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THE APPLICATION OF A DOMAIN DECOMPOSITION METHOD TO SOLVING SINGULAR NEUMANN BOUNDARY PROBLEMS

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Abstract. We consider the problem of solving the very large systems of symmetric and semi-positive definite algebraic equation, arising from the discretization of elliptic problems with Neumann boundary conditions by finite differences or finite elements. The preconditioner developed by J.H. Bramble et al. and based on the domain decomposition method is modified so that the resulted preconditioned conjugate gradient method can be used to solve these singular discrete systems. The condition numbers of the modified methods have been shown still to be $C(1+\ln(H/h))^2$ in R^2. We have tested the several approximate edge matrices, such as probing technique and Fourier approximation, in the numerical experiment. The numerical results have been reported for Neumann boundary condition problems with various coefficients, such as highly varying and jumping coefficients. The results show that the estimation of condition number is fully realized in practice. The modified BPS algorithm is highly parallelizable.

Key Words. domain decomposition, preconditioner, Neumann boundary condition problem, PCG(Preconditioned Conjugate Gradient), finite elements, numerical experiment

AMS subject classifications: 65N20, 65F10.

1. Introduction . The purpose of this paper is to construct a preconditioner for the symmetric and semi-positive definite systems of linear algebraic equations, which result from finite element approximation of elliptic problems with Neumann boundary condition or from the finite difference scheme of pressure in the Navier-Stokes equations. We modify the domain decomposition method which is developed by Bramble, Pasciak and Schalz (BPS) [3]. The modified BPS methods still have the condition number $C(1+\ln(H/h))^2$ for elliptic problems with Neumann boundary condition in R^2. The way of modifying BPS method described in this paper can be used to extend several other parallel algorithms, such as vertex-based method [14] and wirebasket-based method [13, 18, 4, 21], for elliptic problems with various boundary conditions, such as Neumann and mixed boundary conditions. Even though some of the basic idea of the extending way may be found implicitly in [19, 21], explicit expression of the idea and implementation details is needed so that the domain decomposition method can be easily applied to the Neumann boundary problems. Our motivation in modifying BPS method is to design parallel algorithm for Navier Stokes equation. It is well known that we have to solve Laplace equation with 'Neumann like' boundary conditions in each time step, when our scheme for this nonlinear system is based on the velocity and pressure formulation [1]. Without loss of generality, we restrict ourselves to the Neumann boundary value problems in R^2. For this kind problem, the difficulty is how to obtain approximate solution which is orthogonal to the null space (usually consisting of constant functions on the whole domain). The application of an iterative domain decomposition to this singular problem has been discussed in

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Nepomnyaschikh's paper [20]. The iterative method in his paper has convergence rate of geometric progression. A direct parallel algorithm has been developed by Anderson [2] which is only suited to Laplace equation with "Neumann like" boundary condition on the rectangular domain. The extension of general unified theory [15, 14] and vertex space [22] method to this kind singular problem will be discussed in another paper.

There are several difficulties we should overcome when the BPS method is applied to solve Neumann boundary condition problems. In each preconditioner step of BPS algorithm, we first reduce the semi-positive definite problem on the connected domain Ω to Schur complement system on the union of boundaries of non-overlapping sub-domains through solving small size Dirichlet boundary problems on these sub-domains in parallel. Note we let the unknowns on the whole domain boundary be part of unknown of Schur complement system since we would like all subproblems to have the same kind of Dirichlet boundary condition so that the programming can be made easier. This Schur complement system is still singular and the new right hand side still satisfies compatible condition. Then we solve all subproblems obtained by restricting the Schur complement system on all the edges of substructures (exclude of vertex points). All these subproblems have unique solutions. In this paper, we use Bramble [3], Dryja [12, 10] or probing [7, 8, 9] approximate edge matrix. At the same time, we should calculate the approximate solution of symmetric semi-definite problems on the coarse grids with the sub-domains as the elements. This coarse problem results from restricting the Schur complement system on the coarse grid. Note this coarse problem may not be well defined if the right hand side is not properly chosen. In BPS algorithm we have to guarantee that the right hand side satisfies the compatible condition so that we can find the unique coarse problem solution which has zero mean value on the coarse grid. Extend this solution to the edges and the extending result has zero mean value on the whole edges. By extending these two kind solutions of edge subproblems and coarse problem harmonically to the whole domain and summing over all solutions together, we obtain an approximate global 'solution' which is not orthogonal to the null space of the original problem. We subtract the mean value of the approximate global 'solution' from the 'solution' to make its mean value zero. Such modification of BPS remains condition number proportional to $(1+\ln(H/h))^2$ in R^2 which has been proved successful in theory and numerical experiment.

