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FINITE VOLUME METHODS FOR CONVECTION-DIFFUSION PROBLEMS

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ABSTRACT. Derivation, stability and error analysis in both discrete H^1 and L^2 norms for cell-centered finite volume approximations of convection-diffusion problems are presented. Various upwind strategies are investigated. The theoretical results are illustrated by numerical examples.

1. INTRODUCTION

This paper is devoted to the construction of cell-centered finite difference schemes for convection-diffusion second order elliptic equations of divergence type. A main attention is paid to the construction of finite difference schemes with a second order (in a discrete H^1 norm) of approximation and at the same time to provide monotone schemes, i.e., schemes that satisfy a discrete maximum principle. The error estimates we derived are in the natural Sobolev spaces associated with the considered boundary value problem similarly to the finite element method. The upwind strategies for convection-diffusion equations have been used for a long time, but due to their first order of accuracy there have been several attempts to modify them in order to achieve second order of accuracy, cf., e.g., Samarskii [20], see also Axelsson and Gustafson [3]. The central difference approximation of the first derivatives has a disadvantage to require sufficiently small mesh size h in order to guarantee stability of the solution of the discrete problem, but on the other hand it is of second order of accuracy. We investigate a number of modified upwind finite difference strategies which provide both second order of accuracy and that are unconditionally (i.e., not only for sufficiently small $h > 0$) stable (or invertible). There are several known ways in the literature to derive the finite difference discretization schemes. For example one can use direct finite difference approximation by simply replacing the derivatives by divided differences, e.g., Samarskii [20]. In the same book [20] an error estimate of order $O(h^2)$ in the discrete maximum norm under rather demanding assumptions on the solution (to have four continuous derivatives) is derived. A modified upwind finite difference strategy (leading to a second order of accuracy scheme) was also considered in Axelsson and Gustafson [3]. Runchal [19], and also Spalding [22],

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have proposed and numerically tested other upwind finite difference schemes that can be used in both convection dominated and diffusive limits. For one dimensional problems Il'in [13] has proposed finite difference schemes for convection–dominated second order equations and proved an $O(h^2)$ error estimate in the maximum norm. A general treatment of finite difference schemes on triangular meshes was presented in Heinrich [12]. The schemes there are very similar to those obtained by the finite element method and in [12] mostly the selfadjoint case was addressed and in the convection-diffusion case some schemes based on central differences were considered. A general approach for cell-centered finite difference schemes on triangles including local refinement was considered in Vassilevski, Petrova and Lazarov [24]. The error estimates derived in [24] are in a discrete H^1-norm including some superconvergence type estimates on uniform triangulations, namely, $O(h^2)$ error estimate on uniform triangulations. Cell-centered discretizations on tensor-product nonuniform meshes were considered in Weiser and Wheeler [25] with superconvergence type error estimates derived. Similar results only for the Poisson equation were proved in Sũli [23], i.e., H^1-estimates of order $O(h^{1+\alpha})$, $\frac{1}{2} < \alpha \leq 1$. Morton and Sũli [17] considered point-centered finite difference schemes for one dimensional problems and also some hyperbolic equations in two-dimensional domains. Method that is closely related to the finite element one is the finite volume element method proposed and analyzed in Cai [6], Cai, Mandel and McCormick [7], and McCormick [16]; see also some early formulation by Baliga and Patankar [4] that includes the convection-diffusion case. The relationship of the similar box method and the finite element method in the symmetric positive definite case has been investigated by Bank and Rose [5] and in Hackbusch [11]. In Hackbusch [11] second order error estimates in H^1-norm on uniform mesh has been proved.

This paper is devoted to fill-in the lack of results for nonsymmetric equations and cell-centered finite differences. We prove in the present paper for a number of upwind finite difference schemes error estimates in discrete H^1-norm of order $O(h^{m-1})$, $\frac{3}{2} < m \leq 3$ for solution $u \in H^m(\Omega)$. These results can be viewed as natural extension of the results from Ewing, Lazarov, and Vassilevski [9], now for the non-selfadjoint case. In addition, we provide error estimates in L^2-norm elaborating the discrete "Aubin-Nitsche trick" of duality argument proposed in Samarskii, Lazarov, and Makarov [21] and used in the case of finite difference schemes for general self-adjoint elliptic equations in Lazarov, Makarov and Weinelt [15]. For the original duality technique in the finite element method, cf., Aubin [2], Nitsche [18]; which can also be found in Ciarlet [8].

The remainder of the paper is organized as follows. In the subsection 1.1 the boundary value problem is stated and the notations used are introduced in subsection 1.2. The discretization schemes are presented in Section 2. The stability (a priori estimates) and error estimates in H^1-norm are derived in subsection 3.1. The error estimates in L^2-norm are proved in subsection 3.2. Finally, in Section 4, the numerical results are presented.

1.1. Boundary Value Problem. We use the standard notation for Sobolev spaces [1]:

$$W_p^m(\Omega) = \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega),\, |\alpha| \le m\}, \quad m \ge 0,\, 1 \le p \le \infty$$

and $W_2^m(\Omega) = H^m(\Omega)$. The norm in $H^m(\Omega)$ is denoted $\|.\|_{m,\Omega}$ and defined by

$$\|u\|_{m,\Omega} \equiv \left(\sum_{i=0}^m |u|_{i,\Omega}^2\right)^{1/2}, \qquad |u|_{i,\Omega} \equiv \left(\sum_{|\alpha|=i} \|D^\alpha u\|_{0,\Omega}^2\right)^{1/2},$$

$$\|u\|_{m,\infty,\Omega} \equiv \max_{|\alpha|\le m} \sup_{x\in\Omega} |D^\alpha u|,$$

where $\|.\|_{0,\Omega}$ is the standard L^2-norm in Ω. We also use Sobolev spaces with real index $m > 0$ [1].

We consider the following convection-diffusion boundary value problem: find a function $u(x)$ which satisfies the following differential equation and boundary condition:

$$(1) \qquad \begin{cases} \operatorname{div}(-a(x)\nabla u(x) + \underline{b}(x)u(x)) = f(x) & \text{in } \Omega \\ u(x) = 0 & \text{on } \Gamma \end{cases}$$

where $\Omega \subset R^2$ is a bounded domain, $\Gamma = \partial\Omega$, $a(x)$, $f(x)$, and the velocity vector $\underline{b}(x) = (b_1(x), b_2(x))$ are given functions in Ω. We introduce the bilinear form

$$\begin{aligned} a(u,v) &= \int_\Omega a(x) \sum_{i=1}^2 \partial_i u(x) \partial_i v(x)\, dx \\ &\quad + \int_\Omega (\underline{b}(x).\nabla u(x)) v(x)\, dx + \int_\Omega u(x)v(x)(\nabla.\underline{b}(x))\, dx \end{aligned}$$

and the linear form

$$f(v) = \int_\Omega f(x)v(x)\, dx.$$

Here and hereafter ∂_i denotes the partial derivative with respect to x_i.

The problem (1) can also be formulated in the following weak form:

Find $u \in H_0^1(\Omega)$ such that $\quad a(u,v) = f(v) \quad$ for all $v \in H_0^1(\Omega)$.

From

$$\begin{aligned} \int_\Omega (\underline{b}(x).\nabla u(x))u(x)\, dx &= -\int_\Omega \nabla.(\underline{b}(x)u(x))u(x)\, dx \\ &= -\int_\Omega (\nabla.\underline{b}(x))u^2(x)\, dx - \int_\Omega (\underline{b}(x).\nabla u(x))u(x)\, dx \end{aligned}$$

we obtain

$$\int_\Omega (\underline{b}(x).\nabla u(x))u(x)\, dx = -\frac{1}{2}\int_\Omega (\nabla.\underline{b}(x))u^2(x)\, dx$$

and hence

$$(2) \qquad a(u,u) = \int_\Omega a(x)\sum_{i=1}^2 (\partial_i u(x))^2\, dx + \frac{1}{2}\int_\Omega (\nabla.\underline{b}(x))u^2(x)\, dx.$$

Let the coefficients $a(x)$, $\underline{b}(x)$ satisfy the conditions:

(i) $a(x) \geq a_0 > 0$, $a(x) \in W^1_\infty(\Omega)$,

(ii) $(\nabla.\underline{b}(x)) \geq \beta_0 > 0$, $|b_i(x)| \leq \beta_1$, $b_i \in W^1_\infty(\Omega)$

for some positive constant a_0, β_0, β_1. Then we see from (2) that there exists a constant C such that $a(u,u) \geq C\|u\|_{1,\Omega}$, i.e. $a(u,v)$ is a H^1_0-elliptic bilinear form. Then by the Lax-Milgram lemma argument the problem (1) has an unique solution in $H^1_0(\Omega)$.

Remark 1.1. *For the stability analysis (Propositions 2.1, 2.3, 2.5, 2.7) we will need higher smoothness, i.e.,* $b_i(x) \in W^{1+\alpha}_\infty(\Omega)$, $\alpha > 0$.

1.2. Grids and grid functions. We suppose that Ω is a rectangle with sides parallel to the axes x_1 and x_2. Extensions to the case of more general domains can be accomplished using the technique described in Samarskii, Lazarov and Makarov [21, Chapter III, p. 123].

We consider the case of cell-centered grids, which owing to their good conservation properties, are very popular in reservoir simulation, weather prediction, heat transfer etc. We cover the plane R^2 by square cells with sides of length h. The grid points are the centers of the cells (see, Fig. 1). We suppose that the Dirichlet boundary Γ passes through the grid points, as shown in Fig. 1.

The grid points are denoted by $x = (x_1, x_2) = (x_{1,i}, x_{2,j}) = (ih, jh)$, where $i, j = 0, 1, 2, ..., N$ are integer indices. We introduce the following notations for various grids in $\overline{\Omega}$:

$$\overline{\omega} = \Big\{(x_{1,i}, x_{2,j}) \in \overline{\Omega} : i, j = 0, 1, 2, ..., N\Big\};$$

$$\omega = \overline{\omega} \cap \Omega, \quad \gamma = \overline{\omega} \backslash \omega;$$

$$\omega_i^{\pm} = \omega \cup \gamma_i^{\pm}\ , \text{ where } \ \gamma_i^{\pm} = \{x \in \gamma : cos(x_i, \underline{n}) = \pm 1\}\ ,\ i = 1, 2,$$

here $\underline{n}$ is the unit outer normal to the boundary Γ.

Functions defined for $x \in \omega$ are called grid functions. We consistently use the dual notation for the value of the function y at the grid point $x = (x_{1,i}, x_{2,j})$; $y(x) = y(x_{1,i}, x_{2,j}) = y_{i,j}$ and in the points $(x_{1,i}, x_{2,j} \pm h/2) = (x_{1,i}, x_{2,j\pm 1/2})$ and $(x_{1,i} \pm h/2, x_{2,j}) = (x_{1,i\pm 1/2}, x_{2,j})$, $y_{i,j\pm 1/2} = y(x_{1,i}, x_{2,j\pm 1/2})$, $y_{i\pm 1/2,j} = y(x_{1,i\pm 1/2}, x_{2,j})$.

