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B. Engquist

E. Fatemi

S. Osher

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Department of Mathematics

University of California, Los Angeles

Los Angeles, CA. 90024-1555

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B. Engquist, E. Fatemi, and S. Osher [1]
University of California, Department of Mathematics, 405 Hilgard Avenue
Los Angeles, CA 90024-1555

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Proofs to be sent to:
S. Osher
University of California
Department of Mathematics
405 Hilgard Avenue
Los Angeles, CA 90024-1555

Abstract

New numerical methods are derived for the calculation of high frequency asymptotic expansion of the scalar wave equation. The nonlinear partial differential equations defining the terms in the expansion are approximated directly rather than via ray tracing. High resolution numerical algorithms are used to handle discontinuities and new devices are introduced in order to represent the multivalued character of the solution.

1 Introduction

High frequency solutions of different types of the wave equation are of practical importance. The numerical method that we shall develop here is for classical high frequency asymptotic expansions of hyperbolic equations. We shall concentrate here on the scalar linear wave equation in two spatial variables.

$$u_{tt} = c^2 \Delta u = c^2(x,y)(u_{xx} + u_{yy})$$

x and y are the spatial variables, t is time, u is the amplitude of the wave and $c(x,y)$ is the speed of the wave in the medium. Time harmonic solutions of the wave equation, of the form

$$u(x,y,t) = e^{i\omega t} v(x,y),$$

are of special interest. Here ω is the time frequency of the wave as imposed by the boundary conditions. For time harmonic solutions the wave equation is reduced to the Helmholtz equation.

$$\Delta v + \frac{\omega^2}{c^2} v = 0$$

We denote the wavelength of the wave by λ and a typical length of the problem by L. The non-dimensional quantity $k^2 = \frac{\omega^2}{c^2} = \frac{4\pi L^2}{\lambda^2}$ represents the relative size of the wavelength of the wave with respect to the physical size of the problem. Direct numerical solution of the Helmholtz equation for large values of parameter k is difficult. The fundamental difficulty is the fact that the necessary resolution is proportional to $\frac{1}{k}$. For a three dimensional calculation one needs $O(k^3)$ number of points to resolve the solution. We follow a different route based on a classical asymptotic expansion.

High frequency solutions of the scalar wave equation can be approximated by an asymptotic expansion, see e.g. Luneburg, Kline, and Keller [10], [7], [6]. In this approach the solution to the wave equation is expanded in ω in an infinite series. The expansion is substituted in the wave equation and the sequence of the coefficients of ω^n are collected and set to zero. This procedure produces an infinite system of nonlinear partial differential equations that with the boundary conditions determine the expansion. The first and most important term of the expansion is the eikonal equation of the geometrical optics for the phase of the wave. The remaining terms represent amplitude and corrections to the geometrical optics from finiteness of the frequency.

The correspondence between numerical solutions of the wave equation and the eikonal equation is not straightforward. The wave equation is a linear equation but the eikonal equation is a nonlinear equation. Any linear combination of solutions of the wave equation is a solution to the wave equation, but a linear combination of solutions of the eikonal equation does not satisfy the eikonal equation. In order to circumvent this difficulty one has to consider a multivalued phase function. Other classes of solutions for the eikonal equation, not present for the linear equation, are shocks and also expansion waves for the gradient of the solution. Shocks are observed as two opposing waves approach each other and the phase becomes non-unique. Expansion waves are observed in the viscosity solution of the eikonal equation in the regions where geometrical optics predicts a shadow region. Also the asymptotic expansion that we are considering does not necessarily include all solutions of the wave equation. In fact domains with corners or caustics require different expansions in ω [9].

A geometrical theory of diffraction was developed by J. Keller in [6]. According to this theory the phase is a multi-valued function connected through branches. Branches could be surfaces, lines, or points. The branch manifolds are present due to either physical boundaries of the problem, the singularities in the physical boundary, or variation in the index of refraction or its derivatives. All the above singularities are known *a priori* and can be identified for a numerical solution of the problem. In this paper we identify a new kind of branch surface that can be determined only when the eikonal solution is solved. Due to non uniformity of the index of refraction a caustic could develop in the phase. This situation happens for example when a plane wave goes through a convex lens. The singularity is manifested as a discontinuity in the derivative of the phase and is similar to a shock for conservation laws. We identify the shock wave as a branch surface and we develop a numerical method based on shock detection to calculate the branch surface. The information from the branch surface is used to calculate the newly formed phase sheet. Also J. Keller considers grazing rays produced in the shadow region of a cylinder illuminated by a plane wave. The grazing rays encircle the circle in the shadow region. The viscosity solution of the eikonal equation as produced by our numerical procedure is an expansion wave in the shadow region (see section 6). The expansion wave in that region corresponds to the grazing rays as defined by Keller [6].

The traditional way of solving the eikonal equation is by the method of characteristics which in this context is called ray tracing. In this paper we shall explore the application of the modern high resolution algorithms to directly compute the eikonal equation and the other equations defining the terms in the expansion. These modern techniques allow for accurate representation of singularities which is essential in this application. A major part of this paper concerns the approximation of crossing and reflected rays. The standard viscosity solution of the eikonal equation, [4], is not enough and a hierarchy of numerical solutions has to be generated. These new functions represent the multivalued character of the solution and are based on singularity detection.

2 Asymptotic Expansion

It is natural to expand solutions of the wave equation for optical applications around zero wavelength. Here we briefly repeat the standard derivation of the asymptotic expansion in order to introduce our notation and to point out important terms. The solution to the wave equation is expanded in inverse powers of ω in the following form,

$$u(x,y,t) = e^{i\omega\phi(x,y,t)} \sum_{n=0}^{\infty} v_n(x,y,t)(i\omega)^{-n},$$

where u is a complex solution of the wave equation, $\phi(x,y,t)$ is the phase of the wave, and v_n are real functions of time and space. The amplitude of the wave is $|u|^2 = v_0^2 + (v_1^2 + 2v_0v_2)\omega^{-2} + O(\omega^{-4})$. By substituting the above expansion in the scalar wave equation and equating the coefficients of different powers of ω, the partial differential equations for the evolution of ϕ, v_0, v_1, etc are derived. The second time derivative of u is simply calculated to be:

$$u_{tt} = (\phi_t^2 v_0)(i\omega)^2$$
$$+(\phi_{tt} v_0 + 2\phi_t v_{0t} + \phi_t^2 v_1)(i\omega)$$
$$+\sum_{n=0}^{\infty}(\phi_{tt} v_{n+1} + 2\phi_t v_{n+1,t} + \phi_t^2 v_{n+2} + v_{n,tt})(i\omega)^{-n}.$$

Similar expressions for u_{xx} and u_{yy} are obtained by replacing t derivatives by x or y derivatives. Terms of order $(iw)^2$ are collected and the result is the eikonal equation.

$$\phi_t^2 v_0 = c^2(\phi_x^2 + \phi_y^2) v_0$$

In general v_0 is not zero and we take the square root of the above equation. Here we choose the positive root.

$$\phi_t = +c(x,y)|\nabla\phi|$$

Terms of order (iw) result in the equation for evolution of v_0.

$$2\phi_t v_{0,t} + \phi_{tt} v_0 = c^2 \nabla\phi . \nabla v_0 + \Delta\phi v_0$$

By collecting terms of order $(iw)^{-n}$ we get:

$$2\phi_t v_{n+1,t} + \phi_{tt} v_{n+1} + v_{n,tt} = c^2 \nabla\phi \cdot \nabla v_{n+1} + \Delta\phi v_{n+1} + \Delta v_n$$

One can solve for ϕ_t and $v_{n,t}$ and get an infinite system for evolution of the expansion coefficients.

$$\phi_t = c|\nabla\phi|$$
$$v_{0,t} = c^2 \frac{\nabla\phi}{|\nabla\phi|}.\nabla v_0 + \frac{(-\phi_{tt} + c^2\Delta\phi) v_0}{2|\nabla\phi|}$$
$$v_{n+1,t} = c^2 \frac{\nabla\phi}{|\nabla\phi|}.\nabla v_{n+1} + \frac{(-\phi_{tt} + c^2\Delta\phi) v_{n+1}}{2|\nabla\phi|} + \frac{-v_{n,tt} + c^2\Delta v_n}{2|\nabla\phi|}$$

