ p &= P + \bar{P}(t) + \bar{p} + O(\epsilon) \end{aligned}$$

At $x = 0, 1$, V and $\bar{v}$ have boundary layers of thickness $\sqrt{\nu}$ and $\sqrt{\epsilon\nu}$, respectively.

3. Numerical solutions

The presence of sound wave in the solutions of low Mach number Navier-Stokes equation poses a challenge to its numerical simulations. As the sound speed in most cases is much greater than the flow velocity, the time step of any conditionally stable numerical schemes will be severely restricted. Inevitably, the two different time scale must be separated from the equations and then resolved independently. In last section we have shown that the two scales can be separated by an asymptotic expansion, which lead to a series of incompressible equations for the slow scale and a Helmholtz equation for the fast scale. It seems that to solve for the full solution one needs to perform the asymptotic expansion and solve the Helmholtz equation. However, numerically it is disadvantageous to do so unless only one or two terms of the expansion are wanted. Bounded derivative principle suggests, instead, that we may obtain the full solutions by computing with the separated fast and slow waves in the initial conditions. In this section, we will pursuit this idea with finite difference methods. However, we limit ourselves in the slow solutions only. Note that in metrology and oceanography, slow scale is usually the only interest. The resolution of sound wave is a very different issue and is left to another paper.

For the sake of convenience we make a change to our problem. Instead of dealing with no-slip boundary condition in x direction, we assume periodicity in all space dimensions. Such change avoids the resolution of boundary layers, which is not a trivial issue and requires much more efforts.

3.1. The Leap-Frog/Crank-Nicholson method. Two considerations govern our selection of numerical schemes. First, as the our equations are highly unsymmetric, we want the stability restraint to be independent of the Mach number. Second, we need the accuracy of our scheme to be as high as possible. As we will justify later, the Leap-frog/Crank-Nicholson method introduced in this section appears to be the only option which meets both requirements.

Introduce spatial step $\Delta x = \Delta y = h = 1/L$, time step $\Delta t = k$, and denote the approximations by $\mathbf{u}_{i,j}^n \approx \mathbf{u}(ih, jh, nk), p_{i,j}^n \approx p(ih, jh, nk)$. Let z denote any of the variables t, x or y, we define the basic shift operator and central difference operator by

$$E_{\Delta z,z}^j w(z) = w(z + j\Delta z),$$
$$D_{\Delta z,z} = \frac{E_{\Delta z,z} - E_{\Delta z,z}^{-1}}{2\Delta z}.$$

The fourth order accurate numerical differentiation is defined by

$$D_z = \frac{1}{3}(4D_{\Delta z,z} - D_{2\Delta z,z}).$$

In turn we define the fourth order approximations to gradient and Laplacian

$$\nabla_s = \begin{pmatrix} D_x \\ D_y \end{pmatrix},$$
$$\Delta_s = D_{+x}D_{-x} + D_{+y}D_{-y} - \frac{h^2}{6}D_{+x}D_{-x}D_{+y}D_{-y}.$$

Our Leap-Frog/Crank-Nicholson method is formulated as follows.

(34)
$$\begin{cases} \dfrac{\mathbf{u}_{i,j}^{n+1}-\mathbf{u}_{i,j}^{n-1}}{2k}+(\mathbf{u}_{i,j}^{n}\cdot\nabla_s)\mathbf{u}_{i,j}^{n}+\nabla_s(\dfrac{p_{i,j}^{n+1}+p_{i,j}^{n-1}}{2})=\nu\Delta_s(\dfrac{\mathbf{u}_{i,j}^{n+1}+\mathbf{u}_{i,j}^{n-1}}{2})+\mathbf{F}_{i,j}^{n} \\ \epsilon^2\left\{\dfrac{p_{i,j}^{n+1}-p_{i,j}^{n-1}}{2k}+(\mathbf{u}_{i,j}^{n}\cdot\nabla_s)p_{i,j}^{n}\right\}+\nabla_s\cdot(\dfrac{\mathbf{u}_{i,j}^{n+1}+\mathbf{u}_{i,j}^{n-1}}{2})=g_{i,j}^{n}, \\ \quad i=0,1,\dots,L-1,\quad j=0,1,\dots,L-1,\quad n=1,2,\dots, \end{cases}$$

along with the initial conditions

$$\mathbf{u}_{i,j}^{n}=\mathbf{u}(ih,jh,nk),\quad p_{i,j}^{n}=p(ih,jh,nk),\quad n=0,1 \tag{35}$$

and periodic boundary conditions.

