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ABSTRACT

A Vlasov equation for bubbly flow is modified to account for local interactions between bubbles. Fluid equations are deduced in the limit where local interactions cause the system to become locally Maxwellian. The resulting fluid equations are well-posed for sufficiently large temperature. We compute void wave speeds that are found to be in agreement with experiments. In the limit when the temperature is zero, we recover fluid equations previously derived by other investigators. In this limit there are solutions of the equations that blow-up in finite.

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1 Introduction

In a previous paper [1], hence forth denoted Paper I, we derived a Vlasov equation for bubbly flow. The interaction between the bubbles was described in terms of a self-consistent field, the effect of the local interactions was neglected. Local interactions consist of actual collisions (the bubbles touch) and close encounters (their distance is close compared to the average inter-bubble distance). The field during the close encounters is the difference between the actual field and the mean-field. This avoids the double counting of the dipole field. The field responsible for the local interaction decays much faster than the mean field.

In this paper we consider the situation in which these local interactions are important. The effect of collisions relaxes the distribution function to a "local Maxwellian". The system is therefore in local thermodynamic equilibrium and its behavior will be described by a set of fluid equation. As pointed out in Paper I, many investigators have derived fluid equations for bubbly flow by various methods. Quite often, however, they were forced to close the system by physical intuition or *ad-hoc* arguments. Here we shall close our fluid equations with the assumption of local thermodynamic equilibrium.

The plan of the paper is as follows. In Section 2 we discuss the general properties of the collision operator and write a Vlasov-Boltzmann equation for bubbly flow. The fluid equations are derived in Section 3 by taking moments of the Vlasov-Boltzmann equation. The equations are closed by assuming that collisions are dominant and the distribution function is locally Maxwellian. In Section 4 we write the equations in conservation form in one space dimension. We also show that the equations are hyperbolic provided the "temperature" is sufficiently large. By temperature we mean the variance in the bubbles momenta. We conclude this section by comparing a lower bound on the speed of void waves to the experimental results of Biesheuvel & Gorissen. The agreement is favorable. "Frozen" bubble flows are studied in Section 5. Frozen means that the variance of the bubbles momenta is zero. We show that our fluid equations agree with equations derived by Geurst [3], and Pauchon & Smereka [5]. It is also demonstrated that these fluid equations have a similarity solution that blows up in finite time.

2 The Vlasov-Boltzmann Equation

In Paper I, we derived a Vlasov equation for a dilute bubbly fluid that described how the bubbles would move in a self-consistent field; the result was

$$\frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0, \tag{1}$$

where

$$\rho_\ell \boldsymbol{u} = \frac{2}{\tau}\boldsymbol{p} - 9(\nabla\Phi - \frac{1}{3}\boldsymbol{j}), \tag{2}$$

$$\boldsymbol{F} = -\nabla(\boldsymbol{p} \cdot \boldsymbol{u}), \tag{3}$$

$$\Delta\Phi = \nabla \cdot \boldsymbol{j}, \tag{4}$$

$$\boldsymbol{j} = \int \boldsymbol{p} f d\boldsymbol{p}. \tag{5}$$

Here $f(\boldsymbol{x}, \boldsymbol{p}, t)$ is the density function in phase space, $\boldsymbol{x}$ and $\boldsymbol{p}$ are the position and momentum of the bubble, and ρ_ℓ is the liquid density. The velocity, $\boldsymbol{u}$, and the force, $\boldsymbol{F}$, are determined by a self-consistent field which satisfies equation (4). As previously mentioned, Eq. (1) ignores local interactions, which we shall call collisions henceforth. Our goal here is to modify Eq. (1) to include these effects. The resulting equation will be

$$\frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{F} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \left(\frac{\partial f}{\partial t}\right)_c, \tag{6}$$

where the term on the right-hand side accounts for the collisions. The velocity and force given by (2) and (3) are still operative.

Under the assumption that only binary collisions are relevant, the effect of collisions is completely described by a differential cross section,

$$\sigma(\boldsymbol{x}', \boldsymbol{x}_1', \boldsymbol{p}', \boldsymbol{p}_1' \rightarrow \boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{p}, \boldsymbol{p}_1),$$

where $\boldsymbol{x}', \boldsymbol{x}_1', \boldsymbol{p}', \boldsymbol{p}_1'$ are the position and momenta of two bubbles before the collision, and $\boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{p}, \boldsymbol{p}_1$ are the position and momenta of the same bubbles just after the collision.

This function is proportional to the transition probability between the two states. The rate of change of the density function due to collision is given by

$$\left(\frac{\partial f}{\partial t}\right)_c = \int \sigma(\boldsymbol{x}', \boldsymbol{x}_1', \boldsymbol{p}', \boldsymbol{p}_1' \to \boldsymbol{x}, \boldsymbol{x}_1, \boldsymbol{p}, \boldsymbol{p}_1)(f'f_1' - ff_1)d\boldsymbol{x}'d\boldsymbol{x}_1'd\boldsymbol{p}'d\boldsymbol{p}_1'd\boldsymbol{x}_1 d\boldsymbol{p}_1,$$

where $f = f(\boldsymbol{x}, \boldsymbol{p}, t)$, $f_1 = f(\boldsymbol{x}_1, \boldsymbol{p}_1, t)$, $f' = f(\boldsymbol{x}', \boldsymbol{p}', t)$, and $f_1' = f(\boldsymbol{x}_1', \boldsymbol{p}_1', t)$. Since the force between two bubbles is a short-range force then the interaction distance can be considered small compared to the length scale of the variation of $f(\boldsymbol{x}, \boldsymbol{p}, t)$, and only particles at the same position interact. In this case the collisions are described by

$$\left(\frac{\partial f}{\partial t}\right)_c = \int \sigma(\boldsymbol{p}', \boldsymbol{p}_1' \to \boldsymbol{p}, \boldsymbol{p}_1)(f'f_1' - ff_1)d\boldsymbol{p}'d\boldsymbol{p}_1'd\boldsymbol{p}_1, \tag{7}$$

which we recognize as a Boltzmann-type collision operator. We shall refer to (6), with the collision term given by (7), as the Vlasov-Boltzmann equation.

