UCLA
COMPUTATIONAL AND APPLIED MATHEMATICS

Multiresolution Representation of Data, I. Preliminary Report

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June 1993

CAM Report 93-13

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Multiresolution Representation of Data

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This research was supported by grants ONR-N00014-92-J-1890 and NSF-DMS 91-03104 at UCLA and by DOE grant DE-FG02-88ER25053 at the Courant Institute of Mathematical Sciences, NYU.

Abstract.

In this paper we present a general framework for multiresolution representation of data which can be viewed as a generalization of the theory of wavelets. We consider nested sequences of discrete approximations with increasing resolution and show that the set of discrete values at each level of resolution has a multiresolution representation, which consists of the scale-coefficients of all lower levels and the discrete values of the coarsest level in the sequence.

This framework includes discretizations corresponding to unstructured grids in several space-dimensions as well as non-local discretizations, and thus it is general enough to enable us to embed most numerical problems in a multiresolution setting.

1. Introduction

In this paper we present a general framework for multiresolution representation of data, which consists of a sequence of discrete approximations $\{(\mathcal{D}_k, \mathcal{R}_k)\}_{k=0}^{\infty}$ with increasing resolution in k. $\mathcal{D}_k$ is a discretization operator which operates on functions and assigns to them a sequence of discrete values which corresponds to the k-th level of resolution; $\mathcal{R}_k$ is a reconstruction operator which approximates the function from its discrete values of the k-th level of resolution and is required to satisfy

$$\mathcal{D}_k \mathcal{R}_k = I; \tag{1.1}$$

here I is the identity operator in the vector-space of the discrete values.

We show that if the sequence of discretizations $\{\mathcal{D}_k\}$ is nested in the sense that

$$\mathcal{D}_k f = 0 \Rightarrow \mathcal{D}_{k-1} f = 0, \tag{1.2}$$

then there exists an invertible multiresolution transform M such that

$$\mu(\mathcal{D}_k f) = M \cdot \mathcal{D}_k f, \quad \mathcal{D}_k f = M^{-1} \cdot \mu(\mathcal{D}_k f), \tag{1.3a}$$

where

$$\mu(\mathcal{D}_k f) = \begin{pmatrix} d^k(f) \\ \vdots \\ d^1(f) \\ \mathcal{D}_0 f \end{pmatrix} \tag{1.3b}$$

is the multiresolution representation of $\mathcal{D}_k f$, and $d^k(f)$ are the scale-coefficients of the k-th level of resolution. The scale-coefficients $d^k(f)$ are obtained from the prediction error e^k,

$$e^k = \mathcal{D}_k(I - \mathcal{R}_{k-1}\mathcal{D}_{k-1})f \tag{1.4a}$$

by removing the redundancy that is inherent to this set of variables, and is due to the identity

$$\mathcal{D}_{k-1}\mathcal{R}_k e^k = 0, \tag{1.4b}$$

which is a direct consequence of (1.1)-(1.2).

This framework is general enough to include discretizations in unstructured grids in several space-dimensions as well as non-local discretizations (e.g. by global moments); furthermore, it allows for adaptive (data-dependent) reconstruction procedures which are needed to obtain near-optimal rates of data-compression.

Being novices to this field, we are in no position to present a meaningful historical review. We refer the reader to the papers of Daubechies [4], Beylkin, Coifman and Rokhlin [3] and the review paper by Strang [15], all of which had great influence on the research reported in the present paper; it is evident from these papers that the earlier works of Meyer and Mallat deserve special recognition. The primary motivation for the theory of wavelets, as we understand it, is to find a way to represent functions in terms of local decomposition into scales. Conceptually it is linked to Fourier analysis by being an attempt to improve upon the windowed Fourier transform.

The point of view which we pursue in the present paper is somewhat different. We start with a sequence of discretizations corresponding to increasing level of resolution, and use an approximation technique, namely the reconstruction operator $\mathcal{R}_k$, in order to predict the discrete values of the next finer level of resolution by

$$\tilde{f}^{k+1} = \mathcal{D}_{k+1}\mathcal{R}_k(\mathcal{D}_k f). \tag{1.5}$$

We define a "new scale" as the information in $(\mathcal{D}_{k+1} f)$ which cannot be predicted by (1.5) from a lower level of resolution. If $\mathcal{D}_k$ and $\mathcal{R}_k$ are local operators, then this definition of a scale is also local. In order to obtain a multiresolution *representation*, i.e. a one-to-one correspondence between $\mathcal{D}_k f$ to its scale decomposition, we need to remove the redundancy in (1.4).

In [11] we have presented this point of view in the diadic constant coefficient case, which corresponds to the sequence of nested diadic grids

$$X^k = \{x_j^k\}, \quad x_j^k = j \cdot h_k, \quad h_k = 2^{-k} h_0, \quad 0 \leq k < \infty, \tag{1.6a}$$

and discretization by local averages

$$(\mathcal{D}_k f)_i = \langle f, \omega_i^k \rangle, \quad \omega_i^k = \frac{1}{h_k} \omega \left(\frac{x}{h_k} - i \right), \tag{1.6b}$$

where $\omega(x)$ is a solution of a dilation equation

$$\omega(x) = 2 \sum_\ell \alpha_\ell \omega(2x - \ell). \tag{1.6c}$$

We showed that this point of view is a natural generalization of the wavelet theory which allows for a more flexible choice of reconstruction operators and easier handling of boundaries.

In the present paper we remove the requirement of discretization via a dilation equation and thus obtain a general framework which enables us to embed most numerical problems in a multiresolution setting. Once this is done we can improve the efficiency of the numerical solution algorithm by applying data compression to the numerical solution (see [2], [13] and [14]) as well as to the multiresolution representation of the solution operator (see [3], [8], [1]). We can also reorganize the numerical solution algorithm as a multi-scale computation, where we solve the problem directly only in the coarsest level of resolution and then advance from one level to the next finer one by prediction and correction (see [3] and [12]).

The paper is organized as follows:

In Section 2 we describe a general framework for a multiresolution setting which is formulated as a property of the sequence of discretizations $\{\mathcal{D}_k\}$.

In Section 3 we present the main result of this paper: If $\{(\mathcal{D}_k, \mathcal{R}_k)\}_{k=0}^\infty$ is a sequence of discrete approximations with increasing resolution which is nested (1.2), then there exists a multiresolution representation of the form (1.3).

In Section 4 we show how to obtain multiresolution bases for function spaces by first representing $\mathcal{R}_k\mathcal{D}_k f$ in a multiresolution basis and then taking $k \to \infty$. We show that if this limit exists, then $\{(\mathcal{D}_k, \mathcal{R}_k)\}_{k=0}^{\infty}$ has an hierarchical form $\{(\mathcal{D}_k, \hat{\mathcal{R}}_k)\}_{k=0}^{\infty}$ where

$$\hat{\mathcal{R}}_k\mathcal{D}_k f - \hat{\mathcal{R}}_{k-1}\mathcal{D}_{k-1} f$$

involves only the scale-components of the k-th level.

In Section 5 we present examples of multiresolution representations and bases for discretizations by point-value and by cell-average.

In Section 6 we examine the diadic constant coefficient case (1.6) in the context of the new general framework.

In Section 7 we relate the general framework of this paper to the wavelet theory.

In Section 8 we summarize the results of this paper and make some concluding remarks.

2. General Framework

In this section we describe a general framework for multiresolution representation of data. Let $\mathcal{D}$ be a linear operator which is defined in a space of functions $\mathcal{F}$ and takes values in an Euclidean vector-space V of finite dimension J,

$$\mathcal{D} : \mathcal{F} \to V. \tag{2.1a}$$

We say that $\mathcal{D}$ is a discretization operator and refer to J as the resolution of the discretization.

Let $\{\mathcal{D}_k\}$ be a sequence of discretization operators with monotone increasing resolution $\{J_k\}$ and denote

$$\bar{f}^k = \mathcal{D}_k f, \qquad \bar{f}^k \in V^k, \qquad f \in \mathcal{F}, \tag{2.1b}$$

where $\bar{f}^k$ is a column-vector of J_k components.

Definition 2.1. Multiresolution Setting

A sequence $\{\mathcal{D}_k\}$ with monotone increasing resolution is a multiresolution setting if for all k there exists a $J_{k-1} \times J_k$ matrix D_k^{k-1},

$$\text{rank}\,(D_k^{k-1}) = J_{k-1}, \tag{2.2}$$

such that for all $f \in \mathcal{F}$

$$\bar{f}^{k-1} = D_k^{k-1} \bar{f}^k. \tag{2.3}$$

The above relation means that $\bar{f}^{k-1}$ can be evaluated from $\bar{f}^k$ without explicit knowledge of the function f itself. Furthermore it shows that if $\{\mathcal{D}_k\}$ is a multiresolution setting then knowledge of the discretization of f for some level of resolution implies its knowledge for all lower levels of resolution.

In practice it is not easy to check whether a given sequence of discretizations constitutes a multiresolution setting. In order to formulate a verifiable sufficient condition we introduce now the notion of a reconstructible discretization.

Definition 2.2. Reconstructible discretization

We say that the discretization operator $\mathcal{D}$ in (2.19) is reconstructible if there exists a linear operator $\mathcal{R}$

$$\mathcal{R} : V \to \mathcal{F} \tag{2.4a}$$

such that

$$\mathcal{D}R = I \tag{2.4b}$$

where I is the identity operator in V.

We say that $\mathcal{R}$ is a *linear* reconstruction operator and refer to

$$\mathcal{R}(\mathcal{D}f) = \mathcal{R}\bar{f} \in \mathcal{F}$$

as the (approximate) reconstruction of f from $\bar{f}$.

Definition 2.3. Nested sequence of discretization

A sequence $\{\mathcal{D}_k\}$ of discretization operators with monotone increasing resolution is called nested if for all k and $f \in \mathcal{F}$

$$\mathcal{D}_k f = 0 \Rightarrow \mathcal{D}_{k-1} f = 0. \tag{2.5}$$

Now we are ready to formulate our sufficient condition:

Theorem 2.1. *If $\{\mathcal{D}_k\}$ is a nested sequence of reconstructible discretization then it constitutes a multiresolution setting with*

$$D_k^{k-1} = \mathcal{D}_{k-1}\mathcal{R}_k. \tag{2.6}$$

Before proving this theorem we first show the following

Lemma 2.1. *Let $\{\mathcal{D}_k\}$ be a nested sequence of reconstructible discretizations, then*

$$\mathcal{D}_\ell(\mathcal{R}_k\mathcal{D}_k) = \mathcal{D}_\ell \quad \text{for all} \quad \ell \le k. \tag{2.7}$$

Proof: Let $f \in \mathcal{F}$ be any function and define

$$g = (\mathcal{R}_k\mathcal{D}_k)f, \qquad g \in \mathcal{F}.$$

Using the definition 2.2 we get

$$\mathcal{D}_k g = (\mathcal{D}_k\mathcal{R}_k)\mathcal{D}_k f = \mathcal{D}_k f$$

and therefore

$$\mathcal{D}_k(g - f) = 0;$$

Since $\{\mathcal{D}_k\}$ is nested this implies by successive applications of (2.5) that

$$\mathcal{D}_\ell(g - f) = 0, \quad \ell \le k.$$

It follows then that for any $f \in \mathcal{F}$

$$\mathcal{D}_\ell f = \mathcal{D}_\ell g = \mathcal{D}_\ell(\mathcal{R}_k\mathcal{D}_k)f$$

which proves (2.7). □

Proof of Theorem 2.1: First we observe that

$$D_k^{k-1} = \mathcal{D}_{k-1}\mathcal{R}_k : V^k \to V^{k-1}$$

and therefore can be expressed as a $J_{k-1} \times J_k$ matrix. Using Lemma 2.1 we get that

$$D_k^{k-1}\bar{f}^k = \mathcal{D}_{k-1}\mathcal{R}_k(\mathcal{D}_k f) = \mathcal{D}_{k-1}(\mathcal{R}_k\mathcal{D}_k)f = \mathcal{D}_{k-1}f = \bar{f}^{k-1}$$

which proves (2.3). Next let us define P_{k-1}^k

$$P_{k-1}^k = \mathcal{D}_k\mathcal{R}_{k-1} : V^{k-1} \to V^k \tag{2.8}$$

and observe that it can be expressed as a $J_k \times J_{k-1}$ matrix. Using Lemma 2.1 again and the definition 2.2 we get

$$D_k^{k-1}P_{k-1}^k = \mathcal{D}_{k-1}(\mathcal{R}_k\mathcal{D}_k)\mathcal{R}_{k-1} = I \tag{2.9}$$

where I is the identity matrix in V^{k-1}. This shows that

$$\text{rank}(D_k^{k-1}) = \text{rank}(P_{k-1}^k) = J_{k-1}$$

which completes the proof of this theorem. □

Remark 2.1. We observe that since $\bar{f}^{k-1}$ and $\bar{f}^k$ depend only on the discretization operators, the same is true for the matrix D_k^{k-1} in (2.3). It follows therefore that the matrix expression for D_k^{k-1} in (2.6) is the same for any $\mathcal{R}_k$ and thus does not depend at all on the reconstruction operator, but only on its existence.

In the following we present some examples of multiresolution settings. In the first two examples we consider a polygonal domain $B \subset \mathbb{R}^2$ which is triangulated by $T^k = \{T_j^k\}_{j=1}^{J_k}$. Given T^{k-1} we form T^k by dividing each triangle T_j^{k-1} into $\{T_{j,m}^{k-1}\}_{m=1}^3$ by connecting each apex of T_j^{k-1} with its centroid by a straight line; T^k is then defined by arranging these smaller triangles in a sequence with $1 \le j \le J_k$, $J_k = 3J_{k-1}$.

Example 2.1. Cell-averages. Let

$$\mathcal{F} : B \to \mathbb{R}$$

be the space of piecewise-continuous functions in B and define $\bar{f}_i^k$ to be the cell-average of f in T_i^k,

$$(2.10a)\qquad (\mathcal{D}_k f)_i = \bar{f}_i^k = \frac{1}{|T_i^k|}\int_{T_i^k} f dx, \qquad |T_i^k| = \int_{T_i^k} dx.$$

From the additivity of the integral we get that

$$(2.10b)\qquad \bar{f}_j^{k-1} = \left(\sum_{m=1}^{3} |T_{j_m}^k| \bar{f}_{j_m}^k\right) / |T_j^{k-1}|, \qquad T_{j_m}^k = T_{j,m}^{k-1}.$$

We observe that relation (2.10b) implies the existence of the matrix D_k^{k-1} in (2.3) and actually can serve as its definition.

What is the rank of D_k^{k-1}?

Let $f_k \in \mathcal{F}$ be the piecewise-constant function which is defined by

$$(2.11a)\qquad f_k(x) = \bar{f}_j^k \quad \text{for} \quad x \in T_j^k.$$

It is easy to see that

$$(2.11b)\qquad (\mathcal{R}_k \bar{f}^k)(x) = f_k(x)$$

is a reconstruction of $\bar{f}^k$ in the sense of (2.4), and that relation (2.10b) implies that $\{\mathcal{D}_k\}$ is a nested sequence. It follows then from Theorem 2.1 that $\{\mathcal{D}_k\}$ is a multiresolution setting and consequently

$$\operatorname{rank}(D_k^{k-1}) = J_{k-1}.$$

Example 2.2. Pointvalues. Let $X^k = \{x_i^k\}_{i=1}^{I_k}$ denote the vertices in the triangulation T^k, and let $\bar{x}_j^k$ denote the centroid of the triangle T_j^k. It follows from the construction of T^k from T^{k-1} that

$$X^k = X^{k-1} \cup \{\bar{x}_j^{k-1}\}_{j=1}^{J_{k-1}}, \qquad I_k = I_{k-1} + J_{k-1}.$$

Let

$$\mathcal{F} : B \to \mathbb{R}$$

be the space of continuous functions in B and define

$$(2.12a) \qquad (\mathcal{D}_k f)_i = \bar{f}_i^k = f(x_i^k).$$

Since $X^{k-1} \subset X^k$ it follows immediately that $\{\mathcal{D}_k\}$ is a nested sequence of discretization. Let $p_i^k(x)$ be the linear function which iterpolates the appropriate values of $\bar{f}^k$ at the three vertices of T_i^k and define

$$(2.12b) \qquad (\mathcal{R}_k \bar{f}^k)(x) = p_i^k(x) \quad \text{for} \quad x \in T_i^k.$$

Clearly $(\mathcal{R}_k \bar{f}^k)(x)$ is continuous in B and

$$(\mathcal{D}_k \mathcal{R}_k \bar{f}^k)_j = (\mathcal{R}_k \bar{f}^k)(x_j^k) = \bar{f}_j^k;$$

thus $\mathcal{R}_k$ is a reconstruction operator in the sense of (2.4). We conclude by Theorem 2.1 that $\{\mathcal{D}_k\}$ is a multiresolution setting.

Example 2.3. Fourier coefficients. Let $\mathcal{F} = L^2[0,1]$ be the space of square-integrable functions in $[0,1]$ and denote the coefficients of the Fourier-cosine transform of $f \in \mathcal{F}$ by $a_j(f)$,

$$a_j(f) = \int_0^1 f(x) \cos j\pi x dx, \qquad 0 \leq j < \infty.$$

We define

$$\mathcal{D}_k f = \bar{f}^k = (a_0(f), a_1(f), \dots, a_{J_k}(f))^*$$

and note that if $\{J_k\}$ is monotone increasing then $\{\mathcal{D}_k\}$ is a nested sequence. Next we define

$$(\mathcal{R}_k \bar{f}^k)(x) = \bar{f}_0^k + 2 \sum_{j=1}^{J_k} \bar{f}_j^k \cos j\pi x$$

and observe that $\mathcal{R}_k \bar{f}^k \in L^2[0,1]$ and that

$$\mathcal{D}_k \mathcal{R}_k \bar{f}^k = \bar{f}^k.$$

Hence $\{\mathcal{D}_k\}$ is a nested sequence of reconstructible discretization and therefore it constitutes a multiresolution setting.

3. Multiresolution representation

In the previous section we introduced the notion of reconstructible discretization (Definition 2.2) in which $\mathcal{R}$ is a linear operator. We pointed out in Remark 2.1 that $\mathcal{R}$ plays only an auxiliary role in the formulation in the sense that we merely use its existence and not its particular form. In this section, however, $\mathcal{R}$ does play a significant role and it need not be a linear operator. In order to remind us of these differences we shall refer to $\mathcal{R}$ as a reconstruction *procedure* and consider the pair $(\mathcal{D}, \mathcal{R})$.

Definition 3.1. Discrete approximation. We say that $(\mathcal{D}, R)$ is a discrete approximation in $\mathcal{F}$ with resolution J if

$$\mathcal{D} : \mathcal{F} \to V, \quad \mathcal{R} : V \to \mathcal{F}, \tag{3.1a}$$

$$\mathcal{D}R = I, \tag{3.1b}$$

where $\mathcal{F}$ is a space of functions, V is an Euclidean vector-space of finite dimension J and identity I, and $\mathcal{D}$ is a linear operator.

Definition 3.2. Multiresolution sequence of discrete approximations. We say that $\{(\mathcal{D}_k, \mathcal{R}_k)\}$ is a multiresolution sequence of discrete approximations if for all k, $(\mathcal{D}_k, \mathcal{R}_k)$ is a discrete approximation with monotone increasing resolution J_k and $\{\mathcal{D}_k\}$ is a multiresolution setting.

Given $\bar{f}^{k-1} = \mathcal{D}_{k-1} f$ we can get an approximation $\tilde{f}^k$ to $\bar{f}^k = \mathcal{D}_k f$ by

$$\tilde{f}^k = P_{k-1}^k \bar{f}^{k-1}, \qquad P_{k-1}^k = \mathcal{D}_k \mathcal{R}_{k-1}. \tag{3.2a}$$

We refer to P_{k-1}^k as the prediction operator and to

$$e^k = \bar{f}^k - \tilde{f}^k = \bar{f}^k - P_{k-1}^k \bar{f}^{k-1} \tag{3.2b}$$

as the prediction error. In applications to data-compression it is important to minimize the prediction error, and for this purpose we need to consider adaptive (data-dependent) reconstruction procedures. This is the reason why we let $\mathcal{R}_k$, and consequently P_{k-1}^k, be nonlinear operators. In the following lemma we show that relation (2.9) holds even in the nonlinear case.