Assume that the connected polygonal domain Ω is the union of disjoint regions Ω_k, which are either quadrilaterals or triangles;

$$\overline{\Omega} = \bigcup_k \overline{\Omega}_k \qquad \text{and} \quad \Omega_i \bigcap \Omega_j = \emptyset \qquad \text{if } i \neq j.$$

Denote $\Gamma = \bigcup_k \partial\Omega_k$. Let Γ_{ij} be the straight line edge of Ω_k with endpoints v_i and v_j which are vertices on $\partial\Omega_k$. As a model problem for the second order elliptic equation with normal derivative boundary condition, we shall consider the Neumann boundary value problem.

$$(1) \qquad Lu = f \qquad \text{in } \Omega \qquad \text{and} \quad \frac{\partial u}{\partial N} = 0 \qquad \text{on } \partial\Omega$$

where

$$Lv = -\sum_{i,j}^{2} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial v}{\partial x_j})$$

$$\frac{\partial v}{\partial N} = \sum_{i,j}^{2} a_{ij}(x) \frac{\partial v}{\partial x_j} \cos(\vec{n}, \vec{e}_i)$$

with a_{ij} uniformly positive definite, bounded and piecewise smooth on Ω , and $\vec{n}$ the external normal direction of boundary of Ω and $\vec{e}_i$ the unit direction of the ith axis.

This paper is organized as follows. In section 2, we describe the modified BPS algorithm for the Neumann boundary condition problem. We show the condition number of this method grows as $C(1 + \ln(H/h)^2)$ in the following section. In the last section we use Bramble, Dryja or probing edge approximate matrix in the modified BPS method and compare their numerical results. The results of numerical experiment show that the theoretical estimation of condition number for this modified method is correct in practice.

2. The modified BPS method for Neumann boundary problems . In the Sobolev space $H^1(\Omega)$, we introduce a symmetric semi-positive definite bilinear form $A(\cdot,\cdot)$ and the inner product $(\cdot,\cdot)$ in $L^2(\Omega)$ space;

$$A(u,v) = \int_\Omega \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx \tag{2}$$

$$(f,v) = \int_\Omega f v dx. \tag{3}$$

Denote

$$KerA = \{u | u \in H^1(\Omega), A(u,v) = 0, \forall v \in H^1(\Omega)\}$$

For any constant C, it is true that $C \in KerA$. Without loss of generality, we assume $KerA = \{c | c \text{ is any constant}\}$ and that the compatible condition is satisfied

$$(f,v) = 0 \qquad \forall v \in KerA,$$

i.e. f has zero mean value.

The weak corresponding form of Neumann boundary value problem (1) is to find $u \in H^1(\Omega)$ and $u \perp KerA$ such that

$$A(u,v) = (f,v) \qquad \forall v \in H^1(\Omega) \ . \tag{4}$$

which has unique solution, under certain conditions on the bilinear form A and f . Usually $u \perp KerA$ means that the mean value of u is zero.

For problem (4), we introduced two levels of triangulations of Ω. One is the coarse triangulation defined by the substructures Ω_i of diameter $0(H)$. The other is the fine triangulation defined by further dividing the substructures into elements of diameter $0(h)$. Assume that these triangulations are shape regular in the sense common to finite element theory; see Ciarlet [11].

Let $V^h(\Omega)$ be the finite element space of continuous, piecewise linear functions defined on the fine grid. Define a null space by using the same notation

$$KerA = \{v_h | v_h \in V^h, A(v,u) = 0, \forall u \in V^h(\Omega)\}$$

It is still true that every constant C belongs to this $KerA$. $\tilde{V}^h(\Omega)$ is the orthogonal complement space of $KerA$ in $V^h(\Omega)$, i.e.

$$\tilde{V}^h(\Omega) = \{u_h | u_h \in V^h(\Omega) \text{ and } \int_\Omega u_h(x)dx = 0\}.$$

The discrete formula of problem (4) is of the form: Find $u_h \in \tilde{V}^h$ such that

$$A(u_h, v_h) = (f, v_h) \qquad \forall v_h \in \tilde{V}^h(\Omega) \tag{5}$$

This problem is symmetric and positive definite in $\tilde{V}_h(\Omega)$. Hence, we can use preconditioned conjugate gradient method for this kind problem in the space $\tilde{V}_h(\Omega)$. As we know, it is very important to make a 'good' choice for preconditioner B in order to construct an efficient PCG algorithm. The bilinear form $B(\cdot,\cdot)$ should have two properties. Firstly, it should be easy to obtain the solution $w_h \in \tilde{V}^h(\Omega)$ of

$$B(w_h, v_h) = (g, v_h), \qquad \text{for all } v_h \in \tilde{V}^h(\Omega) \tag{6}$$

for a given function g. Secondly, the bilinear form $B(\cdot,\cdot)$ should be spectrally equivalent to the bilinear form $A(\cdot,\cdot)$, i.e. there exist positive constants λ_0 and λ_1 such that

$$\lambda_0 B(v_h, v_h) \le A(v_h, v_h) \le \lambda_1 B(v_h, v_h), \qquad \text{for all } v_h \in \tilde{V}^h(\Omega). \tag{7}$$

