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Abstract

A level set formulation for the solution of the Hamilton-Jacobi equation $F(x, y, u, u_x, u_y) = 0$ is presented where u is prescribed on a set of closed bounded noncharacteristic curves. A time dependent Hamilton-Jacobi equation is derived such that the level set at various time t of this solution is precisely the set of points (x, y) for which $u(x, y) = t$. This gives a fast and simple numerical method for generating the viscosity solution to $F = 0$. The level set capturing idea was first introduced by Osher and Sethian in [12] and the observation that this is useful for an important computer vision problem of this type was then made by R. Kimmel and A.M. Bruckstein in [10] following Bruckstein [1]. Finally we note that an extension to many space dimensions is immediate.

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Introduction.

We are interested in solving a general first order partial differential equation for a function $z = u(x, y)$ of the type:

$$F(x, y, z, p, q) = 0 \tag{0.1}$$

where $p = u_x,\ q = u_y$.

This is a classical problem in P.D.E., in fact the method of characteristics was invented to solve it. Typically we are given Cauchy data on a curve Γ, i.e. for

$$x = x_0(s), \quad y = y_0(s) \tag{0.2a}$$

then

$$z = z_0(s). \tag{0.2b}$$

The data is assumed to be noncharacteristic, i.e. by the chain rule we have:

$$\dot{z}_0(s) = p_0(s)\ \dot{x}_0(s) + q_0(s)\dot{y}_0(s) \tag{0.3}$$

while

$$F(x_0(s), y_0(s), z_0(s), p_0(s), q_0(s)) = 0 \tag{0.4}$$

In order to solve (0.3) and (0.4) locally for smooth $p_0(s), q_0(s)$, the implicit function theorem requires:

$$\dot{y}_0 F_p(x_0, y_0, z_0, p_0, q_0) \neq \dot{x}_0 F_q(x_0, y_0, z_0, p_0, q_0). \tag{0.5}$$

This is the noncharacteristic criterion.

Given (0.5), one then generates characteristic curves via

$$\begin{aligned} \frac{dx}{dt} &= F_p \\ \frac{dy}{dt} &= F_q \\ \frac{dz}{dt} &= pF_p + qF_q \\ \frac{dp}{dt} &= -F_x - pF_z \\ \frac{dq}{dt} &= -F_y - qF_z. \end{aligned} \tag{0.6}$$

The initial data "propagates" along these curves and criterion (0.5) guarantees that we generate a smooth solution locally in time. However, in finite time characteristic curves generally intersect (caustics develop). Fourier Integral Operators (FIO) were developed in the sixties and seventies see e.g. [8] to take care of the resulting multivaluedness (and to do a lot more, of course).

A classical example is the eiconal equation from geometrical optics:

$$p^2 + q^2 = 1. \tag{0.7}$$

If Γ is convex, the solution rapidly develops a caustic. Rather than continuing it as a multivalued solution à la FIO, we may use the recently developed notion of viscosity solutions [3,4,5] for Hamilton-Jacobi equations to continue the solution uniquely as a single-valued uniformly continuous function having "kinks" – i.e. jumps in the first derivative. For most real-world problems this is the appropriate class.

We shall propose an analytic and numerical method for solving (0.1), (0.2) when Γ is a compact set of closed curves dividing R^2 up into Ω and its complement Ω^c, neither of which need necessarily to be compact. We call Ω the "interior" and Ω^c the "exterior".

This method generalizes easily to compact hypersurfaces dividing up R^n into an interior and exterior. In this paper we shall stick to R^2 for simplicity of exposition only.

The present work has three main antecedents. In [12] Osher and Sethian introduced the concept of a level set formulation to propagate curves and surfaces. The problem analyzed there was as follows. We wish to move a closed curve Γ normal to itself with normal velocity u_n. This velocity might be geometrically based, e.g. it might be a function of the curvature of Γ. The level set formulation easily treats self intersections, topological changes, kinks, and higher space dimensions. Theoretical justification for this method (along with a great deal of other very important theory) came later in [2,6,7].

Briefly, one finds a function $\psi(x,y,t)$ so that at $t=0$ we have:

$$\psi(x,y,0)=0 \Leftrightarrow (x,y)\ \varepsilon\ \Gamma \tag{0.8a}$$

$$\psi(x,y,0)>0 \text{ in } \Omega \tag{0.8b}$$

$$\psi(x,y,0)<0 \text{ in } \Omega^c \tag{0.8c}$$

and $\psi(x,y,0)$ is a uniformly continuous and monotonic strictly decreasing function of distance to Γ near Γ which we call $\Gamma(0)$.

