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Investigation of the use of Prandtl/Navier-Stokes Equation Procedures for Two-Dimensional Incompressible Flows

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ABSTRACT. We investigate the technique of combining solutions of the Prandtl equations with solutions of the Navier-Stokes equations to compute incompressible flow around two dimensional bodies. We present computational evidence which shows that if the "obvious" coupling is used to combine the solutions, then the resulting solution is not accurate. We describe an alternate coupling procedure which greatly improves the accuracy of the solutions obtained with the combined equation approach.

1. Introduction

The purpose of this paper is to discuss the accuracy of a procedure used in the computation of the motion of a viscous incompressible fluid past objects in two dimensions. In particular, we will focus on those methods where the Prandtl boundary layer equations are used to describe the flow near an object and the Navier-Stokes equations are used to describe the flow away from an object. A schematic diagram of the situation is presented in Figure 1. In this figure, corresponding to the domain for the computation of flow about a cylinder, the Prandtl equations are solved in an inner region from $r_a \leq r \leq r_a + \delta = r_\delta$ while the full Navier-Stokes are solved in the external region $r > r_\delta$.

One may wonder why a dual equation approach is considered. One reason is that the Prandtl boundary layer equations have a simpler structure than the full Navier-Stokes equations, and so it may be easier to obtain accurate solutions. An example where this aspect has been exploited is in the vortex sheet/vortex blob method [3],[4], [5]. This method is a Lagrangian scheme in which small segments of vorticity (or sheets) are used to approximate the vorticity distribution in the boundary layer region and vortex blobs are used to approximate the vorticity distribution in the external region. The sheet discretization of the Prandtl equations was introduced in order to remove inaccuracies near the boundaries present in vortex blob discretizations of the Navier-Stokes equations.

Another example where a combined equation approach has been used are vortex sheet/potential flow methods. In such methods the flow field is represented as an irrotational flow with embedded vortex sheets. A component of such methods is the

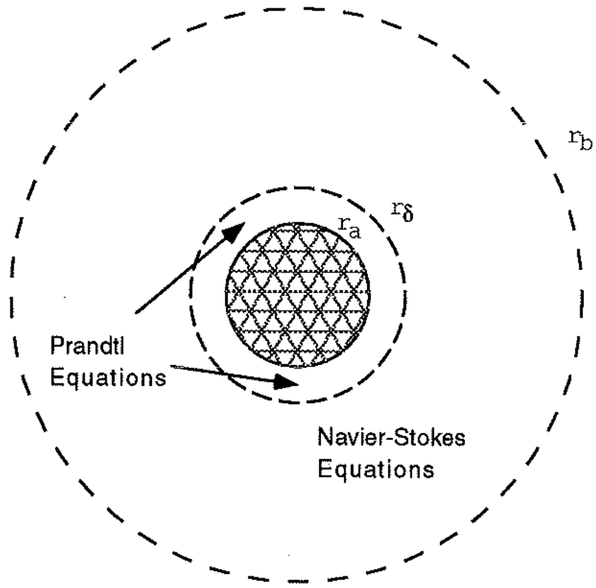

Fig. 1. The decomposition of the domain for the computation of flow about a circular cylinder using a Prandtl/Navier-Stokes equation procedure.

necessity to determine the point at which the vortex sheet attaches to the body (or alternately where the boundary layer separates). Techniques for determining the separation point often utilize solutions of the Prandtl boundary layer equations.

Aside from questions about specific methods, there is the general question about the validity of a procedure which utilizes a combination of the Prandtl and Navier-Stokes equations. This is an interesting question because in many problems of two-dimensional fluid flow the boundary layer separates, and thus the solutions violate the assumptions which are used to derive the Prandtl equations. The fact that the flow separates does not necessarily invalidate the combined equation approach (the Prandtl equations may be applicable even if their method of derivation is inappropriate) but it does suggest that the accuracy is questionable.

