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PRECONDITIONING NON-SYMMETRIC AND INDEFINITE CAPACITANCE MATRIX PROBLEMS IN DOMAIN IMBEDDING

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ABSTRACT. Based on the duality between the interface domain decomposition (or DD) methods and the capacitance matrix (or CM) methods in domain imbedding and on the existing results for preconditioning non-symmetric and indefinite finite element elliptic problems preconditioners of optimal order for the CM problems are constructed. The preconditioning technique explores two–level hierarchical discretization of the imbedded problem; on a coarse grid (of fixed size) and on a fine grid. The major part of the preconditioning then is reduced to solving systems with preconditioners for the capacitance matrix that corresponds to the principal symmetric and coercive part of the elliptic operator. The theory is illustrated with numerical experiments.

1. Introduction. We consider second order elliptic problems in polygonal planar domains Ω with Dirichlet boundary conditions. Polygonal domain Ω is imbedded in a rectangle $\mathfrak{R}$ on which a fast separable elliptic solver can be used. Hence we assume that the coefficients of the elliptic operator allow separation of variables. In a previous paper (cf. [20]) we studied various preconditioners for the capacitance matrix in domain imbedding for self-adjoint and coercive elliptic operators with separable coefficients. In the present paper we extend these results for indefinite and non-symmetric elliptic operators also with separable coefficients. The main difficulty is the indefiniteness while the nonsymmetry can be handled with proper transformation of the separable coefficients. Nevertheless, one may as well want to use the original formulation and thus to deal with non-symmetric problems.

The purpose of the domain imbedding technique is that problems on irregular regions can be solved by minor modification of the software available for standard problems (i.e., on rectangular domains) thus avoiding more complicated data structure needed for the irregular domains. This technique was proposed by Buzbee,

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Dorr, George, and Golub [9] as a direct procedure, and studied later as an iterative process by Proskurowski and Widlund [21], [22], O'Leary and Widlund [19], Astrakhantsev [1], [2], Dryja [11], Börgers and Widlund [6], Nepomnyaschikh [18] and others. In Börgers and Widlund [6] a detailed study of the derivation and comparison of various capacitance matrix techniques and their practical performance was presented. Similar approaches, namely to avoid complicated data structure and to take advantage of already available software for problems on standard domains, have been used in deriving efficient preconditioning techniques for elliptic problems on grids with local refinement, cf., McCormick and Thomas [16], McCormick [17], Bramble, Ewing, Pasciak, and Schatz [7], Ewing, Lazarov, and Vassilevski [14].

The essence of the capacitance matrix technique is that the solution to the original problem can be obtained as a low order modification to the problem on the imbedding rectangle. One can observe a duality between the capacitance matrix technique in domain imbedding and the Schur complement methods in domain decomposition. More precisely, the capacitance matrix can be viewed as the inverse of a Schur complement that appears in the domain decomposition technique (see [20]). Based on this duality we extend here the preconditioning technique in a subspace developed for non-symmetric and indefinite elliptic problems in Vassilevski [27] to the capacitance matrix problems in domain imbedding. The subspace in the case of non-symmetric and indefinite finite element stiffness matrices (as used in Vassilevski [27]) is an orthogonal complement of a vector space corresponding to a coarse (and fixed) finite element discretization space. In the present paper we also use a vector space of a small and fixed size. It corresponds to a set of coarse–grid nodes on the interface across which the original domain is imbedded into a rectangle. As a consequence, one can construct symmetric and positive definite preconditioners for the major principal submatrix of the capacitance matrix (in a two–level hierarchical block form) corresponding to the complement of the thus defined space; namely, to the nodes on the interface that are complementary to the coarse–grid nodes. These preconditioners are based on any available (symmetric and positive definite) preconditioner for the capacitance matrix constructed for the principal symmetric and coercive part (which also has separable coefficients) of the original bilinear form. Such preconditioners were analyzed in our previous paper [20]. In the present paper we derive the Capacitance Matrix Method that is based on a standard block–Choleski factorization with one block in the factorization being the major block corresponding to the complementary node set on the interface and a second block on the diagonal in the factorization that is of small and fixed size. This block can be assembled explicitly in a preprocessing stage at a cost of a fixed number of solvings with the matrix in the imbedding domain.

The outline of the paper is as follows. In §2 the problem is formulated and the capacitance matrix method is briefly outlined. In §3 the change to a two–level hierarchical basis is presented. §4 contains the description of the solution method based on the block–Choleski factorization of the capacitance matrix in the two–level hierarchical block–form. §5 is devoted to the analysis of the properties of the major block of the capacitance matrix and the way of its preconditioning. The main theoretical results are formulated here. Finally, in §6 the numerical results are presented.

2. Problem formulation. Consider the following second order elliptic boundary value problem:

Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that

$$\mathcal{A}(u,\phi) = (f,\phi), \quad \text{for all } \phi \in H_0^1(\Omega), \tag{1}$$

where Ω is a planar polygon and

$$\begin{aligned} \mathcal{A}(u,\phi) &= \int_{\Omega} \{\left(k_1(x)\tfrac{\partial u}{\partial x}\tfrac{\partial \phi}{\partial x} + k_2(y)\tfrac{\partial u}{\partial y}\tfrac{\partial \phi}{\partial y}\right) \\ &\quad + \left(b_1(x)\tfrac{\partial u}{\partial x} + b_2(y)\tfrac{\partial u}{\partial y}\right)\phi - [\lambda_1(x)+\lambda_2(y)]\,u\phi\}dx\,dy. \end{aligned} \tag{2}$$

The coefficients k_1, k_2 are assumed to be positive in Ω, b_1 and b_2 to be smooth, and λ_1 and λ_2 are allowed to be nonnegative in Ω. Problem (1) is solvable with some exceptions for $\lambda = -\lambda_1 - \lambda_2$. We assume that (1) is solvable. Let $\mathfrak{R}$ be a rectangle such that $\mathfrak{R} \supset \Omega$. In our model case shown on Figure 1, $\mathfrak{R} = (0,1)^2$.

We assume that $\mathfrak{R}$ can be triangulated on a right-angled triangular mesh $\mathcal{T}$ such that $\partial\Omega$ consists of edges of triangles from $\mathcal{T}$. This imposes some restriction on the polygonal form of Ω. A treatment of how to construct triangulations for the domain imbedding in a more general case was presented in Börgers and Widlund [6], see also Nepomnyaschikh [18]. Another possibility is to use patch local refinement in the neighborhood of the interface boundary across which Ω is imbedded in the rectangle $\mathfrak{R}$. Then one can use the efficient iterative elliptic solvers developed for problems on patched locally refined grids for generally non-symmetric and indefinite elliptic problems from Ewing, Petrova, and Vassilevski [15] that extend corresponding results for the symmetric case from McCormick and Thomas [16], McCormick [17], Bramble, Ewing, Pasciak, and Schatz [7], see also Ewing, Lazarov, and Vassilevski [14]. Note that these exploit fast elliptic solvers on the patches which are rectangles. However, at this point we shall not go into this subject more specifically.

We assume that the boundary of Ω is aligned with the triangulation $\mathcal{T}$. Let $\mathfrak{W}$ be the finite element space spanned by piecewise linear functions on $\mathcal{T}$, vanishing on $\partial\mathfrak{R}$ and continuous in $\mathfrak{R}$. The subspace $\mathfrak{V}$ of $\mathfrak{W}$ consists of functions that have supports in $\bar{\Omega}$. We denote by $\mathcal{N}$ the node set in $\mathfrak{R}$ corresponding to the triangulation $\mathcal{T}$. The nodes in Ω are denoted by $\mathcal{N}_1$ and the node set on $\Gamma \equiv \partial\Omega \cap \mathfrak{R}$ is denoted by γ.