2.1 Collision model

There are two kinds of local interactions experienced by the bubbles, namely close encounters, which we model as the difference between the actual field and the mean field, and collisions, that we model as elastic hard sphere collisions. This means that the local interaction conserves both energy and momentum. A complete description of the collision operator is obtained by determining the differential cross section. This could be done, in principle, by solving the equations of motion of two bubbles in the presence of the local field (including elastic collisions). However this task is formidable, even if a crude approximation for the local field is used, and it is not necessary in the analysis that follows.

A characteristic length associated to the collision process is the "mean free path". When it is large compared to the size of the physical system under consideration, the collisional effects are then negligible and the flow can be considered collisionless. When the mean free path is much smaller than the macroscopic length scale, then the flow is strongly collisional.

These regimes are characterized by the Knudsen number, κ, given by

$$\kappa = \frac{\text{mean free path}}{\text{macroscopic length scale}}. \tag{8}$$

The collision rate is inversely proportional to κ, and therefore equation (7) can be written as

$$\left(\frac{\partial f}{\partial t}\right)_c = \frac{1}{\kappa} Q(f, f), \tag{9}$$

where Q is the Boltzmann collision operator.

As mentioned previously in Paper I, we do not know the Knudsen number for bubbly flow and therefore consider two limiting cases; $\kappa \to 0$ and $\kappa \to \infty$. The latter case represents the collisionless case and was considered in Paper I. In the discussion that follows we consider the case $\kappa \to 0$. In the regime where $\kappa \to 0$, the behavior of the system can be described by fluid dynamic equations [8, 9]. A formal way to deduce these equations is through the Chapman-Enskog expansion (see, for example, Ref. [9], Ch. 7). Before we do this it is convenient to write the Vlasov-Boltzmann equation in the following form:

$$Df = \frac{1}{\kappa} Q(f, f), \tag{10}$$

where

$$D \equiv \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{F} \cdot \frac{\partial}{\partial \boldsymbol{p}}. \tag{11}$$

2.2 Chapman-Enskog Expansion and Detailed Balance

We expand f as follows

$$f = f_0 + \kappa f_1 + \kappa^2 f_2 + \cdots \tag{12}$$

and substitute it into (10). By equating powers of κ we find

$$\kappa^{-1}: \qquad Q(f_0, f_0) = 0, \tag{13}$$

$$\kappa^0: \qquad Df_0 = 2Q(f_0, f_1). \tag{14}$$

A solution of (13) is called a local Maxwellian. It is determined by the principle of *detailed balance* and the collision invariants [8]. The collision process that we consider conserves momentum, energy, and particle number.

Conservation of momentum is expressed in the form

$$\boldsymbol{p}_1' + \boldsymbol{p}' - \boldsymbol{p}_1 - \boldsymbol{p} = 0. \tag{15}$$

The energy associated to a bubble of speed $\boldsymbol{u}$ and momentum $\boldsymbol{p}$ is given by $e = \frac{1}{2}\boldsymbol{p}\cdot\boldsymbol{u}$ (see Paper I), where $\rho_\ell \boldsymbol{u} = (2/\tau)\boldsymbol{p} - \boldsymbol{w}(\boldsymbol{x})$ and $\boldsymbol{w}(\boldsymbol{x})$ is a mean field (described below). Conservation of energy is therefore written as

$$\boldsymbol{p}_1'\cdot\boldsymbol{u}_1' + \boldsymbol{p}'\cdot\boldsymbol{u}' = \boldsymbol{p}_1\cdot\boldsymbol{u}_1 + \boldsymbol{p}\cdot\boldsymbol{u}. \tag{16}$$

From the expression of $\boldsymbol{u}$, this equation can be written as

$$|\boldsymbol{p}_1'|^2 + |\boldsymbol{p}'|^2 - |\boldsymbol{p}_1|^2 - |\boldsymbol{p}|^2 = 0, \tag{17}$$

since $\boldsymbol{w}(\boldsymbol{x})$ is the same for each bubble before and after collisions. Under very general conditions on the scattering cross section, the following relation holds for a local Maxwellian $f(\boldsymbol{x},\boldsymbol{p},t)$:

$$f_0(\boldsymbol{x},\boldsymbol{p}_1')f_0(\boldsymbol{x},\boldsymbol{p}') = f_0(\boldsymbol{x},\boldsymbol{p}_1)f_0(\boldsymbol{x},\boldsymbol{p}), \tag{18}$$

when $\boldsymbol{p},\boldsymbol{p}',\boldsymbol{p}_1$, and $\boldsymbol{p}_1'$ satisfy (15) and (17). Relation (18) is usually referred to as *detailed balance* (see, for example, Ref. [8], pg. 242) . It follows from (18), (15), and (17) that a local Maxwellian has the form

$$f_0(\boldsymbol{x},\boldsymbol{p},t) = \frac{n}{(2\pi\sigma^2)^{3/2}}\exp\left[-\frac{|\boldsymbol{p}-\bar{\boldsymbol{p}}|^2}{2\sigma^2}\right], \tag{19}$$

where $n = n(\boldsymbol{x},t)$, $\bar{\boldsymbol{p}} = \bar{\boldsymbol{p}}(\boldsymbol{x},t)$, and $\sigma = \sigma(\boldsymbol{x},t)$.

To the lowest order in κ, the evolution of n, $\bar{\boldsymbol{p}}$, and σ is obtained by taking moments of the Vlasov-Boltzmann equation. The system is closed using Eq. (19). The equations obtained in this way are the bubbly flow analogue of the Euler equations derived from the Boltzmann equation in gas dynamics.

