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DOMAIN DECOMPOSITION PRECONDITIONERS FOR CONVECTION DIFFUSION PROBLEMS
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Abstract. We study two domain decomposition based preconditioners for solving singularly perturbed convection diffusion problems. Both preconditioners are based on a partition of the domain into two regions, one where the convection term dominates and on which the convection term is used, and the other where the diffusion term dominates. We consider two different ways of coupling the elliptic and hyperbolic problems, yielding the two preconditioners. One approach proposed by Gastaldi, Quarteroni, and Sacchi-Landriani, enforces continuity of the flux, while another used by Ashby, Saylor and Scroggs enforces continuity of the solution. Numerical results are reported on tests of these preconditioners and of the ILU preconditioner.

1. Introduction. We consider the solution of a singular perturbation scalar convection diffusion problem of the form:

$$(1) \qquad \begin{cases} \epsilon L_d u + L_c u &= f(x,y) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where L_d and L_c are diffusion and convection operators, respectively, defined below:

$$L_d u \equiv -\Delta u, \quad \text{and} \quad L_c u \equiv a(x,y)u_x + b(x,y)u_y.$$

Here ϵ is a positive parameter representing viscosity, and $a(x,y)$, $b(x,y)$ describes the velocity field of the zero viscosity problem. Such systems can arise as a steady state problem or from applying an implicit method to a time dependent problem. Discretization, either by stream line diffusion or upwind finite difference methods, of the above equations yield large, sparse, non-symmetric linear systems which are ill-conditioned. We will focus on solving these linear systems by preconditioned iterative methods such as GMRES, BCG, CGS, QMR, etc, see for instance Saad and Schultz [15] and Freund, Golub and Nachtigal [8].

In order for the preconditioned iterative method to be efficient, each iteration must be inexpensive to implement (each iteration involves solving a linear system with the preconditioner as coefficient matrix), and in addition, the convergence rate of the preconditioned iterative method should be independent of parameters of the discrete problem (such as mesh size h and viscosity ϵ). Most standard domain decomposition and multi-level preconditioners for solving these non-symmetric linear systems, however, require that a non-symmetric problem be solved on a *sufficiently fine* coarse grid during each iteration, in order to maintain an iteration rate which does not deteriorate as $h, \epsilon \to 0$, see [4, 5, 3]. As the cell Reynold's number $\epsilon/h \to 0$, the size of this coarse grid problem becomes correspondingly large, and the solution

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of this coarse grid problem can become expensive and affect the overall complexity of the iterative procedure.

Based on recent studies by Gastaldi, Quarteroni and Sacchi-Landriani [9], and Ashby, Saylor and Scroggs [1], we study in this paper an alternative technique for solving the non-symmetric linear systems arising from convection diffusion problems, based on the use of a two subregion domain decomposition method (primarily to avoid the requirement of solving a sufficiently fine coarse grid problem). The two subregions are chosen as in perturbation theory, see [11], for singular perturbation problems: an elliptic region (which contains the layer and where the elliptic problem dominates) and a hyperbolic region (where the convection term dominates). Our main results can be summarized as follows. For fixed mesh size h on a uniform mesh, both preconditioners we consider yield iteration rates which improve as $\epsilon \to 0$, but which deteriorates mildly for fixed ϵ, as $h \to 0$. However, we expect that these rates of convergence will not deteriorate on locally refined grids (which may be required in the case of singular perturbation problems with boundary or interior layers). We will provide more discussion of this later in the paper.

1.1. Upwind Discretization of the Elliptic Problem. For simplicity we choose our discretization to be the standard 1st order upwind finite difference on a uniform mesh, and this is known to yield a coefficient matrix L^h which is an M-matrix, and which is stable for any choice of ϵ or h. We will denote the grid points on the uniform mesh by $(x_i, y_j) = (ih, jh)$, where h is the mesh size. We denote by u_{ij} and f_{ij} the discrete solution and the forcing term, respectively, at node (i,j):

$$u_{ij} \approx u(x_i, y_j), \quad f_{ij} \equiv f(x_i, y_j).$$

The standard forward and backward difference operators are defined by:

$$\begin{aligned} D_x^+ u_{i,j} &= \tfrac{u_{i+1,j}-u_{i,j}}{h} \\ D_x^- u_{i,j} &= \tfrac{u_{i,j}-u_{i-1,j}}{h} \\ D_y^+ u_{i,j} &= \tfrac{u_{i,j+1}-u_{i,j}}{h} \\ D_y^- u_{i,j} &= \tfrac{u_{i,j}-u_{i,j-1}}{h}. \end{aligned}$$

In order to define the upwind scheme, we will use the notation:

$$a^+(x,y) = \max\{a(x,y), 0\}; \quad a^-(x,y) = \min\{a(x,y), 0\},$$

and similarly for $b(x,y)$. Using these, we approximate:

$$\begin{aligned} (a(x,y)u_x)_{ij} &\approx a^+_{i-\frac{1}{2},j} D_x^- u_{ij} + a^-_{i+\frac{1}{2},j} D_x^+ u_{ij} \\ (b(x,y)u_x)_{ij} &\approx b^+_{i,j-\frac{1}{2}} D_y^- u_{ij} + b^-_{i,j+\frac{1}{2}} D_y^+ u_{ij} \end{aligned}$$

and

$$(-\Delta u)_{ij} \approx \frac{4u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{i,j+1} - u_{i,j-1}}{h^2} = -(\Delta_h u)_{ij}.$$