It is possible to write the equations for evolution of v_n in conservative form by changing the variables. This reformulation for the steady state equation for v_0 is standard and here we extend it for the full time dependent system. Consider the equation for evolution of v_0. We multiply both sides by v_0 and we get:

$$(\phi_t v_0^2)_t = c^2 \nabla \cdot (v_0^2 \nabla\phi)$$

Similarly for higher order terms we multiply both sides by v_{n+1} and we obtain:

$$(\phi_t v_{n+1}^2)_t = c^2 \nabla . (v_{n+1}^2 \nabla \phi) + (-v_{n,tt} + c^2 \Delta v_n) v_{n+1}.$$

We define the new variables as $w_n = \frac{\phi_t v_n^2}{c^2}$, and we derive the transport equations in their conservative form.

$$\phi_t = c|\nabla \phi|$$

$$w_{0,t} = \nabla \cdot (c w_0 \frac{\nabla \phi}{|\nabla \phi|})$$

$$w_{n+1,t} = \nabla \cdot (c w_{n+1} \frac{\nabla \phi}{|\nabla \phi|}) + (\Delta v_n - \frac{v_{n,tt}}{c^2}) v_{n+1}$$

$$v_n = c\sqrt{\frac{w_n}{\phi_t}}$$

The above system consists of the eikonal equation for the phase which is a Hamilton-Jacobi type equation and an infinite hyperbolic system for variables w_n. The system is essentially decoupled, since it can be truncated at any level. The first equation, the eikonal equation, can be solved independently of the others. Once ϕ is obtained, it is used to solve for w_0. Similarly w_n is obtained using the solution of ϕ and w_{n-1}. The equation for w_n has a forcing term $(\Delta v_n - \frac{v_{n,tt}}{c^2})$ which has to be calculated from the previous term, w_{n-1}. If the previous term w_{n-1} is not twice differentiable in space and time, the forcing term is ambiguous. From a numerical point of view, even for smooth solutions, it is important to calculate the forcing term correctly, otherwise error would spread out to higher order terms. The ambiguity in the forcing terms could be resolved by specifying jump conditions along discontinuities of v_n. The proper conditions depend on physical considerations as well as mathematical arguments. The continuity of the phase across branch surfaces supplied us with the necessary boundary conditions for multi-valued solutions of the phase. For v_n the situation is more complicated. For example the amount of reflected energy from a reflecting surface depends on the physical characteristics of the surface. The situation is more complex for other kinds of branch surfaces. In this paper we only considered the situations where all of the v_n are regular. Our numerical procedure is able to handle the situations where some of the v_n may not be regular, but the physical relevance is questionable. For example our solution for a convex lens has a singularity after the focal point but our numerical procedure is stable.

3 Eikonal equation

The first term of the asymptotic expansion is the eikonal equation for the phase of the wave. The eikonal equation can also be derived from the variational problem for light rays. According to the classical geometrical optics light travels from point A to point B along rays, $(x(s), y(s))$, minimizing the following functional:

$$\int_A^B \frac{1}{c(x,y)}\sqrt{x'(s)^2 + y'(s)^2}ds$$

The eikonal equation is the Hamilton-Jacobi equation for the above variational problem. The eikonal equation is a first order nonlinear partial differential equation in three dimensional space and time. In general the solutions of this class of equations are not classical solutions and admit weak solutions. A large class of weak and physically relevant solutions for Hamilton-Jacobi type equations are known as viscosity solutions. In the context of the geometrical optics it seems that multi-valued viscosity solutions are the appropriate class of solutions. The theory for viscosity solutions of Hamilton-Jacobi type equations is developed in [4]. First order monotone numerical schemes that produce viscosity solutions were proven to converge by the same authors. In this work we used the higher order numerical methods developed in [12].

The concept of light rays can be used to derive the eikonal equation. It turns out that these rays are nothing but the the characteristic lines of the eikonal equation. The characteristic lines can be defined as the integral curves of the following vector field,

$$c\frac{\nabla\phi}{|\nabla\phi|}.$$

The characteristic lines are used to determine the boundary conditions for the eikonal equation (also for the transport equations). Let a unit vector normal to the boundary and pointing to the outside of the domain be denoted by $\hat{n}$. We specify boundary conditions for ϕ if $\frac{\nabla\phi}{|\nabla\phi|}\cdot\hat{n}$ is positive. If $\frac{\nabla\phi}{|\nabla\phi|}\cdot\hat{n}$ is negative no boundary conditions are specified.

The eikonal equation is the central part of the asymptotic approximation. In this section we describe two specific solutions of the eikonal equation in one dimensional space. Our numerical examples include computation of the same problems but in two dimensional space. First we consider reflection of a wave from an object and explain how the boundary conditions and the multi-valued nature of the phase is handled. Then we consider the problem of creation of discontinuities in the gradient of the solution and how it is used to calculate the multi-valued solution after appearance of the singularity.

We consider reflection of a plane wave from a plane. The wave is originating from $x = 0$ and reflecting back from $x = 1$, (see figure 1). The phase of the incident wave is denoted by ϕ_1 and the phase of the reflected wave by ϕ_2. The reflected wave , ϕ_2, satisfies the eikonal equation and the following appropriate boundary condition. Note that there is no need for boundary condition at $x = 1$.

$$\phi_{1,t} = |\phi_{1,x}| \qquad \phi_1(0,t) = tH(t)$$

The reflected wave also satisfies the eikonal equation but a boundary condition at the reflecting surface is needed.

$$\phi_{2,t} = |\phi_{2,x}| \qquad \phi_2(1,t) = \phi_1(1,t)$$

The two problems are coupled only at the boundary of the reflecting surface. The boundary condition simply states the fact that the phase is continuous on a branch surface, (here the reflecting surface is the branch surface). Note that the problem is an initial-boundary value problem and the phase has to be defined at time $t = 0$. The phase is not defined in the regions of the space in which there is no wave. Since our boundary conditions imply a zero phase at time zero, we choose the phase to be zero everywhere at time $t = 0$. The explicit time dependent solution of the above problem is :

$$\phi_1(x,t) = (t - |x|)H(t - |x|),$$

$$\phi_2(x,t) = (t - |x-1| - 1)H(t - |x-1| - 1).$$

The Heaviside function, H(x), is defined as $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$.

Next we consider the problem of branch surfaces that are not known *a priori* and are calculated as part of the solution. Consider the eikonal equation in the interval $[0,1]$. We consider two plane waves approaching each other. Appropriate boundary conditions are $\phi(0,t) = tH(t)$ and $\phi(1,t) = tH(t)$. The solution up to time $t = 0.5$ is regular and is explicitly

$$\phi(t,x) = (t - |x|)H(t - |x|) + (t - |x - 1|)H(t - |x - 1|).$$

A singularity is developed in the solution at $t = 0.5$ at point $x = 0.5$. After time 0.5 the viscosity solution as calculated by the numerical method is

$$\phi(t,x) = \max\{(t - |x|)H(t - |x|), (t - |x - 1|)H(t - |x - 1|)\}.$$

At point $x = 0.5$ the derivative of the phase is discontinuous. The singularity is similar to a shock wave for hyperbolic equations. The characteristic lines are directed into the singularity. In fact if we denote the derivative of the phase by $u = \phi_x$, then u satisfies the nonlinear conservation law $u_t - |u|_x = 0$. To keep track of the lost waves we detect the singularity using the direction of $\frac{\nabla\phi}{|\nabla\phi|}$ on two sides of the singular point. We solve a second eikonal equation for the phase of the second wave. The position of the singularity of ϕ_1 is used as position of the boundary for ϕ_2 (see figure 2). We use the continuity of the phase at a branch surface to specify the boundary condition.

$$\phi_2(0.5,t) = \phi_1(0.5,t)$$

The solution for ϕ_2 is:

$$\phi_2(x,t) = (t - |x - 0.5| - 0.5)H(t - |x - 0.5| - 0.5).$$

Figure 3 is a graph of a computed solution of a similar problem. Our graph is a cross section of the two solutions shown in figures six and seven.

4 Transport Equations

The eikonal equation describes the behavior of the phase in the limit of zero wavelength. The amplitude of the wave at that limit can also be determined using the equation for w_0.

The corrections to the amplitude and to the phase due to finiteness of the wavelength can be determined (formally) by computing the remaining terms of the expansion. The accuracy of the approximation is an open question. The system of partial differential equations for w_n is a hyperbolic system with non-constant coefficients. The equations in the system are usually called transport equations since they account for transport of energy.