The first property guarantees that the computational work in each iteration step is small. The second property implies that the condition number κ of corresponding PCG method is less than λ_1/λ_0 . It is well known that the number of steps required to decrease an appropriate norm of the error of a conjugate gradient iteration by a fixed factor is proportional to the condition number κ; see Golub and Van Loan [17]. Therefore, if the condition number κ is a small positive number and slightly depends on the size of grid and substructure, then the resulting algorithm is an efficient method. We are going to construct a preconditioner B so that above two properties could be satisfied in some way. We first introduce an approximate bilinear form $\tilde{A}$ by

$$\tilde{A}_k(u_h, v_h) = \int_{\Omega_k} \sum_{i,j=1}^{2} a_{ij}^k \frac{\partial u_h}{\partial x_i} \frac{\partial v_h}{\partial x_j} dx$$

for each k and then define

$$\tilde{A}(u_h, v_h) = \sum_k \tilde{A}_k(u_h, v_h).$$

Here a_{ij}^k can be chosen as piecewise smooth uniformly positive definite for each Ω_k so that the inequalities

$$C_0 \tilde{A}(u_h, u_h) \le A(u_h, u_h) \le C_1 \tilde{A}(u_h, u_h) \qquad \text{for all } u_h \in V^h(\Omega)$$

are satisfied for positive constants C_0 and C_1 (independent of h, H, Ω_k) and the problem (6) should be easily solvable. From these inequalities, it follows that $Ker\tilde{A} = KerA$. Thus, the problem of finding a preconditioner for A is the same as finding one for $\tilde{A}$.

Denote a subspace

$$V_0^h(\Omega, \Gamma) = \{u_h | u_h \in V^h(\Omega), \qquad u(x) = 0 \text{ on } \Gamma\}$$

which can be represented as the sum of orthogonal subspaces

$$V_0^h(\Omega,\Gamma) = V_0^h(\Omega_1) + V_0^h(\Omega_2) + \cdots + V_0^h(\Omega_n)$$

where

$$V_0^h(\Omega_i) = \{u_h | u_h \in V^h(\Omega),\ u_h = 0,\ x \notin \Omega_i\}.$$

To construct a preconditioner B of A, we first decompose the functions u_h in $\tilde{V}^h(\Omega)$ as $u_h = u_h^I + u_h^B$ where $u_h^I \in V_0^h(\Omega,\Gamma)$ satisfies

$$\tilde{A}_k(u_h^I, v_h) = \tilde{A}_k(u_h, v_h) \qquad \text{for all } v_h \in V_0^h(\Omega_k)$$

for each k , and $u_h^B = u_h$ on Γ and

$$\tilde{A}_k(u_h^B, v_h) = 0 \qquad \text{for all } v_h \in V_0^h(\Omega_k)$$

for all k . We refer to such a function u_h^B as 'discrete $\tilde{A}_k$-harmonic'. It is obvious that such decomposition of function is orthogonal in the $\tilde{A}$-inner product

$$\tilde{A}(u_h, u_h) = \tilde{A}(u_h^I + u_h^B, u_h^I + u_h^B) = \tilde{A}(u_h^I, u_h^I) + \tilde{A}(u_h^B, u_h^B).$$

So we will define $B(\cdot,\cdot)$ by replacing the $\tilde{A}(u_h^B, u_h^B)$ term in above equation. We next further divide the u_h^B as the sum of two functions $u_h^B = u_h^E + u_h^v$ where $u_h^E \in V^h(\Omega)$ is the discrete $\tilde{A}_k$ -harmonic function with zero values at the vertices and $u_h^v \in V^h(\Omega)$ is also the discrete $\tilde{A}_k$-harmonic with linear function values along each edge Γ_{ij} and with the same values as u_h at the vertices. Before defining the preconditioner B , we denote $V_0^h(\Gamma_{ij})$ as the subspace of trace space $V_0^h(\partial\Omega_k)$ whose functions have the supports on the edge Γ_{ij} and introduce an operator $\tilde{l}_0$ defined on each $V_0^h(\Gamma_{ij})$ by

$$< a^{-1}\tilde{l}_0 u_h, v_h >_{\Gamma_{ij}} = < a u_h', v_h' >_{\Gamma_{ij}} \qquad \text{for all } v_h \in V_0^h(\Gamma_{ij})$$

where the prime denotes the differentiation with respect to the arc length s along Γ_{ij}, and the inner product $< \cdot, \cdot >$ on the edge Γ_{ij} is defined as

$$< u_h, v_h >_{\Gamma_{ij}} = \int_{\Gamma_{ij}} u_h v_h ds.$$

By denoting the vertices as v_i or v_j, we could define the preconditioner B by

$$\begin{aligned} B(u_h, v_h) &= \tilde{A}(u_h^I, v_h^I) + \sum_{\Gamma_{ij}} \alpha_{ij} < a^{-1}\tilde{l}_0^{1/2} u_h^E, v_h^E >_{\Gamma_{ij}} \\ &\quad + \sum_{\Gamma_{ij}} \alpha_{ij}(u_h^v(v_i) - u_h^v(v_j))(v_h^v(v_i) - v_h^v(v_j)). \end{aligned} \tag{8}$$