For a given function $y(x)$, $x \in \overline{\omega}$ we use the following discrete inner products and norms:

$$(y,v) = \sum_{x\in\omega} h^2 y(x)v(x), \ \|y\|_{0,\omega} = (y,y)^{\frac{1}{2}};$$

$$(y,v]_s = \sum_{x\in\omega_s^+} h^2 y(x)v(x)\,, \ \|y]|_s = (y,y]_s^{\frac{1}{2}}\,, \quad s = 1, 2\,.$$

We introduce the following finite differences for grid functions $y(x)$:

(i) forward difference $\Delta_1 y_{i,j} = y_{i+1,j} - y_{i,j}$ and divided forward difference $y_{x_1} = \Delta_1 y/h$;

(ii) backward difference $\overline{\Delta}_1 y_{i,j} = y_{i,j} - y_{i-1,j}$ and divided backward difference $y_{\overline{x}_1} = \overline{\Delta}_1 y/h$;

(iii) divided central difference of second order

$$y_{\bar{x}_1 x_1} = \frac{\Delta_1 y_{i,j} - \overline{\Delta}_1 y_{i,j}}{h^2}.$$

Similarly, differences are defined in x_2 and in combination of x_1 and x_2 coordinate directions.

We also introduce the discrete analogues of H^1 and H^2 -norms:

$$|y|^2_{1,\omega} = \|y_{\bar{x}_1}]|^2_1 + \|y_{\bar{x}_2}]|^2_2,$$

$$\|y\|^2_{1,\omega} = |y|^2_{1,\omega} + \|y\|^2_{0,\omega},$$

and

$$|y|^2_{2,\omega} = |y_{\bar{x}_1 x_1}|^2 + 2|y_{\bar{x}_1 \bar{x}_2}|^2 + |y_{\bar{x}_2 x_2}|^2,$$

$$\|y\|^2_{2,\omega} = |y|^2_{2,\omega} + \|y\|^2_{1,\omega}.$$

We will also need the negative norm:

$$\|y\|_{-1,\omega} = \sup_{v \neq 0} \frac{|(y,v)|}{\|v\|_{1,\omega}}.$$

Any grid function $y(x)$ can be considered as an element of a vector space of dimension equal to n, the number of the grid points in ω. In this case, we denote $y(x)$ by $\mathbf{y} \in R^n$ and consider it as an n-dimensional column vector. Then $\mathbf{y}^T$ will be the row vector transpose of $\mathbf{y}$.

2. Discretization schemes

The finite difference approximation is derived from the balance equation. We integrate (1) over each cell e

$$\int_e div[(-a(x)\nabla u(x) + \underline{b}(x)u(x)]\,dx = \int_e f(x)\,dx$$

and then using the Green's formula we get

$$\int_{\partial e} [-a\nabla u.\underline{n} + u\underline{b}.\underline{n}]\,d\gamma = \int_e f(x)\,dx \tag{3}$$

where $\underline{n}$ is the unit outward vector normal to the boundary of e. Splitting $\partial e = s_1^+ \cup s_2^+ \cup s_1 \cup s_2$ (see Fig. 2) this identity can be written in the form:

$$\begin{aligned} \int_{\partial e} W\,d\gamma + \int_{\partial e} V\,d\gamma \;&=\; \int_{s_1^+} W\,d\gamma + \int_{s_1^+} V\,d\gamma - \int_{s_1} W\,d\gamma - \int_{s_1} V\,d\gamma \\ &\quad + \int_{s_2^+} W\,d\gamma + \int_{s_2^+} V\,d\gamma - \int_{s_2} W\,d\gamma - \int_{s_2} V\,d\gamma \end{aligned} \tag{4}$$

where we have denoted by

$$W = -a(\gamma)\nabla u(\gamma).\underline{n} \quad \text{and} \quad V = \underline{b}(\gamma).\underline{n}u(\gamma).$$

FIGURE 1. Cell-centered mesh

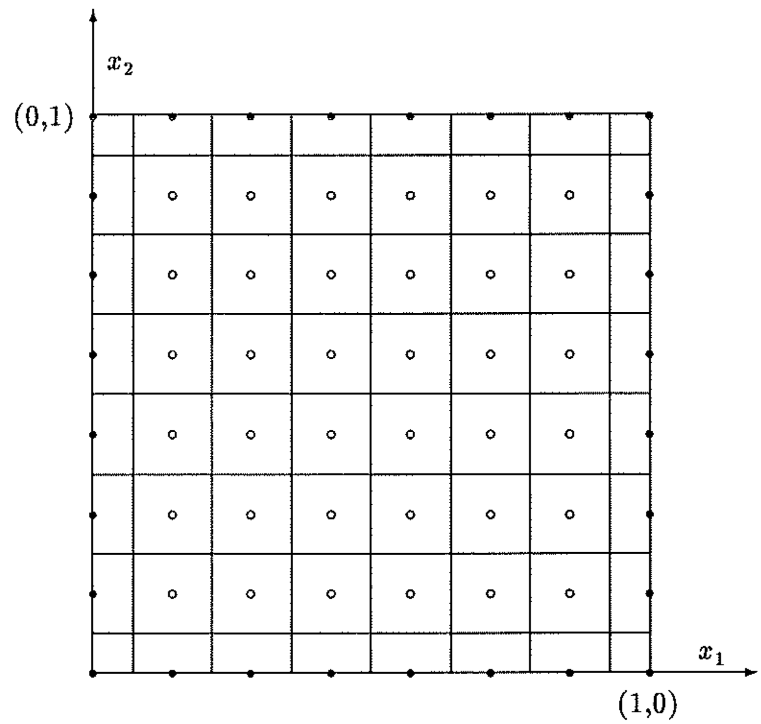
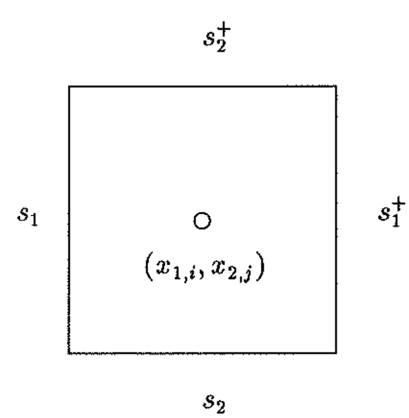


FIGURE 2. cell $e(x)$



In order to construct the finite difference scheme we approximate the balance equation (4). We split the approximation of the balance equation (4) in two parts

$$A^{(2)}y + A^{(1)}y \tag{5}$$

where $A^{(2)}$ is the part arising from the approximation of the second derivatives, and $A^{(1)}$ comes from the approximation of the first derivatives; y is an approximation to the exact solution solution u. We have the expressions

$$\begin{aligned} A^{(2)}y &= \sum_{x\in\omega} [w^+_{1,i,j} - w_{1,i,j} + w^+_{2,i,j} - w_{2,i,j}] \\ A^{(1)}y &= \sum_{x\in\omega} [v^+_{1,i,j} - v_{1,i,j} + v^+_{2,i,j} - v_{2,i,j}] \end{aligned} \tag{6}$$

In these formulae w^+_l, w_l, v^+_l, v_l, $l = 1, 2.$ are some approximations of the corresponding integrals $\int_{s_1^+} W$, $\int_{s_1} W$, $\int_{s_2^+} W$, $\int_{s_2} W$ and $\int_{s_1^+} V$, $\int_{s_1} V$, $\int_{s_2^+} V$, $\int_{s_2} V$, respectively. Now, in order to complete the finite difference scheme we have to express the approximate fluxes w^+_l, w_l, v^+_l, v_l by the approximate values $y(x)$ of the solution $u(x)$ at the grid points. We consider the following approximations:

(1) central difference scheme **CDS**
(2) upwind difference scheme **UDS**
(3) modified upwind difference scheme **MUDS**
(4) Il'in's difference scheme **IDS**

2.1. Central difference scheme (CDS). We call this scheme "central" because of the analogy of $A^{(1)}$ and a central difference approximation of the first derivatives. We first rewrite the fluxes $-a(x)\nabla u(x) = (W_1(x), W_2(x))$ in the form

$$\frac{\partial u}{\partial x_l} = -\frac{W_l(x)}{a(x)}, \qquad x \in \Omega, \quad l = 1, 2\,.$$

Next, we integrate the equation for $l = 1$ along the interval with endpoints $(x_{1,i-1}, x_{2,j})$ and $(x_{1,i}, x_{2,j})$. We get

$$u_{i,j} - u_{i,j-1} = -\int_{x_{1,i-1}}^{x_{1,i}} \frac{W_1(s, x_{2,j})}{a(s, x_{2,j}} \, ds \approx W_{1,i-1/2,j} \int_{x_{1,i-1}}^{x_{1,i}} \frac{ds}{a(s, x_{2,j}}).$$

We can now write the following approximate relations

$$\int_{s_1} W_1(x)\, ds \approx hW_{1,i-1/2,j} \approx -\left(\frac{1}{h}\int_{x_{1,i-1}}^{x_{1,i}} \frac{ds}{a(s, x_{2,j})}\right)^{-1} [u_{i,j} - u_{i-1,j}],$$

$$\int_{s_2} W_2(x)\, ds \approx hW_{2,i,j-1/2} \approx -\left(\frac{1}{h}\int_{x_{2,j-1}}^{x_{2,j}} \frac{ds}{a(x_{1,i}, s)}\right)^{-1} [u_{i,j} - u_{i,j-1}].$$

These approximate relations allow us to define:

$$\begin{aligned} w^+_l(x) \equiv w^+_{l,i,j} &= -k^+_{l,i,j} \, \triangle_l \, y_{i,j} \,, \qquad l = 1, 2 \\ w_l(x) \equiv w_{l,i,j} &= -k_{l,i,j} \overline{\triangle}_l y_{i,j} \,, \qquad l = 1, 2\,, \end{aligned} \tag{7}$$

where

$$k_{1,i,j} = \left(\frac{1}{h} \int_{x_{1,i-1}}^{x_{1,i}} \frac{ds}{a(s, x_{2,j})} \right)^{-1}, \quad k^+_{1,i,j} = k_{1,i+1,j} \tag{8}$$

$$k_{2,i,j} = \left(\frac{1}{h} \int_{x_{2,j-1}}^{x_{2,j}} \frac{ds}{a(x_{1,i}, s)} \right)^{-1}, \quad k^+_{2,i,j} = k_{2,i,j+1}\,.$$

The integrals $\int_{s_l^+} V_l^+(x)\,ds$, $\int_{s_l} V_l(x)\,ds$ can be approximated as follows ($l = 1$)

$$\int_{s_1^+} V_1^+(x)\,ds \approx h b_{1,i+1/2,j} \left[\frac{u_{i,j} + u_{i+1,j}}{2} \right],$$

$$\int_{s_1} V_1(x)\,ds \approx h b_{1,i-1/2,j} \left[\frac{u_{i-1,j} + u_{i,j}}{2} \right],$$

and thus we can define the approximations

$$\begin{aligned}
v^+_{1,i,j} &= B^+_{1,i,j}(y_{i+1,j} + y_{i,j}), \quad B^+_{1,i,j} = \frac{b_{1,i+1/2,j}h}{2},\\
v_{1,i,j} &= B_{1,i,j}(y_{i,j} + y_{i-1,j}), \quad B_{1,i,j} = \frac{b_{1,i-1/2,j}h}{2},\\
v^+_{2,i,j} &= B^+_{2,i,j}(y_{i,j+1} + y_{i,j}), \quad B^+_{2,i,j} = \frac{b_{2,i,j+1/2}h}{2},\\
v_{2,i,j} &= B_{2,i,j}(y_{i,j} + y_{i,j-1}), \quad B_{2,i,j} = \frac{b_{2,i,j-1/2}h}{2}.
\end{aligned} \tag{9}$$

Substituting (7) and (9) in (5) we get **CDS**. This scheme is stable if the local Peclet number satisfies the inequality [14], [20]:

$$\frac{|b_l(.,.)|h}{2k_l(.,.)} \le 1 \tag{10}$$

Obviously this is true only for sufficiently small h. We will not further consider the **CDS** because of its conditional stability.