The leading truncation error of the approximations is

$$O\left(k^2\left(\left|\frac{d^2\mathbf{u}}{dt^2}\right|+\left|\frac{d^2p}{dt^2}\right|\right)+h^4\left(\left|\frac{D^4\mathbf{u}}{D\mathbf{x}^4}\right|+\left|\frac{D^4p}{D\mathbf{x}^4}\right|\right)\right),$$

thus the approximation is meaningful only if the second order time derivatives are bounded independently of Mach number ϵ. In other words, the approximation is valid only for solving the slow solution. Fast wave in the initial conditions, if there is any, must be suppressed before we proceed with our scheme. This initialization process will be discussed in subsequent section.

The existence of the slow solution requires the compatibility condition

$$\sum_{i=0}^{L}\sum_{j=0}^{L}g(ih,jh,t)h^2=0,$$

which is the discrete version of (8).

For deriving the equation for pressure, we rewrite the equations so that all known quantities are put into the right hand side. Also, to make the right hand side bear the same order of magnitude we multiply the momentum equation by k. Thus the equations (34) read

$$\begin{aligned} (I-\nu k\Delta_s)\mathbf{u}^{n+1}+k\nabla_s p^{n+1}&=(I+\nu k\Delta_s)\mathbf{u}^{n-1}-k\nabla_s p^{n-1} \\ &\quad+2k(\mathbf{F}^n-(\mathbf{u}^n\cdot\nabla_s)\mathbf{u}^n)\equiv\mathbf{N}^n, \\ \frac{\epsilon^2}{k}p^{n+1}+\nabla_s\cdot\mathbf{u}^{n+1}&=\frac{\epsilon^2}{k}-\nabla_s\cdot\mathbf{u}^{n-1}+2(g^n-\epsilon^2(\mathbf{u}\cdot\nabla_s)p^n)\equiv M^n. \end{aligned} \tag{36}$$

Under matrix expression the above equations read

$$\begin{pmatrix} I-\nu k\Delta_s & k\nabla_s \\ \nabla_s\cdot & \frac{\epsilon^2}{k} \end{pmatrix}\begin{pmatrix}\mathbf{u}^{n+1}\\ p^{n+1}\end{pmatrix}=\begin{pmatrix}\mathbf{N}^n\\ M^n\end{pmatrix} \tag{37}$$

The equation for p^{n+1} is obtained by eliminating $\mathbf{u}^{n+1}$, which amounts to performing Gaussian elimination to the system,

$$\left(-k\nabla_s\cdot(I-\nu k\Delta_s)^{-1}\nabla_s+\frac{\epsilon^2}{k}I\right)p^{n+1}=M^n-\nabla_s\cdot(I-\nu k\Delta_s)^{-1}\mathbf{N}^n. \tag{38}$$

Pressure is solved first and then used by the first equation of (36) to determine the velocity. It needs to be pointed out here that

$$k\nabla_s \cdot (I - \nu k\Delta_s)^{-1}\nabla_s$$

is ill conditioned and it is not a good approximation to its continuous counterpart

$$k\nabla \cdot (I - \nu k\Delta)^{-1}\nabla.$$

Thus, some technical enhancement should be considered for solving (38). We consider the following preconditioner

$$k(I - \nu k\hat{\Delta}_s)^{-1}\hat{\Delta}_s, \tag{39}$$

for system (38), where $\hat{\Delta}_s$ is the standard fourth order 9-point stensile for Laplacian,

$$\begin{aligned}\hat{\Delta}_s = &\left(I + \frac{h^2}{12}(D_{+x}D_{-x} + D_{+y}D_{-y})\right)^{-1}\\ &\cdot (D_{+x}D_{-x} + D_{+y}D_{-y} + \frac{h^2}{6}D_{+x}D_{-x}D_{+y}D_{-y}).\end{aligned} \tag{40}$$

It has been shown that this is indeed a good preconditioner.