Lemma 3.1. *Let $\{(\mathcal{D}_k, \mathcal{R}_k)\}$ be a multiresolution sequence of discrete approximations, then for any $\bar{f}^{k-1}$ in V^{k-1}*

$$D_k^{k-1}\tilde{f}^k = D_k^{k-1}P_{k-1}^k\bar{f}^{k-1} = \bar{f}^{k-1}. \tag{3.3}$$

Proof: Let $g = \mathcal{R}_{k-1}\bar{f}^{k-1}$, $g \in \mathcal{F}$, and observe that

$$\mathcal{D}_k g = \mathcal{D}_k\mathcal{R}_{k-1}\bar{f}^{k-1} = P_{k-1}^k\bar{f}^{k-1} = \tilde{f}^k.$$

Using (2.3) in the definition of multiresolution setting and (2.4b) it follows that

$$D_k^{k-1}\tilde{f}^k = D_k^{k-1}(\mathcal{D}_k g) = \mathcal{D}_{k-1}g = \mathcal{D}_{k-1}\mathcal{R}_{k-1}\bar{f}^{k-1} = \bar{f}^{k-1},$$

which proves the lemma. □

Corollary 3.1.

$$D_k^{k-1}e^k = D_k^{k-1}(\bar{f}^k - \tilde{f}^k) = \bar{f}^{k-1} - \bar{f}^{k-1} = 0. \tag{3.4}$$

The above relation is an homogeneous system of J_{k-1} linear equation for the J_k components of e^k. Since by Definition 2.1

$$\operatorname{rank}(D_k^{k-1}) = J_{k-1} \tag{3.5}$$

we conclude that e^k can be expressed in terms of $\Delta_k = J_k - J_{k-1}$ independent quantities which we denote by the column-vecotr d^k of Δ_k components. In the following Lemma 3.2 we describe such a tranformation between e^k and d^k which is a generalization of the one which is used for orthogonal wavelets (see [4] and section 7). Before stating the lemma we introduce some notations: let $U \subset V^k$ denote the linear span of the columns of $(D_k^{k-1})^*$, and let W denote its orthogonal complement in V^k; thus

$$V^k = U \oplus W, \quad \dim U = J_{k-1}, \quad \dim W = \Delta_k = J_k - J_{k-1}. \tag{3.6a}$$

Let $\{w_i\}_{i=1}^{\Delta_k}$ be a basis in W and define the $\Delta_k \times J_k$ matrix $G_k^{\Delta_k}$ by

$$(G_k^{\Delta_k})^* = (w_1, \dots, w_{\Delta_k}); \tag{3.6b}$$

thus $G_k^{\Delta k}$ is any $\Delta_k \times J_k$ matrix which satisfies

$$D_k^{k-1}(G_k^{\Delta_k})^* = 0, \quad \text{rank}(G_k^{\Delta_k}) = \Delta_k. \tag{3.6c}$$

Let S_k denote the $J_k \times J_k$ symmetric matrix

$$S_k = (D_k^{k-1})^* D_k^{k-1} + (G_k^{\Delta_k})^* G_k^{\Delta_k}; \tag{3.7a}$$

clearly, S_k is positive definite. To see that observe that for any $v \in V^k$

$$\langle S_k v, v\rangle = \|D_k^{k-1} v\|^2 + \|G_k^{\Delta_k} v\|^2 \geq 0 \tag{3.7b}$$

and that

$$\begin{aligned} \langle S_k v, v\rangle = 0 \Rightarrow D_k^{k-1} v = 0,\ G_k^{\Delta_k} v = 0 \Rightarrow v \ \perp U,\ v \perp W \\ \Rightarrow v \perp V^k \Rightarrow v = 0; \end{aligned} \tag{3.7c}$$

note that the inner-product and norms in (3.7b) are to be interpreted as belonging to the appropriate vector-space.

Lemma 3.2. *Let e^k be the prediction error (3.2), then*

$$e^k = S_k^{-1}(G_k^{\Delta_k})^*\ d^k, \tag{3.8a}$$

where

$$d^k = G_k^{\Delta_k} e^k \tag{3.8b}$$

is a column-vector of $\Delta_k = J_k - J_{k-1}$ components.

Proof: Using (3.4), the invertability of S_k and the definition (3.8b) we get

$$\begin{aligned} e^k = S_k^{-1} S_k e^k &= S_k^{-1}[(D_k^{k-1})^* D_k^{k-1} e^k + (G_k^{\Delta_k})^* G_k^{\Delta_k} e^k] \\ &= S_k^{-1}(G_k^{\Delta_k})^*(G_k^{\Delta_k} e^k) = S_k^{-1}(G_k^{\Delta_k})^* d^k \end{aligned}$$

□

Corollary 3.2.

$$\bar{f}^k = P_{k-1}^k \bar{f}^{k-1} + S_k^{-1}(G_k^{\Delta_k})^* d^k. \tag{3.9}$$

Proof: Use (3.2b) and (3.8a) to rewrite

$$\bar{f}^k = \tilde{f}^k + e^k$$

as (3.9). □

We can interpret relation (3.9) as saying that d^k represents the information which is present in $\bar{f}^k$ and is not predictable from $\bar{f}^{k-1}$ by the reconstruction procedure $\mathcal{R}_k$. Motivated by this interpretation we refer to d^k as the coefficients of the k-th scale in the multiresolution sequence of discrete approximations. We observe that d^k can be expressed directly in terms of $\bar{f}^k$ by writing $\bar{f}^{k-1}$ as $D_k^{k-1}\bar{f}^k$ in (3.2), thus

$$d^k = G_k^{\Delta_k}(I - P_{k-1}^k D_k^{k-1})\bar{f}^k \tag{3.10}$$

where I is the identity matrix in V^k.

Given $\bar{f}^L \in V^L$ we define the column-vector

$$\mu(\bar{f}^L) = \begin{pmatrix} d^L \\ \vdots \\ d^1 \\ \bar{f}^0 \end{pmatrix} \tag{3.11}$$

and observe that it has

$$\sum_{k=1}^{L}(J_k - J_{k-1}) + J_0 = J_L$$

components and thus is also in V^L.

Theorem 3.1. *Let $\{(\mathcal{D}_k, \mathcal{R}_k)\}_{k=0}^L$ be a multiresolution sequence of discrete approximations, then $\mu(\bar{f}^L)$ is a representation of $\bar{f}^L$ in the sense that there is a one-to-one transformation between $\bar{f}^L$ and $\mu(\bar{f}^L)$ which we denote by*

$$\mu(\bar{f}^L) = M\bar{f}^L, \ \bar{f}^L = M^{-1}\mu(\bar{f}^L). \tag{3.12}$$

Proof: Rather than describe this transformation and its inverse, we present the algorithms to carry them out.

$\underline{\mu(\bar{f}^L) = M\bar{f}^L}$ (Encoding)

$$(3.13) \qquad \begin{cases} \text{DO} \quad k = L, \dots, 1 \\ \bar{f}^{k-1} = D_k^{k-1} \bar{f}^k \\ d^k = G_k^{\Delta_k}(\bar{f}^k - P_{k-1}^k \bar{f}^{k-1}) \end{cases}$$

$\underline{\bar{f}^L = M^{-1}\mu(\bar{f}^L)}$ (Decoding)

$$(3.14) \qquad \begin{cases} \text{DO} \quad k = 1, \dots, L \\ \bar{f}^k = P_{k-1}^k \bar{f}^{k-1} + S_k^{-1}(G_k^{\Delta_k})^* d^k. \end{cases}$$

□

The representation $\mu(\bar{f}^L)$ can be viewed as a decomposition of $\bar{f}^L$ into scales, and we refer to it as the multiresolution representation of $\bar{f}^L$. The scale coefficients d^k are derived from the prediction error by removing the redundant information in them by (3.8b). The prediction error itself is a combination of what we intuitively call a "scale-coefficient" (i.e. something which is not predictable from lower levels of resolution by any method) and an approximation error which depends on the "quality" of the particular reconstruction procedure.

Decomposition into scales is useful because it provides a way to analyze the "regularity" of $\bar{f}^L$ and its information contents; the latter leads to data-compression of $\bar{f}^L$ by discarding scale-coefficients which fall below an appropriate tolerance. Given a class of functions $\mathcal{F}$, we first choose a sense of discretization which is appropriate for the particular application, and then generate a refinement sequence which constitutes a multiresolution setting. Once this is done, we still have the freedom to choose a reconstruction procedure. Roughly speaking, the class of reconstruction procedures is as rich as the corresponding class of interpolations (or collocations) for the given discrete data.

Remark 3.1. To make the multiresolution representation computationally useful we want the multiresolution transform (3.13) and its inverse (3.14) to be computationally fast. For this purpose we want D_k^{k-1} and $G_k^{\Delta_k}$ to be banded matrices with width which is uniformly bounded in k, and likewise to choose P_{k-1}^k to be a local operator. In this case we can perform the multiresolution transform and its inverse in $O(J_L)$ operations.

4. Multiresolution bases

In this section we consider the case where $\{\mathcal{R}_k\}$ are linear operators and describe the multiresolution basis in which the components of the multiresolution representation (3.11) are the coordinates.

Let us denote

$$A_m^L = \prod_{k=m}^{L-1} P_k^{k+1} = P_{L-1}^L \cdot \dots \cdot P_m^{m+1}, \; A_L^L = I; \tag{4.1}$$

using algorithm (3.14) we get the following expression for the inverse multiresolution transform

$$\bar{f}^L = \sum_{m=1}^{L} A_m^L S_m^{-1} (G_m^{\Delta_m})^* d^m + A_0^L \bar{f}^0. \tag{4.2}$$

If we assume now that $\{\mathcal{R}_k\}$ are all linear operators we get that P_k^{k+1} is a $J_{k+1} \times J_k$ matrix and A_m^L is thus a $J_L \times J_m$ matrix. Let δ_i^J denote the unit-vector of J components which satisfies

$$(\delta_i^J)_j = \delta_{ij}, \quad 1 \le i, j \le J \tag{4.3a}$$

where δ_{ij} is the Krönecker-δ, and

$$\eta_i^k = S_k^{-1} (G_k^{\Delta_k})^* \delta_i^{\Delta_k}, \; 1 \le i \le \Delta_k = J_k - J_{k-1}, \; \eta_i^k \in V^k, \tag{4.3b}$$

$$\bar{\varphi}_i^{m,L} = A_m^L \delta_i^{J_m}, \; \bar{\psi}_i^{m,L} = A_m^L \eta_i^m = \sum_{j=1}^{J_m} (\eta_i^m)_j \bar{\varphi}_j^{m,L}; \tag{4.3c}$$

observe that $\{\bar{\varphi}_i^{m,L}\}_{i=1}^{J_m}$ and $\{\bar{\psi}_i^{m,L}\}_{i=1}^{\Delta_m}$ are vectors in V^L for all $1 \le m \le L$. Using these definitions it is easy to see that (4.2) can be rewritten as

$$\bar{f}^L = \sum_{m=1}^{L} \sum_{i=1}^{\Delta_m} d_i^m \bar{\psi}_i^{m,L} + \sum_{i=1}^{J_0} \bar{f}_i^0 \bar{\varphi}_i^{0,L}. \tag{4.4a}$$

Combining (4.4a) with Theorem 3.1, we have proved

Theorem 4.1. *Let $\{(\mathcal{D}_k,\mathcal{R}_k)\}_{k=0}^{L}$ be a multiresolution sequence of discrete approximations, then*

$$\mathcal{B}_M = (\{\{\bar{\psi}_i^{m,L}\}_{i=1}^{\Delta_m}\}_{m=1}^{L},\ \{\bar{\varphi}_i^{0,L}\}_{i=1}^{J_0}) \tag{4.4b}$$

is a basis in V^L. The representation of any $\bar{f}^L \in V^L$ in this basis is given by (4.4a), *where the coordinates are the components of the multiresolution representation $\mu(\bar{f}^L)$ in* (3.11); *these coordinates can be computed by algorithm* (3.13).

Next we consider multiresolution bases in function spaces. Let us denote

$$\varphi_i^{m,L} = \mathcal{R}_L \bar{\varphi}_i^{m,L},\ \ \psi_i^{m,L} = \sum_{j=1}^{J_m} (\eta_i^m)_j \varphi_j^{m,L}, \tag{4.5a}$$

and observe that $\varphi_i^{m,L}$ and $\psi_i^{m,L}$ are functions in $\mathcal{F}$. Using the linearity of the reconstruction operator it follows immediately from (4.4a) that for any $f \in \mathcal{F}$

Corollary 4.1.

$$(\mathcal{R}_L \mathcal{D}_L f)(x) = \sum_{m=1}^{L} \sum_{i=1}^{\Delta_m} d_i^m(f) \psi_i^{m,L}(x) + \sum_{i=1}^{J_0} (\mathcal{D}_0 f)_i \varphi_i^{0,L}(x), \tag{4.5b}$$

$$d_i^m(f) = G_m^{\Delta_m}(I - P_{m-1}^m D_m^{m-1})(\mathcal{D}_m f). \tag{4.5c}$$

Definition 4.1. Complete Sequence of Discrete Approximations.

We say that the infinite sequence $\{(\mathcal{D}_k,\mathcal{R}_k)\}_{k=0}^{\infty}$ is complete in $\mathcal{F}$, a Banach space, if for all $f \in \mathcal{F}$

$$\lim_{L\to\infty} \|\mathcal{R}_L \mathcal{D}_L f - f\| = 0. \tag{4.6}$$

We turn now to consider the limit $L \to \infty$ in (4.5). If the multiresolution sequence of discrete approximations is complete, then the LHS of (4.5) converges to f in the sense of the norm in (4.6). We assume now that for all levels of resolution $0 \le k < \infty$

$$\exists \lim_{L\to\infty} \varphi_i^{k,L} = \varphi_i^k \in \mathcal{F},\ 1 \le i \le J_k. \tag{4.7}$$

Later in this section we shall formulate a sufficient condition which ensures that for every $f \in \mathcal{F}$

(4.8a) $$f = \sum_{m=1}^{\infty} \sum_{i=1}^{\Delta_m} d_i^m(f) \psi_i^m + \sum_{i=1}^{J_0} (\mathcal{D}_0 f)_i \varphi_i^0,$$

where

(4.8b) $$\psi_i^m = \sum_{j=1}^{J_m} (\eta_i^m)_j \varphi_j^m$$

and the equality in (4.8a) is to be interpreted in the sense of the norm in (4.6).

Lemma 4.1. *If $\mathcal{D}_\ell$ is a bounded operator, then*

(4.9a) $$\mathcal{D}_\ell \varphi_i^m = \bar{\varphi}_i^{m,\ell}, \; 0 \le \ell \le m.$$

Proof: It follows immediately from Lemma 2.1 and the definition of A_m^k in (4.1) that

$$\begin{aligned} \mathcal{D}_\ell \mathcal{R}_L A_m^L &= \mathcal{D}_\ell \prod_{k=m}^{L-1} (\mathcal{R}_{k+1} \mathcal{D}_{k+1}) \cdot \mathcal{R}_m = \mathcal{D}_\ell \prod_{k=m}^{\ell-2} (\mathcal{R}_{k+1} \mathcal{D}_{k+1}) \cdot \mathcal{R}_m \\ &= \prod_{k=m}^{\ell-1} (\mathcal{D}_{k+1} \mathcal{R}_k) = A_m^\ell. \end{aligned}$$

It follows then from the definition (4.5a) that

(4.9b) $$\mathcal{D}_\ell \varphi_i^{m,L} = \mathcal{D}_\ell \mathcal{R}_L A_m^L \delta_i^{J_m} = A_m^\ell \delta_i^{J_m} = \bar{\varphi}_i^{m,\ell}.$$

Since $\mathcal{D}_\ell$ is bounded we get

$$\mathcal{D}_\ell \varphi_i^m = \lim_{L \to \infty} \mathcal{D}_\ell \varphi_i^{m,L} = \bar{\varphi}_i^{m,\ell}.$$

□

We define now the linear operator $\hat{\mathcal{R}}_k$

(4.10a) $$\hat{\mathcal{R}}_k : V^k \to \mathcal{F}$$

by

(4.10b) $$\hat{\mathcal{R}}_k \bar{f}^k = \sum_{i=1}^{J_k} \bar{f}_i^k \varphi_i^k.$$

Corollary 4.2. $\hat{\mathcal{R}}_k$ *is a reconstruction operator.*

Proof: It follows from (4.9) and $A_m^m = I$, that

$$\mathcal{D}_m \varphi_i^m = \delta_i^{J_m}.$$

Let $\bar{f}^k$ be any vector in V^k, and apply $\mathcal{D}_k$ to both sides of (4.10b)

$$(\mathcal{D}_k \hat{\mathcal{R}}_k)\bar{f}^k = \sum_{i=1}^{J_k} \bar{f}_i^k \mathcal{D}_k \varphi_i^k = \sum_{i=1}^{J_k} \bar{f}_i^k \delta_i^{J_k} = \bar{f}^k.$$

□

Lemma 4.2.

(4.11) $$\varphi_i^{m-1} = \sum_{j=1}^{J_m} (\xi_i^m)_j \varphi_j^m, \ \xi_i^m = P_{m-1}^m \delta_i^{J_{m-1}}.$$

Proof: Using the definition (4.3c) we get

$$\begin{aligned} \bar{\varphi}_i^{m-1,L} &= A_{m-1}^L \delta_i^{J_{m-1}} = A_m^L P_{m-1}^m \delta_i^{J_{m-1}} = A_m^L \xi_i^m \\ &= A_m^L \sum_{j=1}^{J_m} (\xi_i^m)_j \delta_j^{J_m} = \sum_{j=1}^{J_m} (\xi_i^m)_j A_m^L \delta_j^{J_m} = \sum_{j=1}^{J_m} (\xi_i^m)_j \bar{\varphi}_j^{m,L}. \end{aligned}$$

Applying $\mathcal{R}_L$ to both sides of the above relation we get

$$\varphi_i^{m-1,L} = \sum_{j=1}^{J_m} (\xi_i^m)_j \varphi_j^{m,L};$$

taking the limit $L \to \infty$ above we prove (4.11). $\square$

Lemma 4.3.

(4.12a) $\qquad$ (i) $(\hat{\mathcal{R}}_k \mathcal{D}_k)\hat{\mathcal{R}}_{k-1} = \hat{\mathcal{R}}_{k-1};$

(4.12b) $\qquad$ (ii) $\hat{\mathcal{R}}_k(\mathcal{D}_k f) - \hat{\mathcal{R}}_{k-1}(\mathcal{D}_{k-1} f) = \sum_{i=1}^{\Delta_k} d_i^k(f)\psi_i^k$

where ψ_i^k is defined in (4.8b) *and $d_i^k(f)$ are the k-th scale coefficients* (4.5c).

Proof: (i) Denoting $\bar{f}^\ell = \mathcal{D}_\ell f$ and using (4.11) we get

$$\hat{\mathcal{R}}_{k-1}\bar{f}^{k-1} = \sum_{i=1}^{J_{k-1}} \bar{f}_i^{k-1}\varphi_i^{k-1} = \sum_{i=1}^{J_{k-1}} \bar{f}_i^{k-1} \sum_{j=1}^{J_k} (\xi_i^k)_j \varphi_j^k$$
$$= \sum_{j=1}^{J_k} \varphi_j^k \sum_{i=1}^{J_{k-1}} \bar{f}_i^{k-1}(\xi_i^k)_j.$$

Using the definitions of ξ_i^k in (4.11) and the approximation $\tilde{f}^k$ (3.2a) we get

$$\sum_{i=1}^{J_{k-1}} \bar{f}_i^{k-1}(P_{k-1}^k \delta_i^{J_{k-1}})_j = \left(P_{k-1}^k \sum_{i=1}^{J_{k-1}} \bar{f}_i^{k-1}\delta_i^{J_{k-1}}\right)_j = (P_{k-1}^k \bar{f}^{k-1})_j$$
$$= \tilde{f}_j^k.$$

Thus

$$\hat{\mathcal{R}}_{k-1}\bar{f}^{k-1} = \sum_{j=1}^{J_k} \tilde{f}_j^k \varphi_j^k \Rightarrow \mathcal{D}_k(\hat{\mathcal{R}}_{k-1}\bar{f}^{k-1}) = \tilde{f}^k \Rightarrow \hat{\mathcal{R}}_k(\mathcal{D}_k \hat{\mathcal{R}}_{k-1}\bar{f}^{k-1})$$
$$= \sum_{j=1}^{J_k} \tilde{f}_j^k \varphi_j^k = \hat{\mathcal{R}}_{k-1}\bar{f}^{k-1}$$

which proves the first part of the lemma.

$$\text{(ii)} \quad \hat{\mathcal{R}}_k \bar{f}^k - \hat{\mathcal{R}}_{k-1}\bar{f}^{k-1} = \sum_{j=1}^{J_k} \bar{f}_j^k \varphi_j^k - \sum_{j=1}^{J_k} \tilde{f}_j^k \varphi_j^k =$$
$$= \sum_{j=1}^{J_k} e_j^k \varphi_j^k = \sum_{j=1}^{J_k} [S_k^{-1}(G_k^{\Delta_k})^* d^k]_j \varphi_j^k,$$

where we used (3.8a) to express the prediction error e^k in terms of the scale coefficients d^k. Using the definition of η_i^k in (4.3b) we get

$$S_k^{-1}(G_k^{\Delta_k})^* d^k = S_k^{-1}(G_k^{\Delta_k})^* \sum_{i=1}^{\Delta_k} d_i^k \delta_i^{\Delta_k} = \sum_{i=1}^{\Delta_k} d_i^k \eta_i^k.$$

Hence

$$\begin{aligned}\hat{\mathcal{R}}_k \bar{f}^k - \hat{\mathcal{R}}_{k-1} \bar{f}^{k-1} &= \sum_{j=1}^{J_k} \left[\sum_{i=1}^{\Delta_k} d_i^k (\eta_i^k)_j \right] \varphi_j^k = \sum_{i=1}^{\Delta_k} d_i^k \left[\sum_{j=1}^{J_k} (\eta_i^k)_j \varphi_j^k \right] \\ &= \sum_{i=1}^{\Delta_k} d_i^k \psi_i^k.\end{aligned}$$

□

Summing (4.12) from $k = 1$ to $k = L$ we get

$$\hat{\mathcal{R}}_L \mathcal{D}_L f - \hat{\mathcal{R}}_0 \mathcal{D}_0 f = \sum_{k=1}^{L} \sum_{i=1}^{\Delta_k} d_i^k(f) \psi_i^k. \tag{4.13}$$

We refer to $\{(\mathcal{D}_k, \hat{\mathcal{R}}_k)\}_{k=0}^{\infty}$ as the hierarchical form of $\{(\mathcal{D}_k, \mathcal{R}_k)\}_{k=0}^{\infty}$. Combining all previous results we have proved

Theorem 4.2. *Let $\{(\mathcal{D}_k, \mathcal{R}_k)\}_{k=0}^{\infty}$ be a sequence of discrete approximations in a Banach function space $\mathcal{F}$, where $\{\mathcal{D}_k\}$ is a nested sequence of bounded discretization operators.*

(1) *If for all levels of resolution $0 \le k < \infty$*

$$\exists \lim_{L \to \infty} \varphi_i^{k,L} = \varphi_i^k \in \mathcal{F}, \quad 1 \le i \le J_k, \tag{4.14a}$$

then this sequence has an hierarchical form $\{(\mathcal{D}_k, \hat{\mathcal{R}}_k)\}_{k=0}^{\infty}$ which is defined by (4.10).