The transport equations can be written in conservative and nonconservative form. The conservative form is simpler and is more suitable for numerical methods. The equation for the evolution of w_0 in conservative form is:

$$w_{0,t} = \nabla.(cw_0 \frac{\nabla \phi}{|\nabla \phi|}),$$

where $w_0 = \frac{\phi_t v_0^2}{c^2}$. For a time harmonic solution $\phi_t = 1$ and $w_0 = \frac{v_0^2}{c^2}$. The above equation is interpreted as conservation of energy in a tube of light rays. By a tube of rays we mean an area bounded by the characteristic lines of the eikonal equation. Note that the characteristic lines for the eikonal equation and the transport equations are identical. This fact leads to the conclusion that the boundary conditions for the hyperbolic system and the eikonal equation have to be specified at the same part of the boundary. Note that when we have a branch surface we used continuity of the phase to decide the boundary conditions at the branch surface. The situation is more complicated for the boundary conditions for the hyperbolic system at the branch surfaces. Physical arguments as well as mathematical ones have to be considered for specifying the boundary conditions. For example for reflecting surfaces, one has to specify the coefficient of reflection. For phase solutions that have a shock, lower solutions and upper solutions have to be coupled through the fluxes. For branch surfaces due to corners or singularities in the index of refraction or caustics, the form of the asymptotic is different and more investigation is required in these cases; please see [9], and [11].

5 Numerical algorithm

Our numerical algorithm is developed based on the recently devised numerical methods for Hamilton-Jacobi type equations and upwind methods for hyperbolic equations. We developed numerical algorithms to solve the equations both in conservative and nonconservative variables. Here we report only on the numerical algorithm developed for the conservative variables. From a numerical and also theoretical point of view it is the natural way of writing the equations. We consider only the first three equations of the infinite system for ϕ, w_0, and w_1.

$$\phi_t = c|\nabla\phi|$$

$$w_{0,t} = \nabla\cdot(cw_0\frac{\nabla\phi}{|\nabla\phi|})$$

$$w_{1,t} = \nabla\cdot(cw_1\frac{\nabla\phi}{|\nabla\phi|}) + (\Delta v_0 - \frac{v_{0,tt}}{c^2})v_1$$

$$v_0 = c\sqrt{w_0/\phi_t} \qquad v_1 = c\sqrt{w_1/\phi_t}$$

We use third order ENO interpolation and a Godunov type flux to solve the eikonal equation and first order upwind finite difference methods to solve the transport part [12]. We could also have used high order accurate ENO methods for conservation laws to solve the transport equations (see e.g. the references in [12].) We use Δx, Δy, and Δt to denote the mesh size. $\phi^n_{i,j}$ is the numerical approximation to the viscosity solution of the eikonal equation.

$$\phi^n_{ij} = \phi(x_i, y_j, t^n) = \phi(i\Delta x, j\Delta y, n\Delta t)$$

Also we use standard notation for forward, backward, and centered differences.

$$D^+_x\phi_{ij} = \frac{\phi_{i+1,j} - \phi_{i,j}}{\Delta x} \; D^-_x\phi_{ij} = \frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta x} \; D^0_x\phi_{ij} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta x}$$

A Godunov type flux with ENO interpolation is used to solve the eikonal equation.

$$\phi^{n+1}_{ij} = \phi^n_{i,j} + \Delta t\hat{G}(D^{+ENO}_x\phi_{ij}, D^{-ENO}_x\phi_{ij}, D^{+ENO}_y\phi_{ij}, D^{-ENO}_y\phi_{ij})$$

The values of the derivatives, D^{ENO}, are calculated based on third order interpolation with an adaptive stencil [12]. The numerical flux is calculated based on the exact solution of a Riemann problem for the eikonal equation. The flux is denoted by $\hat{G}$ and is defined by

$$\hat{G}(u^+, u^-, v^+, v^-) = \text{ext}_{u\in I(u^-,u^+)}\ \text{ext}_{v\in I(v^-,v^+)}\ H(u,v)$$

$H(u,v) = c\sqrt{u^2+v^2}$ and I(a,b)=[min(a,b), max(a,b)] . The function ext is defined by

$$\text{ext}_{u\in I(a,b)} = \max_{u\in I(a,b)} \quad \text{if} \quad a \le b,$$

$$\text{ext}_{u\in I(a,b)} = \min_{u\in I(a,b)} \quad \text{if} \quad b < a.$$

Note that in general the operations of taking max and min do not commute and the Godunov flux in not always uniquely defined. But for many cases, including our $H(u,v)$, the flux is uniquely defined. The use of the third order ENO interpolation and the exact Riemann solver results in the excellent resolution of the discontinuities in the solution of the phase.

For solving the transport equations a first order upwind scheme is used. In principle various sophisticated schemes developed for nonlinear conservation laws could be used to solve the transport equations. For a qualitative understanding of the solutions and showing the applicability of the approach a first order upwind method is sufficient. The velocity vector field $\frac{\nabla\phi}{|\nabla\phi|}$ was calculated using centered differences.

$$\nabla\phi_{ij} = (D_x^0\phi_{ij}, D_y^0\phi_{ij})$$

$$\left.\frac{\nabla\phi}{|\nabla\phi|}\right|_{i,j} = \frac{\nabla\phi_{ij}}{|\nabla\phi_{ij}|}$$

Once the velocity field is calculated a conservative scheme is used to calculate w_0^{n+1}.

$$w_0^{n+1} = w_0^n + \frac{\Delta t}{\Delta x}(\hat{fx}_{i+\frac{1}{2},j} - \hat{fx}_{i-\frac{1}{2},j}) + \frac{\Delta t}{\Delta y}(\hat{fy}_{i,j+\frac{1}{2}} - \hat{fy}_{i,j-\frac{1}{2}})$$

Where $\hat{fx}$ and $\hat{fy}$ are the numerical fluxes calculated based on a first order upwind method. Define $(\hat{n}_x, \hat{n}_y) = (\left.\frac{\nabla\phi}{|\nabla\phi|}\right|_{i+1,j} + \left.\frac{\nabla\phi}{|\nabla\phi|}\right|_{i,j})/2$. The flux function in the x direction, $\hat{fx}$, is defined as

$$\hat{fx}_{i+\frac{1}{2},j} = c_{i+1,j}w_{0,i+1,j}\hat{n}_x \quad \text{if} \quad \hat{n}_x > 0$$

$$\hat{fx}_{i+\frac{1}{2},j} = c_{i,j}w_{0,i,j}\hat{n}_x \quad \text{if} \quad \hat{n}_x < 0$$

A similar definition is used for calculating the flux function $\hat{fy}$. For solving w_1 we used the same algorithm as above for calculating the divergence term. To calculate the forcing term, $\Delta v_0 - \frac{v_{tt}}{c^2}$, first v_0 and v_1 have to be calculated. v_0 and v_1 were calculated using the numerical Hamiltonian $\hat{G}$ which is equivalent to the numerical ϕ_t.

$$v_{i,j} = c_{ij}\sqrt{w_{ij}/\hat{G}_{ij}}$$

The term $v_{0,tt}$ was calculated using a centered discretization in time.

$$v_{0,tt}^n = \frac{v_{0,ij}^{n+1} - 2v_{0,ij}^n + v_{0,ij}^{n-1}}{2\Delta t^2}$$

Note that the previous values of w_0 have to be stored in order to calculate the time derivative of v_0.