3.2. Stability property of the equations with frozen coefficients. In this section we study the stability properties of equations (34). According to the stability theory we only need to consider the corresponding homogeneous equations with frozen coefficients:

(41)

$$\begin{aligned}&(I - \nu k\Delta_s)\mathbf{u}^{n+1} + k\nabla_s p^{n+1} = (I + \nu k\Delta_s)\mathbf{u}^{n-1} - k\nabla_s p^{n-1} - 2k(\mathbf{U}\cdot\nabla_s)\mathbf{u}^n,\\ &\frac{\epsilon^2}{k}p^{n+1} + \nabla_s \cdot \mathbf{u}^{n+1} = \frac{\epsilon^2}{k}p^{n-1} - \nabla_s \cdot \mathbf{u}^{n-1} - 2(\mathbf{U}\cdot\nabla)p^n,\end{aligned}$$

where $\mathbf{U} = (U, V)^T$ is a constant vector, and all functions are 1-periodic in space. Introduce new variable $q = \epsilon p$ and multiply the divergence equation by $\frac{k}{\epsilon}$, we arrive at

(42)

$$\begin{aligned}&(I - \nu k\Delta_s)\mathbf{u}^{n+1} + \frac{k}{\epsilon}\nabla_s q^{n+1} = (I + \nu k\Delta_s)\mathbf{u}^{n-1} - \frac{k}{\epsilon}\nabla_s q^{n-1} - 2k(\mathbf{U}\cdot\nabla_s)\mathbf{u}^n,\\ &q^{n+1} + \frac{k}{\epsilon}\nabla_s \cdot \mathbf{u}^{n+1} = q^{n-1} - \frac{k}{\epsilon}\nabla_s \cdot \mathbf{u}^{n-1} - 2k(\mathbf{U}\cdot\nabla)q^n.\end{aligned}$$

Under matrix expressions, (42) becomes

$$\begin{aligned}\left[I - k\begin{pmatrix}\nu\Delta_s & -\frac{\nabla_s}{\epsilon}\\ -\frac{\nabla_s\cdot}{\epsilon} & 0\end{pmatrix}\right]\begin{pmatrix}\mathbf{u}\\ q\end{pmatrix}^{n+1} &= -2k(UD_x + VD_y)\begin{pmatrix}\mathbf{u}\\ q\end{pmatrix}^n\\ &+ \left[I + k\begin{pmatrix}\nu\Delta_s & -\frac{\nabla_s}{\epsilon}\\ -\frac{\nabla_s\cdot}{\epsilon} & 0\end{pmatrix}\right]\begin{pmatrix}\mathbf{u}\\ q\end{pmatrix}^{n-1}.\end{aligned} \tag{43}$$

Let

$$Q = \begin{pmatrix}\nu\Delta_s & -\frac{\nabla_s}{\epsilon}\\ -\frac{\nabla_s\cdot}{\epsilon} & 0\end{pmatrix} \quad Q_0 = UD_x + VD_y, \quad \mathbf{w} = \begin{pmatrix}\mathbf{u}\\ q\end{pmatrix},$$

we write (43) as

$$(I - kQ)\mathbf{w}^{n+1} = 2kQ_0\mathbf{w}^n + (I + kQ)\mathbf{w}^{n-1}. \tag{44}$$

Notice that for any vectors $\mathbf{v}, \mathbf{w}$,

$$(\mathbf{v}, Q_0\mathbf{w}) = -(Q_0\mathbf{v}, \mathbf{w}), \quad (\mathbf{w}, Q\mathbf{w}) \le 0.$$

With energy method Kreiss and Oliger[7] showed that when

$$||kQ_0|| \le 1 - \eta, \quad \eta > 0,$$

there is

$$\begin{aligned} \eta(||\mathbf{w}^{n+1}||^2 + ||\mathbf{w}^n||^2) &\le ||\mathbf{w}^{n+1}||^2 + ||\mathbf{w}^n||^2 - 2k(\mathbf{w}^{n+1}, Q_0\mathbf{w}^n) \\ &\le 2(||\mathbf{w}^{n+1}||^2 + ||\mathbf{w}^n||^2). \end{aligned} \tag{45}$$