(2) *If the hierarchical form is complete in $\mathcal{F}$, then*

$$B_M = (\{\{\psi_i^k\}_{i=1}^{J_k}\}_{k=1}^{\infty}, \ \{\varphi_i^0\}_{i=1}^{J_0}) \tag{4.14b}$$

is a basis in $\mathcal{F}$. The expansion of any $f \in \mathcal{F}$ in B_M is given by (4.8)*, where the coordinates are the coefficients of the multiresolution representation of f* (4.5c) *in the original sequence $\{(\mathcal{D}_k, \mathcal{R}_k)\}_{k=0}^{\infty}$.*

Definition 4.2. Hierarchical Sequence of Discrete Approximations

Let $\{(\mathcal{D}_k, \mathcal{R}_k)\}$ be a sequence of discrete approximations where $\{\mathcal{D}_k\}$ is a nested sequence of bounded discretization operators. We say that this sequence is hierarchical if for all k

$$(\mathcal{R}_{k+1}\mathcal{D}_{k+1})\mathcal{R}_k = \mathcal{R}_k. \tag{4.15}$$

Corollary 4.3. *Let $\{(\mathcal{D}_k, \mathcal{R}_k)\}_{k=0}^{\infty}$ be an hierarchical sequence of discrete approximations which is complete in $\mathcal{F}$, then it is in hierarchical form and B_M,* (4.14b) *with*

$$\varphi_i^m = \mathcal{R}_m \delta_i^{J_m}, \ \psi_i^m = \sum_{j=1}^{J_m} (\eta_i^m)_j \varphi_j^m = \mathcal{R}_m \eta_i^m, \tag{4.16}$$

is a basis in $\mathcal{F}$.

Proof: Using induction in (4.15) we get that

$$(\mathcal{R}_\ell \mathcal{D}_\ell)\mathcal{R}_k = \mathcal{R}_k \ \text{ for } \ \ell \geq k.$$

It follows therefore that

$$\mathcal{R}_L A_m^L = \mathcal{R}_L \prod_{k=m}^{L-1} (\mathcal{D}_{k+1}\mathcal{R}_k) = \prod_{k=m}^{L-1} (\mathcal{R}_{k+1}\mathcal{D}_{k+1}) \cdot \mathcal{R}_m = \mathcal{R}_m,$$

and consequently in (4.5a)

$$\varphi_i^{m,L} = \mathcal{R}_L \bar{\varphi}_i^{m,L} = \mathcal{R}_L A_m^L \delta_i^{J_m} = \mathcal{R}_m \delta_i^{J_m} = \varphi_i^m,$$
$$\psi_i^{m,L} = \sum_{j=1}^{J_m} (\eta_i^m)_j \varphi_j^{m,L} = \sum_{j=1}^{J_m} (\eta_i^m)_j \varphi_j^m.$$

Since $\varphi_i^m = \mathcal{R}_m \delta_i^{J_m} \in \mathcal{F}$ and $\varphi_i^{m,L} = \varphi_i^m$ is a constant sequence, we conclude from Part (1) of Theorem 4.2 that the hierarchical form is identical to the original sequence. Part (2) of this theorem follows immediately from the assumption that the given sequence is complete. □

Remark 4.1. It is easy to see that the sequences of discrete approximations in Examples 2.1-2.3 are all hierarchical. The term "hierarchical" is borrowed from the context of finite elements (such as Example 2.2) where $\{\psi_i^m\}$ in (4.16) is called "hierarchical basis" - see [16]. We remark that many of the commonly used finite-element approximations and spectral collocations are hierarchical. However, as will be demonstrated in the next section, this is not true for many approximations that are used in finite-difference methods, and thus the limiting process (4.14a) is actually needed.

5. Examples

In this section we demonstrate the ideas of this paper for two important classes of discretization: pointvalues (Example 2.2 and Example 5.1 in the following) and cell-averages (Example 2.1 and Example 5.2 in the following). In both cases we consider functions which are defined in $[0,1]$ and discretized on the following sequence of nested diadic grids $\{X^k\}_{k=0}^{\infty}$:

$$X^k = \{x_j^k\}_{j=0}^{J_k}, \quad x_j^k = j \cdot h_k, \quad h_k = 2^{-k} h_0, \quad J_k = 2^k J_0, \tag{5.1a}$$

where $h_0 = 1/J_0$ for some integer J_0; observe that

$$X^{k-1} \subset X^k, \qquad X^k - X^{k-1} = \{x_{2j-1}^k\}_{j=1}^{J_{k-1}}. \tag{5.1b}$$

Example 5.1. Pointvalues.

$$\mathcal{D}_k : C[0,1] \to V^k \tag{5.2a}$$

$$\bar{f}_j^k = (\mathcal{D}_k f)_j = f(x_j^k), \quad 0 \le j \le J_k; \tag{5.2b}$$

here $C[0,1]$ is the space of continuous functions in $[0,1]$. Since $x_j^{k-1} = x_{2j}^k$ for $0 \le j \le J_{k-1}$ we get that

$$\bar{f}_j^{k-1} = \bar{f}_{2j}^k = (D_k^{k-1} \bar{f}^k)_j \ , \quad 0 \le j \le J_{k-1} \tag{5.2c}$$

and thus the decimation matrix D_k^{k-1} is given by

$$(D_k^{k-1})_{i,j} = \delta_{2i,j}. \tag{5.2d}$$

The columns of $(D_k^{k-1})^*$ are $\{\delta_{2j}^k\}_{j=0}^{J_{k-1}}$ where δ_j^k denote the unit vector $\delta_j^{J_k}$ in (4.3a). This shows that D_k^{k-1} has a full row-rank and suggests to take in (3.6b)

$$w_j = \delta_{2j-1}^k, \quad 1 \leq j \leq J_{k-1}, \tag{5.3a}$$

which leads to G_k^{k-1} (note change of notation: $\Delta_k = J_{k-1} \to k-1$) that is given by

$$(G_k^{k-1})_{i,j} = \delta_{2i-1,j}. \tag{5.3b}$$

It is easy to see that in this case S_k in (3.7a) is the identity matrix in V^k,

$$S_k = I. \tag{5.3c}$$

The reconstruction procedure $\mathcal{R}_k$ for this discretization is any operator

$$\mathcal{R}_k : V^k \to C[0,1] \tag{5.4a}$$

which satisfies

$$\mathcal{D}_k \mathcal{R}_k \bar{f}^k = \bar{f}^k, \tag{5.4b}$$

i.e.

$$(\mathcal{R}_k \bar{f}^k)(x_j^k) = \bar{f}_j^k, \quad 0 \leq j \leq J_k; \tag{5.4c}$$

this shows that $\mathcal{R}_k$ is any continuous interpolation of the data $\bar{f}^k$ at the grid-points X^k. In the following we use the notation

$$(\mathcal{R}_k \bar{f}^k)(x) = I_k(x; \bar{f}^k) \tag{5.5a}$$

where

$$I_k(x; \bar{f}^k) \in C[0,1], \tag{5.5b}$$

$$I_k(x_j^k; \bar{f}^k) = \bar{f}_j^k, \quad 0 \le j \le J_k, \tag{5.5c}$$

for any $\bar{f}^k \in V^k$.

The predicted values $\tilde{f}^k$ (3.2a),

$$\tilde{f}^k = P_{k-1}^k \bar{f}^{k-1} = \mathcal{D}_k \mathcal{R}_{k-1} \bar{f}^{k-1}, \tag{5.6a}$$

are given by

$$\tilde{f}_i^k = I_{k-1}(x_i^k, \bar{f}^{k-1}). \tag{5.6b}$$

Observe that for $i = 2j$ we have $x_{2j}^k = x_j^{k-1}$ and thus

$$\tilde{f}_{2j}^k = I_{k-1}(x_j^{k-1}; \bar{f}^{k-1}) = \bar{f}_j^{k-1}; \tag{5.6c}$$

consequently, the prediction error e^k (3.2b) satisfies

$$e_{2j}^k = 0, \quad 0 \le j \le J_{k-1}, \tag{5.7a}$$

$$e_{2j-1}^k = \bar{f}_{2j-1}^k - I_{k-1}(x_{2j-1}^k; \bar{f}^{k-1}), \quad 1 \le j \le J_{k-1}. \tag{5.7b}$$

The scale coefficients d^k in (3.8b) are given by

$$d_j^k = (G_k^{k-1} e^k)_j = e_{2j-1}^k, \quad 1 \le j \le J_{k-1}; \tag{5.8a}$$

e^k is recovered from d^k by (3.8a)

$$e^k = (G_k^{k-1})^* d^k,$$

which can be expressed by

$$\begin{cases} e_{2j}^k = 0, & 0 \le j \le J_{k-1} \\ e_{2j-1}^k = d_j^k, & 1 \le j \le J_{k-1}. \end{cases} \tag{5.8b}$$

The periodic case is obtained from the above formulation by replacing $C[0,1]$ in (5.2) and (5.4) with its subclass $\tilde{C}[0,1]$

$$\tilde{C}[0,1] = \{f \mid f \in C[0,1],\ f(0) = f(1)\}. \tag{5.9a}$$

This change implies in (5.2)

$$\bar{f}_0^k = f(0) = f(1) = \bar{f}_{J_k}^k \tag{5.9b}$$

and in (5.4) it imposes the additional requirement

$$I_k(0; \bar{f}^k) = I_k(1; \bar{f}^k). \tag{5.9c}$$

To describe both the periodic and non-periodic case with the same expression we introduce the starting index p,

$$p = \begin{cases} 1 & \text{periodic} \\ 0 & \text{non-periodic} \end{cases}; \tag{5.10a}$$

thus V^k has the dimension $(J_k - p + 1)$ and we denote

$$\bar{f}^k = (\bar{f}_p^k, \ldots, \bar{f}_{J_k}^k)^*, \tag{5.10b}$$

and similarly other quantities in V^k.

Using this notation we now describe the multiresolution transform (3.13) and its inverse (3.14) for both cases as follows:

<u>$\mu(\bar{f}^L) = M\bar{f}^L$</u> (Encoding)

$$\begin{cases} \text{DO}\quad k = L, \ldots, 1 \\ \bar{f}_j^{k-1} = \bar{f}_{2j}^k, \quad p \le j \le J_{k-1} \\ d_j^k = \bar{f}_{2j-1}^k - I_{k-1}(x_{2j-1}^k; \bar{f}^{k-1}), \quad 1 \le j \le J_{k-1}. \end{cases} \tag{5.11}$$

$\underline{\bar{f}^L = M^{-1}\mu(\bar{f}^L)}$ (Decoding)

$$(5.12) \qquad \begin{cases} \text{DO} \quad k = 1, \ldots, L \\ \bar{f}^k_{2j} = \bar{f}^{k-1}_j, \quad p \le j \le J_{k-1} \\ \bar{f}^k_{2j-1} = I_{k-1}(x^k_{2j-1}; \bar{f}^{k-1}) + d^k_j, \quad 1 \le j \le J_{k-1}. \end{cases}$$

We turn now to describe multiresolution bases for the pointvalue discretization. We assume that $\mathcal{R}_k$ is a linear operator and express $I_k(x; \bar{f}^k)$ by

$$(5.13a) \qquad I_k(x; \bar{f}^k) = \sum_{i=p}^{J_k} \bar{f}^k_i u^k_i(x), \quad u^k_i(x) = I_k(x; \delta^k_i);$$

using this in (5.6) we express the prediction matrix P^k_{k-1} by

$$(5.13b) \qquad (P^k_{k-1})_{2i,j} = \delta_{i,j}, \quad (P^k_{k-1})_{2i-1,j} = u^{k-1}_j(x^k_{2i-1}).$$

The functions $\varphi^{m,L}_i(x)$ in (4.5a) are obtained by repeated interpolation of δ^m_i:

$$(5.14a) \qquad \bar{\varphi}^{m,L}_i = \left(\prod_{k=m}^{L-1} P^{k+1}_k \right) \delta^m_i,$$

$$(5.14b) \qquad \varphi^{m,L}_i(x) = I_L(x; \bar{\varphi}^{m,L}_i).$$

Let us assume now that $\varphi^{m,L}_i$ converges in the maximum norm, i.e.

$$(5.14c) \qquad \exists \lim_{L\to\infty} \varphi^{m,L}_i(x) = \varphi^m_i(x) \text{ uniformly for } 0 \le x \le 1.$$

In this case $\varphi^m_i \in C[0,1]$ and

$$(5.14d) \qquad \varphi^m_i(x^k_j) = \bar{\varphi}^{m,k}_i \text{ for } k \ge m.$$

From Lemma 4.2 we get that $\{\varphi^m_i\}$ satisfy the "generalized dilation relation" (4.11)

$$(5.15a) \qquad \varphi^{m-1}_i = \langle P^m_{m-1}\delta^{m-1}_i, \varphi^m \rangle = \langle \delta^{m-1}_i, (P^m_{m-1})^* \varphi^m \rangle = \varphi^m_{2i} + \sum_{j=1}^{J_{m-1}} u^{m-1}_i(x^m_{2j-1})\varphi^m_{2j-1}.$$

The functions $\{\psi_i^m\}$ in (4.8b) are defined in terms of $\{\varphi_i^m\}$ as follows:

$$\text{(5.15b)} \qquad \psi_i^m = \langle \eta_i^m, \varphi^m \rangle = \langle (G_m^{m-1})^* \delta_i^{m-1}, \varphi^m \rangle = \langle \delta_i^{m-1}, G_m^{m-1} \varphi^m \rangle = \varphi_{2i-1}^m.$$

From Lemma 4.3 we get that the interpolation

$$\text{(5.16a)} \qquad \hat{I}_k(x; \bar{f}^k) = \sum_{i=p}^{J_k} \bar{f}_i^k \varphi_i^k(x)$$

is the hierarchical form of $I_k(x; \bar{f}^k)$,

$$\text{(5.16b)} \qquad \hat{I}_L(x, \mathcal{D}_L f) = \hat{I}_0(x; \mathcal{D}_0 f) + \sum_{k=1}^{L} \sum_{j=1}^{J_{k-1}} d_j^k(f) \varphi_{2j-1}^k(x),$$

where

$$\text{(5.16c)} \qquad d_j^k(f) = f(x_{2j-1}^k) - I_{k-1}(x_{2j-1}^k; \mathcal{D}_{k-1} f).$$

When the interpolation $I_k(x; \bar{f}^k)$ is hierarchial to begin with, i.e.

$$\text{(5.17a)} \qquad \tilde{f}^k = \mathcal{D}_k I_{k-1}(\cdot; \bar{f}^{k-1}) \Rightarrow I_k(x; \tilde{f}^k) \equiv I_{k-1}(x; \bar{f}^{k-1})$$

then we get from Corollary 4.3 that

$$\text{(5.17b)} \qquad \hat{I}_k \equiv I_k, \quad \varphi_i^k(x) = I_k(x; \delta_i^k).$$

Example 5.1.1. Piecewise-polynomial interpolation

Let $\mathcal{S}$ denote the stencil

$$\text{(5.18a)} \qquad \mathcal{S} = \mathcal{S}(r,s) = \{-s, -s+1, \dots, -s+r\}, \ r \geq s \geq 0, \ r \geq 1;$$

let $\{L_m(y)\}_{m\in\mathcal{S}}$ denote the Lagrange interpolation polynomials for this stencil

$$L_m(y) = \prod_{\substack{j=-s \\ j\neq m}}^{-s+r} \left(\frac{y-j}{m-j}\right), \quad L_m(i) = \delta_{i,m}, \quad i \in \mathcal{S}, \tag{5.18b}$$

and define

$$q_j^k(x; \bar{f}^k, r, s) = \sum_{m=-s}^{-s+r} \bar{f}_{j+m}^k L_m\left(\frac{x - x_j^k}{h_k}\right). \tag{5.18c}$$

First we consider the periodic case,

$$\bar{f}_{-j}^k = \bar{f}_{J_k-j}^k, \quad \bar{f}_{J_k+j+1}^k = \bar{f}_{j+1}^k, \quad 0 \le j < J_k \tag{5.19a}$$

and define the piecewise-polynomial interpolation $I_k(x; \bar{f}^k)$ of degree r for $0 \le x \le 1$ by

$$I_k(x; \bar{f}^k) = q_j^k(x; \bar{f}^k, r, s) \quad \text{for } x_{j-1}^k \le x \le x_j^k, \ 1 \le j \le J_k. \tag{5.19b}$$

In this case we get in (5.6)

$$\begin{cases} \bar{f}_{2i}^k = (P_{k-1}^k \bar{f}^{k-1})_{2i} = \bar{f}_i^{k-1}, \\ \bar{f}_{2i-1}^k = (P_{k-1}^k \bar{f}^{k-1})_{2i-1} = \sum\limits_{m=-s}^{-s+r} L_m(-1/2) \bar{f}_{i+m}^{k-1} \end{cases} \tag{5.19c}$$

observe that the coefficients of $\bar{f}_{i+m}^{k-1}$ in (5.19c) depend only on m and they vanish if $m < -s$ or $m > -s + r$.

Let us consider now the limiting process $L \to \infty$ in (5.14) where we start by setting $\bar{\varphi}_i^{m,m} = \delta_i^m$ at the points of the m-th grid and then repeatedly apply (5.19c). After k applications we get that $\bar{\varphi}_i^{m,m+k}$ has its support in

$$(2^k - 1)(2s - 1 - 2r) \le j - 2^k i \le (2^k - 1)(2s - 1) \tag{5.20a}$$

and that the values within this support are the same for all i and m; so lets take $i = m = 0$. It follows from (5.20a) that $I_k(x; \bar{\varphi}_0^{0,k})$ has its support in

$$x_{2s-1-2r}^0 + O(2^{-k}) \le x \le x_{2s-1}^0 + O(2^{-k}). \tag{5.20b}$$

If

$$\exists \lim_{k\to\infty} I_k(x;\bar{\varphi}_0^{0,k}) = \varphi_0(x) \tag{5.20c}$$

then the above analysis shows that for all i and m

$$\exists \lim_{k\to\infty} I_k(x;\bar{\varphi}_i^{m,m+k}) = \varphi_i^m(x) = \varphi\left(\frac{x}{h_m} - i\right) \tag{5.20d}$$

where

$$\varphi(x) = \varphi_0(xh_0), \quad \text{support}(\varphi) = [2s-1-2r, 2s-1]. \tag{5.20e}$$

Hence the uniform convergence of the single sequence (5.20c) implies the existence of the hierarchial interpolation $\hat{I}_k$ in (5.16).

The centered stencils $r = 2s-1$ are of particular interest: For $r = 1$, I_k is the piecewise-linear interpolation which is hierarchial - so the limiting process is not needed. Deslauriers and Duboc [5] proved uniform convergence to a continuously differentiable $\varphi(x)$ for $r = 3, 5$ (see Figures 1-2).

It follows from (5.15a) that $\varphi(x)$ satisfies the dilation relation

$$\varphi(x) = \varphi(2x) + \sum_{m=-s}^{-s+r} L_m(-1/2)\varphi(2x+2m+1). \tag{5.20f}$$

Hence the smoothness of the limiting function can be studied from the Fourier transform of (5.20f) as suggested by Daubechies in [4].

In the non-periodic case we modify the centered interpolation $r = 2s-1$ to account for the boundaries by defining $I_k(x;\bar{f}^k)$ in $[x_{j-1}^k, x_j^k]$ as follows:

$$I_k(x;\bar{f}^k) = \begin{cases} q_j^k(x;\bar{f}^k, r, j) & 1 \le j \le s-1 \\ q_j^k(x;\bar{f}^k, r, s) & s \le j \le J_k - s + 1 \\ q_j^k(x;\bar{f}^k, r, j - J_k + r) & J_k - s + 2 \le j \le J_k \end{cases} \tag{5.21}$$

In this case convergence of (5.20c) implies convergence of (5.20d) for $s \le i \le J_m - s + 1$; the convergence of $\varphi_i^{m,L}$ for $0 \le i \le s-1$ and for $J_m - s + 2 \le i \le J_m$ has to be studied independently.

Example 5.1.2. Cubic splines

Let $I_k(x; \bar{f}^k)$ be the unique piecewise-cubic function (i.e. I_k is a cubic polynomial in each $[x_{j-1}^k, x_j^k]$, $1 \le j \le J_k$) which satisfies

(i) $I_k(x_j^k; \bar{f}^k) = \bar{f}_j^k, \quad 0 \le j \le J_k;$

(ii) $\dfrac{d^\ell}{dx^\ell} I_k(x_j^k - 0; \bar{f}^k) = \dfrac{d^\ell}{dx^\ell} I_k(x_j^k + 0; \bar{f}^k), \quad \ell = 1, 2, \ 1 \le j \le J_k - 1;$

(iii) $\dfrac{d}{dx} I_k(0; \bar{f}^k) = s_0, \ \dfrac{d}{dx} I_k(1; \bar{f}^k) = s_1;$

note that requirement (ii) implies $I_k \in C^2[0,1]$.

The periodic case is obtained by specifying data only for $1 \le j \le J_k$ in (i) and adding the requirement $I_k(0; \bar{f}^k) = I_k(1; \bar{f}^k)$ instead; (iii) is replaced by the requirement $\frac{d^\ell}{dx^\ell} I_k(0; \bar{f}^k) = \frac{d^\ell}{dx^\ell} I_k(1; \bar{f}^k)$, $\ell = 1, 2$.

It is easy to see that the cubic-spline interpolation is hierarchial in both the periodic and non-periodic case: Let us denote

$$\tilde{f}_i^k = I_{k-1}(x_i^k; \bar{f}^{k-1}), \ p \le i \le J_k,$$

where p is given by (5.10a); $I_k(x; \tilde{f}^k)$ is the unique cubic spline which interpolates $\tilde{f}^k$ and satisfies the boundary conditions. Since $I_{k-1}(x; \bar{f}^{k-1})$ is a cubic spline which does exactly that, it follows from the uniqueness that $I_k(x; \tilde{f}^k) \equiv I_{k-1}(x; \bar{f}^{k-1})$, and thus I_k is hierarchial by (5.17a). Hence by (5.17b) we get that $\hat{I}_k = I_k$ and thus $I_L(x; \bar{f}^k)$ can be represented by the RHS of (5.16b) with

$$\varphi_i^k(x) = I_k(x; \delta_i^k).$$

In the context of finite element, the set

$$\left(\left\{ \{\varphi_{2i-1}^k\}_{i=1}^{J_{k-1}} \right\}_{k=1}^{L}, \ \{\varphi_i^0\}_{i=p}^{J_0} \right)$$

is referred to as hierarchical basis (see [16]).