Consider finally the bilinear form

$$\begin{aligned} \mathcal{B}(u,\phi) &= \int_{\mathfrak{R}} \{\left(k_1(x)\tfrac{\partial u}{\partial x}\tfrac{\partial \phi}{\partial x} + k_2(y)\tfrac{\partial u}{\partial y}\tfrac{\partial \phi}{\partial y}\right) \\ &\quad + \left(b_1(x)\tfrac{\partial u}{\partial x} + b_2(y)\tfrac{\partial u}{\partial y}\right)\phi - [\lambda_1(x)+\lambda_2(y)]\,u\phi\}dx\,dy. \end{aligned} \tag{3}$$

We denote by A and B the stiffness matrices computed by the bilinear forms $\mathcal{A}(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ using the finite element spaces $\mathfrak{V}$ and $\mathfrak{W}$. Note that those are non-symmetric and possibly indefinite. In the indefinite case (i.e., when $\frac{1}{2}\nabla\cdot\underline{b}+\lambda_1+\lambda_2 > 0$, $\underline{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$) these matrices are invertible only for sufficiently fine meshes.

Remark 1. We note that the elliptic operator that corresponds to the bilinear form $\mathcal{A}(.,.)$

$$\mathfrak{L}u \equiv -\frac{\partial}{\partial x}\left(k_1(x)\frac{\partial u}{\partial x}\right) + b_1(x)\frac{\partial u}{\partial x} - \lambda_1(x)u - \frac{\partial}{\partial y}\left(k_2(x)\frac{\partial u}{\partial y}\right) + b_2(y)\frac{\partial u}{\partial y} - \lambda_2(y)u$$

can be transformed to a self-adjoint one as follows

$$\begin{aligned} p_1(x)p_2(y)\mathfrak{L}u \quad = \quad & -\tfrac{\partial}{\partial x}\left(p_1(x)k_1(x)p_2(y)\tfrac{\partial u}{\partial x}\right) \\ & -\tfrac{\partial}{\partial y}\left(p_1(x)k_2(y)p_2(y)\tfrac{\partial u}{\partial y}\right) \\ & -p_1(x)p_2(y)[\lambda_1(x)+\lambda_2(y)]u. \end{aligned}$$

Here

$$p_1(x) = e^{-\int \frac{b_1(x)}{k_1(x)}\,dx} \quad \text{and} \quad p_2(y) = e^{-\int \frac{b_2(y)}{k_2(y)}\,dy}.$$

However, as we see, the indefiniteness can still be present (if $\lambda_1 + \lambda_2 > 0$). One might as well prefer to work in the original coefficients since separable problems although non-symmetric can be solved equally well as symmetric ones. This is true only for discretizations on uniform meshes and basically for five-point finite difference stencils (or close to that obtained by piecewise linear finite elements).

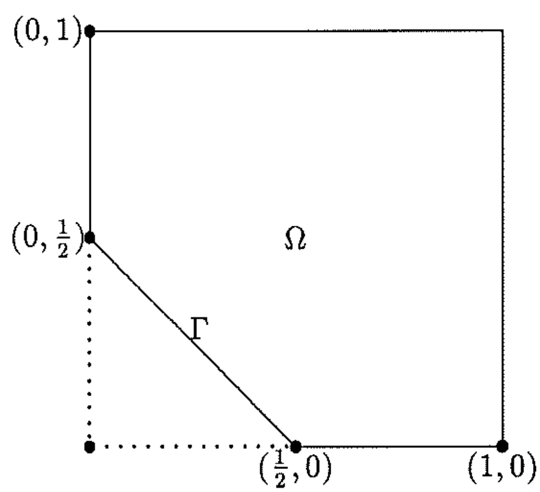

Figure 1. *Polygon Ω imbedded in rectangle $\mathfrak{R} = (0,1)^2$*

As already noted problems with the matrix B can be solved very efficiently based on discrete variants of separation of variables, cf., Buzbee, Golub, and Nielson [8], or on the odd–even cyclic reduction, cf., Schwarztrauber [25], or the generalized marching algorithm, cf., Bank and Rose [4], Bank [5]. The cost is typically $O(N \log N)$ or $O(h^{-2} \log \frac{1}{h})$, where $N = O(h^{-2})$ is the number of nodes in $\mathcal{T}$ and h is the mesh-size.

The Capacitance Matrix Method (or CMM) relies on efficient solvers for problems with the matrix B.

The original problem in Ω after discretization takes the form

$$Ax_1 = \mathbf{b}_1. \tag{4}$$

After imbedding Ω in $\mathfrak{R}$ we consider the problem

$$B\begin{bmatrix}\mathbf{y}_1\\ \mathbf{y}_2\\ \mathbf{y}_0\end{bmatrix} = \begin{bmatrix}\mathbf{b}_1\\ 0\\ \mathbf{b}_0\end{bmatrix} \begin{matrix}\}\ \ \mathcal{N}_1 = \mathcal{N}\cap\Omega\\ \}\ \ \mathcal{N}_2 \equiv (\mathfrak{R}\setminus\bar{\Omega})\cap\mathcal{N},\\ \}\ \ \gamma = \mathcal{N}\cap\Gamma\end{matrix}$$

where $\Gamma = \partial\Omega \setminus \partial\mathfrak{R}$ is the interface boundary across which Ω is imbedded in $\mathfrak{R}$. Note that B admits the following 3×3 DD block form,

$$B = \begin{bmatrix}B_{11} & 0 & B_{10}\\ 0 & B_{22} & B_{20}\\ B_{01} & B_{02} & B_{00}\end{bmatrix} \begin{matrix}\}\ \ \mathcal{N}_1\\ \}\ \ \mathcal{N}_2\,.\\ \}\ \ \gamma\end{matrix}$$

Finally note that $B_{11} = A$. Consider now the following extended matrix,

$$A^E = \begin{bmatrix}A & 0 & B_{10}\\ 0 & B_{22} & B_{20}\\ 0 & 0 & I\end{bmatrix} \begin{matrix}\}\ \ \mathcal{N}_1\\ \}\ \ \mathcal{N}_2\,.\\ \}\ \ \gamma\end{matrix}$$

The problem $A\mathbf{x}_1 = \mathbf{b}_1$ can be rewritten as

$$A^E\begin{bmatrix}\mathbf{x}_1\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}\mathbf{b}_1\\ 0\\ 0\end{bmatrix} \tag{4'}$$

We seek $\mathbf{w}_0$ defined on Γ, such that

$$\mathbf{x} = B^{-1}\begin{bmatrix}\mathbf{b}_1\\ 0\\ \mathbf{b}_0\end{bmatrix} + B^{-1}\begin{bmatrix}0\\ 0\\ \mathbf{w}_0\end{bmatrix}$$

to be the solution of (4'). Introduce now the so–called DD (domain decomposition) Schur complement

$$S = B_{00} - \sum_{i=1}^{2} B_{0,i}B_{i,i}^{-1}B_{i,0}$$

As shown by the present authors in [20] (see also Dryja [11]) the following problem is obtained for $\mathbf{w}_0$

$$S^{-1}\mathbf{w}_0 = -\left.\left(B^{-1}\begin{bmatrix}\mathbf{b}_1\\ 0\\ \mathbf{b}_0\end{bmatrix}\right)\right|_{\gamma}.$$

Let

$$C = S^{-1} = \left(B_{00} - \sum_{i=1}^{2} B_{0i}B_{ii}^{-1}B_{i0}\right)^{-1}.$$

C is called the capacitance matrix. Note that C is non-symmetric and possibly indefinite. S (and C) is invertible for sufficiently fine mesh and in the indefinite case with some exceptions of $\lambda = -\lambda_1 - \lambda_2$. We assume that S (and C) is invertible.

The actions of C are readily available since for any given vector $\mathbf{v}_0$ on Γ,

$$C\mathbf{v}_0 = \left(B^{-1} \begin{bmatrix} 0 \\ 0 \\ \mathbf{v}_0 \end{bmatrix} \right) \Bigg|_{\gamma} .$$

We now present the algorithm for solving problem (3) based on the capacitance matrix method.