Then, $\epsilon L_d + L_c$ is discretized by:

$$\begin{aligned} (L^h u)_{ij} = -\epsilon(\Delta_h u)_{ij} + (a(x,y)u_x)_{ij} + (b(x,y)u_x)_{ij} &= f_{ij} \quad \text{for}(x_i, y_j) \in \Omega \\ u_{ij} &= 0 \quad \text{for}(x_i, y_j) \in \partial\Omega \end{aligned},$$

which yields a linear system:

$$L^h u = f, \text{ where } L^h = \epsilon L_d + L_c, \quad f = \epsilon f_d + f_c,$$

where $L_d = -\Delta_h$, the 5-point Laplacian, L_c is the discretization of the convection term, f_d is the forcing term when $f(x,y) \equiv 0$ with contributions from the boundary data using the discrete Laplacian, and f_c represents the sum of the discrete forcing terms $f(x_i, y_j)$ and the contributions from the boundary values using the convection stencil. To minimize notation, we will use L, L_d, L_c, etc., either to refer to the continuous operator or to its finite difference matrix representation.

2. Convection Operator as Preconditioner. Since both the preconditioners we study will be modifications of the convection component L_c, we first study some properties of it as a preconditioner. One of the advantages of the convection matrix L_c is that for many choices of coefficients a, b, it is invertible and can be permuted to be lower triangular and is thus easily inverted (such as when $a(x,y)$ and $b(x,y)$ are constants). Even for problems in which a and b are varying, it may be possible to find a permutation based on graph theory for which the matrix L_c becomes lower triangular, see for instance [6] (this corresponds to reordering the unknowns consistent with characteristic directions). However, there are choices of a and b (such as for re-circulating flows) for which the matrix L_c becomes singular, and for which it is not reducible to a triangular matrix by permutation. Such cases can be treated by the addition of local diffusion terms to obtain an invertible matrix. The preconditioners of Sections 3 and 4 will be of that form. Apart from invertibility, as the mesh size h is reduced, the rate of convergence of the convection preconditioned system deteriorates. Some of these issues are discussed heuristically in this section.

2.1. Convergence Rate for Convection Preconditioner. From a matrix point of view, the convection preconditioned system $L_c^{-1} L_h$ (when L_c is invertible) has the form:

$$L_c^{-1}(\epsilon L_d + L_c) = \epsilon L_c^{-1} L_d + I.$$

For fixed h, as $\epsilon \to 0$, this preconditioned system approaches I (since L_c and L_d depend only on h), and yields a fast rate of convergence for most iterative methods such as GMRES, BCG, CGS, QMR, etc. However, when h and ϵ are varied, the rate of convergence will depend on h and ϵ, and numerical tests indicate that it deteriorates as $h \to 0$.

In the general case, where L_c may not be invertible (such as for re-circulating flows, i.e, closed characteristic curves), it is possible to obtain an invertible preconditioner by a minor modification of L_c, by the local addition of diffusion terms in the layer region and in the region in the center of the re-circulation. This addition of local diffusion terms, and the technique for solving the preconditioned system will be the main issues we consider in sections 3 and 4. It is hoped that this will yield convergence rates which do not deteriorate when the mesh is refined locally in the diffusion region.

2.2. Approximation Properties of the Convection Problem. We now present a heuristic discussion of the approximation error $u_{ex} - u_c$, when the full convection diffusion problem is replaced by the convection problem:

$$(\epsilon L_d + L_c) u_{ex} = f, \quad \text{replaced by} \quad L_c u_c = f_c.$$

The error $u_{ex} - u_c$ is of interest in assessing the closeness of both problems.

Let us suppose, for simplicity that $f = f_c$, i.e., $f_d = 0$ (this can occur for instance when the Dirichlet data is zero). Then,

$$L_c(u_c - u_{ex}) = f_c - L_c u_{ex} = (\epsilon L_d + L_c)u_{ex} - L_c u_{ex} = \epsilon L_d u_{ex}.$$

Applying L_c^{-1} to both sides, we estimate that:

$$\|u_c - u_{ex}\| \leq \epsilon \|L_c^{-1}\| \|L_d u_{ex}\|.$$

Now, if $\|L_c^{-1}\|$ is uniformly bounded for all h, then the error would depend primarily on $\|L_d u_{ex}\|$. For special choices of boundary conditions, yielding no boundary layers, this quantity may be uniformly bounded independent of h and ϵ and then the error would go to zero as $\epsilon \to 0$. However, u_{ex} generally has boundary layers (whose thickness depends on ϵ) and so the error will in general not approach zero as $\epsilon \to 0$.