We remark that in the periodic case the cubic-spline interpolation is translation invariant in the sense that

$$I_k(x - qh_k; \delta_{j-q}^k) = I_k(x; \delta_j^k) \ \text{ for any integer } \ q.$$

Therefore, for each k, φ_i^k can be described in terms of a single function $\varphi_0^k(x) = I_k(x; \delta_0^k)$ by

$$\varphi_i^k(x) = \varphi_0^k(x - ih_k).$$

Example 5.1.3. Fourier collocation

We start with the non-periodic case and define

$$I_k(x; \bar{f}^k) = \sum_{m=0}^{J_k} a_m(\bar{f}^k) \cos m\pi x \tag{5.22a}$$

where $\{a_m(\bar{f}^k)\}_{m=0}^{J_k}$ are given as the solution of the system of J_k+1 linear equations which is obtained from

$$I_k(x_j^k; \bar{f}^k) = \bar{f}_j^k, \; 0 \le j \le J_k. \tag{5.22b}$$

We observe that the RHS of (5.22a) is a polynomial of degree $\le J_k$ in $y = \cos \pi x$, and that the homogeneous problem for $\bar{f}^k = 0$ in (5.22b) amounts to finding coefficients $a_m(0)$ so that it will have $J_k + 1$ roots. This shows that the trivial solution is the only solution of the homogeneous problem, which in turn implies the uniqueness of the non-homogeneous problem

Next we show that I_k is hierarchial: Let $\tilde{f}^k$ denote

$$\tilde{f}_i^k = I_{k-1}(x_i^k; \bar{f}^{k-1}) = \sum_{m=1}^{J_{k-1}} a_m(\bar{f}^{k-1}) \cos m\pi x_i^k, \qquad 0 \le i \le J_k$$

and observe that

$$I_k(x; \tilde{f}^k) = \sum_{m=0}^{J_k} a_m(\tilde{f}^k) \cos m\pi x$$

with

$$a_m(\tilde{f}^k) = \begin{cases} a_m(\bar{f}^{k-1}) & 0 \le m \le J_{k-1} \\ 0 & J_{k-1} + 1 \le m \le J_k \end{cases}$$

satisfies

$$I_k(x_i^k; \tilde{f}^k) = \tilde{f}_i^k, \; 0 \le i \le J_k,$$

which shows that

$$I_k(x; \tilde{f}^k) \equiv I_{k-1}(x; \bar{f}^{k-1}).$$

The functions $\varphi_j^k(x) = I_k(x; \delta_i^k)$ in this case can be expressed in terms of the discrete Dirichlet kernel (see [6]) as follows:

$$(5.22c) \qquad \varphi_0^k(x) = \frac{1}{J_k} \frac{\sin(J_k \pi x)\cos(\pi x/2)}{2\sin(\pi x/2)}, \quad \varphi_j^k(x) = \varphi_0^k(x - x_j^k).$$

In the periodic case we use

$$(5.23a) \qquad I_k(x; \bar{f}^k) = \sum_{m=0}^{J_{k-1}} b_m(\bar{f}^k)\cos 2\pi m x + \sum_{m=1}^{J_{k-1}-1} c_m(\bar{f}^k)\sin 2\pi m x$$

where the J_k coefficients $\{b_m(\bar{f}^k)\}_{m=0}^{J_{k-1}}$, $\{c_m(\bar{f}^k)\}_{m=1}^{J_{k-1}-1}$ are determined uniquely by

$$(5.23b) \qquad I_k(x_i^k; \bar{f}^k) = \bar{f}_i^k, \ 1 \le i \le J_k.$$

It is easy to see that I_k in (5.23a)-(5.23b) is also hierarchial and the functions $\varphi_j^k(x) = I_k(x; \delta_j^k)$ are given by

$$(5.23c) \qquad \varphi_0^k(x) = \frac{1}{J_k} \frac{\sin(J_k \pi x)\cos \pi x}{\sin \pi x}, \quad \varphi_j^k(x) = \varphi_0^k(x - x_j^k).$$

The standard way to reduce the dimensionality of the Fourier representation (5.22a) is to discard terms $\cos(m\pi x)$ for which $a_m(f)$ is small. The multiresolution representation (5.16b) of the Fourier collocation provides another way to reduce dimensionality, i.e. to discard terms $\varphi_{2j-1}^k(x)$ for which $d_j^k(f)$ is small (see [6]).

Example 5.2. Cell-averages

$$(5.24a) \qquad \mathcal{D}_k : L^1[0,1] \to V^k$$

$$(5.24b) \qquad \bar{f}_j^k = (\mathcal{D}_k f)_j = \frac{1}{h_k} \int_{x_{j-1}^k}^{x_j^k} f(x)dx, \ 1 \le j \le J_k;$$

here $L^1[0,1]$ is the space of functions which are absolutely integrable in $[0,1]$. Since $x_j^{k-1} = x_{2j}^k$, $x_{j-1}^{k-1} = x_{2j-2}^k$, we get

$$(5.24c) \qquad \bar{f}_j^{k-1} = \frac{1}{h_k}\left[\int_{x_{2j-2}^k}^{x_{2j-1}^k} f(x)dx + \int_{x_{2j-1}^k}^{x_{2j}^k} f(x)dx\right] = \frac{1}{2}(\bar{f}_{2j-1}^k + \bar{f}_{2j}^k)$$

and thus the decimation matrix D_k^{k-1} is given by

(5.24d) $$(D_k^{k-1})_{i,j} = \frac{1}{2}(\delta_{2i-1,j} + \delta_{2i,j}).$$

Consequently

$$D_k^{k-1} g = 0 \Rightarrow g_{2i} = -g_{2i-1}, \ 1 \le j \le J_{k-1},$$

and therefore we take in (3.6b)

(5.25a) $$w_j = \frac{1}{2}(\delta^k_{2j-1} - \delta^k_{2j}), \ 1 \le j \le J_{k-1},$$

which leads to

(5.25b) $$(G_k^{k-1})_{i,j} = \frac{1}{2}(\delta_{2i-1,j} - \delta_{2i,j});$$

S_k in (3.7a) turns out to be

(5.25c) $$S_k = \frac{1}{2} I.$$

The reconstruction procedure $\mathcal{R}_k$ for this discretization is any operator

(5.26a) $$\mathcal{R}_k : V_k \to L^1[0,1]$$

which satisfies

(5.26b) $$(\mathcal{D}_k \mathcal{R}_k \bar{f}^k)_j = \frac{1}{h_k} \int_{x^k_{j-1}}^{x^k_j} (\mathcal{R}_k \bar{f}^k)(x)dx = \bar{f}^k_j.$$

Let

(5.27a) $$F(x) = \int_0^x f(y)dy$$

and denote $F^k_j = F(x^k_j), \ 0 \le j \le J_k$; note that $F^k_0 = 0$. The relations

(5.27b) $$\bar{f}^k_j = (F^k_j - F^k_{j-1})/h_k, \quad F^k_j = h_k \sum_{i=1}^{j} \bar{f}^k_i$$

show that knowledge of F^k implies knowledge of $\bar{f}^k$ and vice versa. Let $I_k(x; F^k)$ denote any interpolation function which satisfies (5.5) with respect to F^k and define

$$(5.27c) \qquad (\mathcal{R}_k \bar{f}^k)(x) = \frac{d}{dx} I_k(x; F^k);$$

clearly $\mathcal{R}_k$ in (5.27c) satisfies the requirements in (5.26).

The predicted values $\tilde{f}^k$ (3.2a) are given by

$$(5.28a) \qquad \tilde{f}_i^k = (\mathcal{D}_k \mathcal{R}_{k-1} \bar{f}^{k-1})_i = \frac{1}{h_k} \int_{x_{i-1}^k}^{x_i^k} \frac{d}{dx} I_{k-1}(x; \bar{f}^{k-1}) dx$$
$$= \frac{1}{h_k} [I_{k-1}(x_i^k; F^{k-1}) - I_{k-1}(x_{i-1}^k; F^{k-1})].$$

From (3.4) and (5.24d) we get that the prediction error satisfies

$$(5.28b) \qquad e_{2i}^k = -e_{2i-1}^k, \ 1 \le i \le J_{k-1};$$

consequently the scale coefficients d^k can be expressed by

$$(5.28c) \qquad d_i^k = (G_k^{k-1} e^k)_i = \frac{1}{2}(e_{2i-1}^k - e_{2i}^k) = e_{2i-1}^k = -e_{2i}^k.$$

It follows therefore that

(5.28d)

$$d_i^k(f) = e_{2i-1}^k = \bar{f}_{2i-1}^k - \tilde{f}_{2i-1}^k = \frac{1}{h_k}[F_{2i-1}^k - I_{k-1}(x_{2i-1}^k; F^{k-1})] = \hat{d}_i^k(F)/h_k$$

where $\hat{d}_i^k(F)$ denotes the interpolation error in F, which is also the scale coefficient in (5.7)-(5.8).

We turn now to describe the multiresolution transform (3.13) and its inverse (3.14) in the non-periodic case.

$\mu(\bar{f}^L) = M\bar{f}^L$ (Encoding)

$$(5.29) \qquad \begin{cases} \text{DO } k = L, \dots, 1 \\ \bar{f}_i^{k-1} = \frac{1}{2}(\bar{f}_{2i-1}^k + \bar{f}_{2i}^k), \ 1 \le i \le J_{k-1} \\ d_i^k = [F_{2i-1}^k - I_{k-1}(x_{2i-1}^k; F^{k-1})]/h_k, \ 1 \le i \le J_{k-1}. \end{cases}$$

$\underline{\bar{f}^L = M^{-1}\mu(\bar{f}^L)}$ (Decoding)

$$(5.30)\qquad \begin{cases} \text{DO } k = 1, \dots, L \\ \text{DO } i = 1, \dots, J_{k-1} \\ \bar{f}^k_{2i-1} = [I_{k-1}(x^k_{2i-1}; F^{k-1}) - F^{k-1}_{i-1}]/h_k + d^k_i \\ \bar{f}^k_{2i} = 2\bar{f}^{k-1}_i - \bar{f}^k_{2i-1}. \end{cases}$$

We remark that (5.27b) can be used in order to express the algorithms (5.29)-(5.30) directly in terms of the cell-averages $\bar{f}^k$; in this case the algorithms apply to periodic data by periodic extension of $\bar{f}^k$. When we use the primitive function $F(x)$ (5.27a) we need to have $F(1) = F(0) = 0$ in order to use periodic extension of F^k. Let us denote

$$\bar{f}_{av} = \sum_{j=1}^{J_k} h_k \bar{f}^k_j = \int_0^1 f(x)dx$$

and observe that $F(1) = \bar{f}_{av}$. If the given cell-averages do not satisfy $\bar{f}_{av} = 0$, we subtract the constant value $\bar{f}_{av}$ from them before applying the algorithm and then add it back to the resulting cell-averages, i.e. in (5.29) we subtract $\bar{f}_{av}$ from $\bar{f}^L$ and then add it to the resulting $\bar{f}^0$, and in (5.30) we subtract $\bar{f}_{av}$ from $\bar{f}^0$ and add it to the resulting $\bar{f}^L$.

Next we show that if $\hat{I}_k$ is hierarchial (5.16) then so is the reconstruction (5.27c)

$$(5.31a)\qquad (\hat{\mathcal{R}}_k \bar{f}^k)(x) = \frac{d}{dx}\hat{I}_k(x; F^k).$$

Let us denote

$$(5.31b)\qquad \hat{I}_k(x; F^k) = \sum_{j=1}^{J_k} F^k_j \hat{\varphi}^k_j(x), \quad \hat{\varphi}^k_j(x) = \hat{I}_k(x; \delta^k_i)$$

and assume that $\hat{I}_k$ is hierarchial, i.e.

$$(5.31c)\qquad \hat{I}_k(x; F^k) - \hat{I}_{k-1}(x; F^{k-1}) = \sum_{j=1}^{J_{k-1}} \hat{d}^k_j(F)\hat{\varphi}^k_{2j-1}(x).$$

Differentiating (5.31b)-(5.31c) and using (5.27b) and (5.28c) we get that

(5.32a)
$$(\hat{\mathcal{R}}_k \bar{f}^k)(x) = \sum_{j=1}^{J_k} \bar{f}_j^k \varphi_j^k(x),$$

(5.32b)
$$(\hat{\mathcal{R}}_k \bar{f}^k)(x) - (\hat{\mathcal{R}}_{k-1} \bar{f}^{k-1})(x) = \sum_{j=1}^{J_{k-1}} d_j^k(f) \psi_j^k(x),$$

where

(5.32c)
$$\varphi_j^k(x) = \sum_{i=j}^{J_k} h_k \frac{d}{dx} \hat{\varphi}_i^k(x), \quad \psi_j^k(x) = h_k \frac{d}{dx} \hat{\varphi}_{2j-1}^k(x),$$

Example 5.2.1. Piecewise polynomial reconstruction

Using the piecewise-polynomial interpolation (5.18) for the primitive function $F(x)$ we get from (5.27c) that

(5.33a)
$$(\mathcal{R}_k \bar{f}^k)(x) = \frac{d}{dx} q_j(x; F^k, r, s) = \frac{1}{h_k} \sum_{m=-s}^{-s+r} F_{j+m}^k L'_m \left(\frac{x - x_j^k}{h_k} \right), \quad x_{j-1}^k < x \le x_j^k,$$

and consequently in (5.28a)

(5.33b)
$$\tilde{f}_{2j-1}^k = \frac{1}{h_k} \sum_{m=-s}^{-s+r} (F_{j+m}^{k-1} - F_{j-1}^{k-1}) L_m(-1/2) = \sum_{\ell=-s+1}^{-s+r} \beta_\ell \bar{f}_{j+\ell}^{k-1},$$

(5.33c)
$$\tilde{f}_{2j}^k = 2\bar{f}_j^{k-1} - \tilde{f}_{2j-1}^k = 2\bar{f}_j^{k-1} - \sum_{\ell=-s+1}^{-s+r} \beta_\ell \bar{f}_{j+\ell}^{k-1},$$

where

(5.33d)
$$\beta_\ell = \begin{cases} 2 \sum\limits_{m=\ell}^{-s+r} L_m\left(-\frac{1}{2}\right) & 0 \le \ell \le -s+r \\ -2 \sum\limits_{m=-s}^{\ell-1} L_m\left(-\frac{1}{2}\right) & -s+1 \le \ell \le -1 \end{cases};$$

the coefficients $\{\beta_\ell\}$ depend on the choice of stencil in q_j^k. Note that in the periodic case (5.19) we take the same stencil for all j and k, and therefore we get (in the same way as in (5.20)) that if

(5.34a)
$$\exists \lim_{k \to \infty} \mathcal{R}_k \bar{\varphi}_0^{0,k} = \varphi_0$$

then

$$\exists \lim_{k\to\infty} \mathcal{R}_k \bar{\varphi}_i^{m,m+k} = \varphi_i^m, \quad \varphi_i^m = \varphi\left(\frac{x}{h_m} - i\right), \tag{5.34b}$$

where

$$\varphi(x) = \varphi_0(xh_0), \quad \text{support}(\varphi) = [2(s-r)-1, 2(s-1)]. \tag{5.34c}$$

In Appendix B of [12] we use the continuous differentiability of the limit functions (5.20) for the centered interpolation with $r = 3, 5$ (see [5], [7]), which we denote here by $\hat{\varphi}(x)$, to prove uniform convergence in (5.34) to a continuous $\varphi(x)$. From Lemma 4.2 and (5.33) we get that $\varphi(x)$ satisfies the dilation relation

$$\varphi(x) = 2\varphi(2x) + \sum_{\ell=-s+1}^{-s+r} \beta_\ell[\varphi(2x+2\ell+1) - \varphi(2x+2\ell)]; \tag{5.34d}$$

from (4.8b) and (5.25) we get that

$$\psi_j^k(x) = \psi\left(\frac{2x}{h_k} - j\right), \quad \psi(x) = \varphi(2x+1) - \varphi(2x). \tag{5.34e}$$

The hierarchial form $\hat{\mathcal{R}}_k$ (4.10) which results from the limiting process above is identical to the one obtained by differentiating the hierarchial form of the interpolation $\hat{I}_k$ in (5.31)-(5.32). The relations between $\hat{\varphi}(x)$ and $\varphi(x)$, $\psi(x)$ in (5.32c) can be expressed by

$$\varphi(x) - \varphi(x-1) = \hat{\varphi}'(x), \quad \psi(x) = \hat{\varphi}'(2x+1) \tag{5.35}$$

(see [7] and [12, Appendix B]).

The case $s = r = 1$ corresponds to the piecewise-linear interpolation, which is hierarchial to begin with. Here $\hat{\varphi}(x)$ is the "hat function"

$$\hat{\varphi}(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.36a}$$

which satisfies the dilation equation (5.20f)

$$\hat{\varphi}(x) = \hat{\varphi}(2x) + \frac{1}{2}[\hat{\varphi}(2x-1) + \hat{\varphi}(2x+1)], \quad \text{supp}(\hat{\varphi}) = [-1,1]. \tag{5.36b}$$

The reconstruction $\mathcal{R}_k$ (5.27c) for the cell-averages in this case is the piecewise-constant function

$$(\mathcal{R}_k \bar{f}^k)(x) = \bar{f}_j^k \quad \text{for } x_{j-1}^k < x \le x_j^k. \tag{5.37a}$$

Here

$$\varphi(x) = \chi_{(-1,0]}(x) = \begin{cases} 1 & -1 < x \le 0 \\ 0 & \text{otherwise} \end{cases}, \tag{5.37b}$$

which satisfies the dilation equation (5.34d)

$$\varphi(x) = \varphi(2x) + \varphi(2x+1), \quad \text{supp}(\varphi) = [-1,0]; \tag{5.37c}$$

$\psi(x)$ in (5.34e) is given by
(5.37d)

$$\psi(x) = \varphi(2x+1) - \varphi(2x) = \chi_{(-1,-\frac{1}{2}]}(x) - \chi_{(-\frac{1}{2},0]}(x) = \begin{cases} 1 & -1 < x \le -\frac{1}{2} \\ -1 & -\frac{1}{2} < x \le 0 \\ 0 & \text{otherwise} \end{cases}.$$

The multiresolution basis (4.14b) in this case is the Haar basis.

In Figures 1-4 we present some numerical examples for the limiting process in (5.20c) and (5.34a) for the case of a centered stencil with $r = 3$ and $r = 5$. In Figures 1 and 2 we show the results for the interpolation (5.20c) with $r = 3$ and 5, respectively; part (a) (top) of each figure shows $\varphi_0^{0,6}$ and part (b) (bottom) shows $\varphi_0^{0,1}$. The circles in these figures are the pointvalues of these functions at the gridpoints of X^6; here $h_0 = 1/16$. In Figures 3 and 4 we show the results for the cell-averages (5.34a) with $r = 3$ and 5, respectively. Parts (a) and (c) in each figure show $\varphi_0^{0,6}$ and $\psi_0^{0,6}$, while parts (b) and (d) show $\varphi_0^{0,1}$ and $\psi_0^{0,1}$, respectively; the circles denote the cell-averages of these functions in the intervals of X^6.

We remark that the convergence of the limiting process in both cases is very fast, and the results for $L = 6$ are very close to those of the limit functions.

Remark 5.1. (1) Discretization by cell-averages is particularly useful for the numerical solution of integral equations with integrably singular kernel (see [12]) and for the computation of discontinuous solutions of hyperbolic conservation laws, where it leads to schemes in conservation form (see [13] and [14]).

(2) In this paper we describe reconstruction from cell-averages through interpolation of the primitive function. A more general approach which applies to unstructured grids in several space dimensions is described in [9] and [10].

(3) As we have pointed out in Section 3, the multiresolution transforms can be applied with nonlinear operators $\mathcal{R}_k$. In [6] and [11] we show how to use essentially non-oscillatory (ENO) interpolation techniques, which use an adaptive selection of stencil in (5.19b), in order to improve data-compression of discontinuous data.

6. The Diadic constant-coefficient case

In this section we consider the case of a diadic sequence of uniform one-dimensional grids where D_k^{k-1} (2.3) is a "Töplitz matrix"

$$(D_k^{k-1})_{i,j} = \alpha_{j-2i}, \quad \text{independent of } \ k; \tag{6.1}$$

this case has been studied extensively, primarily within the context of wavelets. In this section we examine it within the general framework for multiresolution representation which was described in the previous sections; later in Section 7 we present the wavelet formulation of this case. Our formulation starts with a finite-dimensional vector-space V (2.1a), and therefore we have to consider the periodic case where the matrix in (6.1) is to be interpreted in a cyclic manner. In order to avoid this awkwardness in notation we consider first the infinite case of grids in $\mathbb{R}$ and functions in $L^2_{\text{loc}}(\mathbb{R})$, and obtain the finite-dimensional case later by taking $\mathcal{F}$ to be periodic functions with period 1, and by describing the multiresolution representation for the finite number of discrete quantities which correspond to $[0, 1]$. Thus D_k^{k-1} and P_{k-1}^k will be treated as infinite matrices; $\{\alpha_\ell\}$ will be taken to be an infinite sequence with compact support, i.e. we assume that α_ℓ are zero when $|\ell| > L$ for some finite integer L.