6 Numerical examples

In this section we discuss several examples computed using the above algorithm. We consider reflection of an incident wave off a sphere. We consider a sphere of radius one at the origin. The source of the wave is at point $(0,0,2)$. We use spherical coordinates and exploit the symmetry of the problem under rotation around the z axis. The eikonal equation in (r,θ) space is written as

$$\phi_t = \sqrt{\phi_{rr}^2 + \frac{\phi_{\theta\theta}}{r^2}}.$$

We denote the ingoing wave by ϕ_1 and the reflected wave by ϕ_2. The physical problem is posed in the entire R^3 but our computational domain is $[1,3] \times [-\pi, \pi]$. The boundary conditions for ϕ_1 are specified as the following. We specify a time harmonic source at point $(2,0)$. which is translated into the boundary condition for the phase,

$$\phi_1(2,0) = tH(t).$$

We use periodic boundary conditions for $\theta = \pi$ and $\theta = -\pi$. The boundary conditions at $r = 1$ and $r = 3$ are determined based on the local direction of the characteristics. If the characteristic line is pointed to the outside of the domain, there is no need for boundary conditions. But if the characteristic line is directed to the inside, we use $\hat{G} = 0$. The direction of the characteristic line is simply determined based on the sign of the normal derivative of the phase. In (r,θ) coordinates, the normal derivative is simply $\frac{d\phi}{d\theta}$. The boundary conditions for the reflected wave , ϕ_2, are specified as the following. For $\theta = \pi$ and $\theta = -\pi$ periodic boundary conditions are specified. For $r = 3$ the boundary condition is again based on the local direction of the characteristic line. The phase of the reflected wave at the surface of the sphere is equal to the incident wave and therefore at $r = 1$ we specify the boundary condition for ϕ_2 according to:

$$\phi_2(1,\theta) = \phi_1(1,\theta).$$

The problem is solved numerically in (r,θ) space and the results are interpolated to a Cartesian grid. A contour plot of the calculated phase of the in-going wave is shown in figure four. One can see that in the in-going wave we calculate the phase even in the shadow region behind the sphere. In the shadow region the solution is composed of two expansion waves starting at the shadow line and meeting on the z axis and creating a shock on the z axis. The expansion waves correspond to the grazing waves as described by J. Keller [6]. The position of the shock corresponds to a local maximum in the amplitude of the wave. Existence of a bright spot behind an opaque sphere illuminated by a plane wave is a well known physical phenomena [2]. If we had calculated the amplitude of the wave on that line, a local maximum would have been observed. A contour plot of the reflected wave is shown in the next figure. The reflected wave also creates two expansion waves and a shock in the shadow region. The solution of this problem can be completed by calculating the hidden solutions generated at the shocks for the incident wave and the reflected wave. In the next example we just describe the general procedure to do that and apply it to a generic problem.

In a nonuniform media the solution of the eikonal equation could develop singularities. These singularities are generated when two wave fronts approach each other. The viscosity solution of the eikonal equation represents only one sheet of the solution and the hidden solutions are ignored. The branch surface of the solution is nothing but the surface of the discontinuity in the gradient of the phase. We consider the eikonal equation in the plane $[-1,1]\times[-1,1]$. We denote by ϕ_1 the original solution and ϕ_2 as the hidden solution. Two time harmonic sources are considered which are implemented by the following boundary conditions.

$$\phi_1(-0.5,0) = (t-0.25)H(t-0.25) \quad \phi_1(0.5,0) = tH(t)$$

The solution of the initial-boundary value problem is unique up to time $t = 0.625$. After that the solution is double valued. The second solution is connected through a branch curve on the shock. The position of the shock is simply the curve defined by $3y^2 + 1/2y - x^2 - 15/64 = 0$. We employ a general shock detection algorithm to detect the position of the branch curve. Once the position is found we use the values of ϕ_1 as boundary condition for ϕ_2. The position of the shock is detected based on the local sign of the derivative of the phase. At each point ϕ_{ij} there are three directions to be checked x, y, and $x-y$ direction. The local forward and backward differences of ϕ_{ij} are calculated in each direction. A shock is detected if in any of the three directions the forward difference is positive and the backward difference is negative. The points $\phi_{1,ij}$ that are detected as a shock are used as boundary points for ϕ_2.

$$\phi_{2,ij} = \phi_{1,ij}$$

This procedure enables us to calculate the hidden solution.

In the next example we compute the solution of the eikonal equation for the shadow of a corner as illuminated by a plane wave. In (x, y) space we consider the square $[-1,1] \times [-1,1]$ excluding the the fourth quadrant. The boundary conditions are specified at $y = -1$.

$$\phi(x,-1,t) = tH(t), \quad \text{for} x < 0$$

For the remaining sides of the square and the sides of the obstacle we specify the boundary conditions based on the direction of the characteristic lines as was described before. The explicit solution in the second and third quadrant simply is a plane wave traveling in the y direction.

$$\phi(x,y,t) = (t-y-1)H(t)$$

In the first quadrant it is a cylindrical wave originating from the corner of the obstacle.

$$\phi(x,y,t) = (t-1-\sqrt{x^2+y^2})H(t-1)$$

The contour plot of the calculated phase is shown in figure six. In principle the complete solution of the above problem is double-valued. We calculated the principal and the second one is a cylindrical wave originating from the corner.

In the next experiment we consider the eikonal equation in a non-homogeneous medium. We compute the distortion of a plane wave as it goes through a concave lens and then a convex lens. Our computational domain in (x, y) space is again the square $[-1, 1] \times [-1, 1]$. A convex lens is defined by defining the speed of the wave, $c(x, y)$, as

$$c(x, y) = 1 \quad \text{if} \quad d > 1$$

$$c(x, y) = 1/2 + 1/2(1/2 - 1/2\cos(\pi d)) \quad \text{if} \quad d < 1$$

d is defined as $d = (x/0.8)^2 + (y/0.3)^2$. Note that $c(x, y)$ is smooth up to the first derivative. We define a concave lens by defining $c(x, y)$ to be

$$c(x, y) = 1 \quad \text{if} \quad D > 1$$

$$c(x, y) = 1/2 + 1/2(1/2 - 1/2\cos(\pi x/s(y))) \quad \text{if} \quad D < 1$$

D is defined as $D = -(x/0.8)^2 + (y/0.3)^2$ and $s(y) = 0.3\sqrt{1 + (y/0.3)^2}$. For boundary conditions we specify the phase at the side $y = -1$

$$\phi(x, -1, t) = tH(t).$$

The contour plot of the phase as it goes through the concave lens is shown in figure seven. As expected the plane wave bends outward. The first term in the expansion of the amplitude , v_0, is calculated and is shown in the next figure. The amplitude is calculated with the numerical algorithm for the conservative variables and then is transformed to obtain v_0.

The contour plot of the phase for the convex lens is shown in the next figure. The convex lens bends the plane wave towards the y axis and the phase becomes non-unique there. Our numerical solution shows the viscosity solution of ϕ and a shock is formed on the y axis starting at the focal point. We are able to calculate the hidden solution using the special algorithm that was described before. The amplitude of the wave, v_0, is shown in the next figure. The calculated v_0 is smooth up to the focal point and it becomes singular on the shock.

7 Conclusion

A general procedure is developed to solve the partial differential equations defining the coefficients of the classic high frequency asymptotic expansion of the scalar wave equation. High order Godunov-ENO schemes are used to solve the eikonal equation for the phase of the solution. The expansion coefficients for the amplitude of the solution are recast in conservative variables and upwind finite difference methods are developed to solve the resulting hyperbolic system. The algorithm can accurately represent the discontinuities in the coefficient.

A new numerical procedure is devised to calculate multivalued solutions of the eikonal equation that are not known *a priori* to be multivalued. The procedure relies on the realization that a shock type discontinuity in the gradient of the phase is a branch surface and using the continuity of the phase across the branch surface to define the boundary conditions for the hidden solution.

In principle a multivalued phase has a multivalued expansion for the amplitude coefficients. The necessary conditions for connecting the multivalued expansions of the amplitude across the branch surfaces are not as simple as the boundary conditions for the phase. The necessary boundary conditions in the branch surfaces due to singularities are *a priori* unknown to us. The same problem is manifested in the solution of the v_1 and higher order terms. The forcing term for w_n contains second order derivatives of the previous term which in general are singular on the branch surfaces. These questions need to be investigated further.