We require that $\Gamma(t)$ evolves so that

$$\psi(x,y,t)=0 \Leftrightarrow (x,y)\ \varepsilon\ \Gamma(t). \tag{0.9a}$$

This means that for $(x(t),y(t))\ \varepsilon\ \Gamma(t)$

$$\frac{d}{dt}\psi(x(t),y(t),t)=0 \tag{0.9b}$$

$$\psi_x x_t+\psi_y y_t+\psi_t=0. \tag{0.9c}$$

Rearranging terms, we arrive at

$$\psi_t=-u_n\sqrt{\psi_x^2+\psi_y^2}. \tag{0.9x}$$

At this point we imagine that u_n is defined throughout R^n, not just on $\Gamma(t)$, and that this is done in a natural way. Thus all level sets of ψ move according to this law.

If u_n is a given function of (x,y), then this is a Hamilton-Jacobi equation and we seek the viscosity solution [4,5]. This has an interesting physical interpretation for flames. Sethian's entropy condition [15] follows for the viscosity solution (see [12] for the proof).

If u_n is the curvature of the level set, the equation becomes:

$$\psi_t=\frac{\psi_{xx}\psi_y^2-2\psi_{xy}\psi_x\psi_y+\psi_{yy}\psi_x^2}{\psi_x^2+\psi_y^2}. \tag{0.10}$$

Thus, we can define the motion of a square via its mean curvature, using (0.10) and following the level set. Again, this was rigorously justified in [2,7] for general curves modulo some unusual exceptions.

As a numerical device this approach has many advantages over tracking. We simply set up a fixed, Eulerian grid, solve (0.9) numerically, and let the plotter find the front. Self intersections, kinks, topological changes, and multispace dimensions are treated routinely. Of course, we have to construct stable, accurate, and efficient methods for (0.9). See [12,13], for a description of such methods.

The second antecedent is [10]. There the authors wished to solve a problem in computer vision. We are given $z(x, y)$ describing the surface of an object which is illuminated by an overhead light source at infinity. In the simplest model the intensity of light $I(x, y)$ is given by

$$I(x,y) - \frac{1}{\sqrt{1+p^2+q^2}} = 0 = F(x,y,z,p,q). \tag{0.11}$$

The shape-from-shading problem is: given $0 < I(x, y) \leq 1$, find $u(x, y)$. This is a very well studied problem, but only recently in [11,14], was the correct theory of viscosity solutions brought to bear. In [10], the authors assumed that they were given a level surface of u, i.e. (0.2) for $z_0 \equiv 0$. What they proposed was to use the methods of [12] to propagate the level surface to generate the solution of (0.11). We now recognize this as a general method for solving (0.1) for Dirichlet data. We shall describe and justify it in the next section.

Finally, the third crucial antecedent came in Bruckstein [1]. There the author transformed the shape from shading problem into a level set propagation P.D.E. and realized the advantages of this formulation. The link with the propagation methods of [12] and the viscosity solution concept came later in [10] for this important problem.

I. Description and Justification of the Method.

Let Γ be a compact set of disjoint closed curves in R^2, dividing R^2 up into an interior Ω and an exterior Ω^c. We wish to solve for $z = u(x, y)$

$$F(x,y,z,p,q) = 0 \tag{1.1a}$$

with Dirichlet data on Γ written locally as

$$x = x_0(s) \tag{1.1b}$$

$$y = y_0(s) \tag{1.1c}$$

$$z = z_0(s) \tag{1.1d}$$

which is noncharacteristic for (1.1a).

We next assume that $z_0(s)$ can be continued into a set containing Γ as a function $w(x, y)$ so that the functon $n(x, y)$ defined by

$$z(x,y) = w(x,y) + n(x,y) \tag{1.2}$$

is the unknown. This has the effect of changing F and setting $z_0 \equiv 0$ in (1.1d). Thus we have the zero level set of the solution to a simple related P.D.E. as boundary data. We continue to call this new P.D.E. F and the new unknown function z.

The noncharacteristic criterion then becomes

$$p_0 F_p + q_0 F_q \neq 0 \tag{1.3}$$

on Γ.