The scheme is hence stable. There is also dissipation due to the viscousity which have not been taken into account. Note that

$$||Q_0|| \le \frac{3}{2h}(|U| + |V|), \tag{46}$$

Thus the restriction on the time step is

$$k < \frac{2h}{3(|U| + |V|)}. \tag{47}$$

When there is no convection terms, i.e, $U = V = 0$, the above scheme is unconditionally stable. This is essential for the numerical approximations to the highly unsymmetric hyperbolic systems. It is shown that the order of any unconditionally stable multistep schemes can not exceed two[10], the proposed LF/CN method is optimal regarding to the accuracy.

In the actual implementation of the scheme (34), we determine time step by freezing the coefficient at their maximal absolute value, i.e,

$$k < \frac{2h}{3\max_{i,j}(|u_{i,j}^n| + |v_{i,j}^n|)}. \tag{48}$$

This condition is not more restrictive then (47) and it works just well.

3.3. Initialization. The preparation of initial conditions is done by the asymptotic expansion (5,11). This process can only be realized by numerical algorithm. Because of the induction of roundoff error, we should not expect to suppress the fast wave up to any order. Nevertheless, to suppress the fast wave up to a few order is achievable. Notice that the initialization requires the time derivatives of the expansion terms of pressure, we will have to compute the solutions of (5, 11) for a few time step so as to generate time difference for approximating time derivatives. We use fourth order Runge-Kutta method to advance the solutions of (5) and (11). The computation can be made as accurate as we want by using arbitrarily small time steps.

Consider the semidiscrete approximation of (5) for $t \geq 0$,

$$\begin{cases} \frac{d\mathbf{U}}{dt} + (\mathbf{U} \cdot \nabla_s)\mathbf{U} + \nabla_s P = \nu \Delta_s \mathbf{U} + \mathbf{F}, \\ \nabla_s \cdot \mathbf{U} = g, (1, P(\cdot, t)) = 0. \end{cases} \tag{49}$$

Here all functions are defined on grid, $\mathbf{U} = \{\mathbf{U}_{i,j}(t)\}$, $P = \{P_{i,j}(t)\}$. Initial conditions are solved from

$$\nabla_s \cdot \mathbf{U}(\cdot, 0) = g(\cdot, 0), \qquad \nabla_s \times \mathbf{U}(\cdot, 0) = \nabla_s \times \mathbf{u}(\cdot, 0) \equiv g'(\cdot), \tag{50}$$

with uniqueness conditions

$$(1, U(\cdot, 0) - u(\cdot, 0)) = 0, \qquad (1, V(\cdot, 0) - v(\cdot, 0)) = 0.$$

Equations (50) lead to

$$\begin{aligned} \Delta_s U &= D_x g + D_y g', \\ \Delta_s V &= D_y g - D_x g'. \end{aligned} \tag{51}$$

At each time step the P is determined by a Laplacian equation

$$\begin{aligned} &\Delta_s P = -\frac{dg}{dt} + \nu \Delta_s g + \nabla_s \cdot (\mathbf{F} - (\mathbf{U} \cdot \nabla_s)\mathbf{U}), \\ &(1, P(\cdot, t)) = 0. \end{aligned} \tag{52}$$

The pressure is unique up to a time dependent function $\bar{P}$, which is determined by

$$\begin{aligned} &(1, p_0 - P(\cdot, 0) - \bar{P}(0)) = 0, \\ &(1, P_t + \bar{P}_t + (\mathbf{U} \cdot \nabla_s)P) = 0. \end{aligned} \tag{53}$$

The solution of (51,52) give us the second order prepared initial conditions such that the first and second order time derivatives of the solution of (49) are bounded independently of ϵ. If we want higher order initialization, we will have to solve for the solutions for a few time step. They can be obtained via fourth order Runge-Kutta method. Note that we eliminate the pressure from the momentum equations by denoting $P = P(\mathbf{U}, \mathbf{F}, g)$. Thus we can rewrite the momentun equations as

$$\frac{d\mathbf{U}}{dt} = -(\mathbf{U} \cdot \nabla_s)\mathbf{U} - \nabla_s P(\mathbf{U}, \mathbf{F}, g) + \nu \Delta_s \mathbf{U} + \mathbf{F} \equiv \mathbf{f}(\mathbf{U}, t). \tag{54}$$