A similar error bound can be derived when the convection matrix L_c is modified by the addition of some local diffusion terms, $M_{d,loc}$, see [9]:

$$M = \epsilon M_{d,loc} + L_c,$$

where $M_{d,loc}$ locally equals L_d in certain subregions, and is zero in the rest of the domain. Two kinds of approximations $M_{d,loc}$ will be considered in sections 3 and 4.

We emphasize that though L_c may often provide an easily invertible preconditioner (depending on a and b), its convergence rate will deteriorate as $h \to 0$, due to the growth in eigenvalues of $L_c^{-1} L_d$.

3. Adapting the Vanishing Viscosity Method of Gastaldi, Quarteroni, and Sacchi-Landriani as Preconditioner. Assume that the domain Ω of the convection diffusion problem (1) is partitioned into two disjoint subdomains Ω_d and Ω_c:

$$\Omega = \Omega_c \cup \Omega_d,$$

where the subdomain Ω_d is chosen to correspond to the *elliptic* or *diffusion* region (containing the boundary or interior layers), and Ω_c corresponds to the *hyperbolic* or *convection* region (in which the convection term is dominant and the solution is smooth). The basic mixed elliptic-hyperbolic approximation to (1) of Gastaldi, Quarteroni, Sacchi-Landriani and Valli [9, 14], is obtained by adding a suitable local diffusion operator $M_{d,loc}$ to the convection term L_c. Accordingly, let us denote a local diffusion operator M_{d,Ω_d} on domain Ω_d as follows:

$$M_{d,\Omega_d} \equiv -\nabla \cdot (m(x,y)\nabla u), \text{ where } m(x,y) = \begin{cases} 1 & \text{in } \Omega_d \\ 0 & \text{in } \Omega - \Omega_d \end{cases}.$$

The mixed elliptic-hyperbolic operator M is then defined by:

$$Mu \equiv \epsilon M_{d,\Omega_d} u + L_c u. \tag{2}$$

In [9], a set of *transmission conditions* are also given for the solution to satisfy on the interface between the elliptic and the hyperbolic regions. In the Appendix, we shall give a heuristic, but perhaps more easily understood, derivation of this set of transmission conditions, by studying a particular discretization scheme near the interface as h tends to zero.

Note, the difference between the original convection diffusion operator of (1) and the mixed elliptic-hyperbolic operator (2) is:

$$(\epsilon L_d + L_c) - (\epsilon M_{d,\Omega_d} + L_c) = \epsilon M_{d,\Omega_c},$$

a diffusion operator on Ω_c.

A heuristic argument can be given, similar to that given in Section 2.2, which explains why the mixed operator M can be a good approximation to L^h. Let u_m denote the solution to the mixed problem. Following Section 2.2, one can easily derive:

$$||u_{ex} - u_m|| \leq \epsilon ||M^{-1}|| ||(M_{d,\Omega_d} - L_d)u_{ex}|| = \epsilon ||M^{-1}|| ||M_{d,\Omega_c} u_{ex}||.$$

As shown above, the term $M_{d,\Omega_c} u_{ex}$ is equal to Δu_{ex} in the interior of Ω_c and vanishes identically on Ω_d. Therefore, if Ω_d is chosen so that the boundary layers are captured properly, then $||M_{d,\Omega_c} u_{ex}||$ is bounded and therefore $||u_{ex} - u_m|| \leq O(\epsilon)$ if we further assume $||M^{-1}||$ is bounded.

One of the advantages of this mixed elliptic-hyperbolic problem is that it can be solved efficiently by a Dirichlet-Neumann type iterative procedure, see [9, 2], having a rate of convergence essentially independent of ϵ and the mesh parameter h, see [9]. The first preconditioner we consider for the full convection diffusion problem is just a simple modification of the mixed elliptic-hyperbolic problem (2). In the next subsection, we shall give a matrix interpretation of this preconditioner.

3.1. Preconditioner M_1 based on mixed elliptic-hyperbolic problem. Let $\Gamma \equiv \partial\Omega_d \cap \partial\Omega_c$ be the interface separating the two subdomains. Based on these subdomains, we introduce a reordering of the unknowns x into two blocks, so that the first block corresponds to the unknowns in the *interior* of Ω_c and the second block corresponds to the unknowns in $\Omega_d \cup \Gamma$:

$$x = (x_1, x_2).$$

Corresponding to this reordering, the matrices L^c and L^d have a blockstructure:

$$L^c = \begin{bmatrix} L^c_{11} & L^c_{12} \\ L^c_{21} & L^c_{22} \end{bmatrix}, \text{ and } L^d = \begin{bmatrix} L^d_{11} & L^d_{12} \\ L^d_{21} & L^d_{22} \end{bmatrix}.$$

The full convection diffusion matrix is then given by:

$$L = \epsilon L_d + L_c = \begin{bmatrix} \epsilon L^d_{11} + L^c_{11} & \epsilon L^d_{12} + L^c_{12} \\ \epsilon L^d_{21} + L^c_{21} & \epsilon L^d_{22} + L^c_{22} \end{bmatrix}.$$