Now we wish to construct a function of three variables $v(x, y, t)$, $t \geq 0$ such that if

$$v(x,y,t) = 0 \text{ then} \tag{1.4a}$$

$$z = u(x,y) = t. \tag{1.4b}$$

Of course any such function will not be unique. However, all of them will satisfy on the level set (1.4):

$$\frac{\partial}{\partial x} v(x,y,u(x,y)) = 0 = v_x + v_t u_x \tag{1.5a}$$

$$\frac{\partial}{\partial y} v(x,y,u(x,y)) = 0 = v_y + v_t u_y \tag{1.5b}$$

Thus, at least formally on this level set:

$$F(x, y, v, \frac{-v_x}{v_t}, \frac{-v_y}{v_t}) = 0. \tag{1.6}$$

We shall choose $v(x, y, 0)$ to be a uniformly continuous function vanishing only for (x, y) on Γ, $v > 0$ in Ω, $v < 0$ in Ω^c and v is a strictly monotone function of distance to Γ near Γ.

The noncharacteristic criterion of (0.3) guarantees that we may invert (1.6) locally for v_t near Γ. To devise a numerical algorithm based on time evolution we need the assumption.

Assumption I. An explicit inversion formula exists for (1.6) near Γ so that the formula

$$v_t + H(x, y, v, v_x, v_y) = 0 \tag{1.7}$$

with $H > 0$ near Γ implies (1.6) near $(x, y) \in \Gamma$, $t = 0$.

We note that H must be homogeneous of degree one in v_x, v_y. We also have a technical assumption: H is a nondecreasing function of v near the initial region. This is required by the theory of viscosity solutions – it basically rules out shocks as possible solutions.

We now have our analytical method for solving (1.1), kinks and all. (Numerical methods may be easily constructed using the results of [12, 13]).

We solve (1.7) on all of R^n (we really only need to do this near $\Gamma(t)$) with uniformly continuous initial data

$$v(x, y, 0) = v_0(x, y) \tag{1.8a}$$

with

$$v_0(x, y) = 0 \text{ iff } (x, y) \in \Gamma \tag{1.8b}$$

$$v_0(x, y) > 0 \text{ iff } (x, y) \in \Omega \tag{1.8c}$$

$$v_0(x, y) < 0 \text{ iff } (x, y) \in \Omega^c. \tag{1.8d}$$

Then, to compute $u(x, y)$ for $(x, y) \in \Omega$ we calculate the level sets via the relation

$$v(x, y, t) = 0 \Leftrightarrow t = u(x, y). \tag{1.9}$$

This allows us to generate $u(x,y)$ by building it up through this level set formulation.

It is clear from the classical method of characteristics - see e.g. [9], that if Γ is a smooth curve and F and H are smooth functions near Γ then the solution to (1.1) is locally (near Γ) the same as (1.9) for $t > 0$ and small. We now claim that the level set generated function (1.9) is a viscosity solution to

$$-1 + H(x, y, u, u_x, u_y) = 0 \tag{1.10}$$

if v is the viscosity solution to (1.7), (1.8).

We now recall the definition of viscosity solution, see e.g. [3].

Definition 1.1. Let $\psi \; \varepsilon \; C^2$ near $(\bar{x}, \bar{y}, \bar{t})$. Suppose $v - \psi$ has a local minimum (maximum) at $(\bar{x}, \bar{y}, \bar{t})$, then v is a viscosity supersolution (subsolution) of (1.7) at this point if

$$\psi_t + H(x, y, v, \psi_x, \psi_y) \geq 0 (\leq 0) \text{ at } (\bar{x}, \bar{y}, t) \tag{1.11}$$

for all such ψ.

Definition 1.2. v is a viscosity solution at this point if it is both a viscosity sub and supersolution.

The fact that $H > 0$ indicates that v is strictly decreasing in t near this point. In fact, if $v - \psi$ has a local maximum there, then

$$\psi_t < -H$$

which means that ψ is strictly decreasing there.