ALGORITHM (CMM)

(i) Solve the problem

$$B\mathbf{y} = \begin{bmatrix} \mathbf{b}_1 \\ 0 \\ \mathbf{b}_0 \end{bmatrix} \begin{matrix} \} & \mathcal{N}_1 \\ \} & \mathcal{N}_2 . \\ \} & \gamma \end{matrix}$$

(ii) Iterate for $\mathbf{w}_0$

$$C\mathbf{w}_0 = -\mathbf{y}_0 .$$

(iii) Solve the problem

$$B\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{w}_0 \end{bmatrix} .$$

(iv) The solution of (4) is

$$\mathbf{x}_1 = \mathbf{y}_1 + \mathbf{v}_1 .$$

3. A change to a two–level hierarchical basis. From now on we focus on step (ii) of the algorithm (CMM). The solution method that we will develop is based on a change of the basis to the two–level hierarchical basis. Let $\tilde{\mathfrak{W}} \subset \mathfrak{W}$ be two nested finite element spaces. Let $\tilde{\mathcal{N}}$ and $\mathcal{N}$ be the corresponding fine and coarse node sets, respectively. And finally let $\tilde{\gamma}$ and γ be the coarse and fine node sets restricted to the interface Γ. Consider the nodal basis functions in $\tilde{\mathfrak{W}}$ and $\mathfrak{W}$,

$$\left\{ \tilde{\phi}_i \right\}_{x_i \in \tilde{\mathcal{N}}}, \quad \text{and} \quad \{\phi_i\}_{x_i \in \mathcal{N}} .$$

The two–level hierarchical basis of $\mathfrak{W}$ is then defined by

$$\left\{ \hat{\phi}_i \right\}_{x_i \in \mathcal{N}} = \left\{ \tilde{\phi}_i \right\}_{x_i \in \tilde{\mathcal{N}}} \cup \{\phi_i\}_{x_i \in \mathcal{N} \setminus \tilde{\mathcal{N}}} .$$

We will first transform the problem into a two–level hierarchical form, exploiting a fixed (and of small dimension) coarse finite element space $\tilde{\mathfrak{W}}$ on a coarse triangulation $\tilde{\mathcal{T}}$ and coarse node set $\tilde{\mathcal{N}}$ in $\mathfrak{R}$ and $\tilde{\gamma}$ on the interface Γ. We assume that $\mathcal{T}$ is obtained by a number of uniform refinement steps of $\tilde{\mathcal{T}}$. Then the transformed capacitance matrix $\hat{C}$ admits a two–by–two block form with respect to the two–level hierarchical partitioning of the node set $\gamma = \tilde{\gamma} \cup (\gamma \setminus \tilde{\gamma})$. The block corresponding to the node set $\tilde{\gamma}$ is of a small (fixed) size. We show that the second block on the diagonal, corresponding to the complementary node set $\gamma \setminus \tilde{\gamma}$ is coercive and can be treated as a slight perturbation (depending upon the coarse grid size and the coefficients of the bilinear form $\mathcal{A}(.,.)$) of a symmetric positive definite block (constructed as previously explained), here for the principal symmetric and coercive part $\mathcal{B}^{(0)}$ of $\mathcal{B}$, i.e., the stiffness matrix corresponding to the bilinear form

$$\mathcal{B}^{(0)}(u, \phi) = \int_{\mathfrak{R}} \left\{ k_1(x) \frac{\partial u}{\partial x} \frac{\partial \phi}{\partial x} + k_2(y) \frac{\partial u}{\partial y} \frac{\partial \phi}{\partial y} \right\} dx\, dy.$$

Consider $\hat{B}$ and $\hat{B}^{(0)}$ that are computed from $\mathcal{B}(.,.)$ and $\mathcal{B}^{(0)}(.,.)$ respectively, using the two–level hierarchical basis functions, i.e,

$$\hat{B} = \left\{ \mathcal{B}(\hat{\phi}_j, \hat{\phi}_i) \right\}, \quad \hat{B}^{(0)} = \left\{ \mathcal{B}^{(0)}(\hat{\phi}_j, \hat{\phi}_i) \right\}.$$

We also need the corresponding coarse–grid stiffness matrices

$$\tilde{B} = \left\{ \mathcal{B}(\tilde{\phi}_j, \tilde{\phi}_i) \right\}, \quad \tilde{B}^{(0)} = \left\{ \mathcal{B}^{(0)}(\tilde{\phi}_j, \tilde{\phi}_i) \right\}.$$

There is a relation between standard nodal basis stiffness matrix B and the matrix $\hat{B}$ computed by the two–level hierarchical basis functions. Namely, for a unit block–triangular matrix

$$J = \begin{bmatrix} I & 0 \\ J_{21} & I \end{bmatrix} \begin{array}{l} \} \tilde{\mathcal{N}} \\ \} \mathcal{N} \setminus \tilde{\mathcal{N}} \end{array}$$

we have

$$\hat{B} = J^T B J.$$

Here $\begin{bmatrix} I \\ J_{21} \end{bmatrix}$ transforms the coefficient vector $\tilde{\mathbf{v}}$ of a function $\tilde{v} \in \tilde{\mathfrak{W}}$ into the coefficient vector of $\tilde{v}$ as a function in $\mathfrak{W} \supset \tilde{\mathfrak{W}}$ with respect to the standard nodal basis of $\mathfrak{W}$. For more details see Yserentant [31]. The transformation of the nodal basis stiffness matrices also induces transformation of their DD Schur complements S to $\hat{S}$. The same holds for their counterparts $S^{(0)}$ and $\hat{S}^{(0)}$, computed for the bilinear form $\mathcal{B}^{(0)}$ (the principal symmetric and coercive part of $\mathcal{B}(.,.)$). Note that $S^{(0)}$ and $\hat{S}^{(0)}$ are symmetric and positive definite. The following relations hold:

$$\hat{S} = J_\Gamma^T S J_\Gamma, \quad \hat{S}^{(0)} = J_\Gamma^T S^{(0)} J_\Gamma, \tag{5}$$

where J_Γ is the transformation matrix from the two–level hierarchical basis coefficient vector of functions restricted to Γ to the standard nodal basis coefficient vector of the same function. Note that this transformation is one–dimensional (it is only on Γ). This relation was observed in Smith and Widlund [24], see also Vassilevski [28]. Because of (5) we can transform problem (ii) to the two–level hierarchical form

$$\hat{C} \hat{\mathbf{w}}_0 = \hat{\mathbf{g}}_0, \tag{6}$$

where

$$\hat{C} = J_\Gamma^{-1} C J_\Gamma^{-T}, \quad \hat{\mathbf{w}}_0 = J_\Gamma^T \mathbf{w}_0, \text{ and } \hat{\mathbf{g}}_0 = -J_\Gamma^{-1} \mathbf{y}_0.$$

Note that the actions of J_Γ^{-1}, J_Γ^{-T} are readily available (cf., Yserentant [31]).

4. Block–factorization of the Capacitance Matrix. In this section we will discuss an efficient solution of the transformed problem (6). The final solution to problem (ii) is then obtained by simply changing back to the original basis, i.e., by computing the product $\mathbf{w}_0 = J_\Gamma^{-T}\hat{\mathbf{w}}_0$.