Let $t = nk$, then the fourth order Runge-Kutta method can be described as

$$\begin{aligned} &\mathbf{k}_1 = \mathbf{f}(\mathbf{U}^n, t), \\ &\mathbf{k}_2 = \mathbf{f}(\mathbf{U}^n + \frac{k}{2}\mathbf{k}_1, t + \frac{k}{2}), \\ &\mathbf{k}_3 = \mathbf{f}(\mathbf{U}^n + \frac{k}{2}\mathbf{k}_2, t + \frac{k}{2}), \\ &\mathbf{k}_4 = \mathbf{f}(\mathbf{U}^n + k\mathbf{k}_3, t + k), \\ &\mathbf{U}^{n+1} = \mathbf{U}^n + \frac{k}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4), \\ &\Delta_s P^{n+1} = -\left(\frac{dg}{dt}\right)^{n+1} + \nu \Delta_s g^{n+1} + \nabla_s \cdot \left(\mathbf{F}^{n+1} - (\mathbf{U}^{n+1} \cdot \nabla_s)\mathbf{U}^{n+1}\right). \end{aligned} \tag{55}$$

The higher order terms of the asymptotic expansion are obtained in a similar way. If p^{th} order initialization is wanted, we need to solve $\mathbf{U}_l, P_l$ iteratively, from

$$
(56)\quad \begin{cases} \dfrac{d\mathbf{U}_l}{dt} + (\mathbf{U}^{(l-1)} \cdot \nabla_s)\mathbf{U}_l + (\mathbf{U}_l \cdot \nabla_s)\mathbf{U}^{(l-1)} + \nabla_s P_l = \nu\Delta_s \mathbf{U}_l + \epsilon^{2l-4}\mathbf{F}_l, \\ \nabla_s \cdot \mathbf{U}_l = g_l, \quad (1, P_l(\cdot, nk)) = 0, \end{cases}
$$

with initial conditions determined by

$$
(57)\qquad \nabla_s \cdot \mathbf{U}_l^0 = g_l^0, \quad \nabla_s \times \mathbf{U}_l^0 = 0, \quad (1, U_l^0) = (1, V_l^0) = 0.
$$

Here,

$$
\begin{aligned}
&\mathbf{U}^{(l-1)} = \mathbf{U} + \sum_{i=1}^{l-1} \epsilon^{2i}\mathbf{U}_i, \quad P^{(l-1)} = P + \sum_{i=1}^{l-1} \epsilon^{2i}(P_i + \bar{P}_i), \\
&\mathbf{F}_1 = 0, \qquad \mathbf{F}_l = -(\mathbf{U}_{l-1} \cdot \nabla)\mathbf{U}_{l-1}, \quad 2 \le l \le m, \\
&g_l = -\{\nabla_t P_{l-1} + \nabla_t \bar{P}_{l-1} + (\mathbf{U}^{(l-1)} \cdot \nabla_s)P_{l-1} + (\mathbf{U}_{l-1} \cdot \nabla_s)P^{(l-1)}\} \\
&\qquad - \epsilon^{2l-2}(\mathbf{U}_{l-1} \cdot \nabla_s)P_{l-1},
\end{aligned}
$$

and $\bar{P}_{l-1}(nk)$ is chosen such that

$$
\begin{aligned}
&(1, g_l(\cdot, nk)) = 0, \\
&\bar{P}_{l-1}(\cdot, 0) = 0.
\end{aligned}
$$

Let

(58)

$$
\begin{aligned}
\mathbf{f}_l &= -(\mathbf{U}^{(l-1)} \cdot \nabla_s)\mathbf{U}_l - (\mathbf{U}_l \cdot \nabla_s)\mathbf{U}^{(l-1)} - \nabla_s P_l(\mathbf{U}_l, \mathbf{F}_l, g_l) + \nu\Delta_s \mathbf{U}_l + \mathbf{F}_l, \\
H_l &= -\nabla_t g_l + \nu\Delta_s g_l + \nabla_s \cdot \left(\mathbf{F}_l - (\mathbf{U}^{(l-1)} \cdot \nabla_s)\mathbf{U}_l - (\mathbf{U}_l^0 \cdot \nabla_s)\mathbf{U}^{0,(l-1)}\right).
\end{aligned}
$$

we summerize the initialization process in the following algorithm. Suppose we need up to p^{th} order time derivatives bounded independently of ϵ.