Similarly, the finite difference approximation to the local diffusion operator M_{d,Ω_d} has the form:

$$M_{d,\Omega_d} = \begin{bmatrix} 0 & 0 \\ 0 & M^d_{22} \end{bmatrix},$$

where M^d_{22} corresponds to the discretization of Laplacian on Ω_d with *Neumann* boundary conditions on Γ:

$$-\Delta u = f, \text{ on } \Omega_d, \quad \text{with } u = 0 \text{ on } \partial\Omega_d \cap \partial\Omega, \ \frac{\partial u}{\partial n} = g \text{ on } \Gamma.$$

And so, the mixed elliptic-hyperbolic discretization matrix M_{gqs} corresponding to (2) is:

$$M_{gqs} = \epsilon M_{d,\Omega_d} + L_c = \begin{bmatrix} L_{11}^c & L_{12}^c \\ L_{21}^c & \epsilon M_{22}^d + L_{22}^c \end{bmatrix}.$$

We define the preconditioner M_1 by setting L_{12}^c to zero:

$$M_1 = \begin{bmatrix} L_{11}^c & 0 \\ L_{21}^c & \epsilon M_{22}^d + L_{22}^c \end{bmatrix}. \tag{3}$$

We note that since $L_{12}^c \neq 0$ in general, M_1 is not the mixed elliptic-hyperbolic approximation, but corresponds to one step in a Dirichlet-Neumann or block Gauss-Seidel type iteration to solve a mixed system. In the case of discretization by spectral methods, Gastaldi, Quarteroni and Sacchi-Landriani [9] showed that the Dirichlet-Neumann iteration to solve the mixed problem converges at a rate independent of N (the order of the spectral method) and ϵ. Heuristically, we expect that in the finite difference case, if M_1 is used as preconditioner for M_{gqs}, it would also converge at a rate independent of h and ϵ.

Note that the system:

$$\begin{bmatrix} L_{11}^c & 0 \\ L_{21}^c & \epsilon M_{22}^d + L_{22}^c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

can be solved in two steps. In step one, solve:

$$L_{11}^c x_1 = f_1,$$

using an ordering which makes the system lower triangular (because it is a purely hyperbolic problem). In step two, solve:

$$(\epsilon M_{22}^d + L_{22}^c) x_2 = f_2 - L_{21}^c x_1.$$

This is potentially a costly step, but an alternate preconditioner may be obtained by replacing $\epsilon M_{22}^d + L_{22}^c$ by a suitably scaled version of the Laplacian or a preconditioner for the Laplacian. Numerical tests of preconditioner M_1 will be presented in section 5.

4. Relation to the Physically Motivated Domain Decomposition of Ashby, Saylor, and Scroggs . We next briefly describe another recently proposed convection diffusion preconditioner by Ashby, Saylor and Scroggs [1], which we also included in our numerical studies. Like the mixed elliptic-hyperbolic problem used in [9, 14], this preconditioner is obtained from the convection matrix L_c by addition of a local diffusion term in a domain Ω_d. But the precise diffusion operator differs from that used in [9]. As before, the domain Ω is partitioned into two subregions Ω_c and Ω_d. It is suitable to choose Ω_c and Ω_d so that they overlap by one grid, in such a way that the interior nodes x_1 in Ω_c and the interior nodes x_2 in Ω_d partition the unknowns into two blocks:

$$x = (x_1, x_2).$$

Corresponding to this partition, the 1st order upwind discretization has the following block structure:

$$L^h = \begin{bmatrix} \epsilon L_{11}^d + L_{11}^c & \epsilon L_{12}^d + L_{12}^c \\ \epsilon L_{21}^d + L_{21}^c & \epsilon L_{22}^d + L_{22}^c \end{bmatrix}, \text{ with } L^h \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix},$$

and the local diffusion matrix used in [1] can be written:

$$M_{d,pmdd} = \begin{bmatrix} 0 & 0 \\ L_{21}^d & L_{22}^d \end{bmatrix}.$$

Note that $M_{d,pmdd}$ is a submatrix of the diffusion problem L_d and differs from the local diffusion term used in [9]. The physically motivated domain decomposition preconditioner M_{pmdd} was defined in [1] for the case of a uni-directional flow field to be:

$$M_{pmdd} = \begin{bmatrix} \epsilon L_{11}^c & 0 \\ \epsilon L_{21}^d + L_{21}^c & \epsilon L_{22}^d + L_{22}^c \end{bmatrix}. \tag{4}$$

This preconditioner is easily inverted in two steps as follows: In step one solve $L_{11}^c x_1 = f_1$. In step 2, solve

$$(\epsilon L_{22}^d + L_{22}^c)x_2 = f_2 - (\epsilon L_{21}^d + L_{21}^c)x_1.$$

The coefficient matrix in step 1 is lower triangular for suitable orderings (again because the problem is hyperbolic), and can be inverted efficiently. In order to reduce the cost of inversion in step 2, Ashby, Saylor and Scroggs approximate the coefficient matrix by an upper triangular matrix, but in our tests we solve this by a direct method. This is potentially the most expensive step of the preconditioning, and as mentioned earlier, its cost can be reduced by replacing $(\epsilon L_{22}^d + L_{22}^c)$ by any suitable preconditioner for the Laplacian on Ω_d.