Now we present a detail description of the process used to solve problem (6) for any given function $g \perp KerA$ (the mean value of function g is zero). In fact, solving the problem (6) is equivalent to finding the corresponding decomposition functions u_h^I and u_h^B. The restriction of function u_h^I on Ω_k could be uniquely determined by solving the small size Dirichlet subproblem with zero boundary condition on Ω_k:

$$\tilde{A}_k(u_h^I, v_h) = (g, v_h) \qquad \text{for all } v_h \in V_0^h(\Omega_k). \tag{9}$$

Therefore, u_h^I could be obtained on the whole domain $\overline{\Omega}$ by solving subproblems on each sub-domain. Since all these subproblems are independent of each other, they can be solved in parallel. With u_h^I now known, the problem reduces to finding u_h^B from following equation:

$$
\begin{aligned}
\sum_{\Gamma_{ij}} \alpha_{ij} < a^{-1} \tilde{l}_0^{1/2} u_h^E, w_h^E > \; &+ \; \sum_{\Gamma_{ij}} \alpha_{ij} (u_h^v(v_i) - u_h^v(v_j))(w_h^v(v_i) - w_h^v(v_j)) \\
&= (g, w_h) - \tilde{A}(u_h^I, w_h^I) \\
&= (g, w_h) - \tilde{A}(u_h^I, w_h) \qquad \text{for all } w_h \in V^h(\Omega).
\end{aligned} \tag{10}
$$

Denote

$$
(\tilde{g}, w_h) = (g, w_h) - \tilde{A}(u_h^I, w_h) \qquad \text{for all } w_h \in V^h(\Omega).
$$

Then $\tilde{g}$ is obviously orthogonal to the subspace $KerA$. Note that the value of $(\tilde{g}, w_h)$ only depends on the value of w_h on each Γ_{ij}. Let w_h be in the subspace of $V^h(\Omega)$ whose elements vanish in the interior mesh points of every Ω_k and all vertices. Then the problem (10) decouples into the independent problems of finding $u_h^E \in V_0^h(\Gamma_{ij})$ such that

$$
\alpha_{ij} < a^{-1} \tilde{l}_0^{1/2} u_h^E, w_h >_{\Gamma_{ij}} = (g, w_h) - \tilde{A}(u_h^I, w_h) \qquad \text{for all } w_h \in V_0^h(\Gamma_{ij}) \tag{11}
$$

on each Γ_{ij}. All these subproblems have unique solutions and could be solved concurrently. In practice we use Dryja approximation matrix [12, 10] or probing edge matrix [7, 8, 9] instead of $\tilde{l}_0^{1/2}$ in above problem. Right now only function u_h^v is left unknown. To determine the function u_h^v , we introduce a subspace of $V^h(\Omega)$ consisting of functions which are linear between the endpoints of each edge Γ_{ij} and vanish at the interior mesh points of each Ω_k. It is clear that for each $w_h \in V_0^h$ in this subspace, the corresponding w_h^E should be zero. In this subspace, the problem (10) reduces to the problem

$$
\sum_{\Gamma_{ij}} \alpha_{ij} (u_h^v(v_i) - u_h^v(v_j))(w_h^v(v_i) - w_h^v(v_j)) = (g, w_h) - \tilde{A}(u_h^I, w_h) \tag{12}
$$

which only has u_h^v as the unknown function. Choose a basis $\Phi_1, \Phi_2, \cdots, \Phi_{N_v}$ in this subspace where N_v is the number of vertices on Γ and $\Phi_i(v_j)$ is one if $i = j$ and zero otherwise. Under this basis, problem (12) reduces to a difference equation on the coarse mesh for the elliptic problem with Neumann boundary condition. Therefore, problem (12) has many solutions when the restricted $\tilde{g}$ on the coarse grid satisfies compatible condition. To find an appropriate solution, we look for the unique solution $\tilde{u}_h^v$ which has zero mean value. Then the values of $\tilde{u}_h^v$ at vertices uniquely determine its values on the edges. Note this extension method should change constant function on the whole coarse grid to the constant function on the whole edges Γ. Extend the sum $\tilde{u}_h^B = \tilde{u}_h^v + u_h^E$ into substructures Ω_k so that

$$
\tilde{A}_k(\tilde{u}_h^B, v_h) = 0 \quad \forall v_h \in V_0^h(\Omega_k) \quad \forall k.
$$

By subtracting the mean value c of $\tilde{u}_h = \tilde{u}_h^B + u_h^I$ from $\tilde{u}_h$, we obtain

$$
u_h = \tilde{u}_h - c = \tilde{u}_h^B + u_h^I - c
$$

which has zero mean value. Hence, for any given function g, the solution u_h of (6) is unique and belongs to $\tilde{V}^h$. Then a preconditioner for the problem (5) in $\tilde{V}^h$ has been well defined.