Let $\{\mathbf{X}^k\}_{k=0}^{\infty}$ be the following sequence of nested diadic grids in $\mathbb{R}$

(6.2a) $$\mathbf{X}^k = \{x_j^k\}_{j=-\infty}^{\infty}, \quad x_j^k = j \cdot h_k, \quad h_k = 2^{-k} h_0, \quad h_0 = 1/J_0$$

where J_0 is some integer, and note that

(6.2b) $$\mathbf{X}^{k-1} \subset \mathbf{X}^k, \quad \mathbf{X}^k - \mathbf{X}^{k-1} = \{x_{2j-1}^k\}_{j=-\infty}^{\infty}.$$

We observe that

(6.2c) $$\{x_j^k\}_{j=0}^{J_k}, \ J_k = 2^k J_0, \quad \text{is a partition of } [0,1].$$

Let $\omega(x)$ be a function of compact support which satisfies

(6.3a) $$\int \omega(x) dx = 1, \quad \int [\omega(x)]^2 dx < \infty$$

and denote

(6.3b) $$\omega_j^k(x) = \frac{1}{h_k} \omega\left(\frac{x}{h_k} - j\right).$$

We define the discretization operator $\mathcal{D}_k$,

(6.3c) $$\mathcal{D}_k : L^2_{\text{loc}}(\mathbb{R}) \to \mathbb{R} \times \mathbb{Z}$$

by

(6.3d) $$(\mathcal{D}_k f)_i = \bar{f}_i^k = \langle f, \omega_i^k \rangle = \int f(x) \omega_i^k(x) dx.$$

Hence $\bar{f}_i^k$ is an average of f over few cells of the k-th grid around x_i^k with a weight function $\omega_i^k(x)$; note that for $f \equiv 1$ we get $\mathcal{D}_k f \equiv 1$ for all k.

We assume now that D_k^{k-1} has the form (6.1), i.e.

(6.4a) $$\bar{f}_i^{k-1} = (D_k^{k-1} \bar{f}^k)_i = \sum_j \alpha_{j-2i} \bar{f}_j^k = \sum_\ell \alpha_\ell \bar{f}_{2i+\ell}^k.$$

Using the definition (6.3d) we get for $f \equiv 1$ that

(6.4b)
$$\sum_{\ell} \alpha_\ell = 1,$$

and that for any $f \in L^2_{\text{loc}}(\mathbb{R})$

$$\left\langle \omega_i^{k-1} - \sum_{\ell} \alpha_\ell \omega_{2i+\ell}^k,\ f \right\rangle = 0$$

which implies that

(6.5a)
$$\omega_i^{k-1} = \sum_{\ell} \alpha_\ell \omega_{2i+\ell}^k.$$

Substituting $y = \frac{x}{h_{k-1}} - j$ in the above relation, we get that ω is the solution of the dilation equation

(6.5b)
$$\omega(y) = 2 \sum_{\ell} \alpha_\ell \omega(2y - \ell).$$

It is shown in [4] and [15] that given $\{\alpha_\ell\}$ which satisfy (6.4b), the dilation equation has a solution ω which is unique up to a multiplicative constant and a shift; however, ω is defined in terms of its Fourier transform and does not necessarily correspond to a function. Daubechies observes in [4] that without imposing additional constraints, the solution ω of the dilation equation is typically a fractal. Strang points out in [15] that $\omega = \delta$, the Dirac distribution, is the solution of the simplest dilation equation

$$\omega(x) = 2\omega(2x).$$

We are interested in $\omega(x)$ which is a generalized "weight function" in the sense that it is concentrated around some point (which we take to be $x = 0$), and that its regularity is not worse than a distribution; we do not need ω to be nonnegative. Daubechies shows in [4] that if $\{\alpha_\ell\}$ has its support in $L_1 \leq \ell \leq L_2$, then $\omega(x)$, the solution of the corresponding dilation equation (6.5b), is likewise supported in an interval of length $L_2 - L_1$. We recall that the solution of the dilation equation is determined up to a multiplicative constant and a shift. We fix the multiplicative constant by (6.3a) and locate it so that its main concentration is placed around $x = 0$. As an example, consider the hierarchy of functions ω^m which is obtained by repeated convolutions with the characteristic function $\chi_{[-\frac{1}{2},\frac{1}{2}]}$,

$$\omega^{m+1} = \omega^m * \chi_{[-\frac{1}{2},\frac{1}{2}]}, \ \omega^0 = \delta. \tag{6.6a}$$

It is easy to see that ω^m has its support in $|x| \le m/2$ and it satisfies a dilation equation (6.5b) with coefficients $\{\alpha_\ell^m\}$, which can be computed by the recursive relation

$$\alpha_\ell^{m+1} = \frac{1}{2}(\alpha_\ell^m + \alpha_{\ell+1}^m), \ \alpha_\ell^0 = \delta_{\ell,0}; \tag{6.6b}$$

observe that $\{\alpha_\ell^m\}$ has its support in $0 \le \ell \le m$ (see [11]).

The simplest and most commonly used discretization is that by pointvalues; this corresponds to $\omega = \delta$, i.e. $m = 0$ in the hierarchy (6.6) (see Example 5.1). The reconstruction procedure in this case is given by any interpolation method of the pointvalue data. This discretization is most suitable for $\mathcal{F}$ of continuous functions. The next weight function in the hierarchy (6.6) is $\omega = \chi_{[-\frac{1}{2},\frac{1}{2}]}$ which corresponds to discretization by cell-averages (see Example 5.2), and is most suitable for $\mathcal{F}$ of piecewise-smooth data with jump-discontinuities. The "hat function", $m = 2$ in the hierarchy (6.6) is most suitable for $\mathcal{F}$ of piecewise-smooth distributions, i.e. piecewise smooth functions with a finite number of δ-singularities (see [11]); such $\mathcal{F}$ is used in vortex methods for the numerical solution of fluid dynamics problems.

Requirement (2.2), which is formulated for the finite-dimensional case, ensures that the J_k components of the prediction error e^k can be expressed in terms of $\Delta_k = J_k - J_{k-1} = J_{k-1}$ quantities $\{d_j^k\}_{j=1}^{J_{k-1}}$; then the counting in (3.11) shows that there is a one-to-one transformation between $\bar{f}^L$ and its multiresolution representation $\mu(\bar{f}^L)$. In [11] we rewrite $D_k^{k-1} e^k = 0$ (3.4) as

$$\sum_\ell \alpha_{2\ell} e_{2(i+\ell)}^k = -\sum \alpha_{2\ell+1} e_{2(i+\ell)+1}^k, \quad 1 \le i \le J_{k-1}, \tag{6.7a}$$

and choose

$$d_j^k = e_{2j-1}^k, \quad 1 \le j \le J_{k-1}. \tag{6.7b}$$

This method is particularly attractive when

$$|\alpha_0| > \sum_{\ell \neq 0} |\alpha_{2\ell}|, \tag{6.7c}$$

which implies that the coefficient matrix in the system of linear equations (6.7a) for the even components of the prediction error is diagonally dominant. In the present paper we choose to remove the redundancy from the prediction error by (3.8) which involves the matrices S_k and G_k^{k-1} (note the change in notation for the diadic case: $\Delta_k \to k-1$). The motivation for this choice is our wish to relate the general framework of the present paper to the more familiar theory of wavelets.

Following Daubechies in [4] we now show:

Lemma 6.1. *Let* G_k^{k-1} *be the infinite matrix*

$$(G_k^{k-1})_{i,j} = (-1)^{j+1} \alpha_{2i-j-1} \tag{6.8a}$$

then

$$\text{(i)} \quad D_k^{k-1}(G_k^{k-1})^* = 0 \tag{6.8b}$$

$$\text{(ii)} \quad S_k = (D_k^{k-1})^* D_k^{k-1} + (G_k^{k-1})^* G_k^{k-1} \tag{6.8c}$$

is a banded Töplitz matrix which is given by

$$(S_k)_{ij} = s(|i-j|), \; s(2m+1) = 0, \; s(2m) = \sum_\ell \alpha_\ell \alpha_{\ell+2m}. \tag{6.8d}$$

Proof:

$$\begin{aligned}\text{(i)} \quad [D_k^{k-1}(G_k^{k-1})^*]_{ij} &= \sum_\ell (-1)^{\ell+1} \alpha_{\ell-2i} \alpha_{2j-\ell-1} \\ &= -\sum_{\ell'} (-1)^{\ell'+1} \alpha_{\ell'-2i} \alpha_{2j-\ell'-1}\end{aligned}$$

where we used the substitution $\ell' = 2(j+i) - \ell - 1$.

(ii) Using $(-1)^{i+j} = (-1)^{|i-j|}$ we can express $(S_k)_{ij}$ in (6.8c) as

$$(S_k)_{ij} = \sum_\ell \alpha_{i-2\ell}\alpha_{i-2\ell+(j-i)} + \sum_\ell (-1)^{|j-i|}\alpha_{2\ell-i-1}\alpha_{2\ell-i-1-(j-i)}.$$

For $j-i=2m$, using $\ell' = i-\ell+m$ in the second sum, we get

$$\begin{aligned}(S_k)_{ij} &= \sum_\ell \alpha_{i-2\ell}\alpha_{i-2\ell+2m} + \sum_{\ell'} \alpha_{i-(2\ell'+1)+2m}\alpha_{i-(2\ell'+1)} \\ &= \left(\sum_{\ell=\text{even}} + \sum_{\ell=\text{odd}}\right)\alpha_\ell\alpha_{\ell+2m} = \sum_\ell \alpha_\ell\alpha_{\ell+2m}.\end{aligned}$$

For $j-i=2m-1$, using $\ell' = i-\ell+m$ in the second sum, we get

$$(S_k)_{i,j} = \sum_\ell \alpha_{i-2\ell}\alpha_{i-2\ell+2m-1} - \sum_{\ell'} \alpha_{i-2\ell'+2m-1}\alpha_{i-2\ell'} = 0.$$

Clearly if $\{\alpha_\ell\}$ has its support in $L_1 \le \ell \le L_2$ we get that $s(m)=0$ for $|m| \ge p = \left[\frac{L_2-L_1+1}{2}\right]$, where [ ] denotes the integer part of the number. □

Algorithm (3.14) for the calculation of the inverse multiresolution transform is well defined for any invertible S_k. However in order to ensure the stability of the Cholesky decomposition $S_k = LL^*$, where L is a lower diagonal matrix, we would like S_k to be diagonally dominant. Hence we consider weight functions $\omega(x)$ which are solutions of the dilation equation (6.5b) with coefficients $\{\alpha_\ell\}_{\ell=L_1}^{L_2}$ that satisfy

(6.9a) $$\sum_{\ell=L_1}^{L_2} (\alpha_\ell)^2 > 2\sum_{q=1}^{p}\left|\sum_{\ell=L_1}^{L_2} \alpha_\ell\alpha_{\ell+2q}\right|, \quad p = \left[\frac{L_2-L_1+1}{2}\right].$$

We remark that this condition is met for $0 \le m \le 4$ in the hierarchy (6.6); it is interesting to note that the method (6.7) of [11] leads to a diagonally dominant matrix for $0 \le m \le 7$ in (6.6). For $m = 0, 1$, as well as for the Daubechies' wavelets, we have the attractive situation where $s(2q) = s(0)\delta_{q,0}$, i.e.,

(6.9b) $$S_k = s(0)I.$$

We turn now to consider the class of linear reconstruction operators $\mathcal{R}_k$ for this discretization:

$$\mathcal{R}_k : \mathbb{R} \times \mathbb{Z} \to L^2_{\text{loc}}(\mathbb{R}) \tag{6.10a}$$

$$\mathcal{D}_k \mathcal{R}_k = I. \tag{6.10b}$$

Definition 6.1. Order of accuracy

We say that the reconstruction $\mathcal{R}_k$ is r-th order accurate if

$$\mathcal{R}_k(\mathcal{D}_k x^m) = x^m \quad \text{for} \quad 0 \leq m \leq r-1. \tag{6.11a}$$

We observe that if $\mathcal{R}_{k-1}$ is r-th order accurate, then the resulting prediction $\tilde{f}^k = P^k_{k-1}\bar{f}^{k-1}$ in (3.2a) is exact for polynomial data of degree less or equal $r-1$,

$$P^k_{k-1}(\mathcal{D}_{k-1}x^m) = \mathcal{D}_k(\mathcal{R}_{k-1}\mathcal{D}_{k-1}x^m) = \mathcal{D}_k x^m, \quad 0 \leq m \leq r-1. \tag{6.11b}$$

Since $f(x) \equiv 1$ implies $\bar{f}^k \equiv 1$ for all k, we say that $\mathcal{R}_k$ is *consistent* if

$$(\mathcal{R}_k \cdot 1)(x) \equiv 1 \tag{6.11c}$$

i.e. $\mathcal{R}_k$ is at least first-order accurate. Using $m = 0$ in (6.11b) we get that the rows of P^k_{k-1} have to satisfy

$$\sum_\ell (P^k_{k-1})_{i,\ell} = 1 \qquad \text{for all } i. \tag{6.11d}$$

Next we show that given a "reasonable" weight function, we can derive r-th order reconstruction procedures from corresponding r-th order interpolation methods; thus we have a large class of reconstruction operators to choose from.

Lemma 6.2. *Let $u(y)$ be such that*

$$u(j) = \delta_{0,j} \tag{6.12a}$$

$$\sum_j (j)^m u(y-j) = y^m \quad \text{for} \quad 0 \leq m \leq r-1; \tag{6.12b}$$

define the matrix B by

$$B_{ij} = \langle \omega(\cdot - i), u(\cdot - j)\rangle = \int \omega(y-i)u(y-j)dy, \tag{6.12c}$$

and denote

$$u_j^k(x) = u\left(\frac{x}{h_k} - j\right). \tag{6.12d}$$

If B is invertible, then

$$\mathcal{R}_k \bar{f}^k = \sum_j \beta_j(\bar{f}^k)u_j^k, \quad \beta(\bar{f}^k) = B^{-1}\bar{f}^k \tag{6.12e}$$

is an r-th order reconstruction.

Proof: We observe that

$$(\mathcal{D}_k u_j^k)_i = \langle \omega_i^k, u_j^k\rangle = \langle \omega(\cdot - i), u(\cdot - j)\rangle = B_{ij},$$

and therefore we get from (6.12e)

$$\begin{aligned}(\mathcal{D}_k \mathcal{R}_k \bar{f}^k)_i &= \sum_j \beta_j(\bar{f}^k)(\mathcal{D}_k u_j^k)_i = \sum_j (B^{-1}\bar{f}^k)_j B_{ij} \\ &= (BB^{-1}\bar{f}^k)_i = \bar{f}_i^k,\end{aligned}$$

which shows that $\mathcal{R}_k$ in (6.12e) satisfies relation (3.1b). Substituting $y = x/h_k$ in (6.12b) we get

$$\sum_j (x_j^k)^m u_j^k(x) = x^m, \quad 0 \le m \le r-1.$$

Applying $\mathcal{D}_k$ to the above relation we see that

$$(x_j^k)^m = \beta_j(\mathcal{D}_k x^m);$$

it follows therefore that for $0 \le m \le r-1$

$$\mathcal{R}_k(\mathcal{D}_k x^m) = \sum_j \beta_j(\mathcal{D}_k x^m)u_j^k = \sum_j (x_j^k)^m u_j^k = x^m.$$

□

Lemma 6.3. *Let $\mathcal{R}_k$ be a linear operator which is translation invariant in the sense that*

$$(\mathcal{R}_k \delta^k_{j-q})(x - qh_k) = (\mathcal{R}_k \delta^k_j)(x) \quad \text{for any integer} \quad q, \tag{6.13a}$$

and denote

$$\gamma^k_\ell = \langle \omega^{k+1}_\ell, \mathcal{R}_k \delta^k_0 \rangle; \tag{6.13b}$$

then the prediction matrix P^{k+1}_k has the constant value γ^k_ℓ on its "subdiagonal" $i - 2j = \ell$, i.e.

$$(P^{k+1}_k)_{i,j} = \gamma^k_{i-2j}. \tag{6.13c}$$

Proof: It follows from (6.13a) for $q = j$ that

$$\begin{aligned}(P^{k+1}_k)_{i,j} &= (\mathcal{D}_{k+1} \mathcal{R}_k \delta^k_j)_i = \langle \omega^{k+1}_i,\ \mathcal{R}_k \delta^k_j \rangle = \langle \omega^{k+1}_i,\ \mathcal{R}_k \delta^k_0(\cdot - jh_k) \rangle \\ &= \langle \omega^{k+1}_i(\cdot + jh_k),\ \mathcal{R}_k \delta^k_0 \rangle = \langle \omega^{k+1}_{i-2j},\ \mathcal{R}_k \delta^k_0 \rangle \\ &= \gamma^k_{i-2j}.\end{aligned}$$

□

Corollary 6.1. (i) *The predicted values $\tilde{f}^{k+1} = P^{k+1}_k \bar{f}^k$ can be computed by*

$$\tilde{f}^{k+1}_{2i} = \sum_\ell \gamma^k_{2\ell} \bar{f}^k_{i-\ell}, \quad \tilde{f}^{k+1}_{2i+1} = \sum_\ell \gamma^k_{2\ell+1} \bar{f}^k_{i-\ell}. \tag{6.14a}$$

(ii) *If the reconstruction is consistent, then*

$$\sum_\ell \gamma^k_{2\ell} = \sum_\ell \gamma^k_{2\ell+1} = 1. \tag{6.14b}$$

In the following we describe the periodic case, which is obtained from the previous formulation for functions in $L^2_{\mathrm{loc}}(\mathbb{R})$ by taking $\mathcal{F}$ to be the periodic extension of functions $f \in L^2[0,1]$, i.e. $f(x+n) = f(x)$ for all integers n. We recall from (6.2c) that for $h_0 = 1/J_0$

$$\text{(6.15a)} \qquad \{X_j^k\}_{j=0}^{J_k}, \ x_j^k = j \cdot h_k, \ h_k = 2^{-k} h_0, \ J_k = 2^k J_0,$$

is a partition of $[0,1]$. The discrete quantities in this case can be expressed by a periodic extension of their values for $1 \le j \le J_k$, e.g.

$$\text{(6.15b)} \qquad \bar{f}_{-j}^k = \bar{f}_{J_k - j}, \ \bar{f}_{J_k+j+1}^k = \bar{f}_{j+1}^k, \ \text{ for } \ 0 \le j \le J_k - 1,$$

and similarly for other discrete quantities.

The multiresolution transform (3.13) in the diadic constant coefficient case can be performed by the following algorithm:

$\underline{\mu(\bar{f}^L) = M\bar{f}^L}$ (Encoding)

(6.16)

$$\begin{cases} & \text{DO} \quad k = L, \ldots, 1 \\ \text{(i)} & \bar{f}_j^{k-1} = \sum_\ell \alpha_\ell \bar{f}_{2j+\ell}^k, \ \ 1 \le j \le J_{k-1}; \\ \text{(ii)} & e_{2j-1}^k = \bar{f}_{2j-1}^k - \sum_\ell \gamma_{2\ell+1}^{k-1} \bar{f}_{j-\ell-1}^{k-1}, \ e_{2j}^k = \bar{f}_{2j}^k - \sum_\ell \gamma_{2\ell}^{k-1} \bar{f}_{j-\ell}^{k-1}, \ 1 \le j \le J_{k-1}; \\ \text{(iii)} & d_j^k = -\sum_\ell \alpha_{2\ell-1} e_{2(j-\ell)}^k + \sum_\ell \alpha_{2\ell} e_{2(j-\ell)-1}^k, \ 1 \le j \le J_{k-1}. \end{cases}$$

From Lemma 6.1 we see that if $\{\alpha_\ell\}$ is of compact support then S_k (6.8c) is banded. In order to express the prediction error e^k in terms of the scale coefficients d^k (3.8a) we have to solve the system of linear equations

$$\text{(6.17a)} \qquad S_k e^k = (G_k^{k-1})^* d^k$$

for the J_k components of e^k. Using property (6.8d) we get that the above system decouples into two separate systems for the odd and even components of e^k, i.e.

$$\text{(6.17b)} \qquad \sum_m s(2m) e_{2(i-m)-1}^k = \sum_\ell \alpha_{2\ell} d_{i+\ell}^k, \ \ 1 \le i \le J_{k-1},$$

$$\text{(6.17c)} \qquad \sum_m s(2m) e_{2(i-m)}^k = -\sum_\ell \alpha_{2\ell-1} d_{i+\ell}^k, \ \ 1 \le i \le J_{k-1}.$$

Let e^{odd} and e^{even} denote the column-vectors of the J_{k-1} odd and even components of e^k, respectively,

$$(6.17d) \qquad e_i^{\text{odd}} = e_{2i-1}^k, \; e_i^{\text{even}} = e_{2i}^k, \; 1 \le i \le J_{k-1},$$

and let $\hat{e}^{\text{odd}}$ and $\hat{e}^{\text{even}}$ denote the RHS of (6.17b) and (6.17c), respectively. Using the periodic extension (6.15b) we rewrite (6.17b)-(6.17c) in the matrix form

$$(6.17e) \qquad \hat{S}e^{\text{odd}} = \hat{e}^{\text{odd}}, \; \hat{S}e^{\text{even}} = \hat{e}^{\text{even}},$$

where $\hat{S}$ is a cyclic banded Töplitz matrix of dimension J_{k-1}, and observe that (6.17e) can be solved in $O(J_{k-1})$ operations. Using these notations we now describe the inverse multiresolution transform (3.14) by the following algorithm:

$\underline{\bar{f}^L = M^{-1}\mu(\bar{f}^L)}$ (Decoding)

(6.18)

$$\begin{cases} & \text{DO} \quad k = 1, \ldots, L \\ \text{(i)} & \hat{e}_j^{\text{odd}} = \sum_\ell \alpha_{2\ell} d_{j+\ell}^k, \; \hat{e}_j^{\text{even}} = -\sum_\ell \alpha_{2\ell-1} d_{j+\ell}^k, \; 1 \le j \le J_{k-1}; \\ \text{(ii)} & e^{\text{odd}} = \hat{S}^{-1}\hat{e}^{\text{odd}}, \; e^{\text{even}} = \hat{S}^{-1}\hat{e}^{\text{even}}; \\ \text{(iii)} & \bar{f}_{2j-1}^k = \sum_\ell \gamma_{2\ell+1}^{k-1} \bar{f}_{j-\ell-1}^{k-1} + e_j^{\text{odd}}, \; \bar{f}_{2j}^k = \sum_\ell \gamma_{2\ell}^{k-1} \bar{f}_{j-\ell}^{k-1} + e_j^{\text{even}}, \; 1 \le j \le J_{k-1}. \end{cases}$$

We observe that if $\{\alpha_\ell\}$ satisfy

$$(6.19a) \qquad \sum_\ell \alpha_\ell \alpha_{\ell+2m} = 0 \quad \text{for} \quad m \ne 0$$

then it follows from (6.8d) that

$$(6.19b) \qquad S_k^{-1} = \frac{1}{s(0)} I$$

and thus the inverse multiresolution transform (6.18) takes the simpler form

$$(6.19c) \qquad \begin{cases} \text{DO} \quad k = 1, \ldots, L \\ \text{DO} \quad j = 1, \ldots, J_{k-1} \\ \bar{f}_{2j-1}^k = \sum_\ell \gamma_{2\ell+1}^{k-1} \bar{f}_{j-\ell-1}^{k-1} + \frac{1}{s(0)} \sum_\ell \alpha_{2\ell} d_{j+\ell}^k \\ \bar{f}_{2j}^k = \sum_\ell \gamma_{2\ell}^{k-1} \bar{f}_{j-\ell}^{k-1} - \frac{1}{s(0)} \sum_\ell \alpha_{2\ell-1} d_{j+\ell}^k. \end{cases}$$

We turn now to consider the limiting process (4.7) in the diadic constant coefficient case.