4.1. Some Differences Between the Two Approaches. One of the differences between the approaches of Gastaldi, Quarteroni and Sacchi-Landriani [9] and of Ashby, Saylor and Scroggs [1] can best be illustrated in the case of uni-directional flow fields (a, b). Namely, M_{pmdd} enforces continuity of the solution across the two subdomains, while M_{gqs} enforces continuity of the flux. In fact, M_{pmdd} corresponds to a Schwarz alternating method (with one grid point overlap), see [13, 10], to solve the convection diffusion problem based on subdomains Ω_c and Ω_d in which the convection-diffusion problem on Ω_c is replaced by the convection problem on Ω_c.

Below, these differences are seen for a simple one dimensional model problem:

$$-\epsilon \frac{d}{dx}\left(m(x)\frac{du}{dx}\right) + \frac{du}{dx} = f(x), \quad \text{on } (0,1) \text{ with } \quad u = 0 \text{ for } x = 0, 1,$$

where:

$$m(x) = \begin{cases} 0 & x \in [0, c] \\ 1 & x \in [c, 1] \end{cases}$$

The preconditioners M_1 and M_{pmdd} for this model problem discretized by the upwind scheme on a grid with 3 nodes each in Ω_c and Ω_d are:

$$M_1 = h^{-2} \begin{bmatrix} h & & & & & \\ -h & h & & & & \\ & -h & h & & & \\ & & -h & \epsilon + h & -\epsilon & \\ & & & -\epsilon - h & 2\epsilon + h & -\epsilon \\ & & & & -\epsilon - h & 2\epsilon + h \end{bmatrix}$$

$$M_{pmdd} = h^{-2} \begin{bmatrix} h & & & & & \\ -h & h & & & & \\ & -h & h & & & \\ & & -\epsilon - h & 2\epsilon + h & -\epsilon & \\ & & & -\epsilon - h & 2\epsilon + h & -\epsilon \\ & & & & -\epsilon - h & 2\epsilon + h \end{bmatrix}.$$

The two matrices differ in the (4,3) and (4,4) entry.

5. Numerical Experiments. We now present the results of numerical tests of the two main preconditioners we have described: M_1 of equation (3), based on the modification of the mixed elliptic-hyperbolic approximation M_{gqs} of [9], and M_{pmdd} of [1], described in equation (4). For comparison purposes, we have also included tests with the standard ILU preconditioner (with natural lexicographical ordering).

Three sample convection diffusion elliptic problems of the following form were considered on the domain $\Omega = [0,1]^2$:

$$-\epsilon \Delta u + a(x,y) u_x + b(x,y) u_y = 0, \quad \text{in } \Omega, \quad u = g(x,y) \text{ on } \partial\Omega.$$

The three choices of coefficients and subdomains were:

1. Uni-directional flow (same as test problem used in [1]):

$$a(x,y) = 0.5, \quad b(x,y) = 1.5, \quad g(x,y) = \begin{cases} 1 & y = 1 \\ 0 & \text{elsewhere} \end{cases}$$

   The subdomains were chosen:

$$\Omega_d = \{(x,y) : y > 1 - (1/6)\}; \ \Omega_c = \Omega - \Omega_d.$$

2. Recirculating flow centered about (0.5, 0.5):

$$a(x,y) = -(y - 0.5), \quad b(x,y) = (x - 0.5), \quad g(x,y) = \begin{cases} 2 & x = 1 \\ 1 & \text{elsewhere} \end{cases}$$

   The subdomains were chosen:

$$\Omega_d = \{(x,y) : |x - 0.5| < 1/12, \text{ or } y > 1 - (1/6), \text{ or } y < 1/6\}; \ \Omega_c = \Omega - \Omega_d.$$

3. Quadrant of re-circulating flow, centered about (0, 0):

$$a(x,y) = y, \quad b(x,y) = -x, \quad g(x,y) = \begin{cases} 2 & x = 1 \\ 1 & \text{elsewhere} \end{cases}$$

   The subdomains were chosen:

$$\Omega_d = \{(x,y) : x > 1 - (1/6), \text{ or } y > 1 - (1/6), \text{ or } y < (1/6)\}; \ \Omega_c = \Omega - \Omega_d.$$

TABLE 1
ILU for Uni-Dir flow (flow 1).

$h \setminus \epsilon$		1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
1/5		4	5	4	4	4	4	3	3	3	2
1/9		5	6	6	6	6	5	4	4	3	3
1/17		8	9	10	10	10	9	6	5	4	4
1/33		10	13	15	16	17	14	11	9	6	5

TABLE 2
ILU for Recirculating flow (flow 2).

$h \setminus \epsilon$		1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
1/5		4	5	5	5	6	6	5	6	5	5
1/9		6	6	7	8	8	9	9	9	9	8
1/17		9	9	11	12	13	16	17	17	16	15
1/33		13	14	15	21	23	30	33	36	37	34

TABLE 3
ILU for Quadrant of recirculating flow (flow 3).