2.2. Upwind difference scheme (UDS). One of the ways to find stable finite difference approximation for convection-diffusion boundary value problem is to use upwind approximation for the first derivatives. In this case, $A^{(1)}$ is defined as in **CDS** and the terms v_1, v_1^+ in $A^{(2)}$ are approximated in the following way:

$$\begin{aligned}
v^+_{1,i,j} &= (B^+_{1,i,j} - |B^+_{1,i,j}|)y_{i+1,j} + (B^+_{1,i,j} + |B^+_{1,i,j}|)y_{i,j},\\
v_{1,i,j} &= (B_{1,i,j} - |B_{1,i,j}|)y_{i,j} + (B_{1,i,j} + |B_{1,i,j}|)y_{i-1,j}.
\end{aligned} \tag{11}$$

In order to investigate the properties of the **UDS** we need the following auxiliary result.

Proposition 2.1. *Let* $\underline{b}(x) \in (W_\infty^{1+\alpha}(\Omega))^2$, $\alpha > 0$ *and* $\nabla.\underline{b}(x) \geq \beta_0$ *for some* $\beta_0 > 0$. *Then there exists* h_0 *such that for* $h \in (0, h_0)$ *the following inequality holds:*

$$[(B_{1,i,j}^+ - B_{1,i,j}) + (B_{2,i,j}^+ - B_{2,i,j})] \geq c_0 h^2 ,$$

where $c_0 = \beta_0 - O(h^\alpha)$, $0 < \alpha \leq 2$.

Proof. Consider the linear functional:

$$l(b_1) := \frac{b_{1,i+1/2,j} - b_{1,i-1/2,j}}{h} - \frac{\partial b_{1,i,j}}{\partial x_1} .$$

This functional is bounded for $b_1 \in W_\infty^{1+\alpha}(\Omega)$, $0 \leq \alpha \leq 2$ and vanishes for all polynomials of second degree. Therefore, by the Bramble-Hilbert lemma argument we get

$$|l(b_1)| \leq C h^\alpha |b_1|_{1+\alpha,\infty,e} .$$

Similar inequality holds for b_2. Multiplying by h^2, using the triangle inequality and the assumption $\nabla.\underline{b} \geq \beta_0$, the desired inequality is obtained. □

Remark 2.1. *The above proposition means that, if the divergence of the vector* $\underline{b}$ *is greater than* $b_0 > 0$, *then the discrete analog of* $\nabla.\underline{b}$, *defined by*

$$\frac{b_{1,i+1/2,j} - b_{1,i-1/2,j}}{h} + \frac{b_{2,i,j+1/2} - b_{2,i,j-1/2}}{h}$$

is also positive for sufficiently small h.

First we will prove that the considered scheme is monotone.

Proposition 2.2. *UDS satisfies the discrete maximum principle and the corresponding matrix* A *is an* M*–matrix.*

Proof. Let $a_{i+k,j+l}$ be the coefficient in front of $y_{i+k,j+l}$, $k, l = -1, 0, 1$ in the finite difference scheme. Then it is enough to check the conditions [12]:

(1) $a_{i,j} > 0$;
(2) $a_{i-1,j}$, $a_{i+1,j}$, $a_{i,j-1}$, $a_{i,j+1}$ are negative;
(3) $a_{i,j} - \sum_{k,l=\pm 1} a_{i+k,j+l} > 0$, i.e., A is strictly diagonally dominant.

We have

(1)

$$a_{i,j} = \sum_{l=1}^{2} [(k_{l,i,j}^+ + k_{l,i,j}) + (B_{l,i,j}^+ - B_{l,i,j}) + |B_{l,i,j}^+| + |B_{l,i,j}|] > 0$$

(2)

$$|B_{l,i,j}| + B_{l,i,j} \geq 0 \Rightarrow -(k_{l,i,j} + |B_{l,i,j}| + B_{l,i,j}) < 0$$

$$B_{l,i,j}^+ - |B_{l,i,j}| \leq 0 \Rightarrow -k_{l,i,j}^+ + B_{l,i,j}^+ - |B_{l,i,j}| < 0$$

(3)

$$a_{i,j} - \sum_{k,l=\pm 1} a_{i+k,j+l} = 2\sum_{l=1}^{2}(B^{+}_{l,i,j} - B_{l,i,j}) \geq 2c_0h^2 > 0\,.$$

□

Let A be the matrix arising from the finite difference approximation. We scale the matrix A and define the linear operator A_h:

$$A_h \,:\, D^0 \to D\,, \quad A_h y = \frac{1}{h^2}\mathrm{Ay}\,,$$

where D is the space of all grid functions and $D^0 = \{y,\, y_{|\gamma} = 0\}$. This means that $\mathbf{y}^T\mathrm{A}\mathbf{y} = (A_h y, y)$.

Now we concentrate on the positive deffinitness of the operator A_h and the matrix A. In Section 1 we showed that the bilinear form, corresponding to the continuous problem (1) is H_0^1–elliptic. In the following proposition we establish that the discrete analog of the bilinear form inherits this property.

Proposition 2.3. *Let $\underline{b}(x) \in (W^{1+\alpha}_{\infty}(\Omega))^2$, $\alpha > 0$ and $\nabla.\underline{b}(x) \geq \beta_0$. Then the matrix A of UDS is a positive real matrix and there exists a constant C such that the following inequality is true:*

$$(A_h y, y) \geq C\|y\|^2_{1,\omega}\ ,\ \textit{for all}\ y \in D^0 = \{y,\, y_{|\gamma} = 0\}.$$

The constant C depends only on the ratio $a(x)/|\underline{b}(x)|$.

Proof. Let $z(x)$ and $y(x)$ be grid functions from D^0. Then

$$(12)\qquad \begin{aligned}(A_h y, z) \;&=\; -\sum_{x\in\omega}\sum_{l=1}^{2}\left[k^{+}_{l,i,j}\triangle_l y_{i,j} - k_{l,i,j}\overline{\triangle}_l y_{i,j}\right] z_{i,j}\\ &\quad+\sum_{x\in\omega}\sum_{l=1}^{2}\left[v^{+}_{l,i,j} - v_{l,i,j}\right] z_{i,j} \equiv I + \sum_{l=1}^{2} J_l.\end{aligned}$$

We transform the sums in formula (12) for $l = 1, 2$ using summation by parts thus obtaining

$$I = \sum_{l=1}^{2}\sum_{x\in\omega} k_{l,i,j}\overline{\triangle}_l y_{i,j}\overline{\triangle}_l z_{i,j}\,.$$

Using (11) we rewrite J_1 in the following way

$$
\begin{aligned}
(13) \qquad J_1 &= \sum_{x\in\omega}[(B^+_{1,i,j} - |B^+_{1,i,j}|)y_{i+1,j} + (B^+_{1,i,j} + |B^+_{1,i,j}|)y_{i,j} \\
&\qquad -(B_{1,i,j} - |B_{1,i,j}|)y_{i,j} - (B_{1,i,j} + |B_{1,i,j}|)y_{i-1,j}]z_{i,j} \\
&= \sum_{x\in\omega}[B^+_{1,i,j}y_{i+1,j} - B_{1,i,j}y_{i-1,j}]z_{i,j} \\
&\quad + \sum_{x\in\omega}\left[B^+_{1,i,j} - B_{1,i,j}\right] y_{i,j}z_{i,j} \\
&\quad - \sum_{x\in\omega}\left[|B^+_{1,i,j}|\triangle_1 y_{i,j} - |B_{1,i,j}|\overline{\triangle}_1 y_{i,j}\right] z_{i,j}.
\end{aligned}
$$

We now transform the first term in the last identity in (13)

$$
\begin{aligned}
\sum_{x\in\omega}\left[B^+_{1,i,j}y_{i,j} - B_{1,i,j}y_{i-1,j}\right] z_{i,j} &= \sum_{x\in\omega}\left[B^+_{1,i,j}y_{i+1,j} - B_{1,i,j}y_{i,j}\right] z_{i,j} \\
&\quad + \sum_{x\in\omega} B_{1,i,j}(y_{i,j} - y_{i-1,j})z_{i,j}
\end{aligned}
$$

Using summation by parts for the first term above we obtain

$$
\sum_{x\in\omega}\left[B^+_{1,i,j}y_{i+1,j} - B_{1,i,j}y_{i-1,j}\right] z_{i,j} = \sum_{x\in\omega} B_{1,i,j}(z_{i,j}\overline{\triangle}_1 y_{i,j} - y_{i,j}\overline{\triangle}_1 z_{i,j}).
$$

Finally we get

$$
\begin{aligned}
(A_h y, z) &= \sum_{l=1}^{2}\sum_{x\in\omega}(k_{l,i,j} + |B_{l,i,j}|)\overline{\triangle}_l y_{i,j}\overline{\triangle}_l z_{i,j} \\
&\quad + \sum_{l=1}^{2}\sum_{x\in\omega} B_{l,i,j}(z_{i,j}\overline{\triangle}_1 y_{i,j} - y_{i,j}\overline{\triangle}_1 z_{i,j}) \\
&\quad + \sum_{l=1}^{2}\sum_{x\in\omega}\left(B^+_{l,i,j} - B_{l,i,j}\right) y_{i,j}z_{i,j}.
\end{aligned}
$$

Letting $z = y$ in the above formula the desired result follows using Proposition 2.1. □

2.3. Modified upwind difference scheme (MUDS). As we will later show the **UDS** is only $O(h)$ accurate. In order to obtain a diagonally dominant matrix and achieve $O(h^2)$ order of accuracy we modify the upwind scheme in the following way [3], (see also [20])

$$
\begin{aligned}
\int_{s_1} b_1 u\, d\gamma &= (B_{1,i,j} - |B_{1,i,j}|)u_{i,j} + (B_{1,i,j} + |B_{1.i,j}|)u_{i-1,j} + O(h) \\
&= I_1 + O(h),
\end{aligned}
$$

$$
\int_{s_1} b_1 u\, d\gamma = B_{1,i,j}(u_{i,j} + u_{i-1,j}) + O(h^2) = I_2 + O(h^2),
$$

$$
\begin{aligned}
\int_{s_1}\left(-a\frac{\partial u}{\partial x_1}+b_1u\right)d\gamma &= -k_{1,i,j}\overline{\triangle}_1 u_{i,j}+I_2+O(h^2)\\
&= -(k_{1,i,j}-|B_{1,i,j}|)\overline{\triangle}_1 u_{i,j}+I_1+O(h^2)\\
&= -\frac{k_{1,i,j}}{1+|B_{1,i,j}|/k_{1,i,j}}\overline{\triangle}_1 u_{i,j}\\
&\quad -\left(k_{1,i,j}-|B_{1,i,j}|-\frac{k_{1i,j}^2}{k_{1,i,j}+|B_{1,i,j}|}\right)\overline{\triangle}_1 u_{i,j}\\
&\quad +I_1+O(h^2)\\
&= -\frac{k_{1,i,j}}{1+|B_{1,i,j}|/k_{1,i,j}}\overline{\triangle}_1 u_{i,j}\\
&\quad +\frac{B_{1,i,j}^2}{k_{1,i,j}+|B_{1,i,j}|}\overline{\triangle}_1 u_{i,j}+I_1+O(h^2)\\
&= -\frac{k_{1,i,j}}{1+|B_{1,i,j}|/k_{1,i,j}}\overline{\triangle}_1 u_{i,j}+I_1+O(h^2).
\end{aligned}
$$