Note that problem (11) and problem (12) are independent. Hence, these two kind subproblems can be solved at the same time. Because all the subproblems have almost the same small size and the computational work for obtaining the solutions to these problem is almost the same small, we have good balance on the working load for each processor when this method is used on the multi-processor computer.

To make the proceed of the inverse of preconditioner B clear, we outline the steps of calculating $u_h \in \tilde{V}^h(\Omega)$ such that

$$B(u_h, v_h) = (g, v_h) \qquad \forall v_h \in \tilde{V}^h(\Omega).$$

where g satisfies compatible condition.

Algorithm:

1. Find $u_h^I \in V_0^h(\Omega, \Gamma)$ such that

$$\tilde{A}_k(u_h^I, v_h) = (g, v_h) \forall v_h \in V_0^h(\Omega_k) \forall k.$$

2. Find u_h^E on each Γ_{ij} by solving one dimensional problem

$$\alpha_{ij} < a^{-1} \tilde{l}_0^{1/2} u_h^E, v_h >_{\Gamma_{ij}} = (g, v_h) - \tilde{A}(u_h^I, v_h) \forall v_h \in V_0^h(\Gamma_{ij}).$$

3. Solve coarse problem to get $\tilde{u}_h^v$ with zero mean value such that equation (12) is true under the same coarse base functions, i.e.

$$\sum_{\Gamma_{ij}} \alpha_{ij} (u_h^v(v_i) - u_h^v(v_j))(w_h^v(v_i) - w_h^v(v_j)) = (g, w_h) - \tilde{A}(u_h^I, w_h).$$

4. Extend the $\tilde{u}_h^v$ to the edges piecewise linearly.
5. Calculate $\tilde{u}_h^B$ such that $\tilde{u}_h^B|_\Gamma = \tilde{u}_h^v + u_h^E$ and

$$\tilde{A}_k(\tilde{u}_h^B, v_h) = 0 \qquad \forall v_h \in V_0^h(\Omega_k) \forall k.$$

6. Compute the mean value c of $u_h^I + \tilde{u}_h^B$ on $\bar{\Omega}$ and let $u_h = u_h^I + \tilde{u}_h^B - c \in \tilde{V}^h(\Omega)$.

3. Theoretical Results. In this section we estimate the condition number of the modified BPS method for Neumann boundary value problem through proving the inequalities in (7). We will use Bramble and others' [3], and Dryja and Widlund, [15]'s approach to prove inequalities in (7).

Since there exist positive constant C_0 and C_1 such that

$$C_0 \tilde{A}(u_h, u_h) \leq A(u_h, u_h) \leq C_1 \tilde{A}(u_h, u_h) \qquad \forall u_h \in \tilde{V}^h(\Omega),$$

it suffices to compare $\tilde{A}(u_h, u_h)$ with $B(u_h, u_h)$. We decompose $u_h \in \tilde{V}^h(\Omega)$ into $u_h = u_h^I + u_h^B$ as the previous section where $u_h^B = \tilde{u}_h^B - c$. Then, we have

$$\tilde{A}(u_h, u_h) = \tilde{A}(u_h^I, u_h^I) + \tilde{A}(u_h^B, u_h^B).$$

From the equality

$$B(u_h, u_h) = \tilde{A}(u_h^I, u_h^I) + B(u_h^B, u_h^B),$$

the proof of the inequalities in (7) will be obtained if the following inequalities are true.

$$\tilde{A}(u_h^B, u_h^B) \leq CB(u_h^B, u_h^B) \tag{13}$$

and

$$B(u_h^B, u_h^B) \leq C(1+\ln(H/h))^2 \tilde{A}(u_h^B, u_h^B). \tag{14}$$

In order to prove these inequalities, we further decompose u_h^B into $u_h^E + u_h^v$ with $u_h^v = \tilde{u}_h^v + c$. Hence, if these inequalities

$$\tilde{A}_k(u_h^B, u_h^B) \leq C \sum_{ij \in \beta_k} \alpha_{ij}(< a^{-1}\tilde{l}_0^{1/2} u_h^E, u_h^E >_{\Gamma_{ij}} + (u_h^v(v_i) - u_h^v(v_j))^2) \tag{15}$$

and

$$\begin{aligned} \sum_{ij \in \beta_k} \alpha_{ij}(< a^{-1}\tilde{l}_0^{1/2} u_h^E, u_h^E >_{\Gamma_{ij}} &+ (u_h^v(v_i) - u_h^v(v_j))^2) \\ &\leq (1+\ln(H/h))^2 \tilde{A}_k(u_h^B, u_h^B), \end{aligned} \tag{16}$$

are satisfied on each substructure, then summing these inequalities with respect to k gives the inequalities in (13) and (14). On each substructure Ω_k, using Bramble and others' [3] results leads

$$\tilde{A}_k(\tilde{u}_h^B, \tilde{u}_h^B) \leq C \sum_{ij \in \beta_k} \alpha_{ij}(< a^{-1}\tilde{l}_0^{1/2} u_h^E, u_h^E >_{\Gamma_{ij}} + (\tilde{u}_h^v(v_i) - \tilde{u}_h^v(v_j))^2)$$

and

$$\begin{aligned} \sum_{ij \in \beta_k} \alpha_{ij}(< a^{-1}\tilde{l}_0^{1/2} u_h^E, u_h^E >_{\Gamma_{ij}} &+ (\tilde{u}_h^v(v_i) - \tilde{u}_h^v(v_j))^2) \\ &\leq (1+\ln(H/h))^2 \tilde{A}_k(\tilde{u}_h^B, \tilde{u}_h^B). \end{aligned}$$