To investigate the combined equation approach we first constructed a finite difference method for solving the complete Navier-Stokes equations for flow about a circular cylinder. The solutions obtained with this code served as benchmarks. We then constructed a finite difference method which utilized the combined equation approach. Specifically a finite difference method which solved the Prandtl equations in the inner region was coupled to a finite difference method which solved the Navier-Stokes equations in the external region. There was the issue of how one couples the solutions to these two sets of equations. Our initial coupling procedure was that used in the vortex sheet/vortex blob method. (This coupling is the "obvious" one.)

We found that the solutions obtained with the combined equation approach differed from those obtained with the full Navier-Stokes equations. (The solutions in both cases were fully converged.) We also found that the solutions obtained with the combined equation approach depended significantly upon the size of the domain where the Prandtl equations were solved. Since the size of the inner region is somewhat arbitrary this behavior was undesirable. For both of these reasons we could only conclude that as implemented, the combined equation approach is not a good idea. After reaching this conclusion, we began to examine procedures which could possibly remove this inaccuracy. We discovered that if one changes the way in which the coupling is performed, then the dual equation approach yields solutions which are close to the solutions of the Navier-Stokes equations. We now provide some of the details of this investigation and indicate how we changed the coupling to yield more accurate solutions. For more information on this investigation one should see [6].

2. The Test Problem and the Results

Our test problem was that of the fluid motion induced by an impulsively started circular. (At $t = 0$ a cylinder at rest is accelerated to unit velocity.) The Navier-Stokes equations were expressed the vorticity form and polar coordinates were used. The equations were thus

$$\frac{\partial \omega}{\partial t} + (u, \frac{v}{r}) \cdot (\frac{\partial \omega}{\partial r}, \frac{\partial \omega}{\partial \theta}) = \frac{1}{\mathbf{Re}} \left(\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \omega}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 \omega}{\partial \theta^2} \right) \tag{1}$$

$$u = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} \qquad v = -\frac{\partial \Psi}{\partial r} \tag{2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial \Psi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = -\omega \tag{3}$$

for $r \geq r_a$ and $0 \leq \theta \leq 2\pi$. Here (u, v) are the radial and tangential components of the velocity respectively, ω is the vorticity, and Ψ is the stream function. The non-dimensionalization is based on the cylinder diameter $2r_a$ and the velocity at infinity; $\mathbf{Re} = \frac{2r_a U_\infty}{\nu}$. Initial and boundary conditions for Ψ and ω which corresponded with the problem of an impulsively started cylinder motion were used.

We computed solutions with a range of Reynolds numbers between 500 and 9,600. Since the conclusions we reached did not depend upon Reynolds number, we choose to report those at Reynolds number 1000. (At this Reynolds number much of the complicated boundary layer features seen in higher Reynolds number flows are present. However it is still possible to resolve Re 1000 flow using a reasonable number of grid points and computing time.)

The numerical method for the full Navier-Stokes equations was a fourth order finite difference scheme. The convective derivative terms and the Laplacian were approximated with fourth order centered differences. (Sufficiently many grid points were used so that the problem of spatial oscillations of the computed solutions was avoided.) The grid was a uniform polar grid from $r = r_a$ to $r = r_b$. r_b was chosen so that the vorticity, except for an exponentially small amount, was contained within the region $r_a \leq r \leq r_b$ over the times which we computed. We implemented