In the last step we have taken into account that $B_1=O(h)$. This heuristic formulae show that if we want to get second order finite difference scheme we should choose w_l^+, w_l, v_l^+, v_l in such a way that they satisfy the following conditions:

$$
w_{l,i,j}^+ + v_{l,i,j}^+ = -\tilde{k}_{l,i,j}^+\triangle_l y_{i,j} + (B_{1,i,j}^+ - |B_{1,i,j}^+|)y_{i+1,j} + (B_{1,i,j}^+ + |B_{1.i,j}^+|)y_{i,j}\,,\ l=1,2,
$$

$$
w_{l,i,j} + v_{l,i,j} = -\tilde{k}_{l,i,j}\overline{\triangle}_l y_{i,j} + (B_{1,i,j} - |B_{1,i,j}|)y_{i,j} + (B_{1,i,j} + |B_{1.i,j}|)y_{i-1,j}\,,\ l=1,2.
$$

We remark here that we split the scheme into two parts only for convenience of the error analysis. Then we define **MUDS** as follows: $A^{(1)}$ is the same as in **CDS** and the expressions w_l, w_l^+ in $A^{(2)}$ are defined by

$$
\begin{aligned}
w_{l,i,j}^+ &= -\tilde{k}_{l,i,j}^+\triangle_l y_{i,j} - |B_{l,i,j}^+|\triangle_l y_{i,j}\,, \quad l=1,2\,,\\
w_{l,i,j} &= -\tilde{k}_{l,i,j}\overline{\triangle}_l y_{i,j} - |B_{l,i,j}|\overline{\triangle}_l y_{i,j}\,, \quad l=1,2\,,
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
\tilde{k}_{l,i,j} &= \frac{k_{1,i,j}}{1+|B_{1,i,j}|/k_{1,i,j}}\,, \quad \tilde{k}_{1,i,j}^+ = \tilde{k}_{1,i+1,j}\,,\\
\tilde{k}_{2,i,j} &= \frac{k_{2,i,j}}{1+|B_{2,i,j}|/k_{2,i,j}}\,, \quad \tilde{k}_{2,i,j}^+ = \tilde{k}_{2,i,j+1}\,.
\end{aligned}
\tag{15}
$$

In the same way as in Proposition 2.2 and Proposition 2.3 it follows.

Proposition 2.4. *MUDS satisfies the discrete maximum principle and the corresponding matrix A is an M–matrix.*

Proposition 2.5. *Let* $\underline{b}(x) \in (W_{\infty}^{1+\alpha}(\Omega))^2$, $\alpha > 0$ *and* $\nabla.\underline{b}(x) \geq \beta_0$. *Then the matrix* A *of the* ***MUDS*** *is a positive real matrix and there exists a constant* C *such that the following inequality is true:*

$$(A_h y, y) \geq C\|y\|_{1,\omega}^2 \text{ , for all } y \in D^0 = \{y,\, y_{|\gamma} = 0\}$$

The constant C *depends only on the ratio* $a(x)/|\underline{b}(x)|$.

2.4. Il'in's difference scheme (IDS). Another approximation we derive in a similar way as in [13]

$$\int_{s_1^+}(-a(\gamma)\frac{\partial u(\gamma)}{\partial x} + b_1(\gamma)u(\gamma))\,d\gamma \approx -\gamma_{1,i,j}^+ \Delta_1 y_{i,j} + B_{1,i,j}^+ y_{i+1,j} + B_{1,i,j}^+ y_{i,j}$$

or

$$w_{l,i,j}^+ = -\gamma_{l,i,j}^+ \,\Delta_l\, y_{i,j} \,, \quad w_{l,i,j} = -\gamma_{l,i,j}\overline{\Delta}_l y_{i,j} \,, \quad l = 1, 2 \tag{16}$$

and v_l^+ , v_l are defined as in **CDS**. We choose the coefficient γ such that the above approximate relation is exact for $u = e^{b_1 x/a}$ when $a(x)$ and $b_1(x)$ are constants. We get

$$\begin{aligned} \gamma_{l,i,j}^+ &= B_{l,i,j}^+ \coth\left(\frac{B_{l,i,j}^+}{k_{l,i,j}^+}\right) \\ \gamma_{l,i,j} &= B_{l,i,j} \coth\left(\frac{B_{l,i,j}}{k_{l,i,j}}\right) \end{aligned} \tag{17}$$

It is easy to see that γ_l^+ and $\gamma_l > 0$ are positive regardless of the sign of b_l. From $|\coth(x)| > 1$ we have $\gamma_l^+ > |B_l^+|$ and $\gamma_l > |B_l|$. Using the same technique as in previous propositions we have:

Proposition 2.6. ***IDS*** *satisfies the discrete maximum principle and the corresponding matrix* A *is an* M*-matrix.*

Proposition 2.7. *Let* $\underline{b}(x) \in (W_{\infty}^{1+\alpha}(\Omega))^2$, $\alpha > 0$ *and* $\nabla.\underline{b}(x) \geq \beta_0$. *Then the matrix* A *of the* ***IDS*** *is a positive real matrix and there exists a constant* C *such that the following inequality is true:*

$$(A_h y, y) \geq C\|y\|_{1,\omega}^2 \text{ , for all } y \in D^0 = \{y,\, y_{|\gamma} = 0\}$$

The constant C *depends only on the ratio* $a(x)/|\underline{b}(x)|$.

Summarizing these approximations we formulate the following discrete problem for (1): find a grid function $y(x)$, which satisfies the finite difference equations:

$$\begin{aligned} \textstyle\sum_{l=1}^2 (w_l^+(x) - w_l(x)) + \sum_{l=1}^2 (v_l^+(x) - v_l(x)) &= \int_e f(x)\,dx \quad \text{in} \quad \omega\,, \\ y(x) &= 0 \qquad\qquad\quad \text{on} \quad \gamma\,, \end{aligned}$$

where w_l, v_l are defined by (7), (14), (16), (9) and (11), respectively. These schemes can be written as systems of linear algebraic equations

$$A\mathbf{y} = \mathbf{f} \tag{18}$$

where the boundary conditions have been eliminated.

3. Stability and error analysis

The stability of problem (18) is a simple consequence of the positive definiteness of the matrix A. Namely, we prove the following lemma.

Lemma 3.1. *For all considered difference schemes the following a priori estimate is valid:*

$$\|y\|_{1,\omega} \le C\|f\|_{-1,\omega}$$

where y is the discrete solution and f is the right-hand side of (18). (The constant C does not depend on y or f.)

Proof. The proof follows from the inequalities based on the the coercivity of the operator A and on the definition of the norm $\|.\|_{-1,\omega}$

$$\|y\|_{1,\omega}^2 \le C(A_h y, y) = C(f, y) \le C\|f\|_{-1,\omega}\|y\|_{1,\omega}\,.$$

□

Remark 3.1. *Since $\|f\|_{-1,\omega} \le \|f\|_{0,\omega}$ and $\|y\|_{0,\omega} \le \|y\|_{1,\omega}$ we also can obtain the following estimate:*

$$\|y\|_{0,\omega} \le C\|f\|_{0,\omega}$$

3.1. Error estimates in discrete H^1–norm. The error analysis presented here is done in the general framework of the methods developed in [21] and [9]. We consider only the case when $a(x) \equiv 1$. Let

$$z(x) = y(x) - u(x)\,,\ x \in \omega$$

be the error of the finite difference method. Substituting $y = z + u$ in (18) we obtain

$$Az = f - Au \equiv \psi \tag{19}$$

Then using (4)–(18) we transform ψ in the following form

$$\sum_{l=1}^{2}\left\{\left[\int_{s_l^+} -\frac{\partial u}{\partial x_l}\,d\gamma - w_l^+\right] - \left[\int_{s_l} -\frac{\partial u}{\partial x_l}\,d\gamma - w_l\right]\right\}$$
$$+\sum_{l=1}^{2}\left\{\left[\int_{s_l^+} b_l u\,d\gamma - v_l^+\right] - \left[\int_{s_l} b_l u\,d\gamma - v_l\right]\right\} \quad \equiv \psi_1 + \psi_2 = \psi\,,$$

where the local truncation error ψ has been split up into two terms:

$$\psi_1 \equiv \sum_{l=1}^{2}\left[\eta_l^+(x) - \eta_l(x)\right]\,,\ \psi_2 \equiv \sum_{l=1}^{2}\left[\mu_l^+(x) - \mu_l(x)\right]\,,$$

$$\eta_l = \int_{s_l} -\frac{\partial u}{\partial x_l}\,d\gamma - w_l\,,\ \mu_l = \int_{s_l} b_l u\,d\gamma - v_l \tag{20}$$

Here ψ_1 is the error of approximation of the first derivatives, and ψ_2 is the error of approximation of the second derivatives.

Note that the components of the local truncation error η_l and μ_l are defined on the shifted grids ω_l^+, $l = 1, 2$. Using summation by parts and the Schwarz inequality, we get

$$\begin{aligned}(\psi_2, z) &= \sum_{l=1}^{2} \sum_{x\in\omega} [\eta_l^+(x) - \eta_l(x)]\, z(x) \\ &= -\sum_{l=1}^{2} \sum_{x\in\omega_l^+} \eta_l(x) \overline{\Delta}_l z(x) \\ &\le \left(\sum_{l=1}^{2} \sum_{x\in\omega_l^+} \eta_l^2(x) \right)^{\frac{1}{2}} \left(\sum_{l=1}^{2} \sum_{x\in\omega_l^+} \overline{\Delta}_l^2 z(x) \right)^{\frac{1}{2}} \\ &\le (\|\eta_1]|_1 + \|\eta_2]|_2)\, \|z\|_{1,\omega}\end{aligned}$$

Likewise

$$(\psi_1, z) \le (\|\mu_1]|_1 + \|\mu_2]|_2)\, \|z\|_{1,\omega} \,.$$

Summarizing these results and using Propositions 2.3, 2.5, 2.7 we obtain the following main result.

Lemma 3.2. *The error $z(x) = y(x) - u(x)$, $x \in \omega$ of all considered finite difference schemes satisfies the a priori estimate*

$$\|z\|_{1,\omega} \le C \sum_{l=1}^{2} (\|\eta_l]|_l + \|\mu_l]|_l) \tag{21}$$

where the components η_l, μ_l, $l = 1, 2$ of the local truncation error are defined by (20) with approximate fluxes w_l^+, w_l, v_l^+, v_l, $l = 1, 2$ determined by (7), (11), (14) and (16) for the ***UDS, MUDS*** *and* ***IDS****, correspondingly. (The constant C does not depend on h or z.)*

In order to use the estimate (21) of Lemma 3.2 we have to bound the corresponding norms of the local truncation error components η_l, μ_l, $l = 1, 2$ defined by (20). These estimates are provided in the lemma given below.

Lemma 3.3. *Let the solution of the problem (1) be H^m-regular, $\frac{3}{2} < m$, and the components of the local truncation error η_l, μ_l, $l = 1, 2$ be defined by (20) with approximate fluxes w_l^+, w_l, v_l^+, v_l, $l = 1, 2$ determined by (7), (11), (14) and (16). Then the following estimates are valid ($l = 1, 2$):*

$$|\eta_l| \le Ch^{m-1} |u|_{m,\bar{e}} \,, \quad \frac{3}{2} < m \le 3, \tag{22}$$

$$|\mu_l| \le \begin{cases} Ch^m \|b_l\|_{1,\infty,\Omega} |u|_{m,\bar{e}} & \text{for } \textbf{MUDS} \text{ and } \textbf{IDS}, \\ C\,[h|b_l|_{0,\infty,\Omega}|u|_{1,\bar{e}} + h^m \|b_l\|_{1,\infty,\Omega}|u|_{m,\bar{e}}] & \text{for } \textbf{UDS}, \end{cases} \tag{23}$$

where $1 < m \le 2$; $\bar{e} = e_{i-1,j} \bigcup e_{i,j}$ for $l = 1$ and $\bar{e} = e_{i,j-1} \bigcup e_{i,j}$ for $l = 2$.