In order to explain the method we first partition $\hat{C}$ into the following two–by–two block form,

$$\hat{C} = \begin{bmatrix} X & U \\ V & W \end{bmatrix} \begin{matrix} \}\ \tilde{\gamma} \\ \}\ \gamma\setminus\tilde{\gamma} \end{matrix}. \tag{7}$$

Similarly, we partition $\hat{S} = \hat{C}^{-1}$

$$\hat{S} = \begin{bmatrix} K & P \\ Q & T \end{bmatrix} \begin{matrix} \}\ \tilde{\gamma} \\ \}\ \gamma\setminus\tilde{\gamma} \end{matrix}.$$

By simple block–matrix manipulations we find

$$W^{-1} = T - QK^{-1}P, \tag{8}$$

i.e., the inverse of the Schur complement $T - QK^{-1}P$ of $\hat{S}$ is a principal submatrix of $\hat{C} = \hat{S}^{-1}$. The solution method will be based on the following exact block–factorization of $\hat{C}$

$$\hat{C} = \begin{bmatrix} Y & U \\ 0 & W \end{bmatrix} \begin{bmatrix} I & 0 \\ W^{-1}V & I \end{bmatrix},$$

where $Y = X - UW^{-1}V$ is the Schur complement of $\hat{C}$. Note that Y is of a small size equal to the number of coarse grid nodes on the interface Γ, i.e, the nodes in $\tilde{\gamma}$. Those are a fixed number $\tilde{m} \geq 1$. Then (6) can be solved based on the above block-factorization of $\hat{C}$ by the usual backward and forward recurrences as follows.

We can rewrite (6) in the form

$$\begin{bmatrix} Y & U \\ 0 & W \end{bmatrix} \begin{bmatrix} I & 0 \\ W^{-1}V & I \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix}, \text{ where } \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{bmatrix} = \hat{\mathbf{g}}_0 = -J_\Gamma^{-1}\mathbf{y}_0, \tag{9}$$

and denote

$$\begin{bmatrix} I & 0 \\ W^{-1}V & I \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix}.$$

Then the backward step in (9) can be carried out as

$$\begin{aligned} W\mathbf{z}_2 &= \mathbf{g}_2, \\ Y\mathbf{z}_1 &= \mathbf{g}_1 - U\mathbf{z}_2, \end{aligned} \tag{10}$$

while the forward step in (9) reads

$$\mathbf{w}_1 = \mathbf{z}_1 \text{ and } \mathbf{w}_2 = \mathbf{z}_2 - W^{-1}V\mathbf{w}_1, \tag{11}$$

where actually the action of W^{-1} in (11) is computed by solving

$$W\mathbf{y}_2 = -V\mathbf{w}_1 \text{ and then letting } \mathbf{w}_2 = \mathbf{z}_2 + \mathbf{y}_2.$$

Finally, $\mathbf{w}_0 = J_\Gamma^{-T} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix}$.

As we can see this algorithm requires the action of Y^{-1}. We assume that Y is assembled explicitly. Then (9) requires one solving with the small block Y (it can be factored or inverted exactly at a negligible cost), two solvings with W and two solvings with $\hat{B}$ (multiplication by U and V).

We need now to present in some detail the explicit generation of Y. Recall that the actions of $\hat{C}$ are readily available for any vector $\mathbf{v}$ on γ,

$$\hat{C}\mathbf{v} = \left(\hat{B}^{-1} \begin{bmatrix} 0 \\ 0 \\ \mathbf{v} \end{bmatrix} \right) \Bigg|_\gamma ,$$

where the right hand side is restricted from $\mathcal{N}$ to γ.

We now write $\hat{B}^{-1}$ in the block form

$$\hat{B}^{-1} = \begin{bmatrix} \star & \star \\ \star & \hat{C} \end{bmatrix} \begin{array}{l} \} \, \mathcal{N} \setminus \gamma \\ \} \, \gamma \end{array}$$

and since (see (7))

$$\hat{C} = \begin{bmatrix} X & U \\ V & W \end{bmatrix} = \begin{bmatrix} Y + UW^{-1}V & U \\ V & W \end{bmatrix} \begin{array}{l} \} \, \tilde{\gamma} \\ \} \, \gamma \setminus \tilde{\gamma} \end{array},$$

we see that

$$\hat{B}^{-1} \begin{bmatrix} 0 \\ \underline{\tilde{\xi}} \\ 0 \end{bmatrix} \begin{array}{l} \} \, \mathcal{N} \setminus \gamma \\ \} \, \tilde{\gamma} \\ \} \, \gamma \setminus \tilde{\gamma} \end{array} = \begin{bmatrix} \star \\ \left(Y + UW^{-1}V\right) \underline{\tilde{\xi}} \\ V\underline{\tilde{\xi}} \end{bmatrix} \begin{array}{l} \} \, \mathcal{N} \setminus \gamma \\ \} \, \tilde{\gamma} \\ \} \, \gamma \setminus \tilde{\gamma} \end{array}.$$

In particular, this shows that we can compute the actions of V. Similarly,

$$\hat{B}^{-1} \begin{bmatrix} 0 \\ 0 \\ \underline{\xi}_0 \end{bmatrix} \begin{array}{l} \} \, \mathcal{N} \setminus \gamma \\ \} \, \tilde{\gamma} \\ \} \gamma \setminus \tilde{\gamma} \end{array} = \begin{bmatrix} \star \\ U\underline{\xi}_0 \\ W\underline{\xi}_0 \end{bmatrix} \begin{array}{l} \} \, \mathcal{N} \setminus \gamma \\ \} \, \tilde{\gamma} \\ \} \, \gamma \setminus \tilde{\gamma} \end{array}.$$

This in turn shows that the actions of U, i.e., the product

$$U\underline{\xi}_0 = \hat{B}^{-1} \begin{bmatrix} 0 \\ 0 \\ \underline{\xi}_0 \end{bmatrix} \Bigg|_{\tilde{\gamma}}$$

are available as well. Since

$$Y\underline{\tilde{\xi}} = -U \left[W^{-1} \left(V\underline{\tilde{\xi}} \right) \right] + X\underline{\tilde{\xi}},$$

the actions of Y are available if we are able to solve systems with the major block W of $\hat{C}$.

Summing it up, Y can be assembled as follows. The ith column of Y is computed by choosing $\underline{\tilde{\xi}} = \underline{\tilde{e}}_i \in \mathbb{R}^{\tilde{m}}$, the ith unit coordinate vector, $i = 1, 2, \ldots, \tilde{m}$. First, $\mathbf{v}_i = \hat{B}^{-1}\left(\begin{bmatrix} 0 \\ \underline{\tilde{e}}_i \\ 0 \end{bmatrix} \begin{matrix} \}\mathcal{N}\setminus\gamma \\ \}\,\tilde{\gamma} \\ \}\,\gamma\setminus\tilde{\gamma} \end{matrix}\right)$ are computed at the cost of $\tilde{m}$ actions of $\hat{B}^{-1}$ (which is based on fast elliptic solvers). We obtain $V\underline{\tilde{e}}_i = \mathbf{v}_i|_{\gamma\setminus\tilde{\gamma}}$ and $X\underline{\tilde{e}}_i = \mathbf{v}_i|_{\tilde{\gamma}}$. Then $\tilde{m}$ solvings with W are needed to compute $\underline{\xi}_{0,i} = W^{-1}\left(V\underline{\tilde{e}}_i\right)$, $i = 1, 2, \ldots, \tilde{m}$. Finally, $\tilde{m}$ additional solvings with $\hat{B}$ are required to compute $U\underline{\xi}_{0,i} = \hat{B}^{-1}\left(\begin{bmatrix} 0 \\ 0 \\ \underline{\xi}_{0,i} \end{bmatrix} \begin{matrix} \}\,\mathcal{N}\setminus\gamma \\ \}\,\tilde{\gamma} \\ \}\,\gamma\setminus\tilde{\gamma} \end{matrix}\right)\Bigg|_{\tilde{\gamma}}$.

The computation of the ith column of Y is completed by $Y\underline{\tilde{e}}_i = -U\underline{\xi}_{0,i} + X\xi_i$.

Thus, the explicit generation of Y requires $2\tilde{m}$ actions of $\hat{B}^{-1}$ (needed for the actions of X and V and of U) and $\tilde{m}$ solvings with W (this dominates the cost of (9) and (10) equal to 2 actions of $\hat{B}^{-1}$ and 2 solvings with W). Problems with W can be solved by iterations in a generalized conjugate gradient method (in the non-symmetric case) with a proper preconditioner. This task is addressed in the next section.