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Figure Captions

Figure 1. Reflection of a 1-D Wave

Figure 2. Interaction of two waves in 1-D

Figure 3. Computed solutions of interacting waves

Figure 4. Contour plot of the phase of the in-going wave

Figure 5. Contour plot of the phase of the reflected wave

Figure 6. Interacting waves

Figure 7. Hidden waves

Figure 8. Contour plot of the phase for a shadow

Figure 9. Contour plot of the phase for a concave lens

Figure 10. Amplitude, v_0, for a concave lens

Figure 11. Contour plot of the phase for a convex lens

Figure 12. Amplitude, v_0, for a convex lens

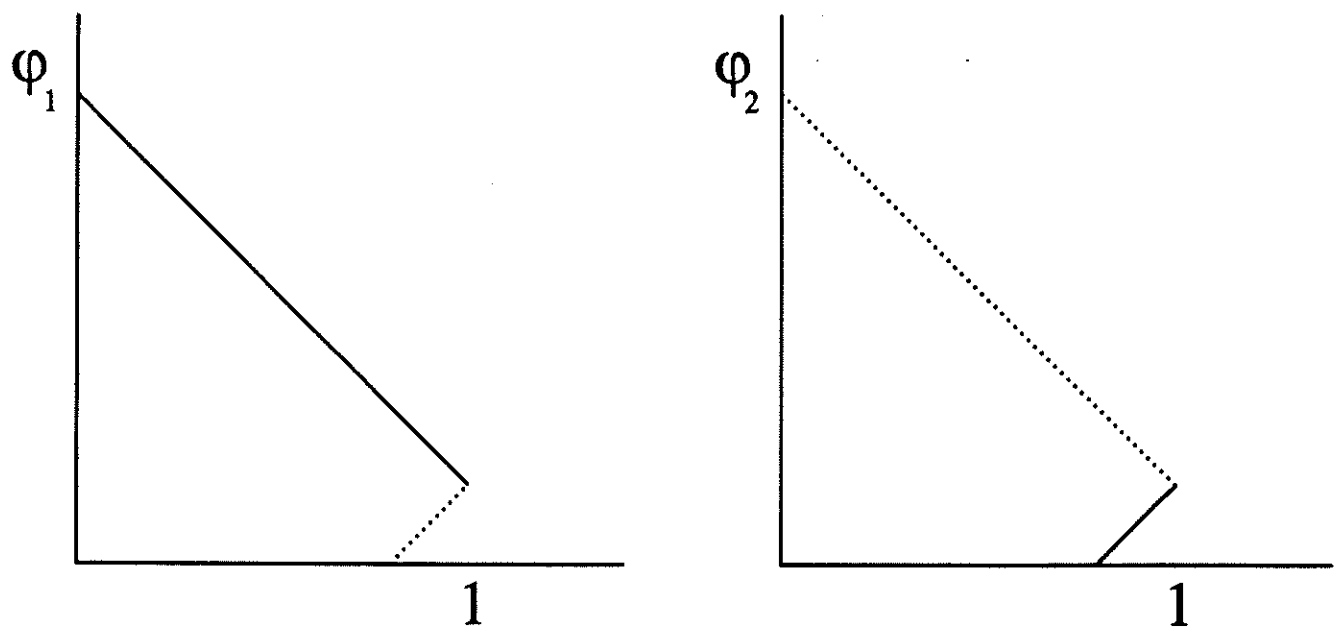


Figure 1

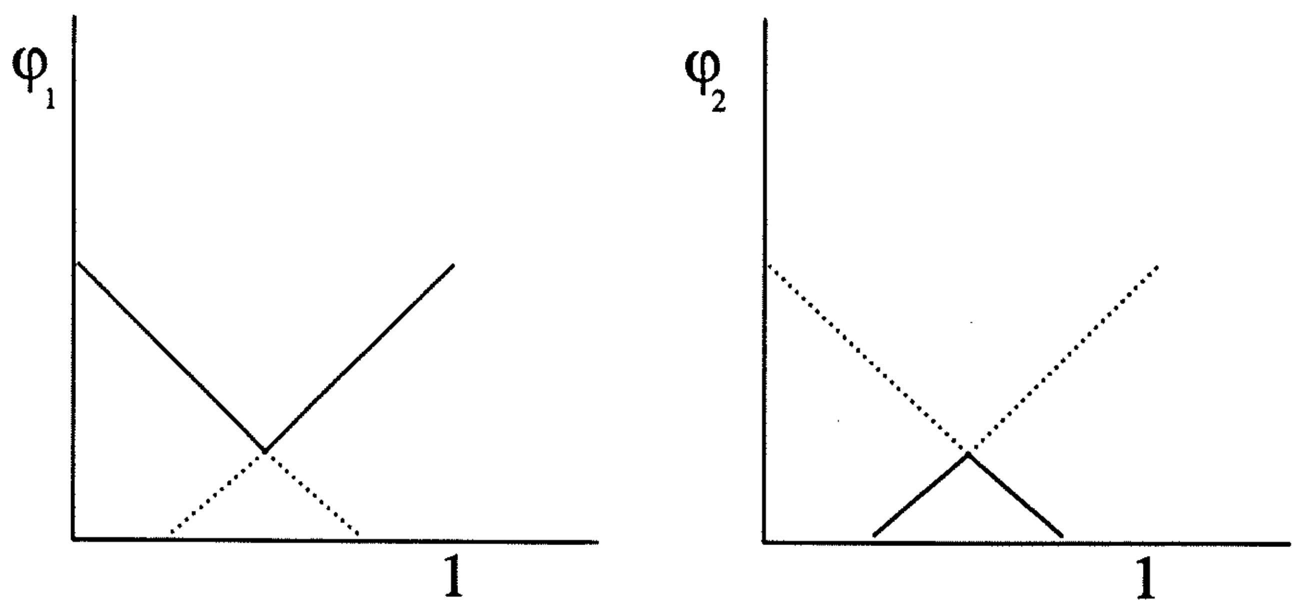


Figure 2

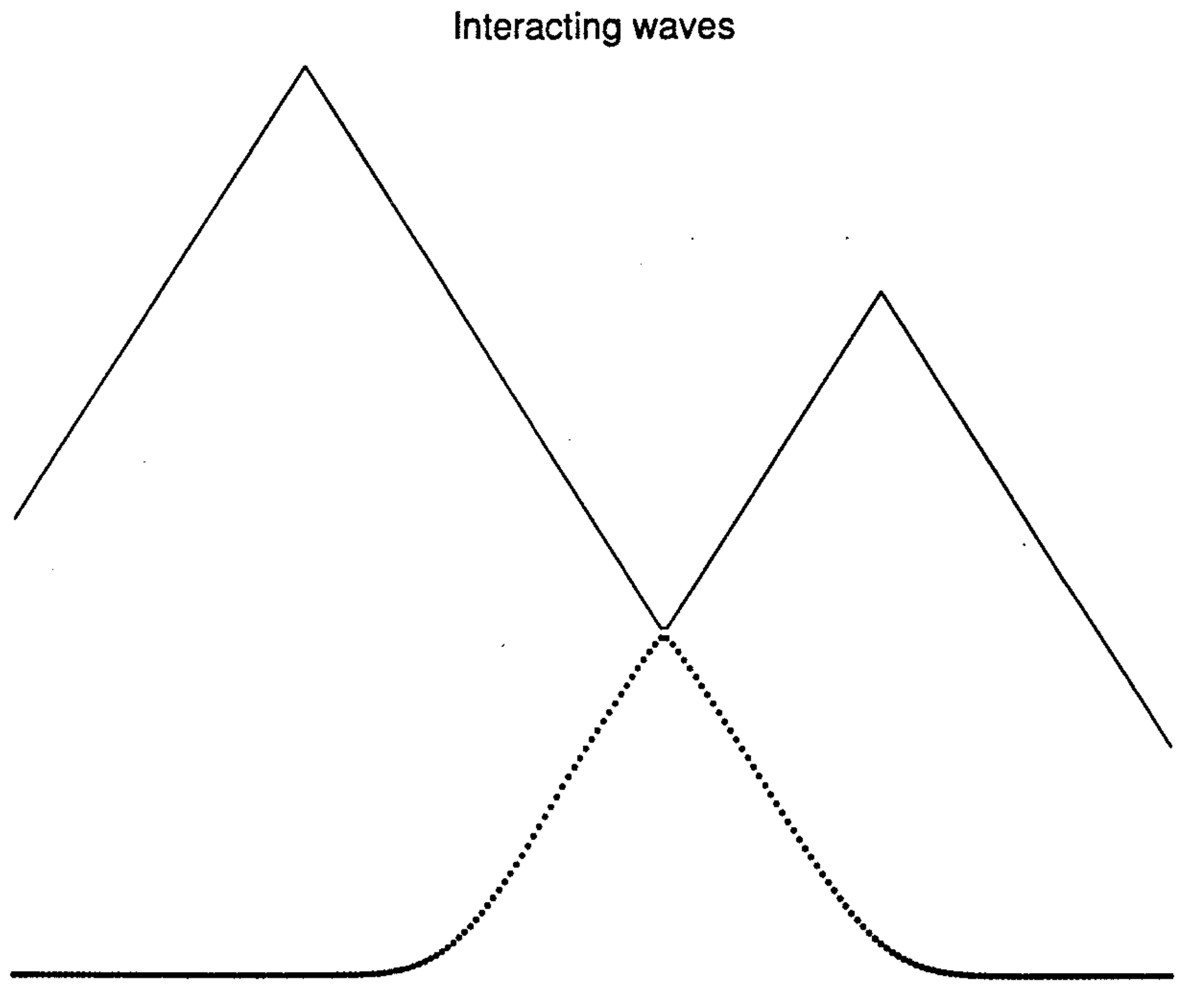
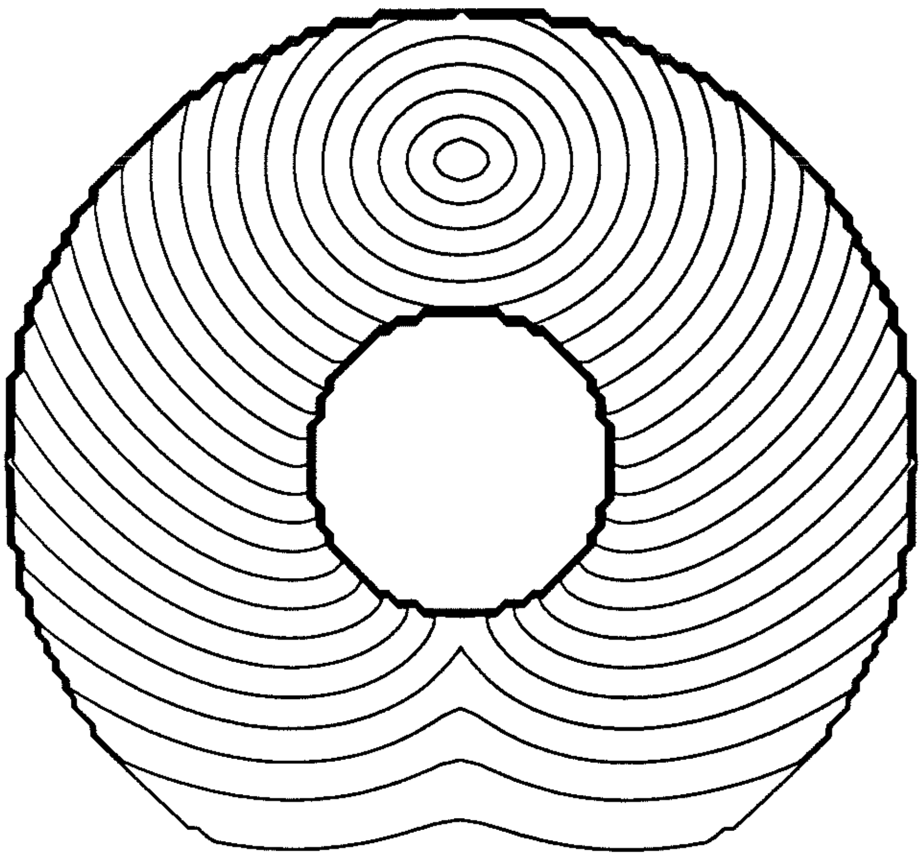