3 Euler Equations

We rewrite Eq. (1) in nondimensional form using the same variables as in Paper I, and make use of the the fact that the vector field $(\boldsymbol{u},\boldsymbol{F})$ is divergence free. The Vlasov-Boltzmann equation in this form is

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x_\alpha}(f u_\alpha) + \frac{\partial}{\partial p_\alpha}(f F_\alpha) = \frac{1}{\kappa}Q(f,f), \tag{20}$$

where

$$u_\alpha = p_\alpha - w_\alpha, \tag{21}$$

$$F_\alpha = p_\beta \frac{\partial w_\beta}{\partial x_\alpha}, \tag{22}$$

with

$$w_\alpha = \frac{3}{2}\left(3\frac{\partial \Phi}{\partial x_\alpha} - j_\alpha\right), \tag{23}$$

$$\frac{\partial^2 \Phi}{\partial x_\alpha \partial x_\alpha} = \frac{\partial j_\alpha}{\partial x_\alpha}, \tag{24}$$

and

$$j_\alpha = \int p_\alpha f dp. \tag{25}$$

We have used Greek letters for Cartesian components and summation is implied over repeated indices.

We multiply (20) by $1, p_\alpha, p_\alpha p_\alpha$ and integrate over $d\boldsymbol{p}$. The collisions conserve bubble number, momentum, and energy so

$$\int Q(f,f) g d\boldsymbol{p} = 0 \tag{26}$$

if $g = 1, p_\alpha$, or $p_\alpha p_\alpha$. Therefore we obtain the following equations:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_\alpha}(n\bar{u}_\alpha) = 0, \tag{27}$$

$$\frac{\partial j_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}[E_{\alpha\beta} - j_\alpha w_\beta] - j_\beta \frac{\partial w_\beta}{\partial x_\alpha} = 0, \tag{28}$$

and

$$\frac{\partial E_{\alpha\alpha}}{\partial t} + \frac{\partial}{\partial x_\beta}[\Psi_\beta - E_{\alpha\alpha} w_\beta] - 2E_{\alpha\beta}\frac{\partial w_\alpha}{\partial x_\beta} = 0, \tag{29}$$

where

$$n(\boldsymbol{x},t) = \int f(\boldsymbol{x},\boldsymbol{p},t) d\boldsymbol{p}, \tag{30}$$

$$j_\alpha(\boldsymbol{x},t) = \int p_\alpha f(\boldsymbol{x},\boldsymbol{p},t) d\boldsymbol{p}, \tag{31}$$

$$n\bar{u}_\alpha = j_\alpha - n w_\alpha, \tag{32}$$

$$E_{\alpha\beta}(\boldsymbol{x},t) = \int p_\alpha p_\beta f(\boldsymbol{x},\boldsymbol{p},t) d\boldsymbol{p}, \tag{33}$$

and

$$\Psi_\alpha(\boldsymbol{x},t) = \int p_\alpha p_\beta p_\beta f(\boldsymbol{x},\boldsymbol{p},t)d\boldsymbol{p}. \tag{34}$$

We also point out that (28) can be put in conservation form

$$\frac{\partial j^\alpha}{\partial t} + \frac{\partial}{\partial x^\beta} F_{\alpha\beta} = 0, \tag{35}$$

with

$$\begin{aligned} F_{\alpha\beta} &= E_{\alpha\beta} + \frac{9}{2}(\widetilde{J}_\alpha \widetilde{J}_\beta - \widetilde{J}_\alpha j_\beta - j_\alpha \widetilde{J}_\beta) + \frac{3}{2} j_\alpha j_\beta \\ &+ \frac{3}{4}\delta_{\alpha\beta}(j \cdot j - 3\widetilde{\boldsymbol{J}} \cdot \widetilde{\boldsymbol{J}}), \end{aligned}$$

where $\widetilde{\boldsymbol{J}} = \nabla\Phi$ (see (24)).

The constitutive relations are obtained by substituting Eq. (19) for f in (33) and (34). We find

$$E_{\alpha\beta} = n(\bar{p}_\alpha \bar{p}_\beta + \sigma^2 \delta_{\alpha\beta}), \tag{36}$$

and

$$\Psi_\alpha = n(\bar{p}_\beta \bar{p}_\beta + 5\sigma^2)\bar{p}_\alpha, \tag{37}$$

with $n\bar{p}_\alpha = j_\alpha$. Equation (27), (28), and (29) can be written, using (36) and (37), as follows

$$\frac{Dn}{Dt} + n\frac{\partial \bar{u}_\alpha}{\partial x_\alpha} = 0, \tag{38}$$

$$\frac{D\bar{p}_\alpha}{Dt} + \frac{1}{n}\frac{\partial(nT)}{\partial x_\alpha} = \bar{p}_\beta \frac{\partial w_\beta}{\partial x_\alpha}, \tag{39}$$

and

$$\frac{DT}{Dt} + \frac{2}{3}T\frac{\partial \bar{u}_\alpha}{\partial x_\alpha} = 0, \tag{40}$$

where $\bar{u}_\alpha = \bar{p}_\alpha - w_\alpha$, $T = \sigma^2$, and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{u}_\alpha \frac{\partial}{\partial x_\alpha}. \tag{41}$$

Equations (38–40) combined with (21), (23), (24), and (32) constitute a closed system of equations for the unknowns n, $\bar{u}_\alpha$, $\bar{p}_\alpha$, w_α and T. Eq. (38) is a statement of bubble conservation. Eq. (39) describes the time evolution of the bubbles' impulse. Finally, (40) is the energy equation.