5. Solving problems with the major block W. From now on our major concern will be to construct preconditioners for the block W of $\hat{C}$. The main idea is that W^{-1} is spectrally equivalent to ${W^{(0)}}^{-1}$, where $W(0)$ is the major block corresponding to the principal symmetric and coercive part $\mathcal{B}^{(0)}$ of the bilinear form $\mathcal{B}$. In fact, W is a perturbation of $W^{(0)}$ and it can be made sufficiently close to $W^{(0)}$ by choosing $\tilde{h}$ (the coarse–grid size) sufficiently small. We need to stress that the spectral equivalence relations we will prove in a moment are independent of the fine mesh size h and no relation is required between $\tilde{h}$ and h. The results are corollaries from the preconditioning technique developed in Vassilevski [27] for non-symmetric and indefinite finite element elliptic problems. Here, we extend these results to the capacitance matrix problems for non-symmetric and indefinite elliptic bilinear forms. We state first some results from Vassilevski [27] for the matrices $\hat{B}$ and $\hat{B}^{(0)}$.

Lemma 1. *Consider the following two–level hierarchical block forms of $\hat{B}$ and $\hat{B}^{(0)}$,*

$$\hat{B} = \begin{bmatrix} \tilde{B} & E \\ F & D \end{bmatrix} \begin{matrix} \}\,\tilde{\mathcal{N}} \\ \}\,\mathcal{N}\setminus\tilde{\mathcal{N}} \end{matrix}, \quad \hat{B}^{(0)} = \begin{bmatrix} \tilde{B}^{(0)} & E^{(0)} \\ F^{(0)} & D^{(0)} \end{bmatrix} \begin{matrix} \}\,\tilde{\mathcal{N}} \\ \}\,\mathcal{N}\setminus\tilde{\mathcal{N}} \end{matrix}.$$

Then for the two–level hierarchical Schur complements

$$Z = D - F\tilde{B}^{-1}E, \quad Z^{(0)} = D^{(0)} - F^{(0)}{\tilde{B}^{(0)}}^{-1}E^{(0)},$$

the following spectral equivalence relations hold

$$\left(1 - O(\tilde{h}^2)\right)\underline{\xi}^T Z^{(0)}\underline{\xi} \le \underline{\xi}^T Z\underline{\xi}, \quad \text{for all } \underline{\xi} \in \mathbb{R}^{n-\tilde{n}},$$

where n and $\tilde{n}$ are the number of nodes in $\mathcal{N}$ and $\tilde{\mathcal{N}}$, respectively. We also have

$$\underline{\xi}^T Z \underline{\zeta} \leq \left(1 + O(\tilde{h})\right) \left[\underline{\xi}^T Z^{(0)} \underline{\xi}\right]^{\frac{1}{2}} \left[\underline{\zeta}^T Z^{(0)} \underline{\zeta}\right]^{\frac{1}{2}}, \quad \text{for all } \underline{\xi},\ \underline{\zeta} \in \mathbb{R}^{n-\tilde{n}}.$$

In particular, we have that Z is coercive. □

Results that provide preconditioners for non-symmetric and/or indefinite elliptic problems were presented in Yserentant [32] for the hierarchical basis method, in Cai and Widlund [10] for additive overlapping domain decomposition methods, and for more general setting in Xu and Cai [29] and Xu [30]. The result from Vassilevski [27] provides way of preconditioning nonsymmetric and indefinite elliptic problems in a reduced form, i.e., in a subspace. The latter result is useful for the reduced capacitance matrix problems since the iterations are performed in a subspace corresponding to unknowns on the interface boundary.

We will also need the following fact from Vassilevski [27] which relates the Schur complements of two spectrally equivalent matrices (positive definite, symmetric and non-symmetric).

Lemma 2. *Consider two block matrices*

$$G = \begin{bmatrix} G_{11} & G_{10} \\ G_{01} & G_{00} \end{bmatrix} \quad \text{and } G^{(0)} = \begin{bmatrix} G_{11}^{(0)} & G_{10}^{(0)} \\ G_{01}^{(0)} & G_{00}^{(0)} \end{bmatrix}.$$

and let G be non-symmetric and $G^{(0)}$ symmetric positive definite. We assume that they satisfy the following spectral equivalence relations

$$\gamma_1 \underline{\zeta}^T G^{(0)} \underline{\zeta} \leq \underline{\zeta}^T G \underline{\zeta}, \quad \text{for all } \zeta,$$

and

$$\underline{\xi}^T G \underline{\zeta} \leq \gamma_2 [\xi^T G^{(0)} \xi]^{\frac{1}{2}} [\zeta^T G^{(0)} \zeta]^{\frac{1}{2}}, \quad \text{for all } \underline{\zeta} \text{ and } \underline{\xi},$$

for some positive constants γ_1, γ_2. Then their Schur complements

$$Y = G_{00} - G_{01} G_{11}^{-1} G_{10}, \quad \text{and} \quad Y^{(0)} = G_{00}^{(0)} - G_{01}^{(0)} {G_{11}^{(0)}}^{-1} G_{10}^{(0)}$$

satisfy the following spectral equivalence relations

$$\gamma_1 \underline{\zeta}_0^T Y^{(0)} \underline{\zeta}_0 \leq \underline{\zeta}_0^T Y \underline{\zeta}_0 \quad \text{for all } \underline{\zeta}_0,$$

and

$$\underline{\xi}_0^T Y \underline{\zeta}_0 \leq \frac{\gamma_2^2}{\gamma_1} [\underline{\xi}_0^T Y^{(0)} \underline{\xi}_0]^{\frac{1}{2}} [\underline{\zeta}_0^T Y^{(0)} \underline{\zeta}_0]^{\frac{1}{2}} \quad \text{for all } \underline{\xi}_0 \text{ and } \underline{\zeta}_0.$$

Proof. First note that since G is coercive ($G^{(0)}$ is symmetric positive definite) the principal submatrix G_{11} is coercive as well and hence invertible. This shows that Y is well-defined.

For $\underline{\zeta} = \begin{bmatrix} \underline{\zeta}_1 \\ \underline{\zeta}_0 \end{bmatrix}$ with $\underline{\zeta}_1$ chosen such that $G_{11}\underline{\zeta}_1 + G_{10}\underline{\zeta}_0 = 0$, i.e., $\underline{\zeta}_1 = -G_{11}^{-1} G_{10} \underline{\zeta}_0$, using the coercivity of G in terms of $G^{(0)}$ and the fact that $G\underline{\zeta} = \begin{bmatrix} 0 \\ Y\underline{\zeta}_0 \end{bmatrix}$, we get

$$\begin{aligned} \underline{\zeta}_0^T Y \underline{\zeta}_0 &= \underline{\zeta}^T G \underline{\zeta} \\ &\geq \gamma_1 \underline{\zeta}^T G^{(0)} \underline{\zeta} \\ &\geq \gamma_1 \inf_{\underline{\zeta}_1} \underline{\zeta}^T G^{(0)} \underline{\zeta} \\ &= \gamma_1 \underline{\zeta}_0^T Y^{(0)} \underline{\zeta}_0 . \end{aligned}$$