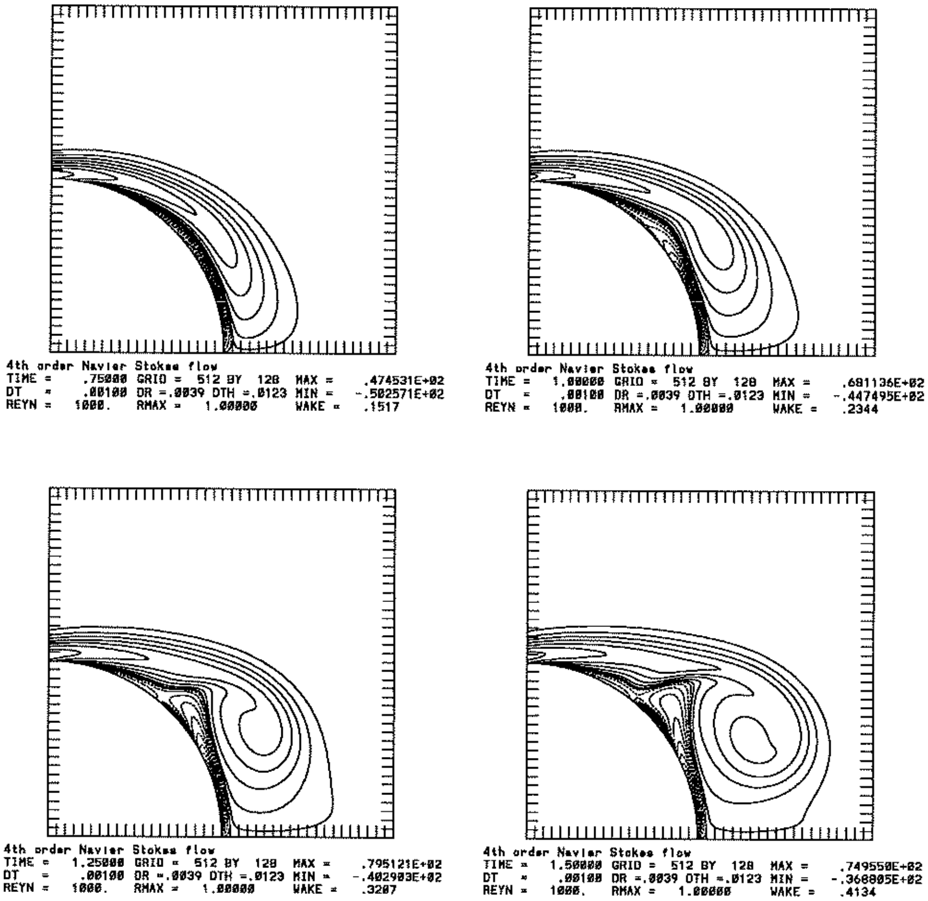

Fig. 2. Solutions of the Navier-Stokes equations at **Re** = 1000. Solid contours indicate negative vorticity, dashed contours positive vorticity.

"infinite domain" boundary conditions for the determination of the stream function (these are described in [2]) as well as high order vorticity boundary conditions at the surface of the cylinder (extending the procedure of [1]). The time-stepping method was explicit fourth order Runge-Kutta. The cylinder radius was 0.5. For a full description of the method and verification of its accuracy, see [6].

The development of the solution for this problem can most easily be seen by considering the dynamics of the vorticity. In Figure 2 we show contours of the vorticity at times t = .75, 1.0, 1.25 and 1.5. Since the flow is symmetric and the most interesting part is near the rear of the cylinder, we just show the upper rear corner. In this figure, and in all the remaining figures, the solid lines correspond to negative vorticity and the dashed lines correspond to positive vorticity. At time $t = .75$ the vortex sheet which forms along the cylinder is just beginning to separate. This process continues as time evolves, and by $t = 1.5$ the boundary layer has clearly separated and acts as a source of vorticity for the interior flow. At this later time we also see the roll-up of the vortex sheet occurring behind the cylinder.

The method utilizing both the Prandtl and Navier-Stokes equations was based on a decomposition of the domain as indicated in Figure 1. In the region external to the boundary layer region the Navier-Stokes equations were approximated in the same way as the full Navier-Stokes equations. In the boundary layer region, that region with $r_a \leq r \leq r_\delta$ the equations solved were the vorticity form of the Prandtl boundary layer equations,

$$\frac{\partial \omega}{\partial t} + (u, \frac{v}{r}) \cdot (\frac{\partial \omega}{\partial r}, \frac{\partial \omega}{\partial \theta}) = \frac{1}{\mathbf{Re}} \frac{\partial^2 \omega}{\partial r^2} \tag{4}$$

$$\frac{\partial v}{\partial r} = \omega \tag{5}$$

$$\frac{\partial (ru)}{\partial r} + \frac{\partial v}{\partial \theta} = 0 \tag{6}$$