4 Properties in One Space Dimension

The fluid equations for bubbly flow in one space dimension in conservation form are

$$\frac{\partial n}{\partial t} + \frac{\partial (n\bar{u})}{\partial x} = 0, \tag{42}$$

$$\frac{\partial}{\partial t}(n\bar{p}) + \frac{\partial}{\partial x}(nT + n\bar{p}\,\bar{u} - \frac{3}{2}n^2\bar{p}^2) = 0 \tag{43}$$

$$\frac{\partial}{\partial t}(n\bar{p}\,\bar{u} + 3nT) + \frac{\partial}{\partial x}[n\bar{u}(\bar{p}^2(1-6n) + 5T)] = 0, \tag{44}$$

and

$$\bar{u} = (1-3n)\bar{p}. \tag{45}$$

An important property of these equations is hyperbolicity, which is deduced by writing (42), (43), and (44) in the form

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0, \tag{46}$$

where $U = (n, \bar{p}, T)^T$ and

$$A = \begin{pmatrix} \bar{u} - 3n\bar{p} & n(1-3n) & 0 \\ T/n - 3\bar{p}^2 & \bar{u} - 3n\bar{p} & 1 \\ -2\bar{p}T & \frac{2}{3}T(1-3n) & \bar{u} \end{pmatrix}. \tag{47}$$

The eigenvalues of A are:

$$\begin{aligned} \lambda_1 &= \bar{p}(1-3n) = \bar{u} \\ \lambda_{2,3} &= \bar{p}(1-6n) \pm \left[\left(\frac{5}{3}T - 3n\bar{p}^2\right)(1-3n)\right]^{\frac{1}{2}}. \end{aligned} \tag{48}$$

The system is hyperbolic if $(\frac{5}{3}T - 3n\bar{p}^2)(1-3n) > 0$, which for dilute bubbly flows, $(n < \frac{1}{3})$, is satisfied if

$$T > T_c \equiv \frac{9}{5}n\bar{p}^2. \tag{49}$$

This result shows that our bubbly flow equations are hyperbolic provided that the "temperature" is sufficiently large. This result is in qualitative agreement with the stability condition found in Paper I, which, using the present notation, can be written as

$$T > T_n = \frac{3\bar{p}^2 n}{1 - 3n}, \tag{50}$$

where we have used a Gaussian distribution as unperturbed solution of equation (1). It is clear that, for a given $\bar{p}$ and n, $T_n > T_c$. This indicates that collisions have a stabilizing effect.

4.1 Comparison with Experiment

The eigenvalues of A given by (48) are interpreted as follows: $\lambda_1 = \bar{u}$ is the speed of the material wave and $\lambda_{2,3}$, if they are real, correspond to propagation of void waves. Experiments that measure void wave speeds using correlation techniques typically measure the fastest wave [2]. In our model the speed of void waves depends on the "temperature", which is not directly available from the experiments; however, it follows form (48) that

$$c = \bar{p}(1 - 6n) \tag{51}$$

is a lower bound on the speed of the fastest void wave. Eq. (51) can be written as

$$c = \frac{(1 - 6n)}{(1 - 3n)}\bar{u}. \tag{52}$$

In order to compare (52) with experimental data, we must recall that our theory has been developed in the zero volumetric flux frame of reference (see discussion in Section 5). In a frame of reference where the volumetric flux is q, Eq. (52) becomes

$$c' = q + \left(\frac{1 - 6n}{1 - 3n}\right)(\bar{u}' - q) \tag{53}$$

where c' and $\bar{u}'$ are the velocities in the new frame of reference.

We shall compare (53) to the experimental results of Biesheuvel & Gorissen [7] who measured void waves of bubbly flows in stagnant water using correlation techniques. They report that the water was stagnant, therefore the volumetric flux is

$$q = \varepsilon\bar{u}, \tag{54}$$

and therefore (53) is

$$c_e = \frac{(1 - 6\varepsilon + 3\varepsilon^2)}{(1 - 3\varepsilon)}\bar{u}, \tag{55}$$

where we have returned to dimensional variables. The speed of the bubbles was also measured by these investigators and was found to obey the relationship

$$\bar{u} = u_\infty (1 - \varepsilon)^2, \tag{56}$$

where u_∞ is the rise speed of a single bubble. Using (56), our expression for (55) becomes

$$\frac{c_e}{u_\infty} = \frac{(1 - 6\varepsilon + 3\varepsilon^2)(1 - \varepsilon)^2}{(1 - 3\varepsilon)}. \tag{57}$$

This expression is plotted along with data from [7] in Figure 1. This curve shows that the expression for c_e lies below all the experimental data in agreement with our theory.

It is remarkable that the experimental data are so close to this curve. It implies that the "temperature" of the bubbly liquid will adjust so that it is close to the transition between well-posed and ill-posed. This suggests that the bubbly mixture is an example of self-organized criticality.

The result of the comparison with the experiment is encouraging, but it lacks a lot of information, such as, for example, an estimate of the "temperature" of the bubble cloud. It would be desirable to make a comparison with other experiments and to check, for example, the range of validity of the kinetic and fluid dynamic models.

5 "Frozen" Bubbly Flows

In this section we will examine the properties of the fluid equations written in Section 4 when $T = 0$. It will be first shown that they are identical to equations obtained by Geurst [3]. We will then prove that these equations have a self-similar blow-up solution.

5.1 Comparison with Previous Work

In Paper I, it was shown that in one space dimension the ambient liquid velocity is

$$v^{\infty} = -\frac{2j}{\rho_{\ell}}, \tag{58}$$

which shows that $v^{\infty} = O(\varepsilon)$, where ε is the void fraction. From (45) (in dimensional form) and (58), we have

$$\varepsilon \bar{u} = -v^{\infty}(1 - 3\varepsilon) \tag{59}$$

that can be written as

$$\varepsilon \bar{u} + (1 - \varepsilon) v^{\infty} = O(\varepsilon^2) \approx 0, \tag{60}$$

where $v^{\infty} = O(\varepsilon)$ has been used. We interpret v^{∞} as the average liquid velocity, $\bar{v}$, and recognize the left-hand side of (60) as the volumetric flux. Therefore we see that, within the approximations used in the derivation of the Vlasov equation, we are in the frame of reference in which the volumetric flux is zero. This is expected in view of the boundary conditions on $\nabla\phi$ at infinity in Paper I ($\nabla\phi = 0$ at infinity).