Proof. Consider first the component $\eta_1(x) = \eta_1(x_{1,i}, x_{2,j})$ for the **UDS**. Then

$$\eta_1(x) = -\int_{s_1} \frac{\partial u}{\partial x_1}\, d\gamma - w_1(x) = -\int_{s_1} \frac{\partial u}{\partial x_1}(x_{1,i-1/2}, \gamma)\, d\gamma + (u_{i,j} - u_{i-1,j}).$$

For a fixed $x \in \omega_1^+$, η_1 is a linear functional of u. Using the imbedding of Sobolev spaces $H^m(\Omega) \subset L_\infty(\Omega)$, $1 < m$, (see for example [1]) we conclude that this functional is bounded in $H^m(\bar{e})$, for $\frac{3}{2} < m$, i.e. $|\eta_1(x)| \leq C\|u\|_{m,\bar{e}}$ for every $u \in H^m(\bar{e})$, $\frac{3}{2} < m$. It is easy to check that η_1 vanishes if u is a polynomial of second degree. Therefore, by the Bramble-Hilbert lemma argument we get that

$$|\eta_1(x)| \leq Ch^{m-1}|u|_{m,\bar{e}}\,, \; \frac{3}{2} < m \leq 3\,. \tag{24}$$

Now we consider η_1 for the **MUDS**. By construction

$$k_1(x) = \left(\frac{1}{1 + |B_{1,i,j}|} + |B_{1,i,j}|\right) = 1 + C_1(x)h^2\,, \quad C_1(x) \sim b_1^2(x).$$

Then

$$\eta_1(x) = -\int_{s_1} \frac{\partial u}{\partial x_1}\, d\gamma - w_1(x) = -\int_{s_1} \frac{\partial u}{\partial x_1}\, d\gamma + (1 + C_1h^2)(u_{i,j} - u_{i-1,j}).$$

We consider $u_{i,j} - u_{i-1,j}$ as a linear functional of u for a fixed $x \in \omega^+$. This functional is bounded in $H^m(\bar{e})$, $1 < m \leq 3$ and vanishes for all polynomials of zero degree. Therefore, by the corollary of the Bramble-Hilbert lemma we get

$$|u_{i,j} - u_{i-1,j}| \leq C(|u|_{1,\bar{e}} + h^{m-1}|u|_{m,\bar{e}})\,,\; 1 < m \leq 3. \tag{25}$$

Hence the estimate (24) is valid in this case as well. Finally for the **IDS** the result follows from the fact

$$B_{1,i,j}coth(B_{1,i,j}) = 1 + \tilde{C}_1(x)h^2\,, \quad \tilde{C}_1(x) \sim b_1^2(x).$$

and the same reasons as in the case for the **MUDS**. In a similar way we can estimate $\eta_2(x)$.

For the component $\mu_1(x)$ let us begin again with the **UDS**. We have,

$$\mu_1(x) = \int_{s_1} b_1 u\, ds - v_1 = l(b_1, u) + \frac{|b_{1,i-1/2,j}|h}{2}\overline{\Delta}_1 u_{i,j} \tag{26}$$

where $l(b_1, u)$ is defined by

$$l(b_1, u) = hb_{1,i-1/2,j}\left[\frac{u_{i,j} + u_{i-1,j}}{2}\right] - \int_s b_1(x_{1,i} - h/2, \gamma)u(x_{1,i} - h/2, \gamma)\,. \tag{27}$$

Now we can estimate the second term in (26) by

$$\frac{|b_{1,i-1/2,j}|h}{2}|u_{i,j} - u_{i-1,j}| \leq Ch|b_1|_{1,\infty,\Omega}\left(|u|_{1,\bar{e}} + h^{m-1}|u|_{m,\bar{e}}\right)\,,\; 1 < m \leq 3.$$

The functional $l(b_1, u)$ is estimated in the following lemma, which concludes the proof for the second component of the truncation error μ_1. We note that for **MUDS** and **IDS** we have only the first term l in the formula (26). □

Lemma 3.4. *If the solution of problem (1) is H^m-regular, $1 < m$, then for the bilinear functional $l(b_1, u)$ defined by (27) the following estimate is valid:*

$$|l(b_1, u)| \le Ch^m \|b_1\|_{1,\infty,\Omega} \|u\|_{m,\bar{e}},\ 1 < m \le 2.$$

Proof. After the change of variables $x_i + s_i h = \gamma_i$ we get the domain $\tilde{E} = \{(s_1, s_2) : -1 < s_1 < 0, |s_2| < \frac{1}{2}\}$ and the functions $\tilde{u}(s_1, s_2) = u(x_1+s_1 h, x_2+s_2 h)$, $\tilde{b}_1(s_1, s_2) = b_1(x_1 + s_1 h, x_2 + s_2 h)$,

$$\begin{aligned} l(b_1, u) = l(\tilde{b}_1, \tilde{u}) &= h\tilde{b}_1(-\frac{1}{2}, 0)\frac{\tilde{u}(0,0) + \tilde{u}(-1,0)}{2} \\ &\quad -h\int_{-1/2}^{1/2} \tilde{b}_1(-\frac{1}{2}, s_2)\tilde{u}(-\frac{1}{2}, s_2)\, ds_2 \end{aligned}$$

We rewrite l in the following way

$$\begin{aligned} l(\tilde{b}_1, \tilde{u}) &= h\tilde{b}_1(-\frac{1}{2}, 0)\left[\frac{\tilde{u}(0,0) + \tilde{u}(-1,0)}{2} - \int_{-1/2}^{1/2} \tilde{u}(-\frac{1}{2}, s_2)\, ds_2\right] \\ &\quad +h\left[\int_{-1/2}^{1/2} [\tilde{b}_1(-\frac{1}{2}, 0) - \tilde{b}_1(-\frac{1}{2}, s_2)]\tilde{u}(-\frac{1}{2}, s_2)]\, ds_2\right] \\ &= h\tilde{b}_1(-\frac{1}{2}, 0)\left[\frac{\tilde{u}(0,0) + \tilde{u}(-1,0)}{2} - \int_{-1/2}^{1/2} \tilde{u}(-\frac{1}{2}, s_2)\, ds_2\right] \\ &\quad +h\int_{-1/2}^{1/2} [\tilde{b}_1(-\frac{1}{2}, s_2) - \tilde{b}_1(-\frac{1}{2}, 0)][\tilde{u}(-\frac{1}{2}, s_2) - \tilde{u}(-\frac{1}{2}, 0)]\, ds_2 \\ &\quad +h\int_{-1/2}^{1/2} [\tilde{b}_1(-\frac{1}{2}, 0) - \tilde{b}_1(-\frac{1}{2}, s_2)]\tilde{u}(-\frac{1}{2}, 0)\, ds \\ &= h\tilde{b}_1(-\frac{1}{2}, 0)p(\tilde{u}) + hc(\tilde{b}_1, \tilde{u}) + h\tilde{u}(-\frac{1}{2}, 0)q(\tilde{b}_1) \end{aligned}$$

where the linear functionals $p(\tilde{u})$, $q(\tilde{b}_1)$ and the bilinear functional $c(\tilde{b}_1, \tilde{u})$ are defined by

$$p(\tilde{u}) = \frac{\tilde{u}(0,0) + \tilde{u}(-1,0)}{2} - \int_{-1/2}^{1/2} \tilde{u}(-\frac{1}{2}, s_2)\, ds_2,$$

$$c(\tilde{b}_1, \tilde{u}) = \int_{-1/2}^{1/2} [\tilde{b}_1(-\frac{1}{2}, s_2) - \tilde{b}_1(-\frac{1}{2}, 0)][\tilde{u}(-\frac{1}{2}, s_2) - \tilde{u}(-\frac{1}{2}, 0)]\, ds_2$$

and

$$q(\tilde{b}_1) := \int_{-1/2}^{1/2} \tilde{b}_1(-\frac{1}{2}, s_2)\, ds_2 - \tilde{b}_1(-\frac{1}{2}, 0).$$

Hence

$$|l(\tilde{b}_1, \tilde{u})| \le h|\tilde{b}_{0,\infty,E}|p(\tilde{u})| + h|c(\tilde{b}_1, \tilde{u})| + h|\tilde{u}_{0,\infty,E}|q(\tilde{b}_1)|.$$

First we consider the linear functional $p(\tilde{u})$. It is bounded for $u \in H^m(E)$, $1 < m$ and vanishes for all polynomials of first degree. Hence $|p(u)| \le Ch^{m-1}|u|_{m,\bar{e}}$, $1 < m \le 2$. Obviously $c(\tilde{b}_1, \tilde{u})$ is a bilinear functional bounded for $(\tilde{b}_1, \tilde{u}) \in W^1_\infty(E) \times H^1(E)$ and vanishes for r, s polynomials of zero degree, i.e., $c(r, \tilde{u}) = 0$ for $\tilde{u} \in H^1(E)$ and $c(\tilde{b}_1, s) = 0$ for $\tilde{b}_1 \in W^1_\infty(E)$. Then by the bilinear variant of the Bramble-Hilbert lemma we have $|c(b_1, u)| \le Ch|b_1|_{1,\infty,\bar{e}}|u|_{1,\bar{e}}$. And finally the linear functional $q(\tilde{b}_1)$ fulfills $q(\tilde{b}_1) = 0$ for all polynomials of first degree and therefore the estimate $|q(b_1)| \le Ch|b_1|_{1,\infty,\bar{e}}$ holds. Combining the above estimates we have

$$|l_1(b_1, u)| \le Ch^m \left[|u|_{m,\bar{e}}|b_1|_{0,\infty,\bar{e}} + h^{2-m}|b_1|_{1,\infty,\bar{e}}(|u|_{1,\bar{e}} + |u|_{0,\infty,\bar{e}})\right].$$

Hence by the imbedding $H^m(\Omega) \subset L^\infty(\Omega)$, $m > 1$ we get the desired assertion. □

Now we are ready to prove the main result of this subsection.