Algorithm of INITIALIZATION:

Solve $\mathbf{U}^0, P^0$ from (51,52)

For $n = 1, 2, \ldots, 4(p-l)$

compute $\mathbf{U}^n, P^n$ from (55)

For $l = p-2, p-3, \ldots, 2, 1$

Compute $\mathbf{U}_l^0, P_l^0$ from

$$
\begin{aligned}
&\nabla_s \cdot \mathbf{U}_l^0 = g_l^0, \quad \nabla_s \times \mathbf{U}_l^0 = 0, \quad (1, U_l^0) = (1, V_l^0) = 0, \\
&\Delta_s P_l^0 = H_l^0
\end{aligned}
$$

For $n = 1, 2, \ldots, 4l$ **compute** $\mathbf{U}_l^n, P_l^n$ **by**

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}_l(\mathbf{U}_l^n, t)\\
\mathbf{k}_2 &= \mathbf{f}_l(\mathbf{U}_l^n + \frac{k}{2}\mathbf{k}_1, t + \frac{k}{2})\\
\mathbf{k}_3 &= \mathbf{f}_l(\mathbf{U}_l^n + \frac{k}{2}\mathbf{k}_2, t + \frac{k}{2})\\
\mathbf{k}_4 &= \mathbf{f}_l(\mathbf{U}_l^n + k\mathbf{k}_3, t + k)\\
\mathbf{U}_l^{n+1} &= \mathbf{U}_l^n + \frac{k}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)\\
\Delta_s P_l^{n+1} &= H_l^{n+1}
\end{aligned}$$

end of INITIALIZATION

Appearently this process of initialization can also be used during the calculations to suppress the fast wave if it exceeds our tolerence.

3.4. Numercal Results. The numerical algorithm is composed of two parts. The first part is the initialization which suppresses the fast wave from the initial data. The second part is the advancement of the Leap-frog/Crank-Nicholson method with the prepared initial conditions. The initialization process can be repeated at later time once the magnitude of fast wave exceeds a set tolerance due to the long term effect of roundoff errors. The framework of the algorithm is displayed below.

Algorithm LF/CN:

I. Initialization:

if *divergence* > *tolerance* **then prepare initial velocity**

Prepare initial pressure

Extrapolate for the initial data of the second time step

II. Advancement in time

Advance with the LF/CN scheme

if *divergence* > *tolerance* **then**

Prepare the velocity

Prepare the pressure

Extrapolate for velocity and pressure of the next time step

if $t \le T$ **go to II**

end of LF/CN

The test data is constructed in the following way. For the continuous equation (1), we compute source functions $\mathbf{F}, g$ from the following distribution of velocity and pressure

$$\begin{aligned} U(x,y,t) &= e^t \sin 2\pi x(1 - \frac{1}{4}\sin 2\pi y), \\ V(x,y,t) &= e^t(1 - \cos 2\pi x)(1 + \frac{1}{4}\cos 2\pi y), \\ P &= 0. \end{aligned}$$

We then run our algorithm LF/CN with the perturbed initial conditions

$$\begin{aligned} u(x,y,0) &= U(x,y,0) + \epsilon \sin 4\pi x(1 - \cos 4\pi y), \\ v(x,y,0) &= V(x,y,0) + \epsilon(1 - \cos 4\pi x)\sin 4\pi y, \\ p &= P. \end{aligned} \tag{59}$$

We take the mach number $\epsilon = 10^{-3}$, viscousity $\nu = 10^{-4}$. The grid size we use is 32×32. The initial time step is determined by

$$0.1 = k\|Q_0\| = \frac{3k \max_{i,j}(|u^0_{i,j}| + |v^0_{i,j}|)}{2h}$$

so that the initial truncation error in time is of the same order as the truncation error in space. Then step size in time is limited by for stability concern

$$0.1 \le k\|Q_0\| = \frac{3k \max_{i,j}(|u^n_{i,j}| + |v^n_{i,j}|)}{2h} \le 0.9.$$

The time interval of our calculation is $0 \le t \le 1$. We take $p = 1$ in the INITIALIZATION algorithm.