$h \setminus \epsilon$		1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
1/5		4	4	5	5	4	5	4	4	4	4
1/9		6	6	6	8	8	7	7	6	6	6
1/17		9	9	11	13	13	13	13	11	10	10
1/33		13	14	16	20	24	25	24	19	17	17

TABLE 4
M_1 (GQS) for Uni-Dir flow (flow 1).

$h \setminus \epsilon$		1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
1/5		15	12	11	9	7	5	4	4	3	3
1/9		36	29	22	18	11	8	6	4	4	3
1/17		87	69	45	32	18	13	8	5	4	4
1/33		*	*	107	64	43	21	12	8	5	4
1/65		*	*	*	*	111	52	20	13	7	5

TABLE 5
M_1 (GQS) for Recirculating flow (flow 2).

$h \setminus \epsilon$		1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
1/5		17	16	15	13	13	11	9	8	6	6
1/9		49	49	42	33	27	20	15	13	10	8
1/17		131	110	87	71	44	37	25	18	15	11
1/33		*	*	*	*	116	89	64	38	25	18
1/65		*	*	*	*	*	*	133	82	49	35

TABLE 6

M_1 (GQS) for Quadrant of recirculating flow (flow 3).

$h \setminus \epsilon$	1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
1/5	13	13	12	12	9	9	7	6	5	4
1/9	35	32	30	24	18	13	9	8	6	5
1/17	106	88	68	55	36	24	17	11	8	6
1/33	*	*	*	139	72	54	33	21	14	10
1/65	*	*	*	*	*	138	75	53	30	18

TABLE 7

M_{pmdd} for Uni-Dir flow (flow 1).

$h \setminus \epsilon$	1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
1/5	13	12	11	9	7	5	4	4	3	3
1/9	30	27	22	16	12	8	6	4	4	3
1/17	61	56	43	32	21	12	8	6	4	4
1/33	140	134	110	65	42	27	11	8	5	4
1/65	*	*	*	*	107	64	21	12	7	5

In all our tests, we used the Bi-Conjugate gradient algorithm [12, 7] to solve the preconditioned system:

$$M^{-1}Lu = M^{-1}f.$$

The stopping criterion was

$$\frac{||r_k||}{||r_0||} \leq 10^{-5},$$

for the Euclidean norm $||.||$, where r_k denotes the kth residual. The number of iterations are tabulated in tables 1 through 9 for various choices of mesh sizes h and viscosity ϵ. An $*$ indicates that the number of iterations exceeded 149. The mesh sizes ranged from:

$$h = 1/5,\ 1/9,\ 1/17,\ 1/33,\ 1/65,$$

while the viscosity ranged from:

$$\epsilon = 1,\ 2^{-1},\ \cdots, 2^{-9}.$$

Tables 1, 2, 3 contain results of the standard ILU preconditioner (in lexicographical ordering) for the 3 test problems. Tables 4, 5, and 6 contain results of tests with the preconditioner M_1 (based on M_{gqs}). Tables 7, 8, and 9 contain results of test with the physically motivated domain decomposition preconditioner M_{pmdd}.

6. Summary and Remarks. The numerical results indicate that the two main preconditioners M_1 and M_{pmdd} improve as $\epsilon \to 0$, but deteriorate as $h \to 0$. In fact, the number of iterations seems to increase at a rate proportional to $O(\epsilon/h)$. This should not be surprising because both preconditioners reduce to the convection operator L_c in Ω_c and therefore as $h \to 0$ with ϵ fixed, we expect the conditioning

TABLE 8
M_{pmdd} for Recirculating flow (flow 2).

$h \setminus \epsilon$	1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
1/5	14	14	14	14	13	11	10	8	6	4
1/9	30	26	24	23	21	17	13	10	9	8
1/17	69	69	66	61	47	39	27	23	15	11
1/33	*	148	138	132	96	77	51	35	25	18
1/65	*	*	*	*	*	*	126	88	50	33

TABLE 9
M_{pmdd} for Quadrant of Recirculating flow (flow 3).

$h \setminus \epsilon$	1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}	2^{-8}	2^{-9}
1/5	15	13	14	12	11	8	7	5	4	3
1/9	31	28	26	25	21	14	10	7	7	5
1/17	72	69	60	52	38	26	16	12	9	6
1/33	*	*	145	119	79	55	32	21	13	9
1/65	*	*	*	*	*	*	72	40	25	16

(or spread of eigenvalues) $\kappa(M^{-1}L^h)$ to behave as $\kappa(I + \epsilon L_c^{-1}L_d) = O(\epsilon/h)$ since $\kappa(L_c^{-1}L_d) = O(h^{-1})$. Thus no convection based preconditioners can be spectrally equivalent to L^h as $h \to 0$. However, it may not be necessary to use a value for h much smaller than ϵ in Ω_c if the boundary layers are captured properly in Ω_d (e.g. with local mesh refinement) and in such situations the convergence rates of these two preconditioners may be quite acceptable.