Pauchon & Smereka [5] have shown that the fluid equations described by Geurst simplify greatly in the zero volumetric flux frame of reference. For massless bubbles they show

$$\begin{aligned} \frac{\partial \varepsilon}{\partial t} &+ \frac{\partial}{\partial x}(gM) = 0, \\ \frac{\partial M}{\partial t} &+ \frac{\partial}{\partial x}(\frac{1}{2} g'(\varepsilon) M^2) = 0, \end{aligned} \tag{61}$$

where

$$M = \frac{\varepsilon(1-\varepsilon)(\bar{u} - \bar{v})}{g(\varepsilon)}, \tag{62}$$

and

$$\frac{1}{g(\varepsilon)} = \frac{\rho_{\ell}}{1-\varepsilon}\left[1 + \frac{m(\varepsilon)}{\varepsilon^2}\right]. \tag{63}$$

Here the prime denotes differentiation with respect to ε; $\bar{u}$ and $\bar{v}$ are the average gas and liquid velocities respectively. $m(\varepsilon)$ is the so-called *exertia* (see

Wallis [4]), and for dilute homogeneous bubbly flows $m(\varepsilon) = \frac{1}{2}\varepsilon$. Equation (58) can be written as

$$\bar{v} = \frac{-2\varepsilon\bar{p}}{\tau\rho_\ell}, \tag{64}$$

which, combined with (59), shows

$$\bar{u} - \bar{v} = \frac{2\bar{p}}{\tau\rho_\ell}(1-2\varepsilon); \tag{65}$$

and it therefore follows that

$$M = \frac{\bar{p}}{\tau} + O(\varepsilon^2), \tag{66}$$

where $m = \frac{1}{2}\varepsilon$ has been used. The expression for g is expanded for small ε to obtain

$$g(\varepsilon) = \frac{1}{\rho_\ell}2\varepsilon(1-3\varepsilon) + O(\varepsilon^3). \tag{67}$$

Therefore, from (66) and (67), neglecting higher order terms, it follows that

$$\begin{aligned}
\frac{\partial\varepsilon}{\partial t} &+ \frac{\partial}{\partial x}\left[\frac{2\varepsilon\bar{p}}{\rho_\ell\tau}(1-3\varepsilon)\right] = 0,\\
\frac{\partial\bar{p}/\tau}{\partial t} &+ \frac{\partial}{\partial x}\left[\left(\frac{\bar{p}}{\tau}\right)^2\frac{(1-6\varepsilon)}{\rho_\ell}\right] = 0,
\end{aligned}$$

which in nondimensional variables are

$$\begin{aligned}
\frac{\partial n}{\partial t} &+ \frac{\partial}{\partial x}n\bar{p}(1-3n) = 0,\\
\frac{\partial\bar{p}}{\partial t} &+ \frac{\partial}{\partial x}\frac{\bar{p}^2}{2}(1-6n) = 0.
\end{aligned} \tag{68}$$

Equation (68) is readily seen to be equivalent to (42) and (43) with $T = 0$.

5.2 Similarity Solution

We look for solutions to system (68) of the form

$$\begin{aligned}
n(x,t) &= \eta(\xi),\\
\bar{p}(x,t) &= (-t)^\alpha\rho(\xi),
\end{aligned} \tag{69}$$

with

$$\xi = \frac{x}{(-t)^{1+\alpha}}$$

and $t < 0$. Equation (69) are substituted into (68) to obtain the ode's

$$\frac{d\eta}{d\xi} = -\alpha g(\eta)\rho/D, \qquad (70)$$

$$\frac{d\rho}{d\xi} = \alpha\rho H/D, \qquad (71)$$

where

$$\begin{aligned} g(\eta) &= \eta(1-3\eta), \\ D &= [H^2 - \frac{1}{2}\rho^2 g g''], \qquad (72) \\ H &= g'(\eta)\rho + (1+\alpha)\xi. \end{aligned}$$

In Appendix A we prove that there is a solution of these ode's which has the following properties when $-1/3 < \alpha < 0$:

1. ρ has a single maximum and decays to zero as $\xi \to \pm\infty$,
2. η is a strictly increasing function bounded between 0 and 1/3.

In particular this means, for a fixed negative value of t, that $n(x,t)$ and $\bar{p}(x,t)$ are positive and bounded functions of x, and therefore they represent an acceptable solution to system (68). The asymptotic behavior of $\rho(\xi)$ and $\eta(\xi)$ is given by

$$\begin{aligned} \rho(\xi) &\sim c_1|\xi|^{\alpha/(1+\alpha)} \\ \frac{\eta(\xi)}{\eta(\xi)-1/3} &\sim c_\pm \exp\left[\frac{c_1\alpha}{1+\alpha}|\xi|^{-1/(1+\alpha)}\right] \end{aligned}$$

where

$$c_\pm = \lim_{\xi\to\pm\infty} \frac{\eta(\xi)}{\eta(\xi)-1/3}.$$

A typical behavior of $\eta(\xi)$ and $\rho(\xi)$ is shown in Figures 2 and 3. Since $\alpha < 0$ it follows that $p(x,t)$ blows up as $t \to 0$.

6 Conclusions

In this paper and its companion [1], we give a kinetic description of incompressible bubbly flow in a liquid. Starting from the Laplace equation for the velocity potential describing the liquid surrounding the bubbles, a Hamiltonian formulation is given describing the motion of N bubbles. The coefficients that appear in the Hamiltonian are very difficult to compute, since they are obtained from the solution of the Laplace equation in a complicated domain. In order to proceed in the treatment an approximation is in order. The major approximation of the model is the so-called point-bubble approximation, according to which the velocity field generated by a bubble is approximated by a dipole field, i.e. the bubble is treated as a point with negligible size. This approximation is formally justified when the distance between two bubbles is large compared to the bubble radius. There is some evidence, however, that the approximation is good also when the distance between the bubbles is comparable to the bubble radius [6].

The derivation of the Vlasov equation is obtained through the Liouville equation and the assumption of total chaos, although we believe it should be possible to derive the Vlasov equation under more general assumptions. The final equation describes the evolution of the density function in phase space for a set of bubbles and is a good approximation only if the total number of bubbles is large enough.

The fluid equations that we derive in this paper are obtained under the assumption of local thermodynamic equilibrium. The agreement with the experiments on the determination of the speed of void waves is quite good. Our model clarifies some difficulties that arose in previous models. It is clear now why some of the fluid models predict complex characteristics and do not give an hyperbolic system of equations. The hyperbolicity of the system depends on the values of the field quantities. In particular, we show that if the variance in the bubbles' velocities is neglected, the effective equations are ill-posed and the zero-temperature case gives rise to a solution that blows-up in finite time. These results are in agreement with previous bubble simulations [6], where the equilibrium state consisted of bubble clusters all with approximately the same speed (and therefore with "low temperature").