Figure 3

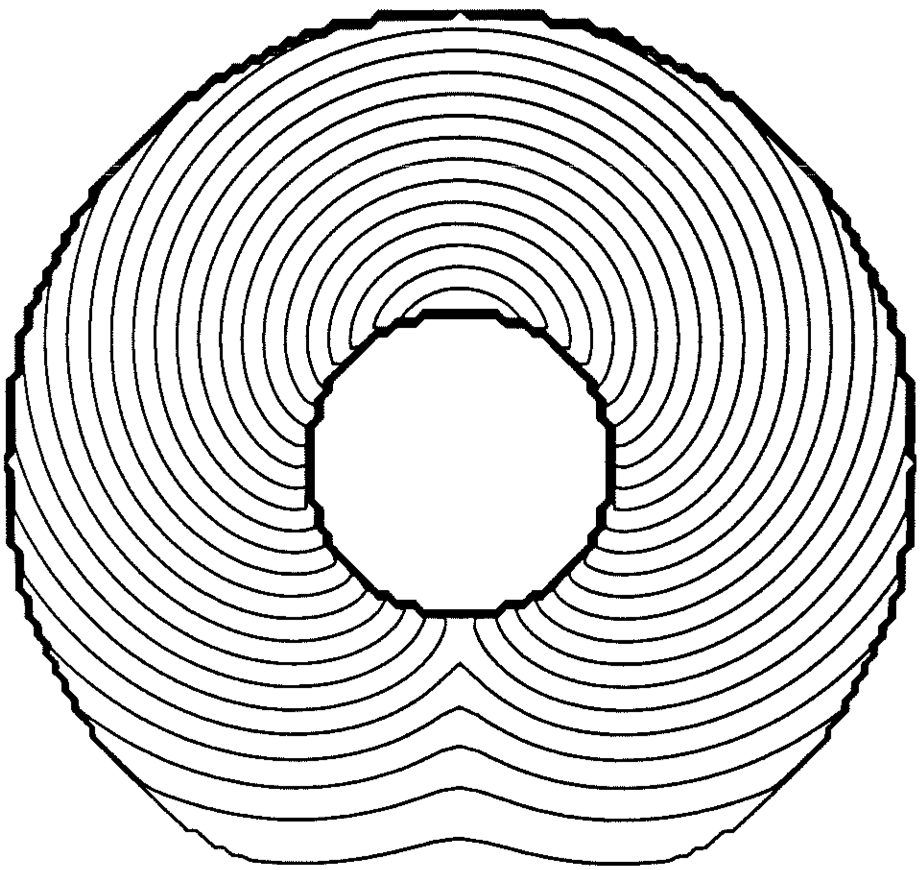
Reflection off a sphere



Ingoing Wave

Figure 4

Reflection off a sphere



Reflected Wave

Figure 5

Interacting Waves