Theorem 3.1. *If the solution $u(x)$ of the problem (1) is H^m-regular, with $\frac{3}{2} < m \le 3$ then:*

*(i) the **MUDS** and the **IDS** defined by (14), (9), (16) and (9) have $O(h^{m-1})$ rate of convergence in the H^1- discrete norm, and*

$$\|y - u\|_{1,\omega} \le Ch^{m-1}\left(1 + h^\delta(\|b_1\|_{1,\infty,\Omega} + \|b_2\|_{1,\infty,\Omega})\right)\|u\|_{m,\Omega},$$

*(ii) the **UDS** defined by (7) and (11) has at most first order of convergence in the H^1-discrete norm, and*

$$\begin{aligned}\|y - u\|_{1,\omega} \le\ & Ch\left(|b_1|_{0,\infty,\Omega} + |b_2|_{0,\infty,\Omega}\right)|u|_{1,\Omega} \\ & + Ch^{m-1}\left(1 + h^\delta(\|b_1\|_{1,\infty,\Omega} + \|b_2\|_{1,\infty,\Omega})\right)\|u\|_{m,\Omega}.\end{aligned}$$

Here

$$\delta = \begin{cases} 1 & \frac{3}{2} < m \le 2, \\ 3 - m & 2 \le m \le 3. \end{cases}$$

Proof. In Lemma 3.3 we have proved the estimates for the components η_l, μ_l, $l = 1, 2$ of the local truncation error . Hence

$$\|\eta_l]|_l = \left(\sum_{x \in \omega_l^+} \eta_l^2(x)\right)^{1/2} \le Ch^{m-1}\left(\sum_{x \in \omega_l^+} |u|^2_{m,\bar{e}}\right)^{1/2} \le Ch^{m-1}|u|_{m,\Omega}.$$

In the same way we get for $\|\mu_l]|_l$

$$\|\mu_1]|_1 \le Ch^m\|b_1\|_{1,\infty,\Omega}\|u\|_{m,\Omega}$$

when **MUDS** or **IDS** are used, and

$$\|\mu_1]|_1 \le Ch^m\|b_1\|_{1,\infty,\Omega}\|u\|_{m,\Omega} + h|b_1|_{0,\infty,\Omega}|u|_{1,\Omega}$$

otherwise. This completes the proof. □

3.2. Error estimates in discrete L^2-norm. Here we elaborate the discrete Aubin-Nitsche "trick" that has been proposed in [21]. In order to simplify our presentation we consider only the case $a(x) \equiv 1$. First, we introduce the following averaging operators [21]:

$$S_i u = \frac{1}{h}\int_{x_i-h/2}^{x_i+h/2} u(x_1, \dots, \xi_i, \dots, x_n)\, d\xi_i$$

$$S_i^+ u = \frac{1}{h}\int_{x_i}^{x_i+h} u(x_1, \dots, \xi_i, \dots, x_n)\, d\xi_i$$

$$S_i^- u = \frac{1}{h}\int_{x_i-h}^{x_i} u(x_1, \dots, \xi_i, \dots, x_n)\, d\xi_i$$

$$T_i = S_i^2 = S_i^+ S_i^- \, , \; T = T_1 T_2$$

Then applying T to the differential equation (1) at any grid point $x \in \omega$ and using the properties:

$$T_i\left(\frac{\partial^2 u}{\partial x_i^2}\right)(x) = u_{\overline{x}_i, x_i}\, , \quad S_i^+\left(\frac{\partial u}{\partial x_i}\right)(x) = u_{x_i}$$

we get

$$-(T_2 u)_{\overline{x}_1, x_1} - (T_1 u)_{\overline{x}_2, x_2} + T_2 S_1^-(b_1 u)_{x_1} + T_1 S_2^-(b_2 u)_{x_2} = Tf(x) \equiv \phi(x). \tag{28}$$

Dividing (18) by h^2 we express the operator A_h in the form

$$\frac{1}{h}(w_1)_{x_1} + \frac{1}{h}(w_2)_{x_2} + \frac{1}{h^2}A^{(1)}y = Tf(x) \equiv \phi(x). \tag{29}$$

Let $z(x) = y(x) - u(x)\, ,\; x \in \omega$ be the error of the finite difference method. Substituting $y = z + u$ in (29) we obtain

$$A_h z = A_h y - A_h u = \phi - A_h u. \tag{30}$$

The right-hand side of (30) is the local truncation error. In order to obtain a priori estimate we represent the local truncation error in a divergence or almost divergence form (depending upon the choice of the difference scheme). Next we rewrite (30) as

$$\begin{aligned} A_h z &= \sum_{l=1}^{2}\left[\frac{1}{h}w_l + (T_{3-l}u)_{\overline{x}_l}\right]_{x_l} + \sum_{l=1}^{2}\left[\frac{1}{h}v_l - T_{3-l}S_l^-(b_l u)\right]_{x_l} \\ &= \sum_{l=1}^{2}(T_{3-l}u - u)_{\overline{x}_l x_l} + \sum_{l=1}^{2}\left[\frac{1}{h}v_l - T_{3-l}S_l^-(b_l u)\right]_{x_l} \\ &\quad + \sum_{l=1}^{2}[-(k_l - 1)u_{\overline{x}_l}]_{x_l}\, . \end{aligned}$$

Finally, we find the expression for the local truncation error

$$A_h z = \eta_{1,\overline{x}_1 x_1} + \eta_{2,\overline{x}_2 x_2} + \mu_{1,x_1} + \mu_{2,x_2} + \xi_{1,x_1} + \xi_{2,x_2} \tag{31}$$

where

$$\eta_l = \begin{cases} T_{3-l}u - u\,, & \text{if } x \in \omega\,, \\ 0\,, & \text{if } x \in \gamma\,, \end{cases} \tag{32}$$

$$\mu_l = T_{3-l}S_l^-(b_l u) - \frac{1}{h}v_l\,, \quad \xi_l = -(k_l - 1)u_{\overline{x}_l}\,,\ x \in \omega^+\,. \tag{33}$$

Let us introduce the solution of the following auxiliary discrete problem

$$\begin{aligned} A_h^T w &= z \quad \text{in } \omega\,, \\ w &= 0 \quad \text{on } \gamma\,. \end{aligned} \tag{34}$$

Note that similarly to the Aubin-Nitsche "trick" w is a solution of a discrete second order problem with a right-hand side the error $z(x)$ of the method. Obviously,

$$(A_h z, w) = (A_h^T w, z) = (z, z) = \|z\|_{0,\omega}^2\,. \tag{35}$$

On the other hand from (31) we get

$$\begin{aligned} (A_h z, w) &= \sum_{l=1}^{2} [(\eta_{l,\overline{x}_l x_l}, w) + (\mu_{l,x_l}, w) + (\xi_{l,x_l}, w)] \\ &= \sum_{l=1}^{2} (\eta_l, w_{\overline{x}_l x_l}) - \sum_{l=1}^{2} \{(\mu, w_{\overline{x}_l}]_l + (\xi, w_{\overline{x}_l}]_l\} \\ &\leq \sum_{l=1}^{2} (\|\eta_l\|_{0,\omega} + \|\mu_l]|_l + \|\xi_l]|_l)(\|w_{\overline{x}_l x_l}\|_{0,\omega} + \|w_{\overline{x}_l}]|_l)\,. \end{aligned} \tag{36}$$

To complete the proof of the a priori estimate we need the following lemma.

Lemma 3.5. *Let* $\underline{b} \in (W_\infty^1)^2$. *Then for the error* $z(x) = y(x) - u(x)$, $x \in \omega$ *of all considered schemes and the solution* w *of the problem (34) the inequalities are valid:*

$$\|w\|_{2,\omega} \leq C_1\|A_h^{(2)}w\|_{0,\omega} \leq C_2\|z\|_{0,\omega} \tag{37}$$

for sufficiently small h.

Proof. Using the definition of $A_h^{(2)}$ and the triange inequality we get

$$\begin{aligned} \|A_h^{(2)}w\|_{0,\omega} &= \|[k_1 w_{\overline{x}_1}]_{x_1} + [k_2 w_{\overline{x}_2}]_{x_2}\|_{0,\omega} \\ &= \|[(1 + C_1(x)h^2)w_{\overline{x}_1}]_{x_1} + [(1 + C_2(x))w_{\overline{x}_2}]_{x_2}\|_{0,\omega} \\ &\geq \|w_{\overline{x}_1 x_1} + w_{\overline{x}_2 x_2}\|_{0,\omega} \\ &\qquad -h^2\|C_{1,\overline{x}_1}w_{\overline{x}_1} + C_1 w_{\overline{x}_1 x_1} + C_{2,\overline{x}_2}w_{\overline{x}_2} + C_2 w_{\overline{x}_2 x_2}\|_{0,\omega} \\ &\geq \|w_{\overline{x}_1 x_1} + w_{\overline{x}_2 x_2}\|_{0,\omega} - D_2 h^2\|w\|_{2,\omega} \end{aligned}$$

Here $k_l = 1\,, C_l = 0\,,\ l = 1, 2$ for the **UDS** and

$$k_l = 1 + C_l(x)h^2\,, \quad C_l(x) \sim b_l^2(x)\,,\ l = 1, 2\,.$$

We use also that C_1, C_2 and $C_{1,\bar{x}_1}$, $C_{2,\bar{x}_2}$ are bounded.

Finally using the equivalence of $\|w_{\bar{x}_1 x_1} + w_{\bar{x}_2 x_2}\|_{0,\omega}$ and $\|w\|_{2,\omega}$ in the space D^0 we obtain

$$\|A_h^{(2)}\|_{0,\omega} \geq \left(D_1 - D_2 h^2\right) \|w\|_{2,\omega} ,$$

where D_1 and D_2 are positive constants. Hence for sufficiently small h the lower bound in (37) is proved.

An upper estimate for $\|A_h^{(2)}\|_{0,\omega}$ is derived by using the standard a priori estimate in $W_2^1(\omega)$, $\|w\|_{1,\omega} \leq C\|z\|_{0,\omega}$. Then

$$\begin{aligned}
\|A_h^{(2)} w\|_{0,\omega} &= \|A_h^{(2)^T} w\|_{0,\omega} = \|(A_h - A_h^{(1)})^T w\|_{0,\omega} \\
&\leq \|A_h^T w\|_{0,\omega} + \|A_h^{(1)^T} w\|_{0,\omega} \\
&\leq \|z\|_{0,\omega} + C\|w\|_{1,\omega} \\
&\leq C\|z\|_{0,\omega} .
\end{aligned}$$

□

Remark 3.2. *Lemma 3.5 is actually a discrete regularity result in $W_2^2(\omega)$ (cf., Hackbusch* [10]*)*

$$\|w\|_{2,\omega} \leq C\|z\|_{0,\omega} .$$

Then (35) and (36) yield

$$\|z\|_{0,\omega}^2 = (A_h z, w) \leq C \sum_{l=1}^{2} (\|\eta_l\|_{0,\omega} + \|\mu_l]|_l + \|\xi_l]|_l) \|z\|_{0,\omega}.$$

Thus, we have proved the following a priori estimate

Lemma 3.6. *The error $z(x) = y(x) - u(x)$, $x \in \omega$ of all considered finite difference schemes satisfies the a priori estimate:*

$$\|z\|_{0,\omega} \leq C \sum_{l=1}^{2} (\|\eta_l\|_{0,\omega} + \|\mu_l]|_l + \|\xi_l]|_l) ,$$

where the components η_l, μ_l and ξ_l, $l = 1, 2$ of the local truncation error are defined by (32) and (33). The constant C does not depend on h or z.

Now we are ready to prove the following basic lemma.