It would be interesting to extend our theory to include several other effects that have been neglected so far; one effect is viscosity. Viscosity should have an effect that will take place on a much longer time scale. It should cool down the bubble distribution by releasing energy into the fluid. The overall effect should then be the switching from the stable to the unstable regime and the clustering. It would be interesting to devise a model that predicts this effect, and possibly to compare the results with experiments. It would also be worth exploring the effect of the bubble size distribution. This would be particularly interesting when gravity and liquid viscosity are included since the rise speed of a bubble depends on its size.

Acknowledgement

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Appendix A

Theorem Let us consider the system of ode's (70–71) with the initial condition

$$\begin{cases} \rho(0) = \rho_0, \\ \eta(0) = 1/6, \end{cases} \tag{73}$$

with $\rho_0 > 0$. The system (70,71,73) has a unique solution $(\eta(\xi), \rho(\xi))$, $\xi \in \boldsymbol{R}$, such that

1. $\rho(\xi)$ has a unique maximum and vanishes at infinity

2. $\eta(\xi)$ is a monotonic increasing function with

$$\lim_{\xi\to-\infty} \eta(\xi) = \eta_- \geq 0, \qquad \lim_{\xi\to\infty} \eta(\xi) = \eta_+ \leq 1/3.$$

Proof From the smoothness of the right hand side of system (70–71) it follows that the solution to the initial value problem (70,71, and 73) exists unique in a neighborhood I_0 of $\xi = 0$. Let I denote the largest of such intervals. Let us consider first the interval $I^+ = I \cap [0, +\infty)$. The derivative of the function H defined in (72) is given by

$$D\frac{dH}{d\xi} = \alpha g'(\eta)\rho H + (1+\alpha)H^2 - \frac{1}{2}(1+3\alpha)gg''\rho^2,$$

where D is defined in (72). At $\xi = 0$ one has

$$H = \frac{d\rho}{d\xi} = 0, \quad \frac{dH}{d\xi} > 0, \quad \frac{d\eta}{d\xi} > 0,$$

and

$$\frac{d^2\rho}{d\xi^2} = -\frac{2\alpha(1+3\alpha)}{\rho_0 g(1/6)g''(1/6)} < 0.$$

This means that at $\xi = 0$ the function $\rho(\xi)$ has a maximum. In the interior of a right neighborhood U_0^+ of $\xi = 0$ it is true that

$$\begin{gathered} \frac{d\rho}{d\xi} < 0, \quad \frac{dH}{d\xi} > 0, \quad \frac{d\eta}{d\xi} > 0, \\ \rho > 0, \quad H > 0, \quad \text{and} \quad 1/6 < \eta < 1/3. \end{gathered} \tag{74}$$

Let U^+ denote the largest of such right neighborhood for which conditions (74) hold. We prove that $U^+ = I^+$. Suppose $U^+ \subset I^+$, $U^+ \neq I^+$, and let $\bar{\xi} \in I^+$ denote the sup of U^+. From the expression of H and $dH/d\xi$ it follows that $H > 0$, $dH/d\xi > 0$ at $\xi = \bar{\xi}$, since $dH/d\xi = 0$ would imply $\eta > 1/3$ or $\rho < 0$. From this it follows that $D > 0$ and the last inequalities are a consequence of Equations (70–71). Let

$$C_1 = \sup_{\xi \in U^+} |\alpha| \frac{H}{D}.$$

Then from Equation (71) it follows $\rho(\bar{\xi}) > \rho_0 \exp(-C_1\bar{\xi}) > 0$ and from (70),

$$\frac{d\eta}{d\xi} = -\frac{\alpha\eta(1-3\eta)\rho}{D},$$

it follows $1/6 < \eta(\bar{\xi}) < 1/3$ since $-\alpha\rho/D$ is a positive and bounded function of ξ in the closure $\overline{U^+}$ of U^+ (since $D > 0$ and $\rho < \rho_0$). Therefore for $\xi \in [0, \bar{\xi}]$ all the conditions (74) hold and, by continuity, they can be extended beyond $\bar{\xi}$, i.e. U^+ can not be a proper subset of I^+, therefore $U^+ = I^+$. We proved that in I^+ it is true that $\rho > 0$, $1/6 < \eta < 1/3$. From this it follows that the local existence theorem can be extended indefinitely, and therefore $I^+ = [0, +\infty)$.

A similar analysis can be extended in $I^- = I \cap (-\infty, 0]$, proving that $I = \boldsymbol{R}$. Furthermore a stationary point for ρ must be a maximum, since

$\rho > 0$ and $0 < \eta < 1/3$, therefore $\rho(\xi)$ is a positive, unimodal distribution and $\eta(\xi)$ is monotonic increasing. The bounds on the limits

$$\lim_{\xi \to \pm\infty} \eta(\xi)$$

are imposed by the uniform estimate on the function $\eta(\xi)$ in $\boldsymbol{R}$. A simple asymptotic analysis shows that

$$\lim_{\xi \to \pm\infty} \rho(\xi) = 0.$$

References

[1] G. Russo and P. Smereka, "A kinetic theory for bubbly flow I: collisionless case", submitted to *Physics of Fluids* (1992).

[2] C. Pauchon and S. Banerjee "Interphase momentum interaction effects in the averaged multifield model. Part I: void wave propagation in bubbly flows", *Int. J. Multiphase Flow*, **12**, 559 (1986).

[3] J. A. Geurst, "Virtual mass effects in two-phase flow", *Physica*, **129A**, 239 (1985).

[4] G. B. Wallis, "Interial coupling in two-phase flow: Macroscopic properties of suspensions in an inviscid fluid", *Multiphase Science and Technology*, **5**, 239 (1989)

[5] C. Pauchon and P. Smereka,"Momentum interactions in dispersed flow: An averaging and a variational approach", *Int. J. Multiphase Flow*, **18**, 65 (1992).