Here u is the normal velocity and v the tangential velocity. These velocity fields are determined from the vorticity by calculating the integrals

$$v(r, \theta) = v_\infty(\theta) + \int_r^\infty \omega(r, \theta)\, dr \tag{7}$$

$$u(r, \theta) = -\frac{1}{r} \int_0^r \frac{\partial v(r, \theta)}{\partial \theta}\, dr \tag{8}$$

The discretization of these equations was similar to that employed for the full Navier-Stokes - i.e. a fourth order centered difference approach. The boundary conditions for the vorticity at the cylinder surface were also discretized in a similar fashion (again see [1]).

An important aspect of the combined equation approach is the determination of the coupling between the two solutions at the interface $r = r_\delta$. When solving the Navier-Stokes equations in the external region one must specify the vorticity at the interface $r = r_\delta$ as well as decide where to specify the stream function boundary condition. When one solves the Prandtl boundary layer equations one treats $r = r_\delta$ as infinity, and thus one needs to specify v_∞, the tangential velocity there. One also needs to specify the boundary conditions for the transport and diffusion of vorticity.

Our initial approach to the coupling was that implicitly used in the vortex sheet/vortex blob method. In such an approach one assumes that the vorticity is continuous across the interface so that any data necessary to close the equations for the transport and diffusion of vorticity for either equation is obtained by using the solutions were obtained with a value of $\delta = \frac{3}{64} \approx 1.5 \frac{1}{\sqrt{\mathbf{Re}}}$. While each solution exhibits a vortex sheet which forms and leaves the surface there are significant quantitative differences. Most noticeable is that the vortex sheet associated with the Prandtl/Navier-Stokes equations separates from the body earlier in time, separates at a different angle, and induces a much larger amount of opposite signed vorticity beneath it. They are clearly not the same solution.

In Figure 4 we show a comparison of the solutions to the Prandtl/Navier-Stokes equations at a fixed time $t = 1.5$, but with values for the width of the inner region of $\delta = \frac{2}{64}$, $\frac{3}{64}$, and $\frac{4}{64}$. The large scale features do not change much for this range of δ, but the small scale features do - in particular the structure of the vorticity which forms beneath the separated sheet. For reference purposes, the cylinder is of radius 0.5, and so each of the tick marks on the figures represent a distance of $\frac{1}{40}$. At $\delta = \frac{4}{64}$ the width is equal to 2.5 tick marks. With larger values of δ the solutions differed even more (in fact with $\delta = \frac{6}{64} \approx \frac{3}{\sqrt{\mathbf{Re}}}$ the solution "blew up" and the computation could not be continued).

These figures lead one to conclude that if one has separated flow, using a combined Prandtl/Navier-Stokes equation is not appropriate. A question remains however, "Can this approach can be fixed ?". After some experimentation, we concluded that the error was in large part due to the manner in which the solutions were coupled at $r = r_\delta$. To obtain better solutions we therefore changed the way in which the solutions were coupled. The coupling described above enforced a continuity of the vorticity and a continuity of tangential velocity. While this coupling is simple and easy to implement, it gives velocity fields whose normal component is not necessarily continuous across the line $r = r_\delta$. It is clear that a continuous velocity field is a reasonable requirement, and we therefore determined a coupling of the two calculations so that both the normal and tangential components of the velocity would be continuous at the interface. The technique for doing this is similar to that used in domain decomposition procedures for solving elliptic partial differential equations. (See the proceedings where [2] appears for references). Essentially, given a vorticity distribution and a tangential velocity along $r = r_\delta$ one can compute a velocity field in the inner region using (7)-(8) and a velocity field in the external region using (2)-(3) with a normal derivative boundary condition on Ψ at $r = r_\delta$. The resulting velocity field will have a continuous tangential velocity by construction, but not necessarily a continuous normal velocity. However, we can ask "Is there a tangential velocity at $r = r_\delta$ so that, after the velocity field is constructed we also get a continuous normal velocity?". It turns out that the answer to this question is "yes" and in fact one can construct a linear system of equations to find it. Our procedure was therefore to construct and solve the set of linear equations which determine this special tangential velocity at $r = r_\delta$. With this tangential velocity one uses (7)-(8) and (2)-(3) to compute the velocity field in the inner and external regions. The net result of this process is the determination of a velocity field which satisfies the differential equations (5)-(6) in the inner region and (2)-(3) in the external region. At the interface, both the tangential and normal velocities are continuous.