Lemma 6.4. *Assume that there exist* $\{\varphi_0^k\} \in L^2[0,1]$ *such that for all* k

(6.20a) $$\lim_{L\to\infty} \|\mathcal{R}_L A_0^L \delta_0^k - \varphi_0^k\|_{L^2[0,1]} = 0$$

and denote

(6.20b) $$\varphi_i^k(x) = \varphi_0^k(x - ih_k);$$

then for all i *and* m

(6.20c) $$\text{(i)} \;\; \exists \lim_{L\to\infty} \|\mathcal{R}_L A_m^L \delta_i^m - \varphi_i^m\|_{L^2[0,1]} = 0,$$

(6.20d) $$\text{(ii)} \;\; \mathcal{D}_m \varphi_i^m = \delta_i^m;$$

(iii) $\{\varphi_0^k(x)\}$ *satisfy the relation*

(6.20e) $$\varphi_0^{k-1}(x) = \sum_\ell \gamma_\ell^{k-1} \varphi_0^k(x - \ell h_k),$$

where γ_ℓ^k *are defined by* (6.13b).

Proof:

$$\text{(i)} \;\; \varphi_i^{m,L} = \mathcal{R}_L A_m^L \delta_i^m = \prod_{k=m}^{L-1} (\mathcal{R}_{k+1}\mathcal{D}_{k+1}) \cdot \mathcal{R}_m \delta_i^m.$$

Using (6.13a) above we get

$$\varphi_i^{m,L}(x) = \prod_{k=m}^{L-1} (\mathcal{R}_{k+1}\mathcal{D}_{k+1}) \cdot (\mathcal{R}_m \delta_i^m)(x) = \prod_{k=m}^{L-1} (\mathcal{R}_{k+1}\mathcal{D}_{k+1})(\mathcal{R}_m \delta_0)(x - ih_m)$$
$$= \varphi_0^{m,L}(x - ih_m).$$

Taking $L \to \infty$ we now get

$$\exists \lim_{L\to\infty} \varphi_i^{m,L}(x) = \varphi_0^m(x - ih_m) = \varphi_i^m(x),$$

which proves the first part of the lemma.

(ii) Using (4.9b) we get that

$$\mathcal{D}_m \varphi_i^{m,L} = \bar{\varphi}_i^{m,m} = \delta_i^m;$$

therefore

$$\begin{aligned} h_m \|\mathcal{D}_m \varphi_i^m - \delta_i^m\|_{\ell_2}^2 &= h_m \|\mathcal{D}_m(\varphi_i^m - \varphi_i^{m,L})\|_{\ell_2}^2 \\ &= h_m \sum_j |\langle \omega_j^m, \varphi_i^m - \varphi_i^{m,L}\rangle|^2 \le h_m \|\omega_j^m\|_{L^2[0,1]}^2 \|\varphi_i^m - \varphi_i^{m,L}\|_{L^2[0,1]}^2 \\ &= \|\omega\|_{L^2(\mathbb{R})}^2 \|\varphi_i^m - \varphi_i^{m,L}\|_{L^2[0,1]}^2 \underset{L\to\infty}{\longrightarrow} 0 \end{aligned}$$

which proves the second part of the lemma.

(iii) It follows from Lemma 4.2 and Lemma 6.3 that

$$\varphi_i^{k-1} = \langle P_{k-1}^k \delta_i^{k-1}, \varphi^k\rangle = [(P_{k-1}^k)^* \varphi^k]_i = \sum_j \gamma_{j-2i}^{k-1} \varphi_j^k = \sum_\ell \gamma_\ell^{k-1} \varphi_{2i+\ell}^k$$

using (6.20b) and $h_{k-1} = 2h_k$ we get that

$$\varphi_o^{k-1}(x - ih_{k-1}) = \sum_\ell \gamma_\ell^{k-1} \varphi_0^k(x - ih_{k-1} - \ell h_k);$$

substituting $y = x - ih_{k-1}$ in the relation above we obtain (6.20e). □

Corollary 6.2. *Let $\varphi_i^k(x)$ be as in* (6.20a) – (6.20b) *and define*

(6.21a) $$\hat{\mathcal{R}}_k(\mathcal{D}_k f) = \sum_j (\mathcal{D}_k f)_j \varphi_j^k;$$

then $\{(\hat{\mathcal{R}}_k, \mathcal{D}_k)\}$ is an hierarchical sequence, i.e.

(6.21b) $$(\hat{\mathcal{R}}_k \mathcal{D}_k)\hat{\mathcal{R}}_{k-1} = \hat{\mathcal{R}}_{k-1}.$$

Proof. Use Corollary 4.2 and the first part of Lemma 4.3. □

We turn now to describe multiresolution bases for function spaces. From the second part of Lemma 4.3 we get

$$\hat{\mathcal{R}}_k \bar{f}^k - \hat{\mathcal{R}}_{k-1} \bar{f}^{k-1} = \sum_{j=1}^{J_k} e_j^k \varphi_j^k = \langle e^k, \varphi^k \rangle, \tag{6.22}$$

where e^k is the prediction error and φ^k denote the column-vector $(\varphi^k)_i = \varphi_i^k$. Let us assume that e^k is represented by d^k of J_{k-1} components as follows:

$$d^k = C_k^{k-1} e^k, \ e^k = E_{k-1}^k d^k, \tag{6.23a}$$

where C_k^{k-1} is a $J_{k-1} \times J_k$ matrix and E_k^{k-1} is a $J_k \times J_{k-1}$ matrix; then

$$\langle e^k, \varphi^k \rangle = \langle E_k^{k-1} d^k, \varphi^k \rangle = \langle d^k, (E_k^{k-1})^* \varphi^k \rangle. \tag{6.23b}$$

Denoting

$$\psi^k(x) = (E_k^{k-1})^* \varphi^k(x) \tag{6.24a}$$

we can express (6.22) as

$$\hat{\mathcal{R}}_k \bar{f}^k - \hat{\mathcal{R}}_{k-1} \bar{f}^{k-1} = \sum_{j=1}^{J_{k-1}} d_j^k \psi_j^k \tag{6.24b}$$

where

$$d^k = C_k^{k-1} e^k = C_k^{k-1} (\bar{f}^k - P_k^{k-1} \bar{f}^{k-1}). \tag{6.24c}$$

Reformulating Theorem 4.2 for the above we now have

Theorem 6.1. *Assume that there exist* $\{\varphi_0^k(x)\}$ *which are square-integrable in* $[0,1]$, *such that*

$$\lim_{L \to \infty} \| \mathcal{R}_L A_0^L \delta_0^k - \varphi_0^k \|_{L^2[0,1]} = 0, \tag{6.25a}$$

and denote

$$(6.25b)\qquad \varphi_j^k = \varphi_0^k(x - jh_k),\ \psi^k = (E_k^{k-1})^*\varphi^k,\ d^k = C_k^{k-1}(\bar{f}^k - P_k^{k-1}\bar{f}^{k-1});$$

then

$$(6.25c)\qquad \text{(i)}\ \hat{\mathcal{R}}_L\bar{f}^L = \sum_{j=1}^{J_L}\bar{f}_j^L\varphi_j^L = \sum_{k=1}^{L}\sum_{i=1}^{J_{k-1}} d_i^k\psi_i^k + \sum_{i=1}^{J_0}\bar{f}_i^0\varphi_i^0.$$

(ii) *If* $\{(\mathcal{D}_k, \hat{\mathcal{R}}_k)\}_{k=0}^{\infty}$ *is complete in* $\mathcal{F}$, *then for any* $f \in \mathcal{F}$

$$(6.25d)\qquad f = \sum_{k=1}^{\infty}\sum_{i=1}^{J_{k-1}} d_i^k\psi_i^k + \sum_{i=1}^{J_0}\bar{f}_i^0\varphi_i^0.$$

In the present paper we use in (6.24a)

$$C_k^{k-1} = G_k^{k-1},\ E_{k-1}^k = S_k^{-1}(G_k^{k-1})^*,$$

and thus Theorem 6.2 applies to this case with

$$(6.26a)\qquad \psi^k = G_k^{k-1}S_k^{-1}\varphi_k,\ d^k = G_k^{k-1}e^k.$$

ψ_i^k above can be expressed by

$$(6.26b)\qquad \psi_i^k(x) = \sum_{\ell}(-1)^{\ell+1}\alpha_{\ell-1}\tilde{\varphi}_{2i-\ell}^k(x),\ \tilde{\varphi}^k = S_k^{-1}\varphi_k.$$

7. Wavelet formulation

In this section we consider the diadic constant coefficient case of the previous section in the special circumstances where in addition to (6.1) we also have that P_{k-1}^k is a *constant* "Toplitz matrix"

$$(7.1)\qquad (P_{k-1}^k)_{i,j} = \gamma_{i-2j},\ (D_k^{k-1})_{i,j} = \alpha_{j-2i},\ \text{independent of } k$$

(see Examples 5.1.1 and 5.2.1).

First we present this case in the context of the general framework of this paper, where (7.1) is obtained by taking $\mathcal{R}_k$ to be both translation invariant and independent of k, and show the extra structure that results from this assumption. Then we present the wavelet point of view which is based on the observation that in this special case, the design of a multiresolution representation reduces to a choice of coefficients $\{\alpha_\ell\}$ and $\{\gamma_\ell\}$ subject to appropriate algebraic conditions.

We assume now that $\mathcal{R}_k$, in addition to being translation invariant, is also the same for all levels of resolution, i.e.

$$(\mathcal{R}_{k+1}\delta_0^{k+1})(x) = (\mathcal{R}_k\delta_0^k)(2x) \quad \text{for all} \quad k. \tag{7.2a}$$

In this case the coefficients $\{\gamma_\ell^k\}$ in (6.13b) are the same for all levels of resolution,

$$\gamma_\ell^{k+1} = \langle \omega_\ell^{k+1},\ \mathcal{R}_{k+1}\delta_0^{k+1}\rangle = \langle \omega_\ell^k,\ \mathcal{R}_k\delta_0^k\rangle = \gamma_\ell^k, \tag{7.2b}$$

and we show the following:

Theorem 7.1. *Assume that $\mathcal{R}_k$ is translation invariant* (6.13a) *and independent of k* (7.2a). *If there exists $\varphi(x) \in L^2(\mathbb{R})$ such that*

$$\lim_{L\to\infty} \|\mathcal{R}_L A_0^L \delta_0^0 - \varphi_0\| = 0,\ \ \varphi(x) = \varphi_0(xh_0) \tag{7.3}$$

then

(i) *$\varphi(x)$ satisfies the dilation equation*

$$\varphi(x) = \sum_\ell \gamma_\ell \varphi(2x-\ell) \tag{7.4}$$

(ii)

$$\hat{\mathcal{R}}_k \bar{f}^k = \sum_j \bar{f}_j^k \varphi_j^k, \quad \varphi_j^k = \varphi\left(\frac{x}{h_k} - j\right) \tag{7.5}$$

is hierarchial.

(iii) *If* $\{(\mathcal{D}_k, \hat{\mathcal{R}}_k)\}_{k=0}^{\infty}$ *is complete in* $\mathcal{F}$, *then for any* $f \in \mathcal{F}$

$$f = \sum_{k=1}^{\infty} \sum_i d_i^k \psi_i^k + \sum_i \bar{f}_i^0 \varphi_i^0 \tag{7.6a}$$

where

$$d_i^k = G_k^{k-1}(\mathcal{D}_k f - P_{k-1}^k \mathcal{D}_{k-1} f), \tag{7.6b}$$

$$\psi_i^k(x) = \sum_\ell (-1)^{\ell+1} \alpha_{\ell-1} \tilde{\varphi}_{2i-\ell}^k(x), \quad \tilde{\varphi}^k = S_k^{-1} \varphi^k. \tag{7.6c}$$

Proof:

$$\varphi_i^{m,L} = \mathcal{R}_L A_m^L \delta_i^m = \prod_{k=m}^{L-1} (\mathcal{R}_{k+1} \mathcal{D}_{k+1}) \cdot \mathcal{R}_m \delta_i^m.$$

Using (7.2a) we get

$$\begin{aligned}\varphi_0^{m,L}(2x) &= \prod_{k=m+1}^{L-1} (\mathcal{R}_{k+1} \mathcal{D}_{k+1}) \cdot (\mathcal{R}_{m+1} \mathcal{D}_{m+1}) \cdot (\mathcal{R}_{m+1} \delta_0^{m+1})(x) \\ &= \prod_{k=m+1}^{L-1} (\mathcal{R}_{k+1} \mathcal{D}_{k+1}) \cdot (\mathcal{R}_{m+1} \delta_0^{m+1})(x) = \varphi_0^{m+1,L}(x).\end{aligned}$$

It follows then by induction that

$$\varphi_0^{m,L}(x) = \varphi_0^{0,L}(2^m x)$$

and therefore (7.3) implies that

$$\exists \lim_{L\to\infty} \varphi_0^{m,L} = \varphi_0^m, \ \varphi_0^m(x) = \varphi(x/h_m).$$

Now we can apply Lemma 6.4 and Theorem 6.1 in order to complete the proof of Theorem 7.1. □

Corollary 7.1. *If*

$$\sum_\ell \alpha_\ell \alpha_{\ell+2m} = 0 \text{ for } m \neq 0 \tag{7.7a}$$

then $\psi_i^k(x)$ in Theorem 7.1 satisifes

$$\psi_i^k(x) = \psi\left(\frac{x}{2h_k} - i\right), \tag{7.7b}$$

where $\psi(x)$, the "mother wavelet function", is

$$\psi(x) = \left[\sum_\ell (-1)^{\ell+1}\alpha_{\ell-1}\varphi(2x+\ell)\right] \Big/ \sum_\ell \alpha_\ell^2. \tag{7.7c}$$

Proof: From (7.7a) and (6.8d) we get that

$$S_k^{-1} = \frac{1}{s(0)}I,\ \tilde{\varphi}^k = \frac{1}{s(0)}\varphi^k,\ s(0) = \sum_\ell \alpha_\ell^2.$$

It follows then from (7.6c) that

$$\psi\left(\frac{x}{2h_k} - i\right) = \frac{1}{s(0)}\sum_\ell (-1)^{\ell+1}\alpha_{\ell-1}\varphi\left(\frac{x}{h_k} - 2i + \ell\right) = \frac{1}{s(0)}\sum_\ell (-1)^{\ell+1}\alpha_{\ell-1}\varphi_{2i-\ell}^k$$
$$= \sum_\ell (-1)^{\ell+1}\alpha_{\ell-1}\tilde{\varphi}_{2i-\ell}^k = \psi_i^k(x).$$

□

In this paper we obtain a multiresolution representation from any sequence of discrete approximations $\{(\mathcal{D}_k, \mathcal{R}_k)\}$ for which $\{\mathcal{D}_k\}$ is nested (Definition 2.3). This is a very general concept which applies to unstructured grids (Examples 2.1 and 2.2) as well as to nonlocal discretizations (Example 2.3). The matrices D_k^{k-1} and P_{k-1}^k are obtained from the sequence of discrete approximations by

$$D_k^{k-1} = \mathcal{D}_{k-1}\mathcal{R}_k,\ P_{k-1}^k = \mathcal{D}_k\mathcal{R}_{k-1}; \tag{7.8a}$$

the matrix $G_k^{\Delta_k}$ relates only to D_k^{k-1} and is chosen so that

$$D_k^{k-1}(G_k^{\Delta_k})^* = 0,\ \text{rank}(G_k^{\Delta_k}) = \Delta_k. \tag{7.8b}$$

We observe that the multiresolution transform (3.13) and its inverse (3.14) are expressed in terms of these matrices without reference to $\mathcal{D}_k$ and $\mathcal{R}_k$. Therefore we can formulate requirements for the existence of the multiresolution representation (3.11) directly in terms of the matrices D_k^{k-1}, P_k^{k-1}, $G_k^{\Delta_k}$ as follows: Find such matrices so that

$$D_k^{k-1} P_{k-1}^k = I \tag{7.9}$$

and $G_k^{\Delta_k}$ satisfies (7.8b); observe that the above relation implies

$$\operatorname{rank}(D_k^{k-1}) = \operatorname{rank}(P_{k-1}^k) = J_{k-1}. \tag{7.10}$$

In spite of its simplicity, (7.9) is not a useful guideline for design of multiresolution algorithms in complicated situations such as bounded domains with unstructured grids. We note that if the underlying discretization has the property that

$$f(x) \equiv 1 \Rightarrow \bar{f}^k \equiv 1 \quad \text{for all} \quad k, \tag{7.11a}$$

then we have to add the consistency requirements

$$\sum_\ell (D_k^{k-1})_{i,\ell} = 1, \ \sum_\ell (P_{k-1}^k)_{i,\ell} = 1, \ \forall i. \tag{7.11b}$$

We turn now to consider the special case where both matrices D_k^{k-1} and P_{k-1}^k are constant Töplitz matrices which are defined in terms of $\{\alpha_\ell\}$ and $\{\gamma_\ell\}$ (7.1), respectively. In this case

$$(D_k^{k-1} P_{k-1}^k)_{i,j} = \sum_m \alpha_{m-2i}\gamma_{m-2j} = \sum_\ell \alpha_\ell \gamma_{\ell+2(i-j)}$$

and therefore (7.9) is equivalent to the following condition on the choice of $\{\alpha_\ell\}$ and $\{\gamma_\ell\}$

$$\sum_\ell \alpha_\ell \gamma_{\ell+2m} = \delta_{0,m}. \tag{7.12a}$$

The consistency relation (7.11) implies that

$$\sum_\ell \alpha_\ell = 1 \tag{7.12b}$$

$$\sum_\ell \gamma_{2\ell} = \sum_\ell \gamma_{2\ell+1} = 1. \tag{7.12c}$$

We refer to the algebraic approach of finding $\{\alpha_\ell\}$ and $\{\gamma_\ell\}$ subject to conditions (7.12) as the wavelet formulation. In the following we assume that both $\{\alpha_\ell\}$ and $\{\gamma_\ell\}$ are of compact support.

The consistency relation (7.12c) implies that P_{k-1}^k is at least first-order accurate in the sense of (6.11b). In Lemma 7.1 we formulate conditions on the matrix

$$Q = P_{k-1}^k D_k^{k-1} \tag{7.13a}$$

for the prediction to be r-th order accurate

Lemma 7.1. *The prediction is r-th order accurate in the sense that*

$$\bar{f}_i^k = \sum_{m=0}^{r-1} b_m (x_i^k)^m, \ \bar{f}^{k-1} = D_k^{k-1}\bar{f}^k \Rightarrow P_{k-1}^k \bar{f}^{k-1} = \bar{f}^k, \tag{7.13b}$$

if and only if

$$\sum_p Q_{i,i+p} = 1, \ \sum_p Q_{i,i+p} p^\ell = 0, \ 1 \le \ell \le r-1. \tag{7.13c}$$

Proof: Proving (7.13b) amount to showing that

$$Q_k \bar{f}^k = \bar{f}^k \quad \text{for} \quad \bar{f}_i^k = \sum_{m=0}^{r-1} b_m (x_i^k)^m,$$

which is true if and only if

$$\sum_j Q_{i,j}(x_j^k)^m = (x_i^k)^m \quad \forall i, \ 0 \le m \le r-1. \tag{7.13d}$$

Using the binomial expansion and the substitution $j = i + p$ we get

$$\begin{aligned}
\sum_j Q_{i,j}(x_j^k)^m &= (h_k)^m \sum_p Q_{i,i+p}(i+p)^m \\
&= (h_k)^m \sum_p Q_{i,i+p} \sum_{\ell=0}^m \binom{m}{\ell} i^{m-\ell} p^\ell \\
&= (x_i^k)^m \sum_p Q_{i,i+p} + (h_k)^m \sum_{\ell=1}^m \binom{m}{\ell} i^{m-\ell} \left[\sum_p Q_{i,i+p} p^\ell\right].
\end{aligned}$$

For $m = 0$ we get that (7.13d) is true if and only if $\sum_p Q_{i,i+p} = 1$. For $m \geq 1$ we get that if the proposition of this lemma is true for $0 \leq \ell \leq m-1$, then

$$\sum_j Q_{i,j}(x_j^k)^m = (x_i^k)^m + (h_k)^m \sum_p Q_{i,i+p}p^m,$$

which shows that it is also true for $0 \leq \ell \leq m$; by induction we get that it is true for $0 \leq \ell \leq r-1$. □

We remark that for the discretizatin (6.3) we get that for any $\{b_m\}_{m=0}^{r-1}$ there exists $\{a_m\}_{m=0}^{r-1}$ such that

$$\left[\mathcal{D}_k\left(\sum_{m=0}^{r-1} a_m x^m\right)\right]_i = \sum_{m=1}^{r-1} b_m(x_i^k)^m \quad \forall i,$$

and vice versa. Therefore (7.13c) implies r-th order accuracy in the sense of (6.11b).