Lemma 3.7. *If the solution u of the problem (1) with constant coefficient $a(x)$ is $H^m(\Omega)$–regular, $1 < m \leq 2$ then the components of the local truncation error η_l and μ_l, $l = 1, 2$, defined by (32) and (33), respectively, satisfy the following estimates:*

(i)

$$\|\eta_l\|_{0,\omega} \leq Ch^m \|u\|_{m,\Omega} ,$$

(ii)

$$\|\mu_l]|_l \leq \begin{cases} Ch^m \|b_l\|_{1,\infty,\Omega} \|u\|_{m,\Omega} & \textit{for } \textbf{\textit{MUDS}} \textit{ and } \textbf{\textit{IDS}} \\ C\left(h|b_l|_{0,\infty,\Omega}|u|_{1,\Omega} + h^m \|b_l\|_{1,\infty,\Omega}\|u\|_{m,\Omega}\right) & \textit{for } \textbf{\textit{UDS}}, \end{cases}$$

(iii)

$$\|\xi_1]|_1 \leq \begin{cases} Ch^2\|u\|_{m,\Omega} & \textit{for MUDS and IDS} \\ 0 & \textit{for UDS,} \end{cases}$$

Proof. Consider $e_{i,j} = \{(x_1, x_2) : x_{1,i-1} \leq x_1 \leq x_{1,i+1}, x_{2,j-1} \leq x_2 \leq x_{2,j+1}\}$. We begin with **UDS**. To obtain (i) we rewrite (32) in the form

$$\eta_1 = u(x_{1,i}, x_{2,j}) - \int_{-1}^{1}(1 - |s_2|)u(x_{1,i}, x_{2,j} + s_2)\, ds_2.$$

We have that η_1 is a linear functional of $u(x)$, bounded for $u \in H^m(\Omega)$, $1 < m \leq 2$. This functional vanishes for all polynomial of first degree. Therefore, by Bramble-Hilbert lemma argument we get

$$|\eta_1(x)| \leq Ch^{m-1}|u|_{m,e}, \; 1 < m \leq 2. \tag{38}$$

$$\|\eta_1\|_{0,\omega} = \left(\sum_{x\in\omega} \eta_1^2(x)h^2\right)^{\frac{1}{2}} \leq Ch^m|u|_{m,e}.$$

We note that in this case $\xi_1(x) \equiv 0$. Now, let us take the component $\eta_1(x)$ for the **MUDS** and the **IDS**. In both schemes the coefficients $\tilde{k}_1(x)$ and $\gamma_1(x)$ are perturbations of the coefficient $k_1(x) \equiv 1$ of the **UDS** with a term of order $O(h^2)$. More precisely,

$$\tilde{k}_1(x) = \frac{1}{1 + |b_1(x)h/2|} + \frac{|b_1(x)|h}{2} = 1 + C_1h^2 \qquad \textbf{(MUDS)}$$

and

$$\gamma_1(x) = \frac{b_1(x)h}{2}\coth\left(\frac{b_1(x)h}{2}\right) = 1 + \tilde{C}_1h^2 \qquad \textbf{(IDS)}.$$

Since

$$\xi_1(x) = -(k_1(x) - 1)_{\bar{x}_1} = -C_1h^2\frac{[u_{i,j} - u_{i,j-1}]}{h}$$

we have

$$|\xi(x)| \leq Ch(|u|_{1,e} + h^{m-1}|u|_{m,e}), \; 1 < m \leq 2$$

and hence

$$\|\xi_1]|_1 \leq Ch^2\|u\|_{m,\Omega}, 1 < m \leq 2.$$

(ii): For the second component $\mu_1(x)$ we proceed in the same way as in Lemma 3.3. First, we need the equality (see [21]):

$$T_2S_1^-(b_1u)(x_{1,i}, x_{2,j}) =$$
$$\int_{-1}^{1}(1 - |s_2|)\left[\int_{-1}^{0} b_1(x_{1,i} + s_1, x_{2,j} + s_2)u(x_{1,i} + s_1, x_{2,j} + s_2)\, ds_1\right] ds_2.$$

Now, let us consider the component for the **MUDS** and **IDS**

$$\mu_1(x) = T_2S_1^-(b_1u)(x_{1,i}, x_{2,j}) - \frac{b_{1,i-1/2,j}}{2}[u_{i,j} + u_{i-1,j}]$$

We can represent μ_1 in the following way

$$\mu_1(x) = b_{1,i-1/2,j}p(u) + c(b_1, u) + u_{i,j}q(b_1)$$

where

$$\begin{aligned} p(u) &= \int_{-1}^{1}(1-|s_2|)\left[\int_{-1}^{0} u(x_{1,i}+s_1, x_{2,j}+s_2)\,ds_1\right] ds_2 \\ &\quad - \frac{[u(x_{1,i}, x_{2,j}) + u(x_{1,i}-h, x_{2,j})]}{2}, \end{aligned}$$

$$\begin{aligned} c(b_1, u) &= \int_{-1}^{1}(1-|s_2|)\left[\int_{-1}^{0} u(x_{1,i}+s_1, x_{2,j}+s_2) - u(x_{1,i}, x_{2,j})\right] \\ &\quad \times [b_1(x_{1,i}+s_1, x_{2,j}+s_2) - b_1(x_{1,i}, x_{2,j})\,ds_1]\,ds_2 \end{aligned}$$

and

$$\begin{aligned} q(u) &= \int_{-1}^{1}(1-|s_2|)\left[\int_{-1}^{0} b_1(x_{1,i}+s_1, x_{2,j}+s_2) - b_1(x_{1,i}, x_{2,j})\,ds_1\right] ds_2 \\ &\quad - b_1(x_{1,i}-h/2, x_{2,j}). \end{aligned}$$

We have the estimates:

$$\begin{aligned} |p(u)| &\le Ch^{m-1}|u|_{m,e}, \ 1 < m \le 2, \\ |c(b_1, u)| &\le Ch|b_1|_{1,\infty,e}|u|_{1,e}, \\ |q(u)| &\le Ch|b_1|_{1,\infty,e}, \end{aligned}$$

Hence

$$|\mu_1(x)| \le Ch^{m-1}\|b_1\|(|u|_{m,e} + h^{2-m}(|u|_{1,e} + |u|_{0,\infty,e})).$$

For **UDS** we have to add the error of the term $-|b_{1,i,j}|u_{\bar{x}_1}$ which is $h\left(|b_1|_{0,\infty,e}(|u|_{1,e} + h^{m-1}|u|_{m,e})\right)]$.
Combining the above results we obtain the assertions of the lemma. □

From Lemma 3.6 and Lemma 3.7 we get immediately

Theorem 3.2. *If the solution of problem (1) is H^m-regular, $1 < m \le 2$ then:*
*(i) the **MUDS** and **IDS** defined by (14), (9), (16) and (9) have $O(h^m)$ rate of convergence in the L^2– discrete norm, i.e.,*

$$\|y-u\|_{0,\omega} \le Ch^m\left(1 + \|b_1\|_{1,\infty,\Omega} + \|b_2\|_{1,\infty,\Omega}\right)\|u\|_{m,\Omega}$$

*(ii) the **UDS** defined by (7) and (11) has at most first order of convergence in the L^2–discrete norm, i.e.,*

$$\begin{aligned} \|y-u\|_{0,\omega} &\le Ch\left(|b_1|_{0,\infty,\Omega} + |b_2|_{0,\infty,\Omega}\right)|u|_{1,\Omega} \\ &\quad + Ch^m\left(1 + \|b_1\|_{1,\infty,\Omega} + \|b_2\|_{1,\infty,\Omega}\right)\|u\|_{m,\Omega}. \end{aligned}$$

Remark 3.3. *The technique used in Subsection 3.1 and 3.2 directly gives the same estimates for the **CDS**, when this scheme is stable, i.e., when (10) holds.*

4. Numerical results

In this section on the basis of model test examples we study the error behavior of our three schemes (**UDS**, **MUDS**, and **IDS**) in both H^1 and L^2 discrete norms.

We consider

$$(39)\qquad \begin{cases} div(-\varepsilon\nabla u(x) + \underline{b}(x)u(x)) &= f(x)\,, \quad \text{in } \Omega\,, \\ u(x) &= 0\,, \quad \text{on } \Gamma\,, \end{cases}$$

and for velocity vector $\underline{b}$ we choose

$$b_1 = (1 - x\cos\alpha)\cos\alpha\,,\ b_2 = (1 - y\sin\alpha)\sin\alpha\,,$$

where the angle is $\alpha = 15^0$.

Problem 1. *$f(x,y)$ is chosen such that the solution is*

$$u(x,y) = x(1-x)y(1-y)e^{d(x+2y)} \text{ for } d = 0 \text{ or } d = 1.$$

In tables 1-6 we display the error for smooth solutions without boundary layer behavior. In the first and the second rows we show the $L^2(\omega)$ and $H^1(\omega)$-norms of the error $z = y - u$ and "numerical" rate of convergence β, i.e., h^β. Our computational experiments clearly show that **MUDS** and **IDS** exhibit second order of convergence both in L^2 and H^1-norms for problems with moderate convection (i.e., not too small $\varepsilon > 0$) ; the factor β is in the range of 1.822–1.995, correspondingly. For these problems **UDS** is only first order accurate: β is between 0.947-1.260. For highly dominating convection all schemes show about first order of accuracy. The results for $\varepsilon = 10^{-2}$, 10^{-5} show that all considered schemes are stable.

Problem 2.

$$f(x,y) = div(\underline{b}, \nabla u_0)\,,\ u_0(x,y) = x^2y(1-y)$$

Here u_0 is the solution of the equation (39) when $\varepsilon = 0$. In Tables 7-9 we show $\|y - u_0\|_{0,\overline{\omega}}$, where $\overline{\omega}$ is a grid in $\overline{\Omega} = [0, 7/8] \times [0,1]$, i.e., away from the boundary layer. This gives us a reasonable information since for small ε the function u_0 is close to the exact solution of problem 2, except within the boundary layer. In fact we have an estimate $\|u - u_0\|_{0,\overline{\Omega}} \le C\varepsilon$. Our experiments show very weak dependence of the numerical solution with respect to $\varepsilon \to 0$ in $\overline{\Omega}$. This means that if we use more sophisticated method near the boundary layer, e.g., local refinement, defect–correction, in combination with the proposed schemes outside the layer we can get better results.

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TABLE 1. **UDS** , $\alpha = 15^0$, $d = 0$

ϵ/N		16	32	64	128	256
1	L^2	$0.389.10^{-3}$	$0.198.10^{-3}$	$0.100.10^{-3}$	$0.503.10^{-4}$	$0.252.10^{-4}$
	β	0.947	0.974	0.986	0.991	0.997
	H^1	$0.154.10^{-2}$	$0.859.10^{-3}$	$0.454.10^{-3}$	$0.233.10^{-3}$	$0.118.10^{-3}$
	β	0.699	0.842	0.920	0.962	0.982
10^{-2}	L^2	$0.149.10^{-1}$	$0.811.10^{-2}$	$0.425.10^{-2}$	$0.218.10^{-2}$	$0.110.10^{-2}$
	β	0.780	0.878	0.932	0.963	0.987
	H^1	$0.633.10^{-1}$	$0.462.10^{-1}$	$0.288.10^{-1}$	$0.163.10^{-1}$	$0.868.10^{-2}$
	β	0.298	0.454	0.682	0.821	0.909
10^{-5}	L^2	$0.233.10^{-1}$	$0.135.10^{-1}$	$0.737.10^{-2}$	$0.388.10^{-2}$	$0.200.10^{-2}$
	β	0.667	0.787	0.873	0.926	0.956
	H^1	$0.110.10^{0}$	$0.779.10^{-1}$	$0.505.10^{-1}$	$0.305.10^{-1}$	$0.180.10^{-1}$
	β	0.338	0.498	0.625	0.727	0.761