The results of this approach are seen in Figure 5. It is clear that the improved coupling removes much of the error seen previously. While there are still slight dif-

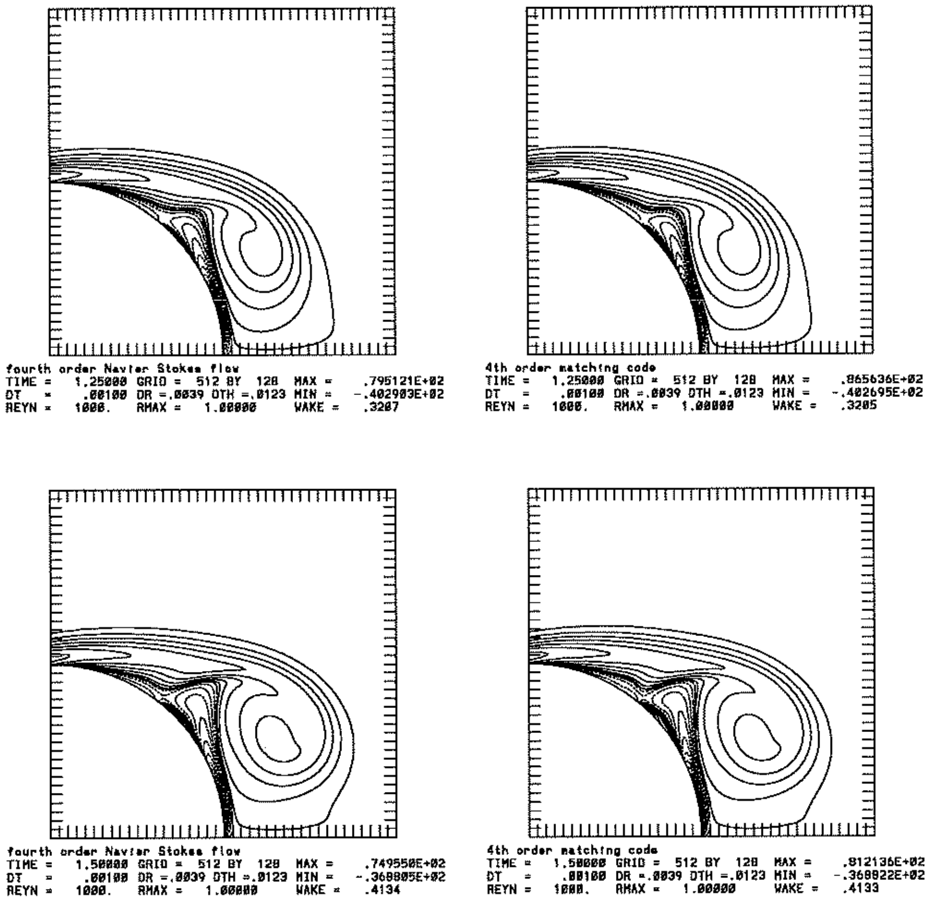

Fig. 5. A comparison of the solutions of the Navier-Stokes equations (left) with the solutions of Prandtl/Navier-Stokes equations (right) at **Re** $= 1000$ with $\delta = \frac{3}{32}$. The Prandtl/Navier-Stokes equations were coupled so that both components of the velocity field were continuous across $r = r_\delta$.

ferences, there is good general agreement. Moreover, as can be deduced from Figure 6 we see that the improved method has only a weak dependence on the thickness of the inner region. With large values of δ the solutions were still significantly different from the Navier-Stokes equations, but this is to be expected, because, away from the body, the terms neglected in the Navier-Stokes equations in order to derive the Prandtl equations become important.