We judge the quality of the computed slow solution by the magnitude of the *divergence* $= \nabla \cdot \mathbf{u} - g$, which also reflects the strength of the fast wave in the pressure. For comparison purpose, we also provide the results by LF/CN without the initialization process. The results are given in Figure 1,2,3 and 4, where the maximum norm of the divergence at each time step is plotted.

It takes 1381 time steps to reach $t = 1$. It can be seen that without suppressing the sound wave from the initial condition, the divergence is of order $O(10^{-3}) = O(\epsilon)$. This magnitude implies that the sound wave in pressure is at least of order $O(1)$. Under the situation, the resolution of the pressure is no longer reliable, unless we considerably decrease the time step. However, through the initialization the sound wave is effectively suppressed, and it stays in the order of $o(1)$. The maximum norm of the pressure given in Figure 3 and Figure 4, respectively, is consistent with the magnitude of the divergence.

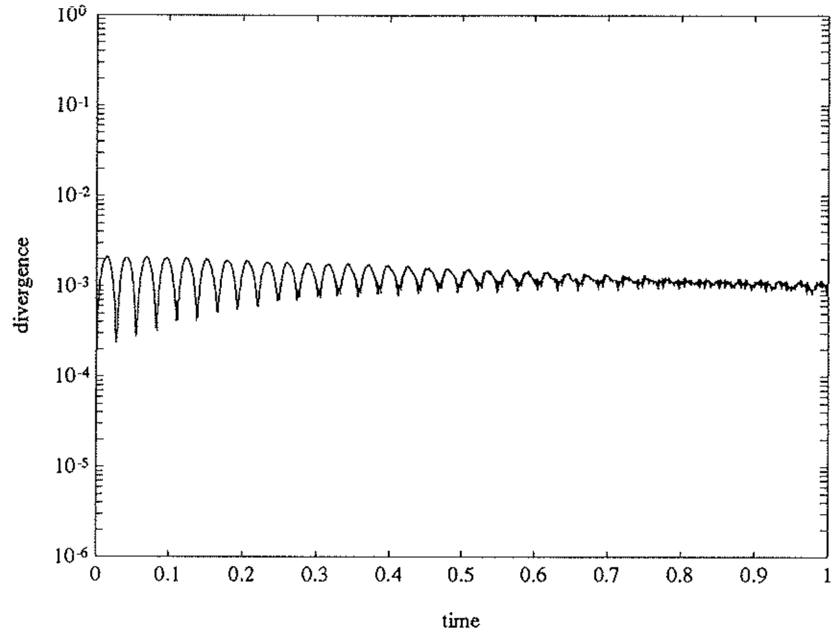

Figure 1. Without initialization.