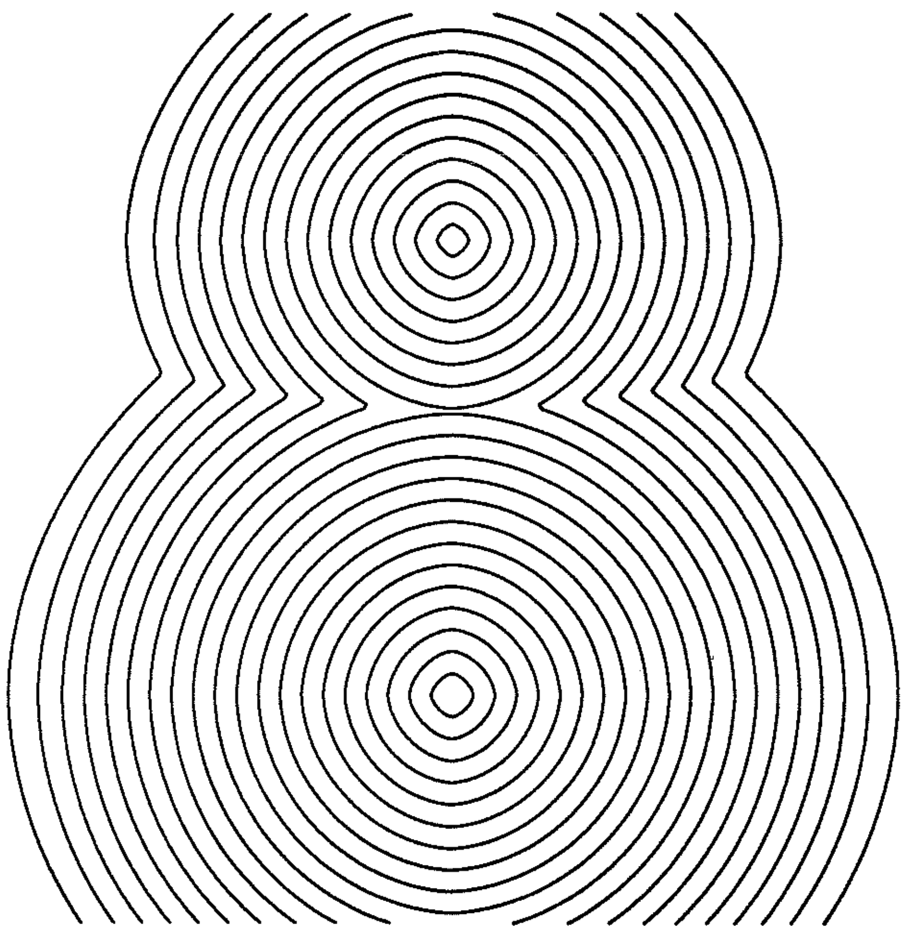


Figure 6

Hidden Waves

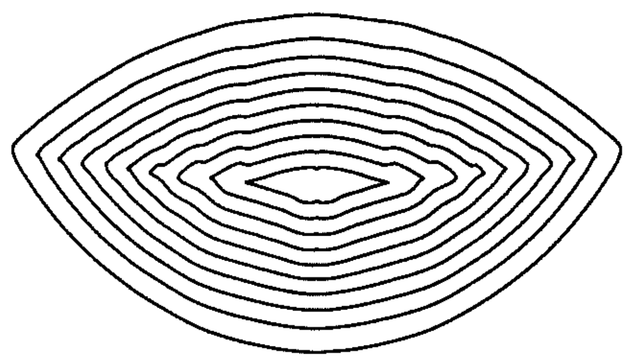


Figure 7

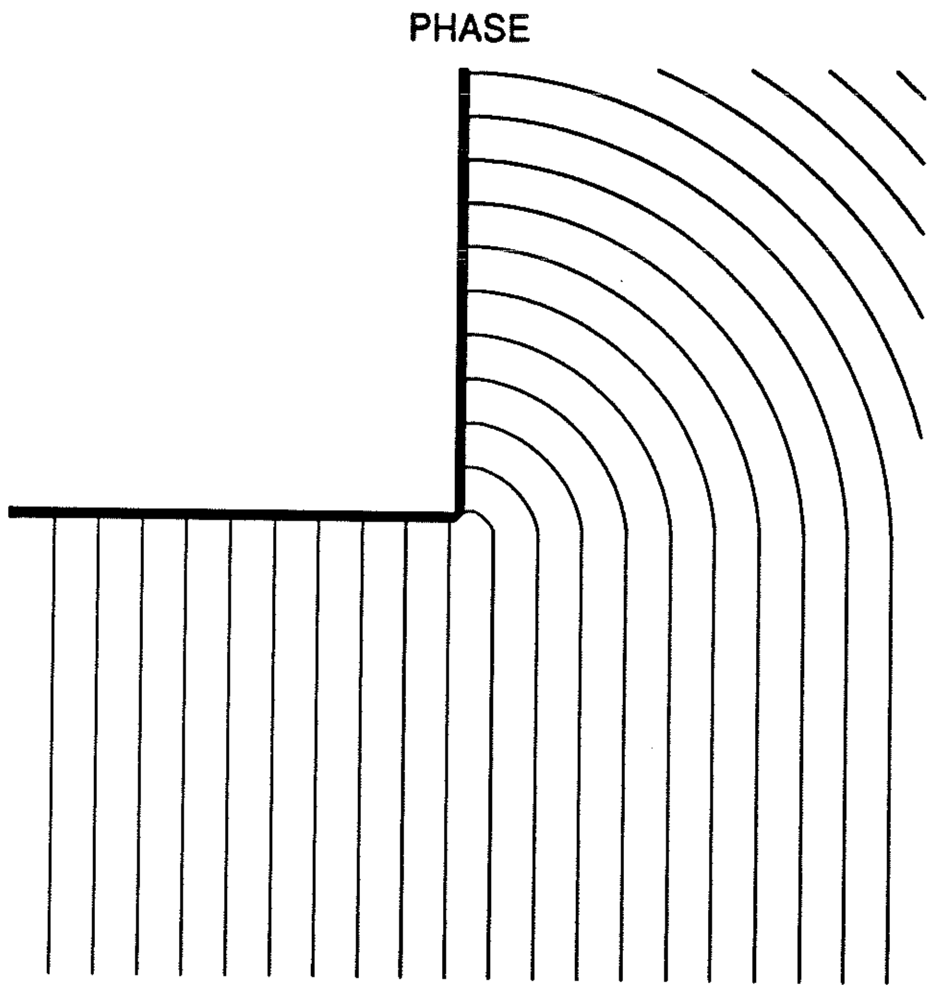


Figure 8

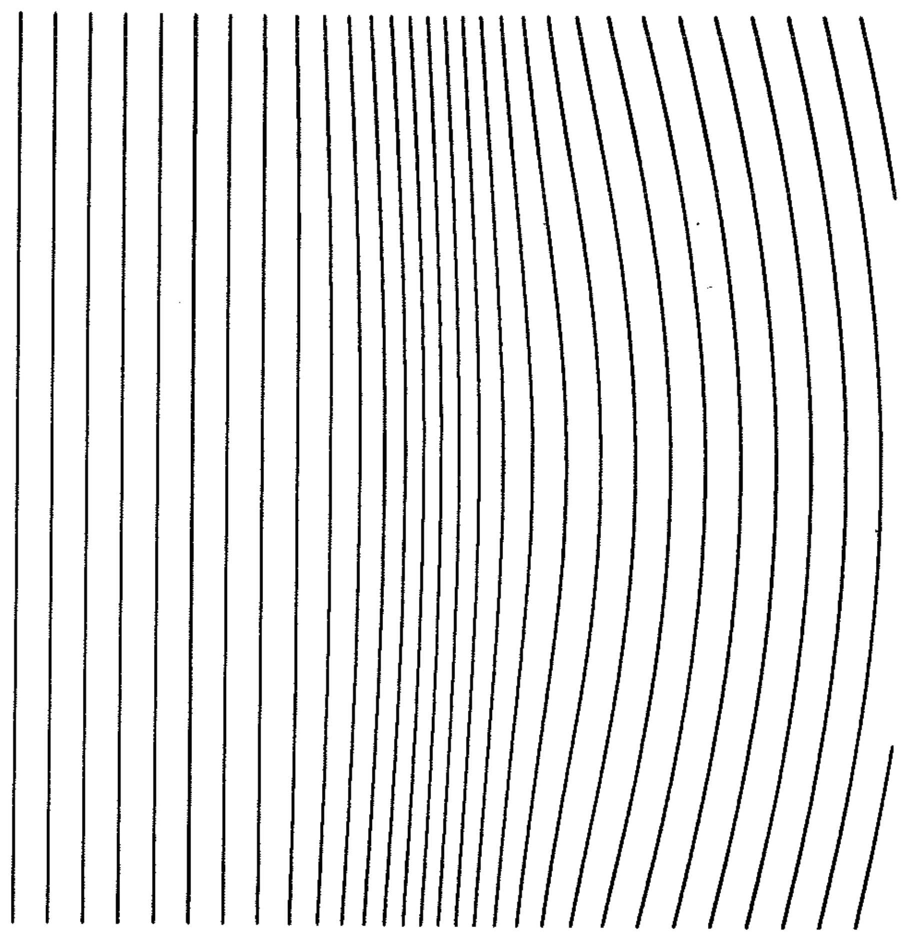