We take ψ so that

$$v(x, y, t) \leq \psi(x, y, t) \text{ near } (\bar{x}, \bar{y}, \bar{t}) \tag{1.12a}$$

and

$$v(\bar{x}, \bar{y}, \bar{t}) = \psi(\bar{x}, \bar{y}, \bar{t}). \tag{1.12b}$$

Then, for t satisfying $\bar{t} < t < \bar{t} + \epsilon$ for $\epsilon > 0$ small,

$$\begin{aligned} v(\bar{x}, \bar{y}, t) - v(\bar{x}, \bar{y}, \bar{t}) &\leq \psi(\bar{x}, \bar{y}, t) - \psi(\bar{x}, \bar{y}, \bar{t}) \\ &= \psi_t(\bar{x}, \bar{y}, \bar{\bar{t}})(t - \bar{t}) \text{ (where } \bar{t} < \bar{\bar{t}} < t) \\ &< -\bar{H}(t - \bar{t}). \end{aligned} \tag{1.13}$$

Thus v is uniformly strictly decreasing for $t > \bar{t}$. This is true for all such $\bar{t}$ and t in any neighborhood in which $v(x, y, t)$ is a viscosity solution. Thus there exists an increasing uniformly continuous inverse function h such that

$$h(v(x,y,t)) = u(x,y) - t. \tag{1.14}$$

What remains to be shown is that u is a viscosity solution to

$$-1 + H(x, y, u, u_x, u_y) = 0. \tag{1.14}$$

This follows directly from Theorem (5.2) of [2] under the hypothesis that H is independent of u. Thus we make that assumption for theoretical purposes only and conclude. (This key Theorem of [2] was motivated by problems involving motion of level sets such as those described in [12]).

II. Examples.

Given the shape-from-shading problem described above, with a remote generally non overhead light source whose direction cosines, are $(\alpha, \beta, -\gamma)$ for $\gamma > 0$, with respect to the normal to the surface $z = u(x, y)$ we wish to solve:

$$I(x,y)\sqrt{1 + u_x^2 + u_y^2} - \alpha u_x - \beta u_y - \gamma = 0 \tag{2.1a}$$

with

$$u = 0 \text{ on } \Gamma = \partial\Omega. \tag{2.1b}$$

The noncharacteristic criterion is satified if $\gamma\sqrt{1 + p^2 + q^2} \neq I$. Equation (1.6) becomes in this case

$$I(x,y)\sqrt{1 + \frac{v_x^2}{v_t^2} + \frac{v_y^2}{v_t^2}} + \frac{\alpha v_x}{v_t} + \frac{\beta v_y}{v_t} - \gamma = 0. \tag{2.2}$$

This can be inverted to obtain our version of (1.7)

(2.3)

$$v_t + \frac{(\mathrm{sign}(I^2 - \gamma^2))\gamma(\alpha v_x + \beta v_y) + I\sqrt{v_x^2(-\beta^2 + 1 - I^2) + v_y^2(-\alpha^2 + 1 - I^2) + 2\alpha\beta v_x v_y}}{|\gamma^2 - I^2|} = 0.$$

Note that if, for example, $\gamma = 1 \Leftrightarrow \alpha = \beta = 0$, the resulting overhead formula:

$$v_t + \frac{I}{\sqrt{1-I^2}}\sqrt{v_x^2 + v_y^2} = 0 \tag{2.4}$$

gives difficulties near $I = 1$. This is inherent in the problem [11,14]. In the general case the method has problems because of a possibly negative quantity under the square root sign, unless $I \le \gamma$. If this inequality fails we must require that the gradients satisfy:

$$v_x^2(1 - I^2 - \beta^2) + v_y^2(1 - I^2 - \alpha^2) + 2\alpha\beta v_x v_y \ge 0 \tag{2.5a}$$

if

$$I > \gamma. \tag{2.5b}$$

This is, of course, required at the zero level set of $v(x, y, t)$ from (2.1a) using (2.3) and the related (1.5a,b), but it does present some numerical difficulties.

Example 2. Control-optimal cost determination

$$-(\sin y)u_x + (\sin x)u_y + |u_y| - \frac{1}{2}\sin^2 y - (1 - \cos x) = 0 \tag{2.6a}$$

$$u = 0 \text{ on } \Gamma, \text{ which is noncharacteristic, which means :} \tag{2.6b}$$

$$\frac{1}{2}\sin^2 y + (1 - \cos x) \neq 0 \text{ on } \Gamma. \tag{2.6c}$$

We are led to:

$$v_t + \frac{|v_y| + (\sin x)v_y - (\sin y)v_x}{\frac{1}{2}\sin^2 y + (1 - \cos x)} = 0 \tag{2.7}$$

and the quantity $H(x, y, u_x, u_y)$ defined above is assumed to be strictly positive near Γ. (Of course, if it strictly negative, everything works with a different initialization, just reversing the inequalities in (1.8c-d)). $\sqcap$

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