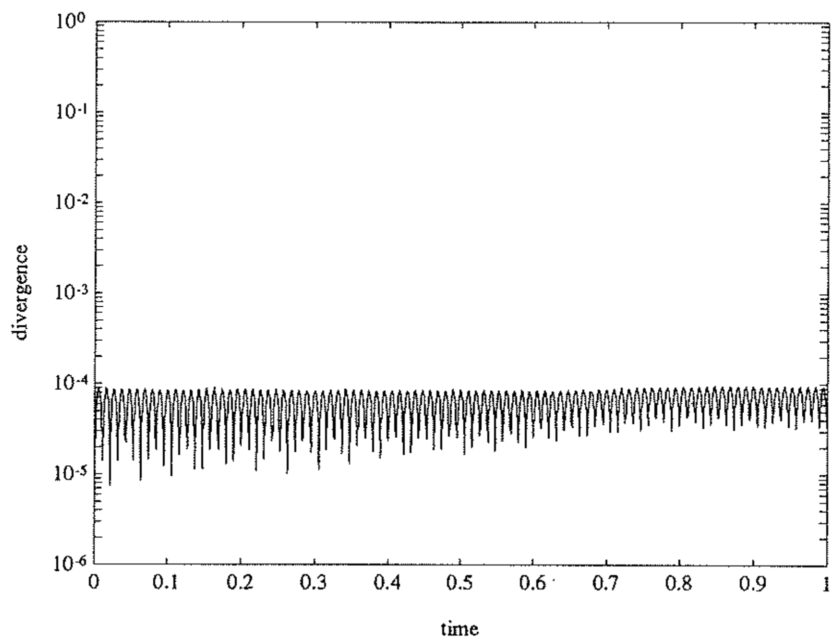

Figure 2. With initialization

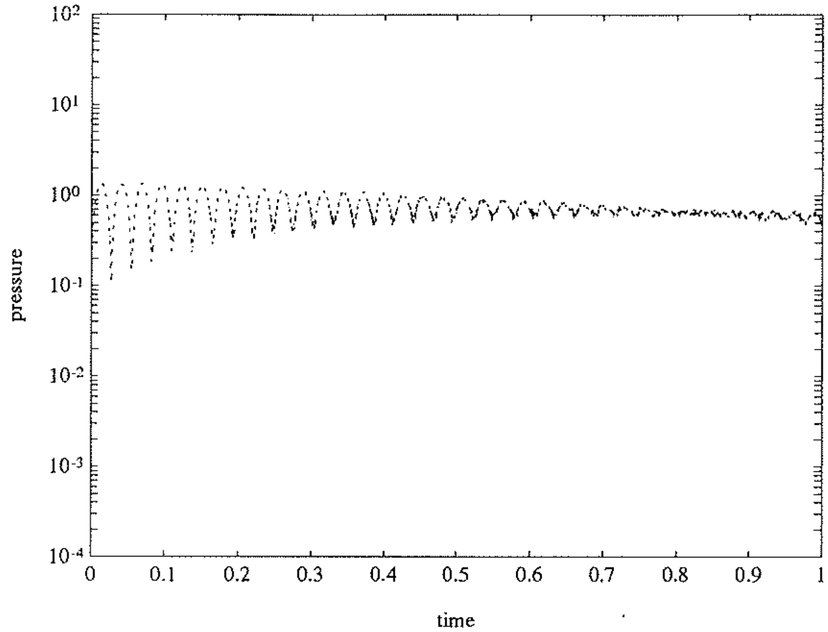

Figure 3. Without initialization.

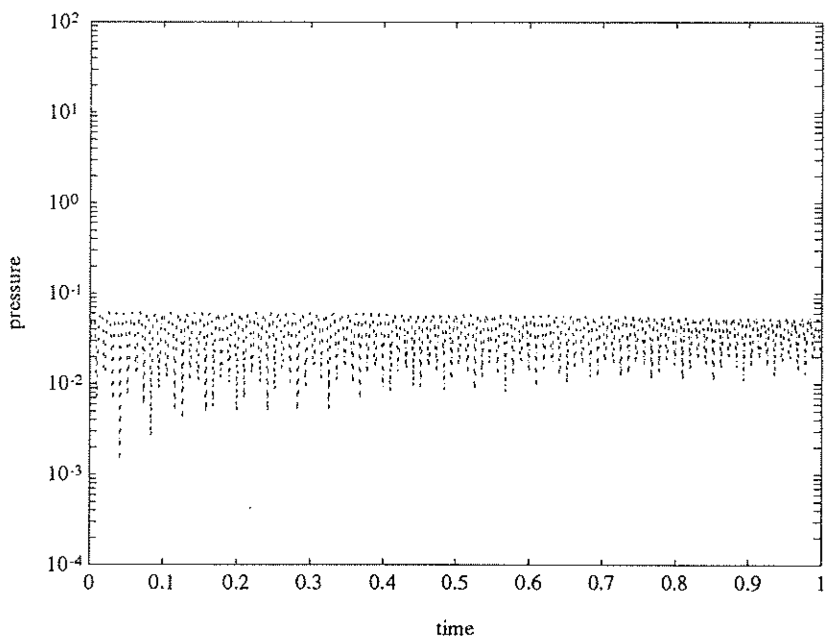

Figure 4. With initialization

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