We used a main minimization property of Schur complements of symmetric positive definite matrix $G^{(0)}$. For the second desired inequality for $\underline{\xi} = \begin{bmatrix} \underline{\xi}_1 \\ \underline{\xi}_0 \end{bmatrix}$ arbitrary and $\underline{\zeta} = \begin{bmatrix} \underline{\zeta}_1 \\ \underline{\zeta}_0 \end{bmatrix}$ in the subspace $G_{11}\underline{\zeta}_1 + G_{10}\underline{\zeta}_0 = 0$, using the boundedness of G in terms of $G^{(0)}$ and the fact that $G\underline{\zeta} = \begin{bmatrix} 0 \\ Y\underline{\zeta}_0 \end{bmatrix}$, we get

$$\begin{aligned} \underline{\xi}_0^T Y \underline{\zeta}_0 &= \underline{\xi}^T G \underline{\zeta} \\ &\leq \gamma_2 [\underline{\xi}^T G^{(0)} \underline{\xi}]^{\frac{1}{2}} [\underline{\zeta}^T G^{(0)} \underline{\zeta}]^{\frac{1}{2}} \\ &\leq \tfrac{\gamma_2}{\sqrt{\gamma_1}} [\underline{\xi}^T G^{(0)} \underline{\xi}]^{\frac{1}{2}} [\underline{\zeta}^T G \underline{\zeta}]^{\frac{1}{2}} \\ &= \tfrac{\gamma_2}{\sqrt{\gamma_1}} [\underline{\xi}^T G^{(0)} \underline{\xi}]^{\frac{1}{2}} [\underline{\zeta}_0^T Y \underline{\zeta}_0]^{\frac{1}{2}} . \end{aligned}$$

Note now that the left–hand side of the inequality is independent of $\underline{\xi}_1$. Hence

$$\begin{aligned} \underline{\xi}_0^T Y \underline{\zeta}_0 &\leq \tfrac{\gamma_2}{\sqrt{\gamma_1}} \left[\inf_{\underline{\xi}_1} \underline{\xi}^T G^{(0)} \underline{\xi} \right]^{\frac{1}{2}} [\underline{\zeta}_0^T Y \underline{\zeta}_0]^{\frac{1}{2}} \\ &= \tfrac{\gamma_2}{\sqrt{\gamma_1}} [\underline{\xi}_0^T Y^{(0)} \underline{\xi}_0]^{\frac{1}{2}} [\underline{\zeta}_0^T Y \underline{\zeta}_0]^{\frac{1}{2}} . \end{aligned}$$

Letting $\underline{\xi}_0 = \underline{\zeta}_0$ we get the inequality

$$[\underline{\zeta}_0^T Y \underline{\zeta}_0]^{\frac{1}{2}} \leq \frac{\gamma_2}{\sqrt{\gamma_1}} [\underline{\zeta}_0^T Y^{(0)} \underline{\zeta}_0]^{\frac{1}{2}} .$$

Substituting last inequality in the preceding one the final result follows

$$\underline{\xi}_0^T Y \underline{\zeta}_0 \leq \frac{\gamma_2^2}{\gamma_1} [\underline{\xi}_0^T Y^{(0)} \underline{\xi}_0]^{\frac{1}{2}} [\underline{\zeta}_0^T Y^{(0)} \underline{\zeta}_0]^{\frac{1}{2}} . \quad \square$$

We next note that

$$Z^{-1} = \begin{bmatrix} W & \star \\ \star & \star \end{bmatrix} \begin{matrix} \} \ \gamma \setminus \tilde{\gamma} \\ \} \ \left(\mathcal{N} \setminus \tilde{\mathcal{N}}\right) \setminus (\gamma \setminus \tilde{\gamma}) \end{matrix}$$

and similarly

$$Z^{(0)^{-1}} = \begin{bmatrix} W^{(0)} & \star \\ \star & \star \end{bmatrix} \begin{matrix} \} \ \gamma \setminus \tilde{\gamma} \\ \} \ \left(\mathcal{N} \setminus \tilde{\mathcal{N}}\right) \setminus (\gamma \setminus \tilde{\gamma}) \end{matrix}.$$

This is the case since Z^{-1} and $Z^{(0)^{-1}}$ are principal submatrices of $\hat{B}^{-1}$ and $\hat{B}^{(0)^{-1}}$, respectively. (Note that Z and $Z^{(0)}$ are Schur complements of $\hat{B}$ and $\hat{B}^{(0)}$, respectively.) And the same is valid for W and $W^{(0)}$, i.e., they must be principal submatrices of Z^{-1} and $Z^{(0)^{-1}}$, respectively.

As a consequence of Lemma 1 and Lemma 2 we get the following result

Theorem 1. *The following spectral equivalence relations hold*

$$\left(1 - O(\tilde{h})\right) \underline{\xi}_0^T W^{(0)} \underline{\xi}_0 \leq \underline{\xi}_0^T W \underline{\xi}_0, \quad \textit{for all } \underline{\xi}_0 \in \mathbb{R}^{m-\tilde{m}},$$

and

$$\underline{\xi}_0^T W \underline{\zeta}_0 \leq \left(1 + O(\tilde{h}^2)\right) \left[\underline{\zeta}_0^T W^{(0)} \underline{\zeta}_0\right]^{\frac{1}{2}} \left[\underline{\xi}_0^T W^{(0)} \underline{\xi}_0\right]^{\frac{1}{2}}, \quad \textit{for all } \underline{\xi}_0 \textit{ and } \underline{\zeta}_0 \in \mathbb{R}^{m-\tilde{m}}.$$

Here m and $\tilde{m}$ denote the number of nodes in γ and $\tilde{\gamma}$, respectively.

Proof. Since W^{-1} and $W^{(0)^{-1}}$ are Schur complements of Z and $Z^{(0)}$, respectively, Lemma 1 and Lemma 2 imply the following spectral equivalence relations

$$(1 - O(\tilde{h}^2)) \underline{\zeta}_0^T W^{(0)^{-1}} \underline{\zeta}_0 \leq \underline{\zeta}_0^T W^{-1} \underline{\zeta}_0, \quad \text{for all } \underline{\zeta}_0,$$

and

$$\underline{\xi}_0^T W^{-1} \underline{\zeta}_0 \leq (1 + O(\tilde{h}))[\underline{\xi}_0^T W^{(0)^{-1}} \underline{\xi}_0]^{\frac{1}{2}} [\underline{\zeta}_0^T W^{(0)^{-1}} \underline{\zeta}_0]^{\frac{1}{2}}, \quad \text{for all } \underline{\xi}_0 \text{ and } \underline{\zeta}_0.$$

Consider $X = W^{(0)^{\frac{1}{2}}} W^{-1} W^{(0)^{\frac{1}{2}}}$. Note that symmetric positive definite square root of $W^{(0)}$ exists since $W^{(0)}$ is symmetric and positive definite. Denote $\|\underline{\xi}_0\|^2 = \underline{\xi}_0^T \underline{\xi}_0$. The coercivity estimate for W^{-1} in terms of $W^{(0)^{-1}}$ implies

$$(1 - O(\tilde{h}^2)) \|\underline{\xi}_0\|^2 \leq \underline{\xi}_0^T X \underline{\xi}_0 \leq \|\underline{\xi}_0\| \|X \underline{\xi}_0\|.$$

Hence for $\underline{\xi}_0 = X^{-1} \underline{\zeta}_0$ we get

$$\|X^{-1} \underline{\zeta}_0\| \leq (1 + O(\tilde{h}^2)) \|\underline{\zeta}_0\|,$$

which is equivalent to

$$\underline{\xi}_0^T W^{(0)^{-\frac{1}{2}}} W W^{(0)^{-\frac{1}{2}}} \underline{\zeta}_0 \leq (1 + O(\tilde{h}^2)) \|\underline{\zeta}_0\| \|\underline{\xi}_0\|, \quad \text{for all } \underline{\xi}_0 \text{and } \underline{\zeta}_0.$$