Figure 9

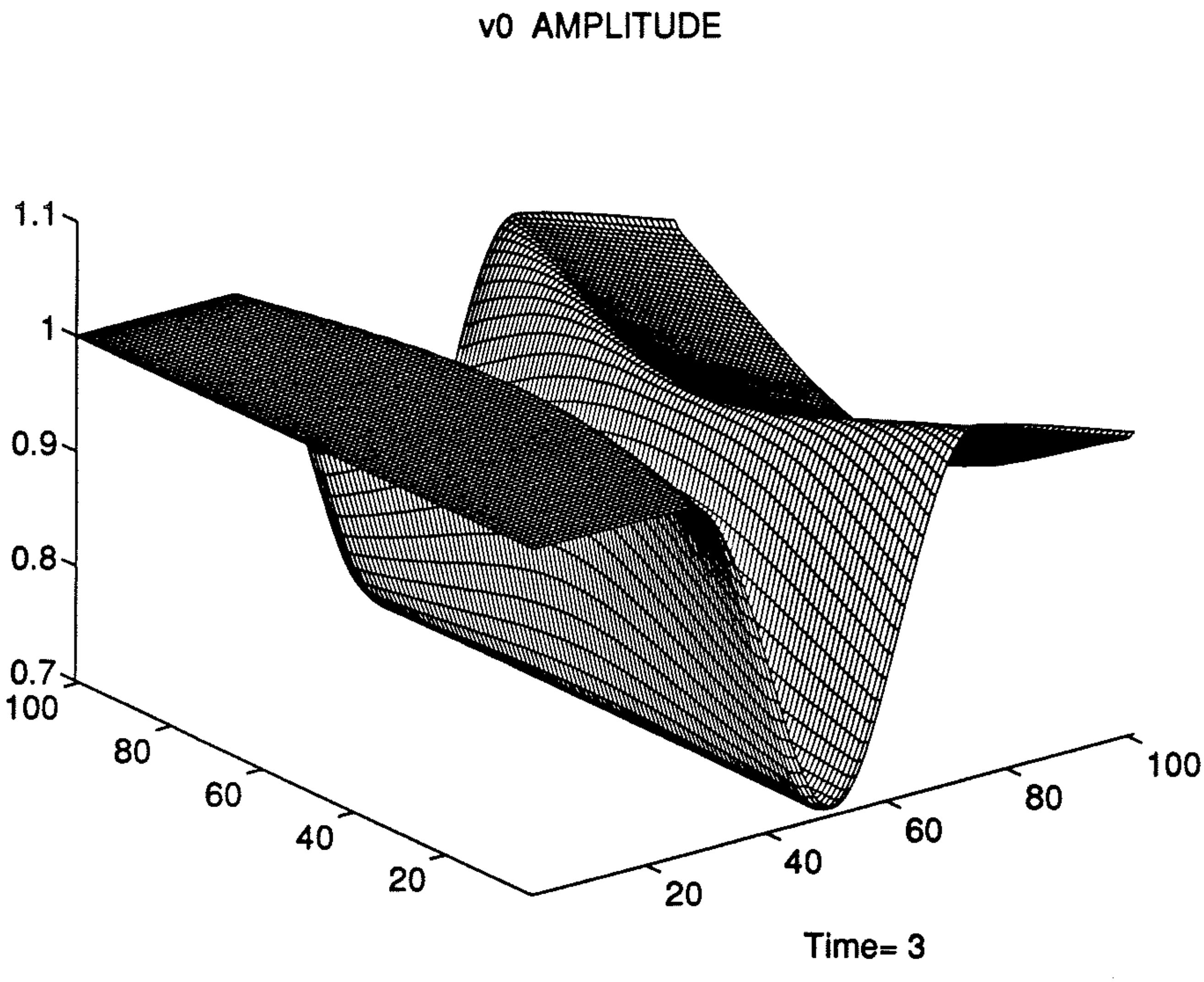


Figure 10

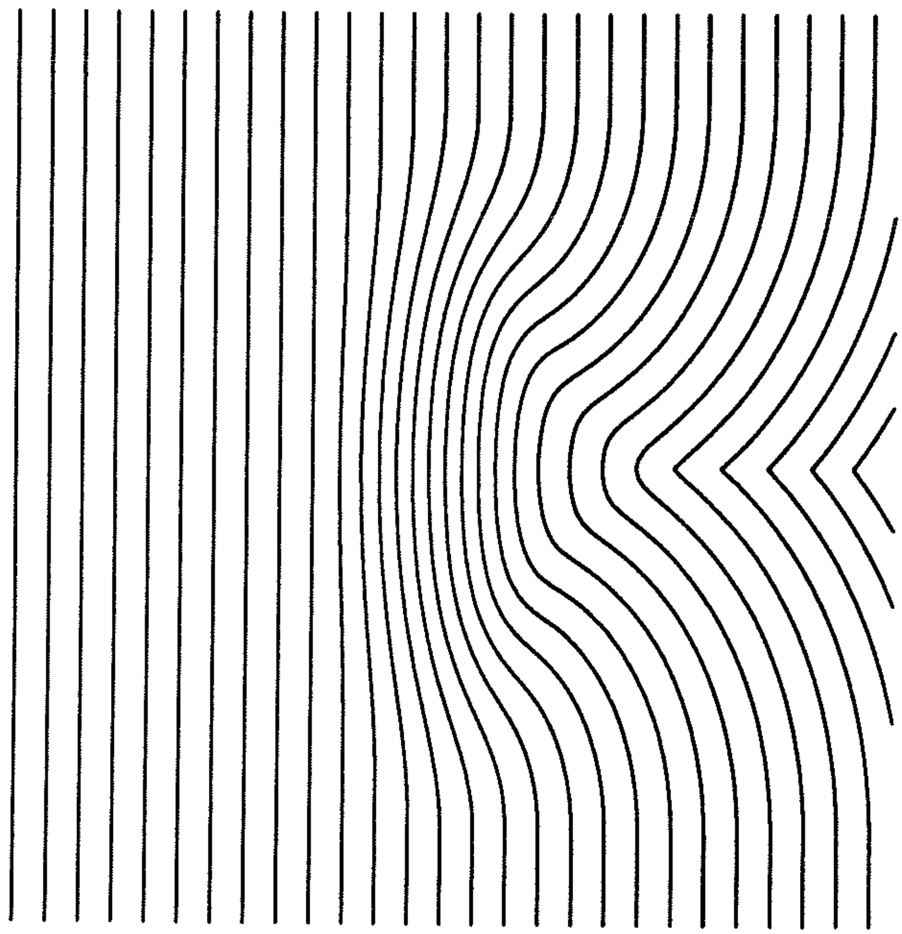


Figure 11

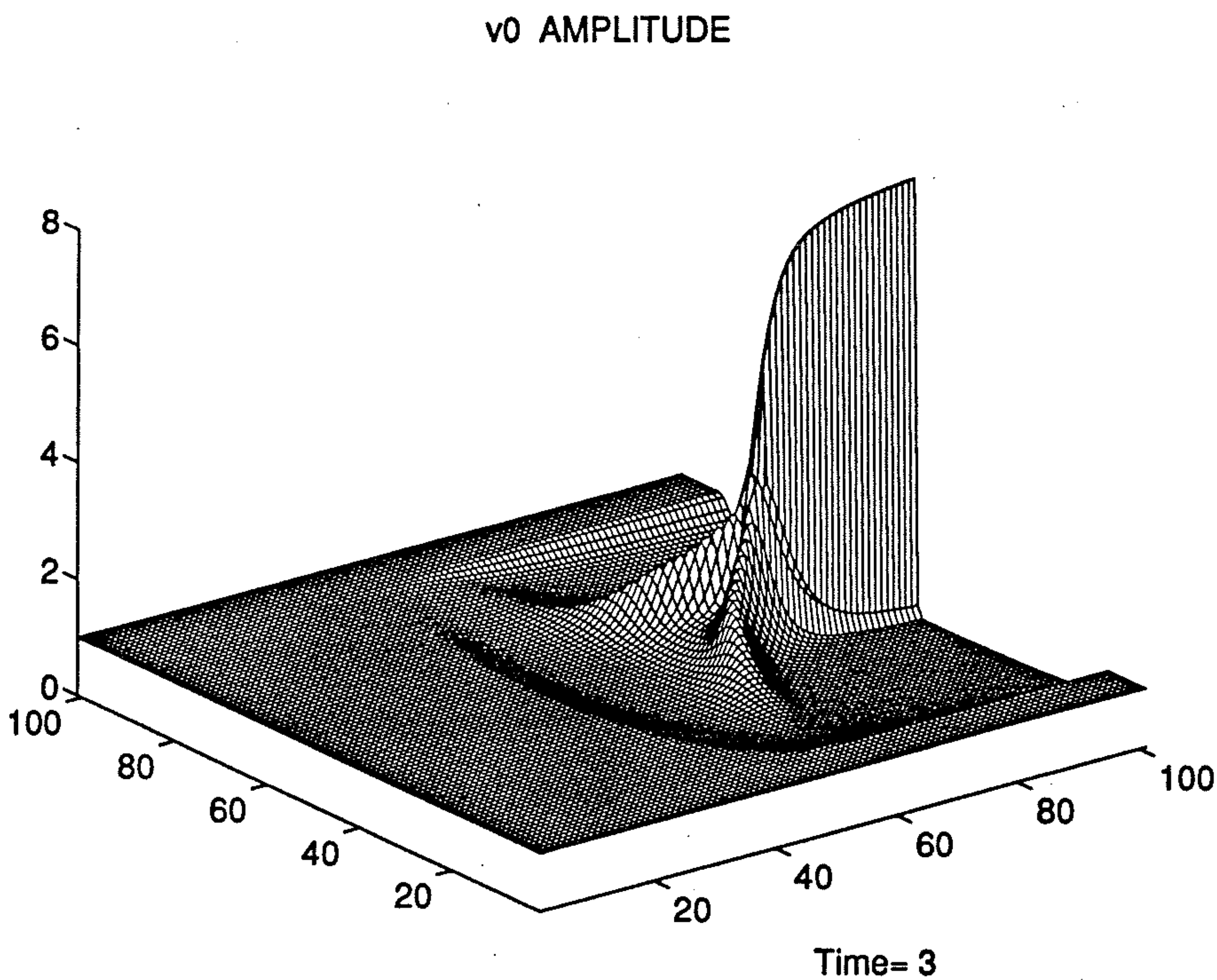


Figure 12