[6] P. Smereka, "On the dynamics of bubbles in a periodic box", *J. Fluid Mech.*, **254**, 79 (1993)

[7] A. Biesheuvel and W. C. M. Gorissen, "Void fraction disturbances in a uniform bubbly liquid", *Int. J. Multiphase Flow*, **16**, 211 (1990).

[8] R. L. Liboff, "Introduction to the theory of kinetic equations", Wiley, New York, 1969.

[9] S. Chapman and T. G. Cowling," The mathematical theory of non-uniform gases", Cambridge University Press, Cambridge, 1970.

Figure Captions

Figure 1 The void wave speed for air bubbles rising in stagnant water. The solid line is the lower bound on the wave speed given by Eq.(55) and the circles are experimental date from Ref. [7].

Figure 2 The profile of the similarity solution, computed by the numerical solution of Eqs. (70) and (71). Here η is plotted against ξ.

Figure 3 Same as Figure 2 except here ρ is plotted against ξ.

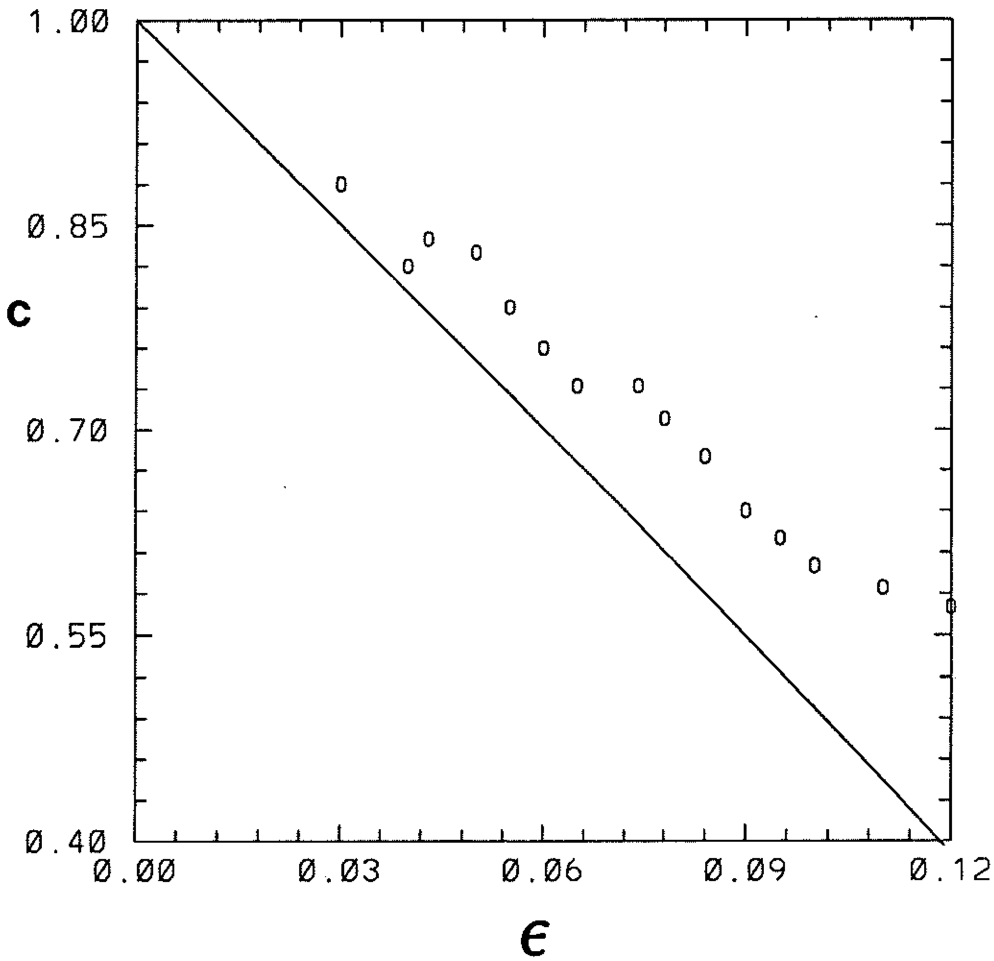

Figure 1
Russo & Smereka II

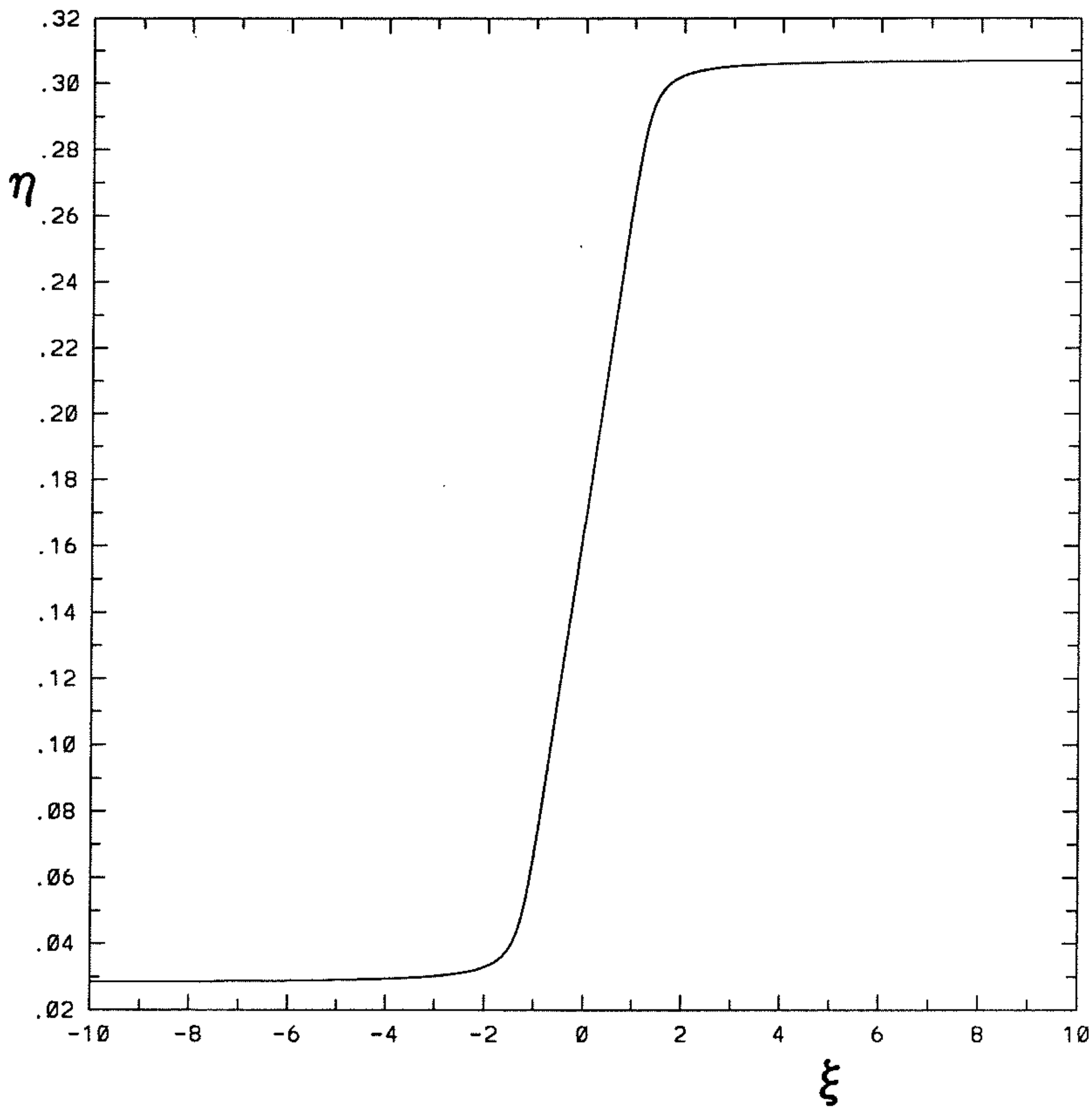

Figure 2
Russo & Smereka II

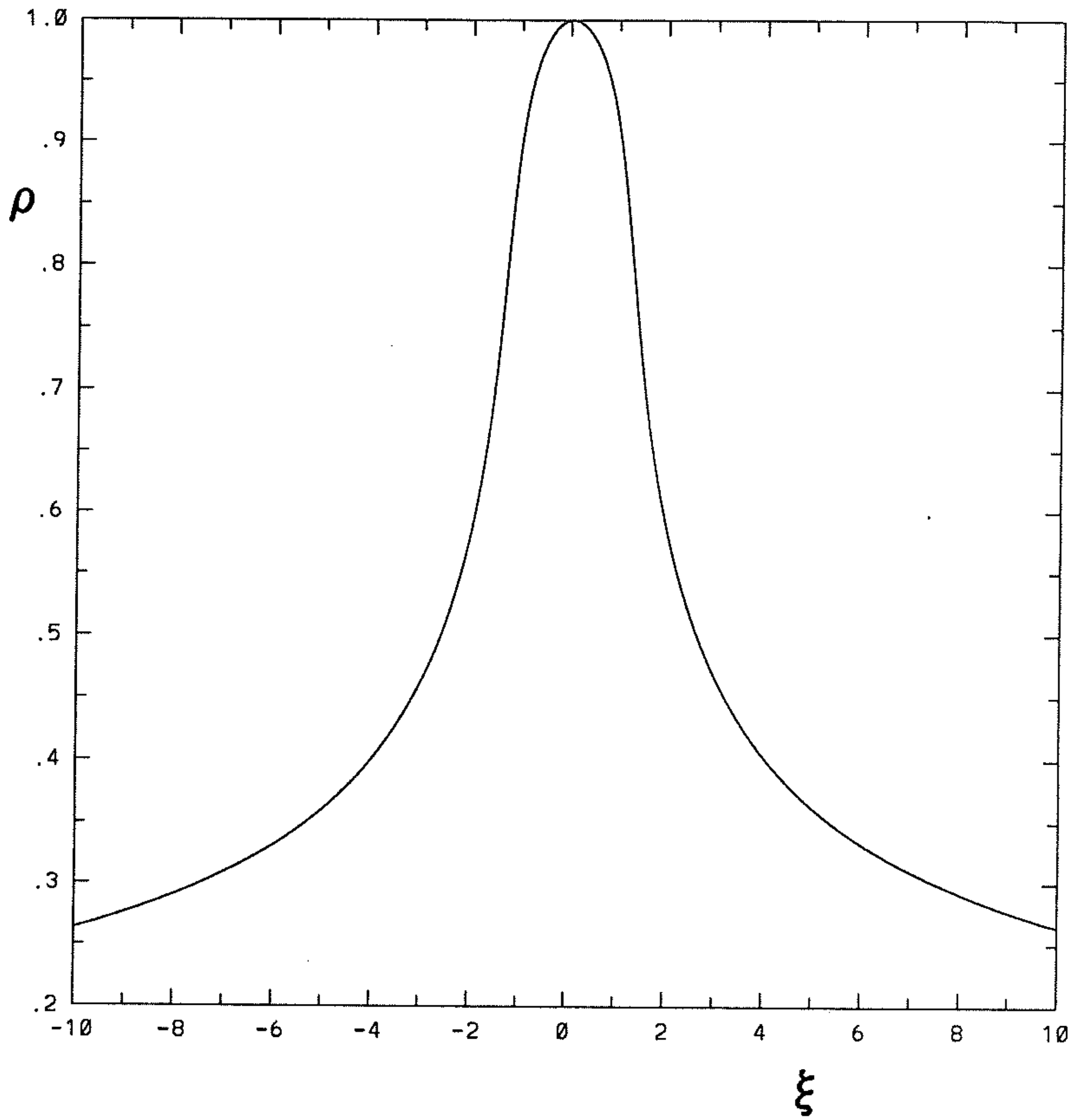

Figure 3
Russo { Smeraka II