One can also observe that there is little difference between the performance of the preconditioners M_1 and M_{pmdd}. Note that the ILU preconditioner performed uniformly well in the uni-directional flow case, since the nodes where aligned with the characteristic directions. It especially works well for the case of large diffusion, unlike the other two preconditioners. This seems to indicate that if a suitable algorithm is available for permuting the unknowns along the characteristic directions, then ILU could provide a very effective preconditioner.

Appendix

A. Heuristic derivation of transmission boundary conditions for the mixed elliptic-hyperbolic problem. This appendix is included to provide a heuristic derivation of the transmission boundary conditions associated with the mixed elliptic-hyperbolic problem, see [9]. The solution of a standard elliptic problem, such as:

$$-\nabla \cdot (d(x,y)\nabla u) + L_c u = f, \quad u = 0 \text{ on } \partial\Omega, \tag{5}$$

where

$$d(x,y) \equiv \begin{cases} \eta & \text{in } \Omega_c \\ \epsilon & \text{in } \Omega_d \end{cases}$$

satisfies the following *transmission boundary conditions*:

$$\begin{aligned} u^+ &= u^- && \text{on } \Gamma \\ \eta \tfrac{\partial u^-}{\partial n} - (a,b)\cdot \vec{n}_1 u^- &= \epsilon \tfrac{\partial u^+}{\partial n} - (a,b)\cdot \vec{n}_1 u^+ && \text{on } \Gamma, \end{aligned}$$

where $+$ refers to the limit approaching Γ from Ω_d, and $-$ refers to the limit from Ω_c, and $\vec{n}_1$ is the outward normal to Γ from Ω_c. Heuristically, as $\eta \to 0$, problem (5) approaches the mixed elliptic-hyperbolic problem (2) and the flux transmission conditions approach:

$$(6) \qquad -(a,b)\cdot\vec{n}_1 u^- = \epsilon\frac{\partial u^+}{\partial n} - (a,b)\cdot\vec{n}_2 u^+, \quad \text{on } \Gamma.$$

However, it is not true that $u^+ = u^-$ on Γ, see [9], i.e., the solution can be discontinuous across parts of the interface. In fact, the transmission conditions associated with the mixed problem (2) are different from the standard elliptic case, because the solution of the mixed problem need not be continuous across the interface Γ (it will in general be continuous only across the inflow boundary of Ω_c). However, the flux ($\vec{n}\cdot(m\nabla u - (a,b)u)$) will be continuous across the interface. Taking into account this possible discontinuity in u, the valid transmission boundary condition for (2) was shown in [9] to be:

$$(7) \qquad \begin{array}{rcll} 0 & = & \epsilon\frac{\partial u^+}{\partial n}, & \text{on } \Gamma_{in} \\ -u^-(a,b)\cdot\vec{n}_1 & = & \epsilon\frac{\partial u^+}{\partial n} - u^+(a,b)\cdot\vec{n}_1, & \text{on } \Gamma_{out} \end{array}$$

where

$$\begin{array}{l} \Gamma_{in} = \{(x,y)\in\Gamma : (a(x,y),b(x,y))\cdot\vec{n}_1 < 0\} = \text{ inflow boundary },\\ \Gamma_{out} = \{(x,y)\in\Gamma : (a(x,y),b(x,y))\cdot\vec{n}_1 > 0\} = \text{ outflow boundary }. \end{array}$$

Here we present a heuristic derivation of these using the 1st order upwind finite difference approximation of problem (2).

To simplify this heuristic derivation, we will make the following assumptions about the domain:

$$\Omega = [0,1]^2, \quad \text{with } \Omega_c \equiv [0,c]\times[0,1], \quad \text{and } \Omega_d \equiv [c,1]\times[0,1],$$

for some $c \in (0,1)$. For this choice of subdomains, the outward normal on Γ for Ω_c is $\vec{n}_1 = (1,0)$. If u is smooth within the interior of each subdomain, by construction the upwind discretization M^h of the mixed problem (2) satisfies:

$$\begin{array}{ll} M^h u = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + O(h), & (x_i,y_j)\in \text{ interior } \Omega_c \\ M^h u = -\epsilon\Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + O(h), & (x_i,y_j)\in \text{ interior } \Omega_d \end{array}.$$

We now consider the difference approximations on the interface Γ. Suppose (x_i, y_j) is a node on the inflow boundary Γ_{in}, i.e.,

$$\vec{n}_1\cdot(a,b) < 0.$$

Since $\vec{n}_1 = (1,0)$, it follows that $a < 0$ in the neighborhood of (x_i, y_j). So the upwind finite difference approximation yields:

$$(M^h u)_{ij} = \begin{array}{c} \frac{\epsilon}{h^2}(2u_{ij} - 0.5u_{i,j+1} - 0.5u_{i,j-1} - u_{i+1,j}) \\ -\frac{1}{h}a_{i+1/2,j}(u_{ij} - u_{i+1,j}) \\ +\frac{b^+_{i,j-1/2}}{h}(u_{ij} - u_{i,j-1}) - \frac{b^-_{i,j+1/2}}{h}(u_{ij} - u_{i,j+1}) \end{array} = f_{ij},$$

which when multiplied by h results in:

$$(8)\qquad h(M^h u)_{ij} = \begin{array}{c} \frac{\epsilon}{h}\left(2u_{ij} - 0.5u_{i,j+1} - 0.5u_{i,j-1} - u_{i+1,j}\right) \\ -a_{i+1/2,j}(u_{ij} - u_{i+1,j}) \\ +b^{+}_{i,j-1/2}(u_{ij} - u_{i,j-1}) - b^{-}_{i,j+1/2}(u_{ij} - u_{i,j+1}) \end{array} = hf_{ij}.$$