Finally, letting $\underline{\zeta}_0 := W^{(0)^{\frac{1}{2}}} \underline{\zeta}_0$ and $\underline{\xi}_0 := W^{(0)^{\frac{1}{2}}} \underline{\xi}_0$ the desired boundedness inequality for W in terms of $W^{(0)}$ follows. For the coercivity one, letting $\underline{\xi}_0 := W^{(0)} \underline{\xi}_0$ and $\underline{\zeta}_0 := W \underline{\xi}_0$ in the boundedness estimate for W^{-1}, we get

$$\underline{\xi}_0^T W^{(0)} \underline{\xi}_0 \leq (1 + O(\tilde{h}))[\underline{\xi}_0^T W^T W^{(0)^{-1}} W \underline{\xi}_0]^{\frac{1}{2}} [\underline{\xi}_0^T W^{(0)} \underline{\xi}_0]^{\frac{1}{2}},$$

which implies

$$(1-O(\tilde{h}))\underline{\xi}_0^T W^{(0)} \underline{\xi}_0 \le \underline{\xi}_0^T W^T W^{(0)^{-1}} W \underline{\xi}_0.$$

Finally from the coercivity estimate for W^{-1} letting $\underline{\zeta}_0 = W\underline{\xi}_0$ and using the last inequality we get

$$\underline{\xi}_0^T W \underline{\xi}_0 = \underline{\xi}_0^T W^T \underline{\xi}_0 \ge (1-O(\tilde{h}^2))\underline{\xi}_0^T W^T W^{(0)^{-1}} W \underline{\xi}_0 \ge (1-O(\tilde{h}))\underline{\xi}_0^T W^{(0)} \underline{\xi}_0,$$

which is the desired coercivity estimate for W. □

Corollary 1. Let $\hat{M}^{(0)}$ be an optimal order symmetric positive definite preconditioner for the symmetric and coercive capacitance matrix $\hat{C}^{(0)}$ (for examples see, e.g., Proskurowski and Vassilevski [20]). Let $\hat{M}^{(0)} = \begin{bmatrix} \star & \star \\ \star & \Lambda \end{bmatrix}$, i.e., $\Lambda = \begin{bmatrix} 0 \\ I \end{bmatrix}^T \hat{M}^{(0)} \begin{bmatrix} 0 \\ I \end{bmatrix}$, where $\begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{matrix} \} \ \tilde{\gamma} \\ \} \ \gamma \setminus \tilde{\gamma} \end{matrix}$. Then Λ gives an optimal order preconditioner for W in the GMRES method. The convergence factor is at least

$$\left[1 - \left(\frac{1}{\varkappa}\frac{\delta_1}{\delta_2}\right)^2\right]^{\frac{1}{2}},$$

where $\delta_1 = 1 - O(\tilde{h})$ and $\delta_2 = 1 + O(\tilde{h})$ are the constants in the spectral equivalence relations between W and $W^{(0)}$ from Theorem 1 and $\varkappa = Cond\left(\hat{M}^{(0)^{-1}}\hat{C}^{(0)}\right)$. In particular, we see that for $\tilde{h}$ sufficiently small the convergence of the method will be close to that of the symmetric positive definite case.

Proof. First note that Λ is spectrally equivalent to $\hat{W}^{(0)}$ since both are the same principal submatrices of symmetric positive definite matrices, which are spectrally equivalent. Moreover we have $\mathrm{Cond}\left(\Lambda^{-1}W^{(0)}\right) \le Cond\left(\hat{M}^{(0)^{-1}}\hat{C}^{(0)}\right) = \varkappa$. The rest of the proof follows from Theorem 1 and a basic convergence rate estimate of GMRES based on coercivity and boundedness of the preconditioned matrix $\Lambda^{-1}W$ which we now verify. Denote by $\lambda_{\min}$ and $\lambda_{\max}$ the extreme eigenvalues of $\Lambda^{-1}W^{(0)}$. We have from the coercivity of W

$$\underline{\zeta}_0^T W \underline{\zeta}_0 \ge (1-O(\tilde{h}))\underline{\zeta}_0^T W^{(0)} \underline{\zeta}_0 \ge \lambda_{\min}(1-O(\tilde{h}))\underline{\zeta}_0^T \Lambda \underline{\zeta}_0.$$

which shows the coercivity of W in terms of Λ. Similarly from the boundedness of W in terms of $W^{(0)}$ from Theorem 1 we get

$$\begin{aligned} \underline{\xi}_0^T W \underline{\zeta}_0 &\le (1+O(\tilde{h}))[\underline{\xi}_0^T W^{(0)} \underline{\xi}_0]^{\frac{1}{2}}[\underline{\zeta}_0^T W^{(0)} \underline{\zeta}_0]^{\frac{1}{2}} \\ &\le \lambda_{\max}(1+O(\tilde{h}))[\underline{\xi}_0^T \Lambda \underline{\xi}_0]^{\frac{1}{2}}[\underline{\zeta}_0^T \Lambda \underline{\zeta}_0]^{\frac{1}{2}}, \end{aligned}$$

which shows the boundedness of W in terms of Λ. Then the convergence rate estimate of GMRES from Saad and Schultz [23] (that is a steepest descent type estimate derived earlier in Eisenstat, Elman and Schultz [13]) we get the following bound of the convergence factor

$$\left[1-\left(\frac{\lambda_{\min}}{\lambda_{\max}}\frac{\delta_1}{\delta_2}\right)^2\right]^{\frac{1}{2}} \leq \left[1-\left(\frac{1}{\varkappa}\frac{\delta_1}{\delta_2}\right)^2\right]^{\frac{1}{2}}. \quad \square$$

Remark 2. The preconditioner Λ defined in Corollary 1 can be implemented as follows, provided that the actions of $\hat{M}^{(0)^{-1}}$ on vectors are available. Consider the following block–form of $\hat{M}^{(0)^{-1}}$,

$$\hat{M}^{(0)^{-1}} = \begin{bmatrix} U & F \\ G & \Delta \end{bmatrix} \begin{array}{l} \} \, \tilde{\gamma} \\ \} \, \gamma \setminus \tilde{\gamma} \end{array}.$$

We have, since Λ is a principal submatrix of $\hat{M}^{(0)}$, $\Lambda = \left(\Delta - GU^{-1}F\right)^{-1}$, i.e., the inverse of a Schur complement of a matrix is a principal submatrix of the inverse matrix. Hence

$$\Lambda^{-1} = \Delta - GU^{-1}F.$$

The actions of Λ^{-1} can be computed based on this expression since, as the actions of $\hat{M}^{(0)^{-1}}$ are available, the actions of Δ, G, and F are available as well.

Thus the process of computing $\mathbf{z}_2 = \Lambda^{-1}\mathbf{r}_2$ for any given $\mathbf{r}_2$ is as follows:

(i) Compute $\begin{bmatrix} F\mathbf{r}_2 \\ \Delta\mathbf{r}_2 \end{bmatrix} = \hat{M}^{(0)^{-1}} \begin{bmatrix} \mathbf{0} \\ \mathbf{r}_2 \end{bmatrix}$,

(ii) Solve for $\mathbf{w}_1$: $U\mathbf{w}_1 = F\mathbf{r}_2$,

(iii) Compute $G\mathbf{w}_1$: $\begin{bmatrix} \star \\ G\mathbf{w}_1 \end{bmatrix} = \hat{M}^{(0)^{-1}} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{0} \end{bmatrix}$,

(iv) Compute final solution $\mathbf{z}_2 = \Delta\mathbf{r}_2 - G\mathbf{w}_1$.

We can assemble U explicitly (this requires $\tilde{m}$ actions of $\hat{M}^{(0)^{-1}}$) and factor it exactly in a preprocessing stage since U is of a small size $\tilde{m} \times \tilde{m}$, where $\tilde{m}$ was the number of coarse grid nodes on the interface Γ, i.e., the nodes in $\tilde{\gamma}$. Therefore, one action of Λ^{-1} can be computed at a cost of two actions of $\hat{M}^{(0)^{-1}}$ (one action needed for Δ and F, and another action needed for G) plus one (inexpensive) action of U^{-1} (solving systems with U based on its exact factorization). One should point out that the cost of the action of $\hat{M}^{(0)^{-1}}$ is negligible in comparison with the one of $\hat{B}^{-1}$ in §4.