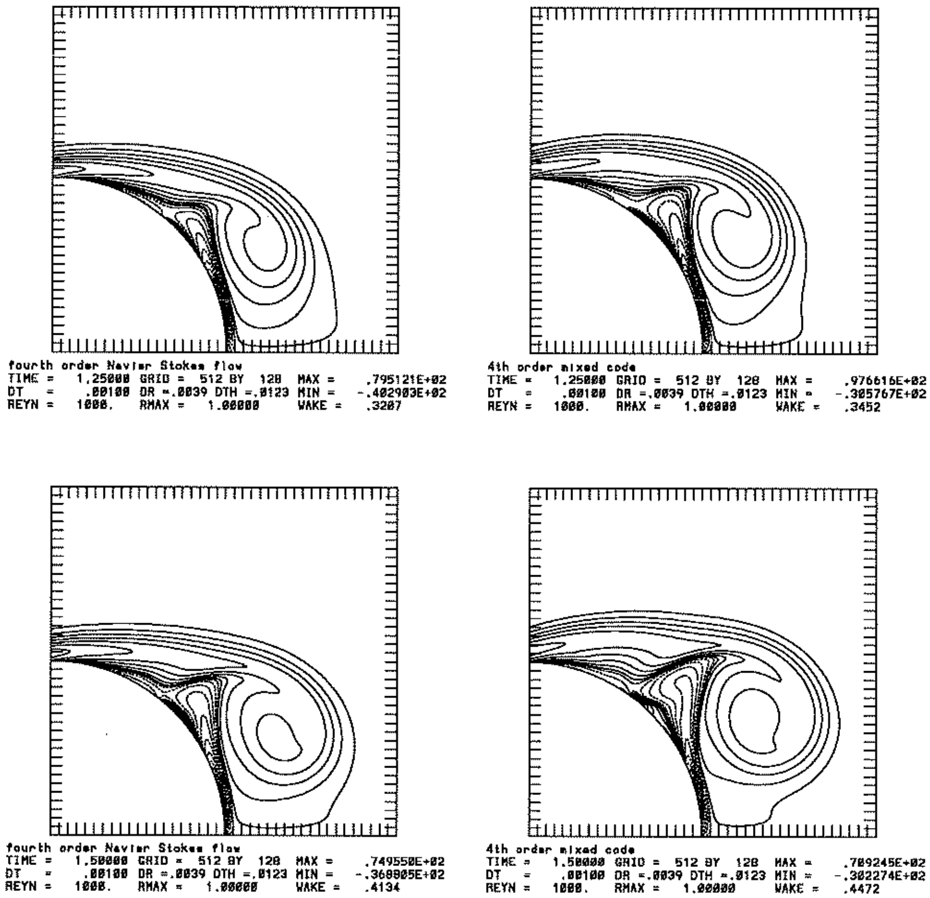

Fig. 3. A comparison of the solutions of the Navier-Stokes equations (left) with the solutions of Prandtl/Navier-Stokes equations (right) at $\mathbf{Re} = 1000$ with $\delta = \frac{3}{32}$.

vorticity on "the other side" of the interface. This external velocity was obtained by differentiating the solution to (3) with the boundary condition $\Psi = 0$ at the cylinder surface $r = r_a$. The tangential velocity v_∞ needed for the Prandtl equations was just taken to be the external velocity evaluated at the point $r = r_\delta$. The net result of this process was a velocity field which satisfies the differential equations (5)-(6) in the inner region and (2)-(3) in the external region. At the interface, the tangential velocity is continuous.