Now, since on $\Omega_d \cup \Gamma$ the discrete solution u^h is the solution of:

$$(9)\qquad \begin{cases} M^h u^h = & -\epsilon \Delta u + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + O(h) = f & \text{in } \Omega_d \\ u^h = & u & \text{on } \Gamma, \\ u^h = & 0 & \text{on } \partial\Omega_d - \Gamma, \end{cases}$$

we expect heuristically that elliptic regularity results hold and that the solution on $\Omega_d \cup \Gamma$ is smooth (Note: this would not hold on $\Omega_c \cup \Gamma$ since the problem on Ω_c is not elliptic and u_1 depends only on the data on Γ_{in}). Then, by applying Taylor series expansions to the discretization on Γ, we obtain that on Γ_{in}:

$$\begin{aligned} \tfrac{\epsilon}{h}\left(2u_{ij} - 0.5u_{i,j+1} - 0.5u_{i,j-1} - u_{i+1,j}\right) &= -\epsilon\tfrac{\partial u}{\partial n} + O(h) \\ -a_{i+1/2,j}(u_{ij} - u_{i+1,j}) &= O(h) \\ +b^{+}_{i,j-1/2}(u_{ij} - u_{i,j-1}) - b^{-}_{i,j+1/2}(u_{ij} - u_{i,j+1}) &= O(h) \\ hf_{ij} &= O(h) \end{aligned}$$

Using these in (8),

$$(10)\qquad \Longrightarrow \qquad -\epsilon\frac{\partial u}{\partial n} = 0 + O(h), \qquad \text{on } \Gamma_{in}.$$

Similarly, we consider the difference stencil on the outflow boundary Γ_{out}, i.e., on grid points $(x_i, y_j) \in \Gamma$ such that $a(x_i, y_j) > 0$, and obtain that:

$$(M^h u)_{ij} = \begin{array}{c} \frac{\epsilon}{h^2}\left(2u_{ij} - 0.5u_{i,j+1} - 0.5u_{i,j-1} - u_{i+1,j}\right) \\ +\frac{1}{h}a_{i-1/2,j}(u_{ij} - u_{i-1,j}) \\ +\frac{b^{+}_{i,j-1/2}}{h}(u_{ij} - u_{i,j-1}) - \frac{b^{-}_{i,j+1/2}}{h}(u_{ij} - u_{i,j+1}) \end{array} = f_{ij},$$

which when multiplied by h results in:

$$(11)\qquad \begin{array}{c} \frac{\epsilon}{h}\left(2u_{ij} - 0.5u_{i,j+1} - 0.5u_{i,j-1} - u_{i+1,j}\right) \\ +a_{i-1/2,j}(u_{ij} - u_{i-1,j}) \\ +b^{+}_{i,j-1/2}(u_{ij} - u_{i,j-1}) - b^{-}_{i,j+1/2}(u_{ij} - u_{i,j+1}) \end{array} = hf_{ij}.$$

Again, using heuristically, the smoothness of u on $\Omega_d \cup \Gamma$, we obtain that:

$$\begin{aligned} \tfrac{\epsilon}{h}\left(2u_{ij} - 0.5u_{i,j+1} - 0.5u_{i,j-1} - u_{i+1,j}\right) &= -\epsilon\tfrac{\partial u}{\partial n} + O(h), \\ +a_{i-1/2,j}(u_{ij} - u_{i-1,j}) &= a_{i,j}u^{+} - a_{i,j}u^{-} + O(h) \\ +b^{+}_{i,j-1/2}(u_{ij} - u_{i,j-1}) - b^{-}_{i,j+1/2}(u_{ij} - u_{i,j+1}) &= O(h) \\ hf_{ij} &= O(h) \end{aligned}$$

Substituting these in (11), we obtain:

$$(12)\qquad \Longrightarrow \qquad -\epsilon\frac{\partial u^{+}}{\partial n} + au^{+} = au^{-}, \qquad \text{on } \Gamma_{out}.$$

(10) and (12) provide the transmission boundary conditions for the flux. Next, we note that for the upwind difference approximation, the solution in Ω_c depends on the data u on Γ_{in}, and by properties of difference scheme, the solution can be expected to be continuous as Γ_{in} is approached from Ω_c, i.e.,

$$u^- = u^+, \qquad \text{on } \Gamma_{in}. \tag{13}$$

Since the solution in Ω_c does not depend on the boundary values on Γ_{out}, it will in general not be be continuous across Γ_{out}. Thus, heuristically, equations (10), (12) and (13) together yield an $O(h)$ approximation to the transmission conditions (7) derived in [9].

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