We finally note that actually we have available the actions of $M^{(0)^{-1}}$, where $M^{(0)}$ is a preconditioner for $C^{(0)}$ in the standard nodal basis. Then, since $\hat{M}^{(0)} = J_\Gamma^{-1} M^{(0)} J_\Gamma^{-T}$, the actions of $\hat{M}^{(0)^{-1}}$ can be computed based on the relation

$$\hat{M}^{(0)^{-1}} = J_\Gamma^T M^{(0)^{-1}} J_\Gamma.$$

In other words, we need to perform some change of the basis (on Γ only) using the transformation matrices J_Γ and J_Γ^T defined in (5). This transformation is based on linear interpolation (for the actions of J_Γ) along the edges (of coarse–grid triangles) that cover Γ.

6. Numerical experiments. We report some numerical experiments performed on a Sparc 2 station in Fortran. A double precision version of the subroutines BLKTRI and SINT from the current version of [26] obtained via Netlib were used for solving separable problems in a rectangle and as a real FFT, and the subroutine GCGLS (generalized conjugate gradient least squares) which implements the version of GMRES in the form proposed in [3].

In the experiments as the polygonal region Ω we choose the unit square $R = (0,1)^2$ with the corner $\{(x,y) : y < \frac{1}{2} - x\}$ cut off. The example corresponds to the model bilinear form (2) with the coefficients chosen as

$$k_1(x) = e^x,\ \ k_2(y) = e^y,\ \ b_1(x) = c_2 sin(2\pi x),\ \ b_2(y) = c_2 sin(2\pi y),$$
$$\lambda_1(x) = c_3 e^x,\ \ \lambda_2(y) = c_3 e^y$$

Table 1. Performance for the problem with $c_2 = c_3 = 1$

n^2	$\tilde{m}$	$\|\text{error}\|_2$	N_p	N_s	t_p	t_s
15^2	1	0.173D-04	11	22	0.24	0.48
15^2	3	0.173D-04	21	14	0.45	0.31
31^2	1	0.418D-05	11	32	1.24	3.77
31^2	3	0.416D-05	29	27	3.39	3.19
31^2	7	0.418D-05	49	22	5.59	2.60
63^2	1	0.106D-05	11	38	6.20	21.55
63^2	3	0.981D-06	29	32	16.28	18.24
63^2	7	0.974D-06	49	27	27.42	15.44
63^2	15	0.950D-06	72	18	40.43	10.42
127^2	1	0.405D-02	11	44	29.75	120.11
127^2	3	0.219D-06	29	38	78.41	104.22
127^2	7	0.105D-05	46	32	124.23	87.72
127^2	15	0.792D-06	78	27	210.62	74.24
127^2	31	0.269D-06	136	22	367.59	60.83

The exact solution is $u(x,y) = \sin x \sin y$ with the corresponding values for the boundary condition and the source function f. The stopping criterion for the generalized conjugate gradient iterations to solve the capacitance matrix equation (step (ii) in the Algorithm CMM) is when the norm $\left(\underline{\zeta}_0^T \Lambda^{-1} \underline{\zeta}_0\right)^{\frac{1}{2}}$ of the current residual $\underline{\zeta}_0$ decrease by 10^{-6} from its initial value (with the zero initial guess for the solution). Here Λ is the preconditioner defined in Corollary 1 on the basis of $M^{(0)}$ (see Remark 2) and equal to the inverse of the square root of the discrete second derivative $-\frac{d^2(.)}{ds^2}$ along the interface boundary Γ, i.e., on the node set γ. This preconditioner (more precisely $M^{(0)^{-1}}$) was proposed by Dryja [12] for the symmetric positive definite DD Schur complements $S^{(0)}$ (see (5)). For more details we refer to our previous paper [20].

We denote the number of meshpoints in each coordinate direction by n, and by $\tilde{m}$ and m the number of coarse and fine meshpoints on the separator Γ, equal to the dimension of the capacitance matrix $\hat{C}$ and its principal submatrix X of (7). N_p and N_s are the number of BLKTRI solves for the preprocessing and solving stages, respectively, and t_p and t_s are the CPU-time (in sec) for the preprocessing and solving stages, respectively. In all experiments $m = \frac{n-1}{2}$. Note that the dominant cost of the whole algorithm is the cost of the separable solver BLKTRI called $(N_p + N_s)$–times while the cost of the preconditioner and the transformation is of lower order with respect to n, i.e., it is negligible.

Table 2. Performance for the problem with $c_2 = c_3 = 10$

n^2	$\tilde{m}$	$\|error\|_2$	N_p	N_s	t_p	t_s
15^2	1	0.365D-04	11	22	0.24	0.48
15^2	3	0.365D-04	21	14	0.46	0.33
31^2	1	0.872D-05	11	32	1.29	3.72
31^2	3	0.873D-05	33	32	3.79	3.74
31^2	7	0.873D-05	49	22	5.59	2.58
63^2	1	0.213D-05	11	44	6.26	25.05
63^2	3	0.214D-05	33	32	18.45	18.19
63^2	7	0.215D-05	49	32	27.42	18.23
63^2	15	0.218D-05	84	22	47.09	12.71
127^2	1	0.631D-01	11	51	29.90	139.90
127^2	3	0.263D-05	29	44	78.78	120.89
127^2	7	0.676D-06	49	38	133.17	104.62
127^2	15	0.165D-05	96	32	260.83	88.24
127^2	31	0.555D-06	136	22	369.90	61.18

We can see from the data that the accuracy of the method is clearly $O(h^2)$ (the error for $n = 127$ is affected by the chosen stopping criterion) and largely insensitive with respect to the value of $\tilde{m}$, the number of auxiliary coarse mesh nodes on Γ (unless the ratio $\frac{m}{\tilde{m}}$ is too large), even if $\tilde{m} = 1$. For moderately nonsymmetric and indefinite problems as in Tables 1 and 2, the cost of preprocessing grows linearly with $\tilde{m}$ (averaging 4 to 6 $\hat{B}$–solves per iteration, which is consistent with the results in [20]) and thus is much higher than for the symmetric definite problems.

The performance deteriorates somewhat for strongly indefinite problems, although the iterations converge even if the algorithm detects that the major block W (see (7), (8)) of $\hat{C}$ is non-coercive (denoted by "*" in Table 3). This is the case when the coarse–grid is not fine enough. Such behavior is explained in Saad and Schultz [23]; namely, that the GMRES converges even in the presense of eigenvalues of the matrix (in our case $\Lambda^{-1}W$) in the left part of the complex plane.

Table 3. Performance for the problem with $c_2 = 0$ & $c_3 = 100$

n^2	$\tilde{m}$	$\|\text{error}\|_2$	N_p	N_s	t_p	t_s
15^2	1	0.102D-04	11	22	0.24	0.48
15^2	3	0.102D-04	39	14	0.81	0.32
31^2	1	0.562D-05	22	72	2.50	8.23 *
31^2	3	0.571D-05	60	38	6.82	4.38
31^2	7	0.373D-05	77	27	8.75	3.14
63^2	1	0.554D-06	16	86	8.93	48.65 *
63^2	3	0.571D-06	74	51	41.89	29.01 *
63^2	7	0.112D-04	87	38	48.68	21.55
63^2	15	0.398D-05	105	22	58.96	12.73
127^2	1	0.245D-05	16	100	43.49	273.03 *
127^2	3	0.113D-04	74	72	200.28	196.17 *
127^2	7	0.525D-04	77	51	209.28	140.01
127^2	15	0.207D-05	137	32	373.86	88.96
127^2	31	0.655D-05	214	22	582.26	61.73

Note that in all tables when increasing the coarse–grid accuracy (i.e., choosing large $\tilde{m}$) the convergence of the method improves. However this gives rise to a higher cost of the preprocessing stage. When a series of the same problems with different right–hand sides are to be solved (the case when the presented method is to be recommended) the cost of the preprocessing stage can be neglected.

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