When D_k^{k-1} and P_{k-1}^k are the constant matrices in (7.1) we get that

$$Q_{i,i+p} = \begin{cases} \sum\limits_m \gamma_{2m}\alpha_{2m+p} & \text{for } i = \text{ even} \\ \sum\limits_m \gamma_{2m+1}\alpha_{2m+1+p} & \text{for } i = \text{ odd.} \end{cases} \tag{7.14a}$$

This shows that the consistency conditions (7.12) imply that

$$\sum_p Q_{i,i+p} = 1 \quad \forall i,$$

and that the prediction is r-th order accurate, $r \geq 2$, if and only if

$$\sum_p p^\ell \sum_m \gamma_{2m}\alpha_{2m+p} = 0, \ \sum_p p^\ell \sum_m \gamma_{2m+1}\alpha_{2m+1+p} = 0, \ 1 \leq \ell \leq r-1. \tag{7.14b}$$

We refer the reader to Appendix A where we derive a necessary and sufficient condition for (7.14b).

Following Daubechies in [4] we now consider the limiting process $L \to \infty$ for

$$\varphi_i^{m,L}(x) = \sum_j (\bar{\varphi}_i^{m,L})_j \chi_{[x_{j-1}^L, x_j^L]}(x), \tag{7.15a}$$

where χ_I is the characteristic function of the interval I, and $\bar{\varphi}_i^{m,L}$ is defined as in (4.3c), i.e.

$$\bar{\varphi}_i^{m,L} = A_m^L \delta_i^m, \ A_m^L = \prod_{k=m}^{L-1} P_k^{k+1}. \tag{7.15b}$$

As in Examples 5.1.1 and 5.2.1 we get that if

$$\exists \lim_{L\to\infty} \varphi_0^{0,L} = \varphi_0(x) \tag{7.15c}$$

then

(i) $\exists \lim_{L\to\infty} \varphi_i^{m,L} = \varphi_i^m, \quad \varphi_i^m = \varphi\left(\frac{x}{h_m} - i\right), \quad \varphi(x) = \varphi_0(xh_0);$

(ii) support (φ) is the same as that of $\{\gamma_\ell\}$;

(iii) φ satisfies the dilation equation (7.4).

At this point we can switch to our formulation by defining

$$(\mathcal{D}_k f)_i = \langle f, \omega_i^k \rangle, \quad \omega_i^k = \frac{1}{h_k}\omega\left(\frac{x}{h_k} - j\right), \tag{7.16a}$$

where ω is the solution of the dilation equation (6.5), and

$$(\mathcal{R}_k \bar{f}^k)(x) = \sum_i \bar{f}_i^k \varphi_i^k(x). \tag{7.16b}$$

In order to prove that $\mathcal{R}_k$ is a reconstruction, i.e.

$$\mathcal{D}_k \mathcal{R}_k = I \tag{7.16c}$$

we have to show that

$$\langle \omega_i^m, \ \varphi_j^m \rangle = \delta_{ij}, \ \forall i, j, m. \tag{7.16d}$$

To do so we express $\omega_i^k(x)$ as the L^2 limit

$$\omega_i^m(x) = \frac{1}{h_m} \lim_{L\to\infty} \omega_i^{m,L}(x), \tag{7.17a}$$

where, as in (7.15), we define

$$\bar{\omega}_i^{m,L} = \prod_{k=m}^{L-1} (D_{k+1}^k)^* \delta_i^m, \tag{7.17b}$$

$$\omega_i^{m,L}(x) = \sum_j (\bar{\omega}_i^{m,L})_j \chi_{[x_{j-1}^L, x_j^L]}(x),$$

and observe that

$$\begin{aligned} \frac{1}{h_m}\langle \omega_i^{m,L},\ \varphi_j^{m,L}\rangle_{L^2} &= \langle \bar{\omega}_i^{m,L},\ \bar{\varphi}_j^{m,L}\rangle_{\ell^2} \\ &= \langle \prod_{k=m}^{L-1} (D_{k+1}^k)^* \delta_i^m,\ \prod_{k=m}^{L-1} P_k^{k+1}\delta_j^m\rangle_{\ell^2} \\ &= \langle \delta_i^m,\ D_{m+1}^m \cdots D_L^{L-1} P_{L-1}^L \cdots P_m^{m+1}\delta_j^m\rangle_{\ell^2} = \langle \delta_i^m,\ \delta_j^m\rangle_{\ell^2} \\ &= \delta_{i,j}; \end{aligned} \tag{7.17c}$$

the last part follows from (7.9). Taking the limit $L \to \infty$ in (7.17c) we obtain (7.16d).

Since $\varphi(x)$ satisfies a dilation equation, we get that $\{(\mathcal{D}_k,\ \mathcal{R}_k)\}$ is an hierarchial sequence of discrete approximations; hence (7.6) in Theorem 7.1 follows from Corollary 4.3.

Example 7.1. Piecewise-polynomial reconstruction

From Example 5.1.1 we get that

$$\begin{cases} \alpha_\ell = \delta_{\ell,0} \\ \gamma_0 = 1,\ \gamma_{2m+1} = L_{-m}(-1/2),\ -s+r \le m \le -s, \end{cases} \tag{7.18}$$

where $L_m(y)$ is defined by (5.18b), is a solution of the system of algebraic equations (7.12) and (7.14) for $1 \le \ell \le r-1$.

From Example 5.2.1 we get that

$$\begin{cases} \alpha_\ell = \frac{1}{2}(\delta_{\ell,0} + \delta_{\ell,-1}) \\ \gamma_{2m+1} = \beta_{-m},\ \gamma_{2m} = 2\delta_{m,0} - \beta_{-m},\ -s+1 \le m \le -s+r, \end{cases} \tag{7.19}$$

where β_m is defined by (5.33d), is also a solution of the above mentioned system of algebraic equations.

Example 7.2. Daubechies' wavelets

In the framework of this paper, Daubechies' compactly supported orthonormal wavelets, are characterized by the choice

$$P_{k-1}^{k} = 2(D_k^{k-1})^* \iff \gamma_\ell = 2\alpha_\ell. \tag{7.20a}$$

In this case (7.9) becomes

$$2D_k^{k-1}(D_k^{k-1})^* = I, \tag{7.20b}$$

which is equivalent to the condition (7.12a)

$$\sum_\ell \alpha_\ell \alpha_{\ell+2m} = \frac{1}{2}\delta_{m,0}; \tag{7.20c}$$

note that this implies in (6.9b)

$$S_k = \frac{1}{2}I. \tag{7.20d}$$

The consistency conditions (7.12b)-(7.12c) are now equivalent to

$$\sum_\ell \alpha_{2\ell} = \sum_\ell \alpha_{2\ell+1} = 1/2. \tag{7.21a}$$

The conditions for r-th order accuracy (7.14b) for $r \geq 2$ become

$$\sum_p p^\ell \sum_m \alpha_{2m}\alpha_{2m+p} = 0, \ \sum_p p^\ell \sum_m \alpha_{2m+1}\alpha_{2m+1+p} = 0, \ 1 \leq \ell \leq r-1. \tag{7.21b}$$

In Appendix A we show that

$$\sum_p (-1)^p p^\ell \alpha_p = 0, \ 1 \leq \ell \leq r-1, \tag{7.21c}$$

is a necessary condition for (7.21b) to be true; Strang [15] argues that this is also a sufficient condition.

Daubechies' construction in [4] is based on the Fourier transform of the limiting process in (7.15) and its goal is to ensure convergence and to increase the regularity of its limit $\varphi(x)$, which is the solution to the dilation equation (7.4); observe that in the orthonormal wavelet case, the dilation equation for $\omega(x)$ (6.5) is the same as (7.4). For each r, this construction results in a unique set of $2r$ coefficients $\{\alpha_\ell^r\}_{\ell=0}^{2r-1}$ for which the limiting process converges to φ^r with increasing regularity in r. This set of $2r$ coefficients solves the system of $2r$ algebraic equations which is composed of the $(r-1)$ equations of (7.20c) for $1 \le m \le r-1$, the 2 equations of (7.21a) and the $(r-1)$ equations of (7.21c); the condition $\Sigma\alpha_\ell^2 = \frac{1}{2}$ is derivable from this system by using the orthogonality condition in $(\underset{\ell}{\Sigma}\alpha_{2\ell})^2 + (\underset{\ell}{\Sigma}\alpha_{2\ell+1})^2 = \frac{1}{2}$. For $r = 1$ we get from this system $\alpha_0 = \alpha_1 = \frac{1}{2}$ which corresponds to the Haar basis; note that this is identical to the cell-average algorithm 5.2.1 with $r = 1$. For $r = 2$ we get from this system

$$\text{(7.21d)} \quad \alpha_0 = (1 \mp \sqrt{3})/8, \ \alpha_1 = (3 \mp \sqrt{3})/8, \ \alpha_2 = (3 \pm \sqrt{3})/8, \ \alpha_3 = (1 \pm \sqrt{3})/8$$

which is identical to Daubechies' result. Strang [15] seems to conjecture that Daubechies' construction is equivalent to solving this system of 2r algebraic equations.

We observe that Daubechies' approach couples the discretization with the reconstruction by (7.20a) and thus increasing r implies better reconstruction but for a different discretization which depends on r.

We remark that it follows from (7.20d) and Corollary 7.1 that

$$\text{(7.22a)} \qquad f = \sum_{k=1}^{\infty}\sum_i d_i^k \psi_i^k + \sum_i \bar{f}_i^0 \varphi_i^0$$

where ψ_i^k are obtained by

$$\text{(7.22b)} \qquad \psi_i^k(x) = \psi\left(\frac{x}{2h_k} - i\right),$$

from a "mother wavelet function" $\psi(x)$ which is defined by

$$\text{(7.22c)} \qquad \psi(x) = 2\sum_\ell (-1)^{\ell+1}\alpha_{\ell-1}\varphi(2x+\ell).$$

In Figures 5-6 we present the results of the limiting process (7.15) for Daubechies' wavelets for $r = 3$ and 5, respectively. Part (a) in each figure shows $\varphi_0^{0,6}$ while part (b) shows $\psi_0^{0,6}$. The coefficients $\{\alpha_\ell^r\}$ in these numerical experiments are the Daubechies' coefficients [4] divided by $\sqrt{2}$ (to account for the different normalization).

Remark 7.1. Observe that the roles of discretization and reconstruction in the wavelet formulation are interchangeable in the following sense: If $(\{\alpha_\ell\}, \{\gamma_\ell\})$ is a solution to (7.12), and in addition

$$\sum_\ell \alpha_{2\ell} = \sum_\ell \alpha_{2\ell+1},$$

then $(\{\bar{\alpha}_\ell\}, \{\bar{\gamma}_\ell\})$

$$\bar{\alpha}_\ell = \frac{1}{2}\,\gamma_\ell, \qquad \bar{\gamma}_\ell = 2\alpha_\ell$$

is also a solution to (7.12) corresponding to

$$(\mathcal{D}_k f)_j = \langle f,\, \bar{\omega}_j^k\rangle, \quad \bar{\omega}_j^k(x) = \frac{1}{h_k}\,\bar{\omega}\left(\frac{x}{h_k} - j\right),$$
$$(\mathcal{R}_k \bar{f}^k)(x) = \sum_j \bar{f}_j^k\ \bar{\varphi}_j^k(x), \quad \bar{\varphi}_j^k(x) = \bar{\varphi}\left(\frac{x}{h_k} - j\right),$$

where $\bar{\omega}(x)$ and $\bar{\varphi}(x)$ satisfy

$$\bar{\omega}(x) = \sum_\ell\ \gamma_\ell\,\bar{\omega}(2x-\ell), \qquad \int \bar{\omega}(x)dx = 1,$$
$$\bar{\varphi}(x) = 2\sum_\ell\ \alpha_\ell\,\bar{\varphi}(2x-\ell), \qquad \int \bar{\varphi}(x)\bar{\omega}(x)dx = 1.$$

8. Summary and concluding remarks

In this paper we have presented a general framework which enables us to embed most numerical problems in a multiresolution setting. This framework consists of a sequence of discrete approximations $\{(\mathcal{D}_k,\ \mathcal{R}_k)\}$ with increasing resolution, where $\{\mathcal{D}_k\}$ is nested, i.e.

(8.14a) $$\mathcal{D}_k\mathcal{R}_k = I,$$

(8.1b) $$\mathcal{D}_k f = 0 \Rightarrow \mathcal{D}_{k-1} f = 0.$$

This framework allows for discretization in unstructured grids as well as for nonlocal discretization; it also allows for adaptive (data-dependent) reconstruction procedures.

Using $\{(\mathcal{D}_k,\ \mathcal{R}_k)\}$ we define the decimation matrix D_k^{k-1} and the prediction matrix P_{k-1}^k by

$$D_k^{k-1} = \mathcal{D}_{k-1}\mathcal{R}_k, \tag{8.2a}$$

$$P_{k-1}^k = \mathcal{D}_k\mathcal{R}_{k-1}. \tag{8.2b}$$

Using these matrices we have shown that $\bar{f}^L$, the finest-level discrete values, can be represented by $\mu(\bar{f}^L)$ which consists of the scale coefficients $\{d^k\}_{k=1}^L$, and the coarsest-level discrete values $\bar{f}^0$. In (3.13) we have presented the multiresolution transform $\mu(\bar{f}^L) = M\bar{f}^L$, and in (3.14) its inverse.

The fundamental property in this framework is

$$D_k^{k-1}P_{k-1}^k = I \Rightarrow \operatorname{rank}(D_k^{k-1}) = \operatorname{rank}(P_{k-1}^k) = J_{k-1}. \tag{8.3a}$$

This property implies that the prediction error e^k satisfies

$$D_k^{k-1}e^k = 0, \tag{8.3b}$$

and thus can be represented in terms of the $\Delta_k = J_k - J_{k-1}$ scale coefficients d^k. There are many ways to express the transformations (6.23) between these two sets of variables; in this paper we used a generalized version of the wavelet technique (3.6)-(3.8), which leads to an orthogonal decomposition in the diadic constant coefficient case.

The multiresolution representation (3.11) enables us to obtain data compression by replacing $\mu(\bar{f}^L)$ with a truncated $\hat{\mu}(\bar{f}^L)$

$$\hat{\mu}(\bar{f}^L) = \mathbf{tr}_\varepsilon \cdot \mu(\bar{f}^L) = \begin{pmatrix} \hat{d}^L \\ \vdots \\ \hat{d}^1 \\ \bar{f}^0 \end{pmatrix}, \tag{8.4a}$$

where

$$\hat{d}_j^k = \begin{cases} 0 & |d_j^k| \le \varepsilon_k \\ d_j^k & \text{otherwise} \end{cases} . \tag{8.4b}$$

The crucial numerical issue is the stability of the data-compression procedure. We would like to formulate conditions on $\mathcal{D}_k$, $\mathcal{R}_k$ and ε_k so that

$$\|\bar{f}^L - \hat{f}^L\| \le C \cdot \varepsilon, \quad \hat{f}^L = M^{-1} \cdot \mathbf{tr}_\varepsilon(M\bar{f}^L), \tag{8.5a}$$

where C is independent of L. Let $\hat{e}^k$ denote

$$\hat{e}^k = S_k^{-1}(G_k^{\Delta_k})^*(d^k - \hat{d}^k), \tag{8.5b}$$

then we get from (4.2) that

$$\bar{f}^L - \hat{f}^L = \sum_{m=1}^{L} A_m^L \hat{e}^m; \tag{8.5c}$$

this shows that

$$\|A_m^L \hat{e}^m\| \le C\varepsilon_m, \ \sum_{m=1}^{L} \varepsilon_m \le \varepsilon, \tag{8.5d}$$

is a sufficient condition for (8.5a); in the diadic constant coefficient case

$$\varepsilon_m = \varepsilon \cdot 2^{m-(L+1)} \tag{8.5e}$$

seems to be a natural choice (see [11] and [12]).

By Lax's equivalence theorem, convergence is equivalent to stability + consistency. Therefore convergence of the limiting process

$$\mathcal{R}_L A_m^L \delta_i^m \underset{L\to\infty}{\longrightarrow} \varphi_i^m, \tag{8.6a}$$

also implies stability of the data-compression procedure (see [12]).

If this process is convergent in a strong sense, then φ_i^m satisfies the relation (4.11)

$$\varphi_i^{m-1} = \langle P_{m-1}^m \delta_i^{m-1}, \varphi^m \rangle; \tag{8.6b}$$

in the wavelet case this relation implies the dilation equation (7.4). Relation (8.6b) also implies that

$$\hat{\mathcal{R}}_k \bar{f}^k = \sum_i \bar{f}_i^k \varphi_i^k \tag{8.7a}$$

is hierarchial, and thus

$$\hat{\mathcal{R}}_k \bar{f}^k - \hat{\mathcal{R}}_{k-1} \bar{f}^{k-1} = \sum_i d_i^k \psi_i^k \tag{8.7b}$$

where

$$\psi_i^k = \langle S_k^{-1} (G_k^{\Delta_k})^* \delta_i^{\Delta_k},\ \varphi^k \rangle, \tag{8.7c}$$

$$d_i^k = G_k^{\Delta_k} (\bar{f}^k - P_{k-1}^k \bar{f}^{k-1}). \tag{8.7d}$$

Denoting

$$\Phi^k = \text{span}\{\varphi_i^k\}, \quad \Psi^k = \text{span}\{\psi_i^k\} \tag{8.8a}$$

we get from (8.7) that

$$\Phi^k = \Psi^k \oplus \Phi^{k-1}, \tag{8.8b}$$

which implies

$$\Phi^L = \Psi^L \oplus \cdots \oplus \Psi^1 \oplus \Phi^0; \tag{8.8c}$$

in the wavelet case this direct sum is an orthogonal decomposition.

The basic relation which implies additional structure is (2.7)

$$\mathcal{D}_m(\mathcal{R}_k \mathcal{D}_k) = \mathcal{D}_m \quad \text{for} \quad m \leq k. \tag{8.9a}$$

From this relation we get that

$$\text{(8.9b)} \qquad \begin{aligned} &\mathcal{D}_k \varphi_i^k = \delta_i^k \\ &\mathcal{D}_m(\hat{\mathcal{R}}_k \bar{f}^k - \hat{\mathcal{R}}_{k-1} \bar{f}^{k-1}) = 0 \quad \text{for} \quad m \le k-1; \end{aligned}$$

the latter implies

$$\text{(8.9c)} \qquad \mathcal{D}_m \psi_i^k = 0 \quad \forall i, \quad m \le k-1.$$

When $\mathcal{D}_k$ is given by (6.3)

$$\text{(8.10a)} \qquad (\mathcal{D}_k f)_i = \langle \omega_i^k, f \rangle$$

then (8.9b)-(8.9c) become

$$\text{(8.10b)} \qquad \langle \omega_i^k, \ \varphi_j^k \rangle = \delta_{ij}$$

$$\text{(8.10c)} \qquad \langle \omega_i^k, \ \psi_j^m \rangle = 0 \quad \text{for} \quad m \le k-1.$$

Denoting

$$\text{(8.11a)} \qquad \Omega^k = \text{span}\{\omega_i^k\}$$

the above relations can be expressed by:

(8.11b) $\quad \Omega^k$ and Φ^k are bi-orthonormal,

(8.11c) $\quad \Psi^m \perp \Omega^k$ for $m \le k-1$.

For Daubechies' orthonormal wavelets we also have that $\Phi^k = \Omega^k$ is an orthonormal system, in which case we get that

$$\{\{\psi_i^m\}\}_{m=1}^L$$

is an orthonormal basis for $\hat{\mathcal{R}}_L \bar{f}^L - \hat{\mathcal{R}}_0 \bar{f}^0$ (see [11]).

Our opinion is that these orthogonality relations should be regarded as a consequence rather than essence.

Appendix A.

Conditions (7.14b) can be written as

(A.1)
$$\sum_{\bar m}\gamma_{\bar m}\sum_p p^\ell\alpha_{\bar m+p}=\sum_{\bar m}\gamma_{\bar m}\sum_p\alpha_p(p-\bar m)^\ell$$
$$=\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^k\left[\sum_{\bar m}\gamma_{\bar m}\bar m^k\right]\left[\sum_p\alpha_p p^{\ell-k}\right],\ 1\le\ell\le r-1,$$

where $\bar m$ = even or $\bar m$ = odd. Let us denote

(4.2a)
$$C_j^{\text{odd}}=\sum_m\gamma_{2m+1}(2m+1)^j,\ C_j^{\text{even}}=\sum_m\gamma_{2m}(2m)^j$$

(A.2b)
$$A_j=\sum_m\alpha_m m^j,$$

and observe that the consistency relations (7.12b)-(7.12c) imply

(A.2c)
$$A_0=C_0^{\text{odd}}=C_0^{\text{even}}=1.$$

Using this notation we can rewrite (A.1) as

(A.3)
$$\sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^kC_k^{\text{even}}A_{\ell-k}=0,\ \sum_{k=0}^{\ell}\binom{\ell}{k}(-1)^kC_k^{\text{odd}}A_{\ell-k}=0,\ 1\le\ell\le r-1.$$

Subtracting the second equation from the first we get that

(A.4a)
$$(-1)^{\ell+1}(C_\ell^{\text{even}}-C_\ell^{\text{odd}})A_0=\sum_{k=0}^{\ell-1}\binom{\ell}{k}(-1)^k(C_k^{\text{even}}-C_k^{\text{odd}})A_{\ell-k}.$$

Since $A_0=1$, (A.3b) shows that

(A.4b)
$$C_k^{\text{even}}=C_k^{\text{odd}}\text{ for }0\le k\le\ell-1\Rightarrow C_\ell^{\text{even}}=C_\ell^{\text{odd}};$$

since by (A.2c) it is true for $\ell=1$, we get by induction that it is true for $1\le\ell\le r-1$. This result can be expressed by

(A.4c)
$$\sum_m(-1)^m m^\ell\gamma_m=0\ \text{ for }\ 1\le\ell\le r-1.$$

Denoting

$$(A.5a) \qquad 2C_j = \sum_m m^j \gamma_m = C_j^{\text{odd}} + C_j^{\text{even}}$$

we get that

$$(A.5b) \qquad C_j = C_j^{\text{odd}} = C_j^{\text{even}}$$

and thus we get the additional condition

$$(A.5c) \qquad \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k C_k A_{\ell-k} = 0 \quad \text{for} \quad 1 \leq \ell \leq -r-1.$$

Each of the conditions (A.4c) and (A.5c) by itself is only a necessary condition, but together they do constitute a sufficient condition for (A.1) to be true.