For any constant c, we can prove that

$$\tilde{A}_k(\tilde{u}_h^B + c, \tilde{u}_h^B + c) \leq C \sum_{ij \in \beta_k} \alpha_{ij}(< a^{-1}\tilde{l}_0^{1/2} u_h^E, u_h^E >_{\Gamma_{ij}} + (\tilde{u}_h^v(v_i) + c - (\tilde{u}_h^v(v_j) + c))^2)$$

and

$$\begin{aligned} \sum_{ij \in \beta_k} \alpha_{ij}(< a^{-1}\tilde{l}_0^{1/2} u_h^E, u_h^E >_{\Gamma_{ij}} &+ (\tilde{u}_h^v(v_i) + c - (\tilde{u}_h^v(v_j) + c))^2) \\ &\leq (1+\ln(H/h))^2 \tilde{A}_k(\tilde{u}_h^B + c, \tilde{u}_h^B + c). \end{aligned}$$

Therefore, it is obvious that the inequalities in (15) and (16) follow from these inequalities. Summing the inequalities in (15) and (16) over all sub-domains gives the inequalities in (13) and (14). Hence, we have following estimation on condition number.

THEOREM 3.1. *The above preconditioner B satisfies : for all $u_h \in \tilde{V}^h(\Omega)$*

$$\frac{C_0}{(1+\ln^2(H/h))} B(u_h, u_h) \leq A(u_h, u_h) \leq C_1 B(u_h, u_h) \tag{17}$$

for positive constants C_0 and C_1 which are independent of h and H. Thus, the condition number of corresponding preconditioned conjugate gradient method grows at most like $\kappa \leq C(1+\ln^2(H/h))$ as h tends to zero.

4. Numerical Experiment. In this section, we will replace the preconditioner by some approximate matrices and show numerical results of such modified BPS algorithms for a scalar, seconder order, self-adjoint elliptic equation with Neumann boundary conditions on square domain $\Omega = (0,1) \times (0,1)$.

$$\begin{cases} Lu = -\nabla \cdot (\alpha(x,y)\nabla u(x,y)) = f(x,y) & \text{in } \Omega \\ \frac{\partial u}{\partial N} = 0 & \text{on } \partial\Omega. \end{cases} \tag{18}$$

As we know, the stiffness matrix of edge problem (11) could be spectrally approximated by Bramble matrix [3], by Dryja matrix [12, 10] , by Golub and Mayers matrix [16], by Chan's exact matrix [5], or by probing matrix [6, 7, 8, 9]. All these spectral approximation matrices except probing matrix are only suitable to piecewise constant coefficient elliptic problems. Extending these approximate method to variable coefficient elliptic problems needs to multiply these matrices by the square root of diagonal matrix of stiffness matrix on the right and left. The two approximation edge matrices we consider here are Bramble matrix and probing matrix.

If we have $n-1$ nodes on one edge E_{ij}, Bramble matrix corresponding to the edge stiffness matrix resulted from Laplace equation could be written as

$$\tilde{S}_{E_{ij}} = \tilde{D}WDW^{-1}\tilde{D} \qquad \text{where } W_{st} = \sqrt{\frac{2}{n}}\sin(\frac{st\pi}{n}) \tag{19}$$

and where D is a diagonal matrix with entries

$$D_{ss} = \sqrt{\frac{(2 - 2cos(\pi s/(n)))(4 + 2cos(\pi s/(n))}{6}}.$$

Here positive diagonal matrix $\tilde{D}$ is determined by the square root of the diagonal elements of the stiffness matrix resulted from discrete problem (5) on the fine grid. Note that W is a discrete sin transform, and so $\tilde{S}_{E_{ij}}$ can be inverted in $0(n \ln n)$ operations.