One free parameter in this Prandtl/Navier-Stokes approach is δ the width of the inner region. In our experiments we used values on the order of $\frac{1}{\sqrt{\mathbf{Re}}}$, the

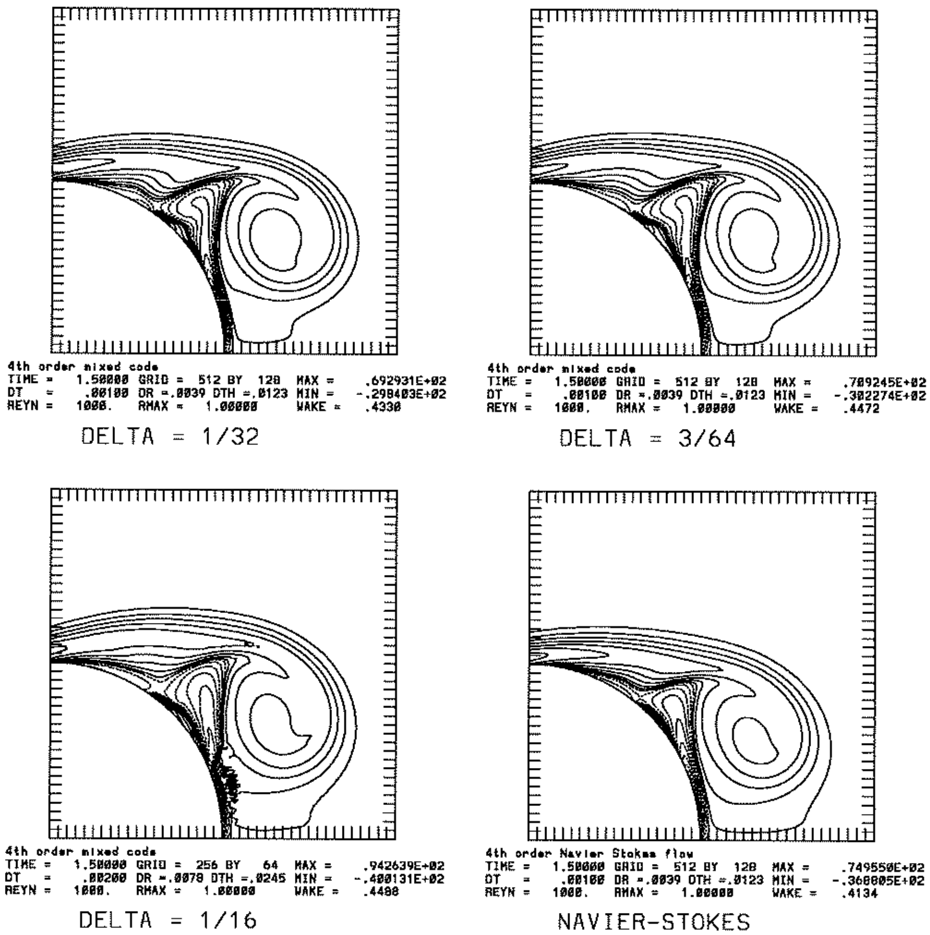

Fig. 4. Solutions of the Prandtl/Navier-Stokes equations for values of the inner region width $\delta = \frac{2}{64}, \frac{3}{64}$, and $\frac{4}{64}$.

order of the size of the boundary layer [7]. This choice for δ is that which has been previously used in calculations with a vortex sheet/vortex blob method.

At early times the solutions for the Navier-Stokes equations and those obtained with the combined Prandtl/Navier-Stokes equations did not differ appreciably. This is to be expected since the boundary layer had not separated, and it is reasonable to expect that approximations made in deriving the Prandtl boundary layer equations are appropriate. At later times the situation is rather different. In Figure 3 we show a comparison of the vorticity distributions obtained with the Prandtl/Navier-Stokes equations verses the solution obtained with the full Navier-Stokes equations. These