TABLE 2. **MUDS** , $\alpha = 15^0$, $d = 0$

ϵ/N		16	32	64	128	256
1	L^2	$0.213.10^{-4}$	$0.567.10^{-5}$	$0.146.10^{-5}$	$0.372.10^{-6}$	$0.940.10^{-7}$
	β	1.822	1.909	1.957	1.973	1.985
	H^1	$0.818.10^{-4}$	$0.239.10^{-4}$	$0.649.10^{-5}$	$0.169.10^{-5}$	$0.431.10^{-6}$
	β	1.559	1.775	1.881	1.941	1.971
10^{-2}	L^2	$0.102.10^{-1}$	$0.416.10^{-2}$	$0.148.10^{-2}$	$0.468.10^{-3}$	$0.134.10^{-3}$
	β	1.100	1.294	1.491	1.661	1.804
	H^1	$0.436.10^{-1}$	$0.240.10^{-1}$	$0.101.10^{-1}$	$0.347.10^{-2}$	$0.104.10^{-2}$
	β	0.609	0.861	1.249	1.541	1.738
10^{-5}	L^2	$0.233.10^{-1}$	$0.135.10^{-1}$	$0.736.10^{-2}$	$0.387.10^{-2}$	$0.198.10^{-2}$
	β	0.667	0.787	0.875	0.927	0.967
	H^1	$0.110.10^{0}$	$0.784.10^{-1}$	$0.511.10^{-1}$	$0.309.10^{-1}$	$0.174.10^{-1}$
	β	0.338	0.489	0.618	0.728	0.820

TABLE 3. **IDS** , $\alpha = 15^0$, $d = 0$

ϵ/N		16	32	64	128	256
1	L^2	$0.169.10^{-4}$	$0.451.10^{-5}$	$0.116.10^{-5}$	$0.295.10^{-6}$	$0.740.10^{-7}$
	β	1.840	1.906	1.959	1.975	1.995
	H^1	$0.650.10^{-4}$	$0.189.10^{-4}$	$0.511.10^{-5}$	$0.133.10^{-5}$	$0.338.10^{-6}$
	β	1.578	1.782	1.887	1.942	1.976
10^{-2}	L^2	$0.860.10^{-2}$	$0.288.10^{-2}$	$0.816.10^{-3}$	$0.213.10^{-3}$	$0.540.10^{-4}$
	β	1.253	1.578	1.819	1.937	1.980
	H^1	$0.366.10^{-1}$	$0.166.10^{-1}$	$0.557.10^{-2}$	$0.158.10^{-2}$	$0.420.10^{-3}$
	β	0.786	1.141	1.575	1.818	1.911
10^{-5}	L^2	$0.233.10^{-1}$	$0.133.10^{-1}$	$0.736.10^{-2}$	$0.387.10^{-2}$	$0.198.10^{-2}$
	β	0.667	0.787	0.875	0.927	0.967
	H^1	$0.110.10^{0}$	$0.770.10^{-1}$	$0.511.10^{-1}$	$0.309.10^{-1}$	$0.175.10^{-1}$
	β	0.338	0.515	0.592	0.728	0.820

TABLE 4. **UDS** , $\alpha = 15^0$, $d = 1$

ϵ/N		16	32	64	128	256
1	L^2	$0.232.10^{-2}$	$0.102.10^{-2}$	$0.470.10^{-3}$	$0.223.10^{-3}$	$0.113.10^{-3}$
	β	1.260	1.186	1.118	1.040	0.981
	H^1	$0.930.10^{-2}$	$0.451.10^{-2}$	$0.218.10^{-2}$	$0.106.10^{-2}$	$0.545.10^{-3}$
	β	0.984	1.044	1.049	1.040	0.960
10^{-2}	L^2	$0.486.10^{-1}$	$0.267.10^{-1}$	$0.141.10^{-1}$	$0.725.10^{-2}$	$0.368.10^{-2}$
	β	0.769	0.864	0.921	0.960	0.978
	H^1	$0.228.10^{0}$	$0.170.10^{0}$	$0.110.10^{0}$	$0.637.10^{-1}$	$0.345.10^{-1}$
	β	0.291	0.423	0.628	0.788	0.885
10^{-5}	L^2	$0.719.10^{-1}$	$0.408.10^{-1}$	$0.219.10^{-1}$	$0.114.10^{-1}$	$0.585.10^{-2}$
	β	0.690	0.817	0.898	0.942	0.963
	H^1	$0.329.10^{0}$	$0.215.10^{0}$	$0.138.10^{0}$	$0.847.10^{-1}$	$0.496.10^{-1}$
	β	0.536	0.614	0.640	0.704	0.772

TABLE 5. MUDS , $\alpha = 15^0$, $d = 1$

ϵ/N		16	32	64	128	256
1	L^2	$0.768.10^{-3}$	$0.204.10^{-3}$	$0.526.10^{-4}$	$0.134.10^{-4}$	$0.337.10^{-5}$
	β	1.830	1.911	1.956	1.973	1.991
	H^1	$0.302.10^{-2}$	$0.900.10^{-3}$	$0.246.10^{-3}$	$0.646.10^{-4}$	$0.165.10^{-4}$
	β	1.531	1.747	1.871	1.929	1.969
10^{-2}	L^2	$0.291.10^{-1}$	$0.117.10^{-1}$	$0.415.10^{-2}$	$0.130.10^{-2}$	$0.374.10^{-3}$
	β	1.101	1.315	1.495	1.675	1.797
	H^1	$0.132.10^{0}$	$0.721.10^{-1}$	$0.306.10^{-1}$	$0.106.10^{-1}$	$0.319.10^{-2}$
	β	0.677	0.872	1.236	1.529	1.732
10^{-5}	L^2	$0.719.10^{-1}$	$0.407.10^{-1}$	$0.218.10^{-1}$	$0.114.10^{-1}$	$0.576.10^{-2}$
	β	0.690	0.821	0.901	0.935	0.985
	H^1	$0.329.10^{0}$	$0.216.10^{0}$	$0.138.10^{0}$	$0.840.10^{-1}$	$0.432.10^{-1}$
	β	0.536	0.607	0.646	0.716	0.822

TABLE 6. IDS , $\alpha = 15^0$, $d = 1$

ϵ/N		16	32	64	128	256
1	L^2	$0.752.10^{-3}$	$0.200.10^{-3}$	$0.515.10^{-4}$	$0.131.10^{-4}$	$0.330.10^{-5}$
	β	1.828	1.911	1.957	1.975	1.989
	H^1	$0.296.10^{-2}$	$0.883.10^{-3}$	$0.242.10^{-3}$	$0.634.10^{-4}$	$0.162.10^{-4}$
	β	1.532	1.745	1.867	1.932	1.968
10^{-2}	L^2	$0.227.10^{-1}$	$0.755.10^{-2}$	$0.212.10^{-2}$	$0.553.10^{-3}$	$0.138.10^{-3}$
	β	1.277	1.588	1.832	1.939	2.002
	H^1	$0.100.10^{0}$	$0.454.10^{-1}$	$0.153.10^{-1}$	$0.443.10^{-2}$	$0.116.10^{-2}$
	β	0.880	1.139	1.569	1.788	1.933
10^{-5}	L^2	$0.718.10^{-1}$	$0.407.10^{-1}$	$0.218.10^{-1}$	$0.114.10^{-1}$	$0.576.10^{-2}$
	β	0.690	0.819	0.901	0.935	0.985
	H^1	$0.329.10^{0}$	$0.216.10^{0}$	$0.138.10^{0}$	$0.840.10^{-1}$	$0.475.10^{-1}$
	β	0.536	0.607	0.646	0.716	0.822

TABLE 7. **UDS** , $\alpha = 15^0$, **boundary layer**

ϵ/N		16	32	64	128	256
10^{-3}	L^2	$0.427.10^{-2}$	$0.252.10^{-2}$	$0.147.10^{-2}$	$0.894.10^{-3}$	$0.594.10^{-3}$
	β	0.622	0.761	0.778	0.717	0.590
	H^1	$0.414.10^{-1}$	$0.276.10^{-1}$	$0.172.10^{-1}$	$0.109.10^{-1}$	$0.744.10^{-2}$
	β	0.332	0.585	0.682	0.658	0.551
10^{-4}	L^2	$0.393.10^{-2}$	$0.225.10^{-2}$	$0.122.10^{-2}$	$0.645.10^{-3}$	$0.342.10^{-3}$
	β	0.641	0.805	0.883	0.920	0.920
	H^1	$0.365.10^{-1}$	$0.233.10^{-1}$	$0.134.10^{-1}$	$0.733.10^{-2}$	$0.396.10^{-2}$
	β	0.380	0.648	0.798	0.870	0.888
10^{-5}	L^2	$0.391.10^{-2}$	$0.223.10^{-2}$	$0.119.10^{-2}$	$0.621.10^{-3}$	$0.318.10^{-3}$
	β	0.642	0.810	0.906	0.938	0.965
	H^1	$0.361.10^{-1}$	$0.229.10^{-1}$	$0.130.10^{-1}$	$0.700.10^{-2}$	$0.364.10^{-2}$
	β	0.387	0.657	0.817	0.893	0.943

TABLE 8. **MUDS** , $\alpha = 15^0$, **boundary layer**

ϵ/N		16	32	64	128	256
10^{-3}	L^2	$0.392.10^{-2}$	$0.224.10^{-2}$	$0.122.10^{-2}$	$0.682.10^{-3}$	$0.340.10^{-3}$
	β	0.640	0.807	0.877	0.839	0.652
	H^1	$0.364.10^{-1}$	$0.233.10^{-1}$	$0.135.10^{-1}$	$0.790.10^{-2}$	$0.525.10^{-2}$
	β	0.381	0.644	0.787	0.773	0.415
10^{-4}	L^2	$0.390.10^{-2}$	$0.223.10^{-2}$	$0.119.10^{-2}$	$0.618.10^{-3}$	$0.315.10^{-3}$
	β	0.643	0.806	0.906	0.945	0.972
	H^1	$0.361.10^{-1}$	$0.228.10^{-1}$	$0.130.10^{-1}$	$0.696.10^{-2}$	$0.361.10^{-2}$
	β	0.384	0.663	0.811	0.901	0.947
10^{-5}	L^2	$0.390.10^{-2}$	$0.223.10^{-2}$	$0.119.10^{-2}$	$0.618.10^{-3}$	$0.315.10^{-3}$
	β	0.643	0.806	0.906	0.945	0.972
	H^1	$0.360.10^{-1}$	$0.229.10^{-1}$	$0.130.10^{-1}$	$0.696.10^{-2}$	$0.361.10^{-2}$
	β	0.388	0.653	0.806	0.901	0.947

TABLE 9. **IDS , $\alpha = 15^0$, boundary layer**

ϵ/N		16	32	64	128	256
10^{-3}	L^2	$0.390.10^{-2}$	$0.222.10^{-2}$	$0.116.10^{-2}$	$0.594.10^{-3}$	$0.376.10^{-3}$
	β	0.643	0.813	0.936	0.966	0.660
	H^1	$0.361.10^{-1}$	$0.227.10^{-1}$	$0.124.10^{-1}$	$0.665.10^{-2}$	$0.454.10^{-2}$
	β	0.384	0.669	0.872	0.899	0.551
10^{-4}	L^2	$0.390.10^{-2}$	$0.223.10^{-2}$	$0.119.10^{-2}$	$0.619.10^{-3}$	$0.314.10^{-3}$
	β	0.643	0.806	0.906	0.943	0.979
	H^1	$0.360.10^{-1}$	$0.229.10^{-1}$	$0.130.10^{-1}$	$0.697.10^{-2}$	$0.360.10^{-2}$
	β	0.384	0.653	0.817	0.899	0.953
10^{-5}	L^2	$0.390.10^{-2}$	$0.223.10^{-2}$	$0.119.10^{-2}$	$0.619.10^{-3}$	$0.315.10^{-3}$
	β	0.643	0.806	0.906	0.943	0.975
	H^1	$0.360.10^{-1}$	$0.229.10^{-1}$	$0.130.10^{-1}$	$0.697.10^{-2}$	$0.361.10^{-2}$
	β	0.384	0.653	0.817	0.899	0.949

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