In the probing method, we assume that the restriction of Schur complement on each edge E_{ij} can be approximately represented by a tridiagonal matrix, since the coupling value between two nodes on the edge will quickly decay to zero when the distance of the two nodes on the edge increases. On the other hand, it is well known that the multiplication of capacitance matrix with vector can be easily implemented. However, it is very expensive computational work to find each entries of Schur complement matrix. Hence, we construct an approximate symmetric tridiagonal matrix for the restriction on the edge of Schur complement through multiplying the stiffness matrix (restriction of capacitance matrix on the edge E_{ij}) with zero extension vectors of two vectors $(1,0,1,0,\cdots)^T$ and $(0,1,0,1,\cdots)^T$. The following matrix multiplication gives us an intuitive idea how the probing method works:

$$\begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & b_3 & \\ & & b_3 & b_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ b_1 + b_2 & a_2 \\ a_3 & b_2 + b_3 \\ b_3 + b_4 & a_4 \\ \vdots & \vdots \end{pmatrix}.$$

$\alpha = 300$	$\alpha = 10^{-4}$	$\alpha = 31400$	$\alpha = 5$
$\alpha = 0.05$	$\alpha = 6$	$\alpha = 0.07$	$\alpha = 2700$
$\alpha = 10^{6}$	$\alpha = 0.1$	$\alpha = 200$	$\alpha = 9$
$\alpha = 1$	$\alpha = 6000$	$\alpha = 4$	$\alpha = 140000$

FIG. 1. *Discontinuous Coefficient Function* $\alpha(x, y)$

The detail description of probing technique can be found in [8]. It is well known that the computational work for inverting the symmetric tridiagonal matrix with size $n \times n$ is about $0(n)$ or $0(n \ln n)$ when we use forward and backward substitution method.

The problem (12) on the coarse grid can be simply replaced by the original problems (5) discretized on the coarse grid. Since the problem is on the coarse grid, the number of unknowns is much less than that of unknowns on fine grid. Hence, it will not need much computational work to get the approximate solution with zero mean value on the coarse grid for problem (5).

We use the modified BPS method for this problem with Bramble matrix or probing matrix as approximation to the edge stiffness matrix. These two modified BPS algorithms will be denoted as BPS and P-BPS methods respectively. In the following tables we list the condition number κ calculated by approximate method and the number of iterations required to deduce the 2-norm of the residue by a factor of 10^{-5} for various grid size $h = 1/N$ and sub-domain sizes $H = 1/N_c$. The initial guess was chosen to be zero vector, and the right hand side was chosen to have zero mean value.

Example 1. Let $\alpha(x, y) \equiv 1$ on the whole domain Ω. The operator L becomes Laplace operator $-\triangle$. In table 1, we summarize the numerical results of the modified BPS algorithms for this simple problem. We consider these results as the benchmark of these modified BPS algorithms. From table 1, we notice that BPS algorithm and P-BPS algorithm have almost same condition numbers for various sizes of fine and coarse grids. This result shows the probing edge matrices are as good as other approximate edge matrices.

TABLE 1
The Modified BPS Algorithms for Laplace Operator $-\triangle$

Fine Grid	Coarse Grid	Rate	BPS		P-BPS	
$N \times N$	$N_c \times N_c$	H/h	Iter.	κ	Iter.	κ
32×32	2×2	16	19	18.05	16	12.16
32×32	4×4	8	17	12.15	14	8.00
32×32	8×8	4	14	7.22	12	5.46
32×32	16×16	2	11	4.02	9	3.09
64×64	2×2	32	21	24.49	19	17.91
64×64	4×4	16	19	17.99	16	12.34
64×64	8×8	8	17	12.36	14	8.42
64×64	16×16	4	14	7.23	12	5.44
64×64	32×32	2	11	4.08	10	3.11
128×128	2×2	64	24	32.79	21	33.69
128×128	4×4	32	21	24.28	19	17.83
128×128	8×8	16	19	17.91	16	11.89
128×128	16×16	8	16	11.95	14	8.49
128×128	32×32	4	14	7.09	12	5.46
128×128	64×64	2	11	4.05	10	3.12
256×256	2×2	128	24	41.09	28	59.84
256×256	4×4	64	23	31.64	22	31.95
256×256	8×8	32	21	23.94	18	17.69
256×256	16×16	16	18	17.45	16	12.22
256×256	32×32	8	16	11.89	14	8.40
256×256	64×64	4	14	7.09	12	5.47
256×256	128×128	2	11	4.07	10	3.12

Example 2. In this example, we choose variable coefficient $\alpha(x,y) = 1 + 10(x^2 + y^2)$ in Neumann boundary problem and show the numerical results in Table 2.

TABLE 2
The Modified BPS Algorithms for $\alpha(x,y) = 1 + 10(x^2 + y^2)$

Fine Grid	Coarse Grid	Rate	BPS		P-BPS	
$N \times N$	$N_c \times N_c$	H/h	Iter.	κ	Iter.	κ
32×32	2×2	16	22	22.51	17	14.61
32×32	4×4	8	17	12.18	14	8.31
32×32	8×8	4	15	7.34	12	5.44
32×32	16×16	2	11	4.02	10	3.12
64×64	2×2	32	25	34.38	20	19.41
64×64	4×4	16	20	17.70	16	12.03
64×64	8×8	8	17	12.15	15	8.49
64×64	16×16	4	15	7.33	12	5.48
64×64	32×32	2	11	4.08	10	3.13
128×128	2×2	64	24	35.95	23	32.95
128×128	4×4	32	21	23.86	19	17.60
128×128	8×8	16	19	17.45	17	12.27
128×128	16×16	8	16	11.90	14	8.35
128×128	32×32	4	14	7.09	12	5.51
128×128	64×64	2	11	4.05	10	3.12
256×256	2×2	128	26	50.60	28	58.48
256×256	4×4	64	24	32.05	22	32.10
256×256	8×8	32	21	24.30	17	17.30
256×256	16×16	16	18	17.51	16	12.16
256×256	32×32	8	16	11.91	14	8.42
256×256	64×64	4	14	7.13	12	5.46
256×256	128×128	2	11	4.06	10	3.12