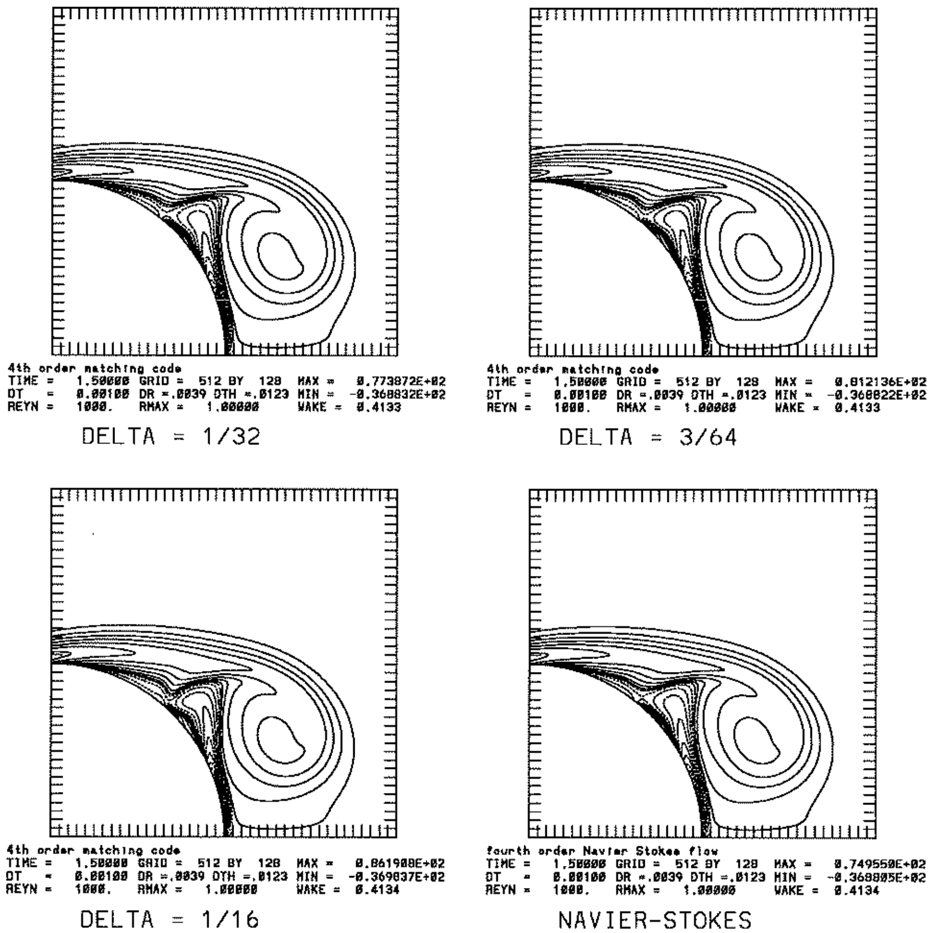

Fig. 6. Solutions of the Prandtl/Navier-Stokes equations for values of the inner region width $\delta = \frac{2}{64}$, $\frac{3}{64}$, and $\frac{4}{64}$. The Prandtl/Navier-Stokes equations were coupled so that both components of the velocity field were continuous across $r = r_\delta$.

3. Conclusion

We have described an investigation in which we sought to determine the accuracy of using a Prandtl/Navier-Stokes equation approach to computing flows about bodies in two dimensions. From our initial experiments in which a simple coupling was used, it was clear that when separation is present, the combined equations yield solutions which are significantly different from those obtained with the Navier-Stokes equations. We have also shown that if one improves the coupling, then the solutions to the Prandtl/Navier-Stokes equations appear to be acceptable, even in cases in which there is separation. The computational experiments we presented were carried out at Re=1000, but experiments at higher Reynolds numbers indicate that our conclusions are not sensitive to the Reynolds number. Another aspect of this work is that it provides evidence that the inaccuracies incurred with the use of a Prandtl/Navier-Stokes approach are problem specific. (When the flow wasn't separated the simple coupling approach was o.k..) Since this is the case, we cannot immediately generalize this work to Prandtl/Navier-Stokes approaches to flows where separation occurs at a sharp edge. It is our expectation that simple matching is sufficient, but further numerical work (which we are in the process of carrying out) should provide information regarding this question.

Acknowledgements

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