Example 3. More difficult problem is considered in this example. Table 3 gives us the numerical results for Neumann boundary problem with highly varying coefficients $\alpha(x,y) = e^{10xy}$. Comparing table 3 with other tables, we found that iteration numbers and condition numbers in table 3 are larger than that of other tables even though condition numbers in table 3 satisfy the estimation $c(1+\ln^2(H/h))$. The reason for this is the highly varying coefficients make the preconditioner worse in a certain way. Since the condition number and iteration number of P-BPS method are less than that of BPS method, probing edge matrix is better than the other scaling Fourier edge approximate matrices. Our numerical experiment have been shown that our choice of $\tilde{D}$ in (25) is much better than $\alpha_{ij}I$ suggested in [3] for this kind problem.

TABLE 3
The Modified BPS Algorithms for $\alpha(x,y) = e^{10xy}$

Fine Grid	Coarse Grid	Rate	BPS		P-BPS	
$N \times N$	$N_c \times N_c$	H/h	Iter.	κ	Iter.	κ
32×32	2×2	16	17	65.20	17	42.83
32×32	4×4	8	17	17.97	14	11.63
32×32	8×8	4	14	8.00	12	5.93
32×32	16×16	2	11	4.14	10	3.21
64×64	2×2	32	22	83.13	19	56.49
64×64	4×4	16	21	24.72	17	16.61
64×64	8×8	8	19	13.30	15	9.11
64×64	16×16	4	15	7.48	12	5.44
64×64	32×32	2	11	4.13	10	3.10
128×128	2×2	64	24	101.22	24	83.61
128×128	4×4	32	22	31.72	17	22.36
128×128	8×8	16	20	18.78	15	12.92
128×128	16×16	8	16	12.07	15	8.72
128×128	32×32	4	15	7.40	12	5.59
128×128	64×64	2	11	4.08	9	3.09
256×256	2×2	128	32	119.78	27	141.88
256×256	4×4	64	25	39.69	22	38.50
256×256	8×8	32	22	25.00	17	18.60
256×256	16×16	16	20	17.69	17	12.28
256×256	32×32	8	17	11.96	15	8.57
256×256	64×64	4	15	7.29	12	5.47
256×256	128×128	2	11	4.04	10	3.12

Example 4. In this last example, the coefficient is a highly discontinuous function which is piecewise constant on the sixteen regions pictured as Fig.1. This kind coefficient was considered in Bramble and others' paper [3]. For this problem, we should divide the domain into sub-domains in such a way so that the discontinuous should only happen along the boundaries of sub-domains. We list the results in table 4 so that they can be compared with the results in Bramble's paper [3]. From results in table 1 and table 4, we notice that the modified BPS methods for different coefficient have almost same condition numbers and iteration numbers even though the coefficients change drastically across the boundaries of sub-domains.

TABLE 4
The Modified BPS Algorithms for Discontinuous Coefficient

Fine Grid	Coarse Grid	Rate	BPS		P-BPS	
$N \times N$	$N_c \times N_c$	H/h	Iter.	κ	Iter.	κ
32×32	4×4	8	15	12.59	14	8.84
32×32	8×8	4	14	7.17	12	5.60
32×32	16×16	2	11	3.99	9	3.04
64×64	4×4	16	18	18.54	16	12.80
64×64	8×8	8	16	11.86	14	8.84
64×64	16×16	4	14	7.15	12	5.43
64×64	32×32	2	11	4.14	10	3.13
128×128	4×4	32	21	24.29	17	18.36
128×128	8×8	16	18	18.24	15	12.72
128×128	16×16	8	16	12.03	13	8.58
128×128	32×32	4	14	7.20	12	5.46
128×128	64×64	2	11	4.10	10	3.12
256×256	4×4	64	23	32.50	22	32.92
256×256	8×8	32	21	24.57	17	17.98
256×256	16×16	16	18	17.44	14	12.05
256×256	32×32	8	16	11.80	14	8.50
256×256	64×64	4	14	7.10	12	5.46
256×256	128×128	2	11	4.06	10	3.12

In summary, we note that the convergence rates of the modified BPS algorithms keep almost same when we change the coefficient from constant to piecewise constant in Neumann boundary problem. This phenomenon shows that the condition number and iteration number of the modified BPS algorithms are independent of constant or piecewise constant coefficient. These numerical results in all tables also demonstrate that the condition numbers of modified BPS methods grow as $C(1+\ln^2(H/h))$ with C bounded by 5 at the most.

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