In the case of Daubechies' orthonormal wavelets $\gamma_\ell = 2\alpha_\ell$, and then (A.4c) is (7.21c). However, for the associated prediction to be r-th order accurate we also need to have (A.5c), i.e.

$$(A.6) \qquad \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k A_k A_{\ell-k} = 0, \ 1 \leq \ell \leq r-1.$$

For $\ell =$ odd (A.6) is an identity, and therefore it is not an extra condition. For $\ell = 2$ we get

$$A_0 A_2 - 2(A_1)^2 + A_2 A_0 = 0 \Rightarrow A_2 = (A_1)^2;$$

we verified that this condition is satisfied by the set $\{\alpha_\ell^r\}_{\ell=0}^{2r-1}$ in [4] for $r = 3, 4, 5$.

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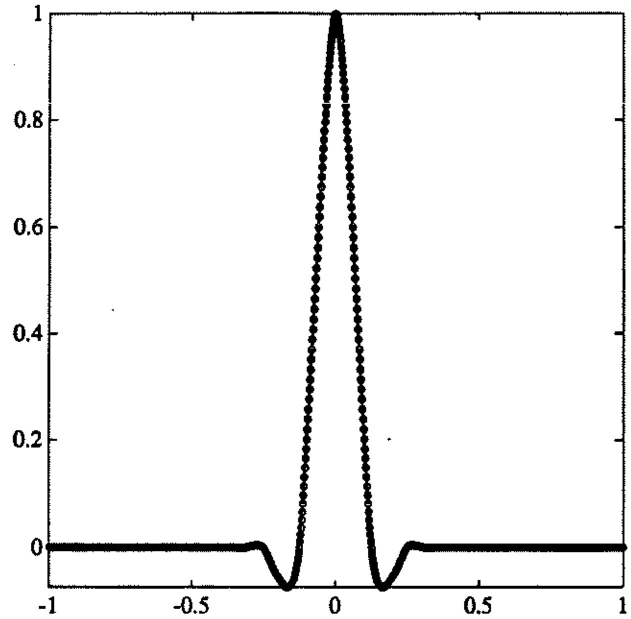

Fig. 1a. $\varphi_0^{0,6}(r=3)$

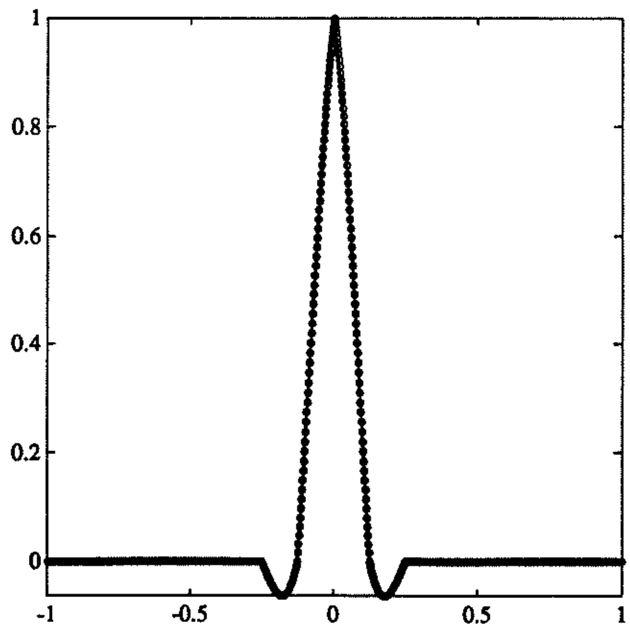

Fig. 1b. $\varphi_0^{0,1}(r=3)$

Figure 1. Limiting process for pointvalues with centered interpolation, $r=3$.

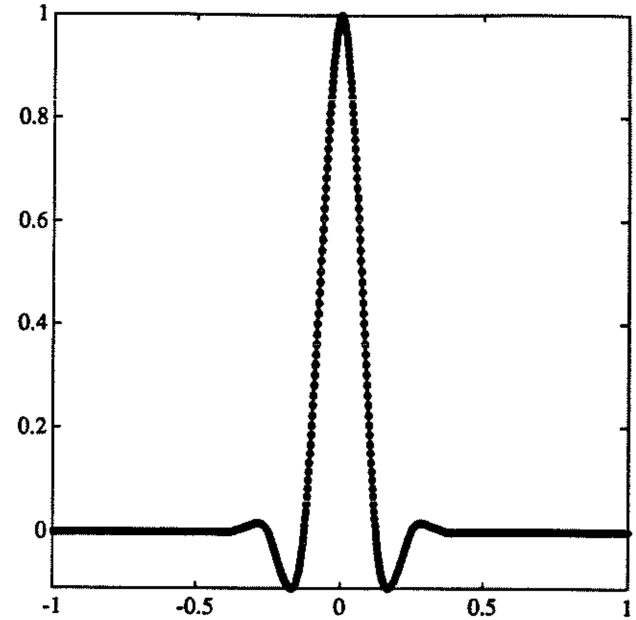

Fig. 2a. $\varphi_0^{0,6}(r=5)$

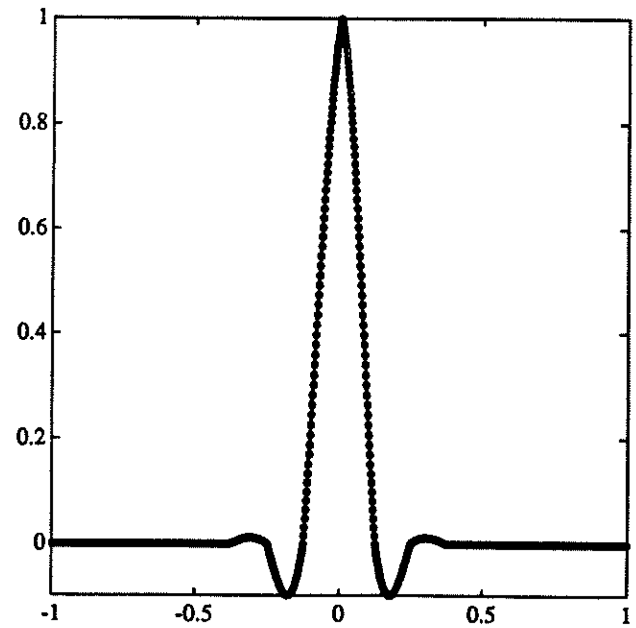

Fig. 2b. $\varphi_0^{0,1}(r=5)$

Figure 2. limiting process for pointvalues with centered interpolation, $r=5$.

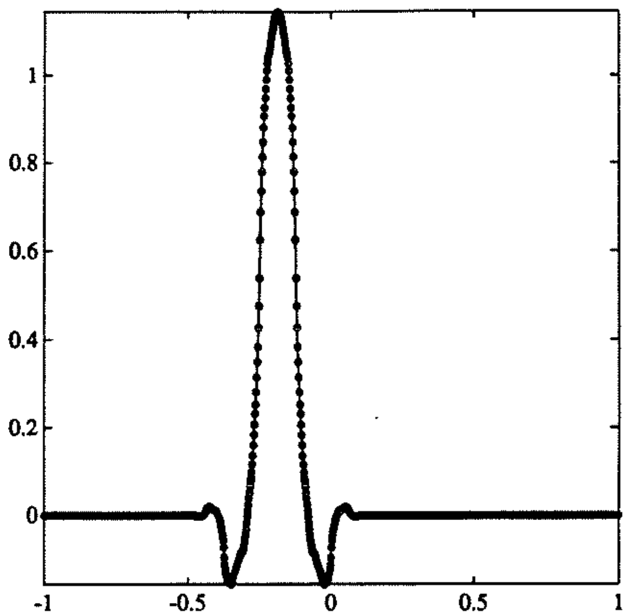

Fig. 3a. $\varphi_0^{0,6}(r=3)$

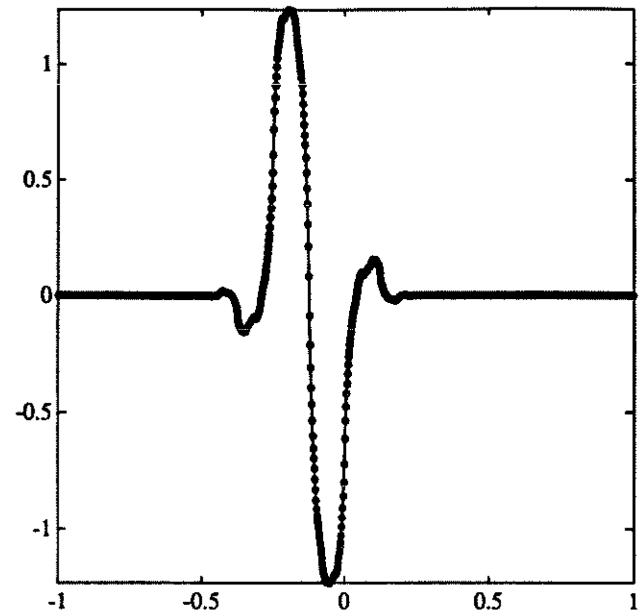

Fig. 3c. $\psi_0^{0,6}(r=3)$

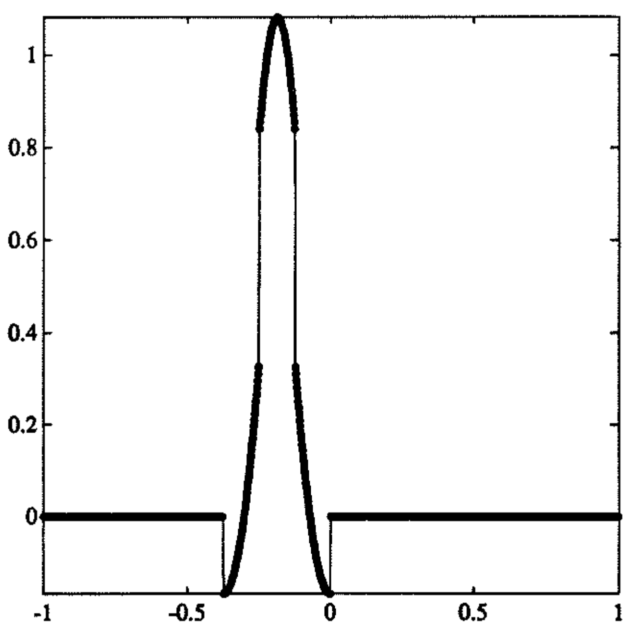

Fig. 3b. $\varphi_0^{0,1}(r=3)$

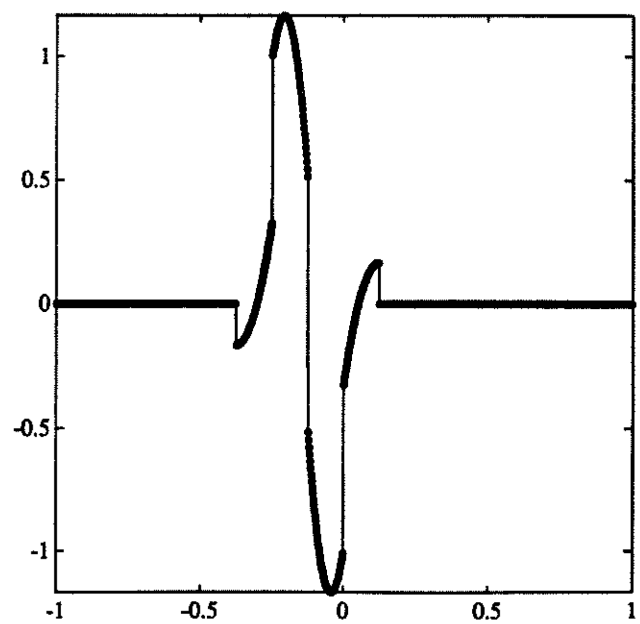

Fig. 3d. $\psi_0^{0,1}(r=3)$

Figure 3. Limiting process for cell-averages with centered reconstruction, $r = 3$.

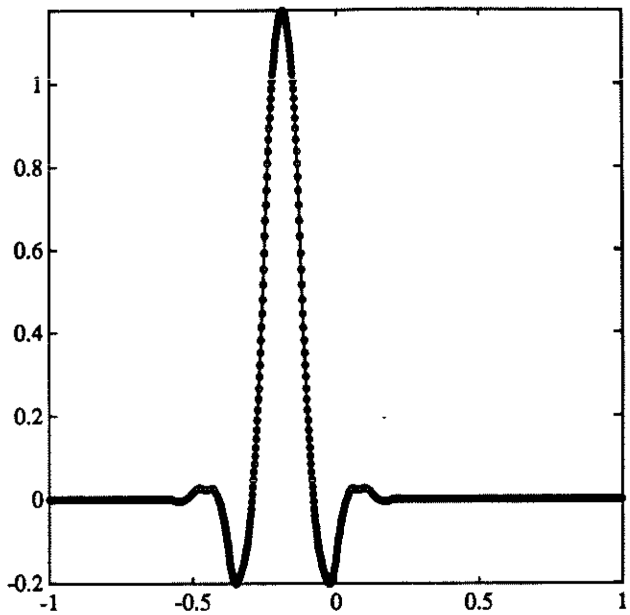

Fig. 4a. $\varphi_0^{0,6}(r=5)$

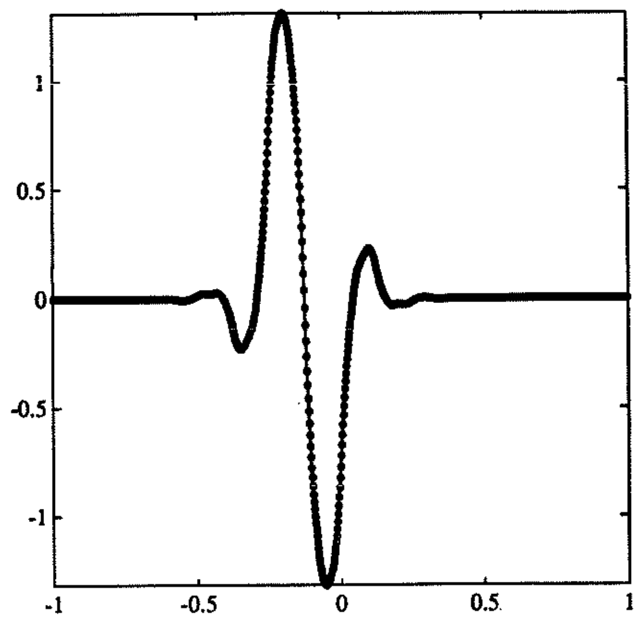

Fig. 4c. $\psi_0^{0,6}(r=5)$

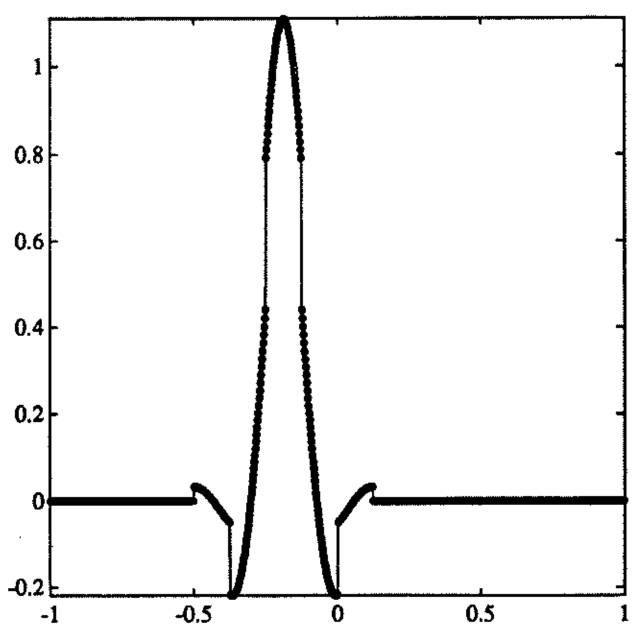

Fig. 4b. $\varphi_0^{0,1}(r=5)$

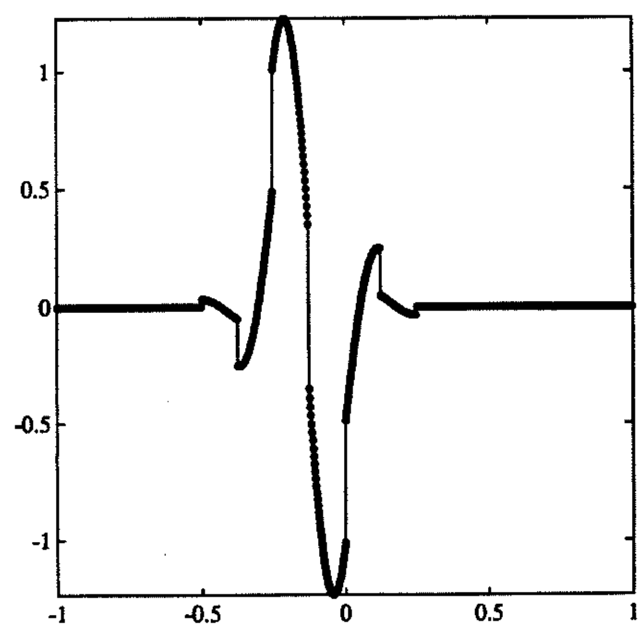

Fig. 4d. $\psi_0^{0,1}(r=5)$

Figure 4. Limiting process for cell-averages with centered reconstruction, $r=5$.

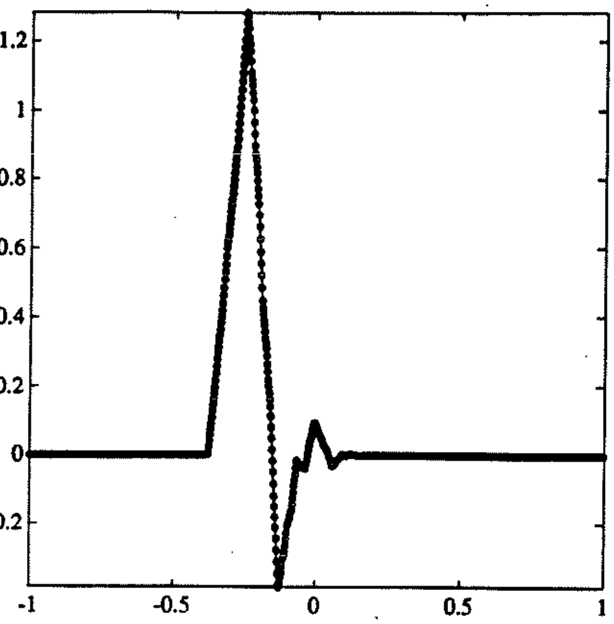

Fig. 5a. $\varphi_0^{0,6}(r=3)$

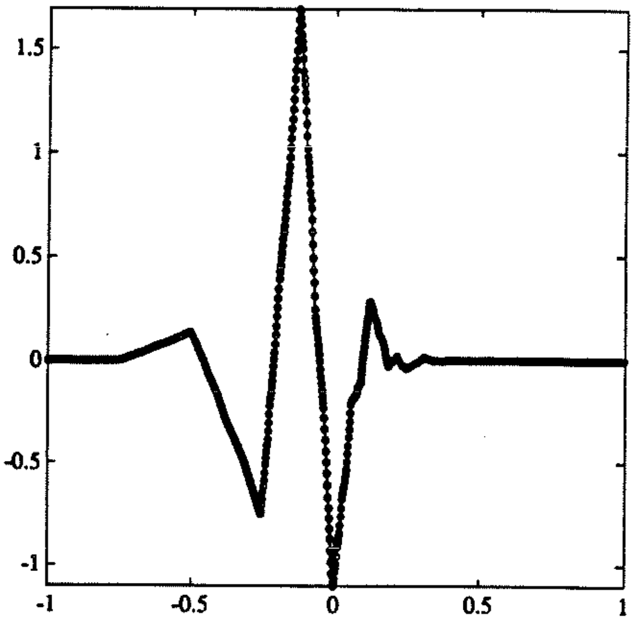

Fig. 5b. $\psi_0^{0,6}(r=3)$

Figure 5. Limiting process for Daubechies' wavelets, $r=3$.

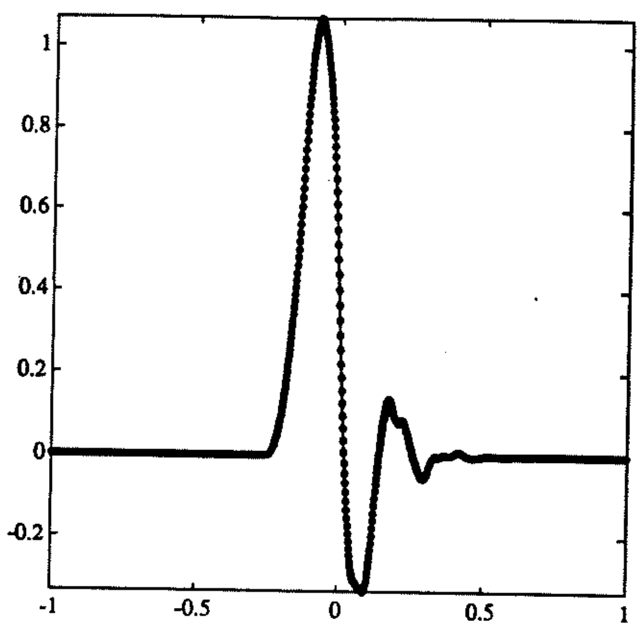

Fig. 6a. $\varphi_0^{0,6}(r=5)$

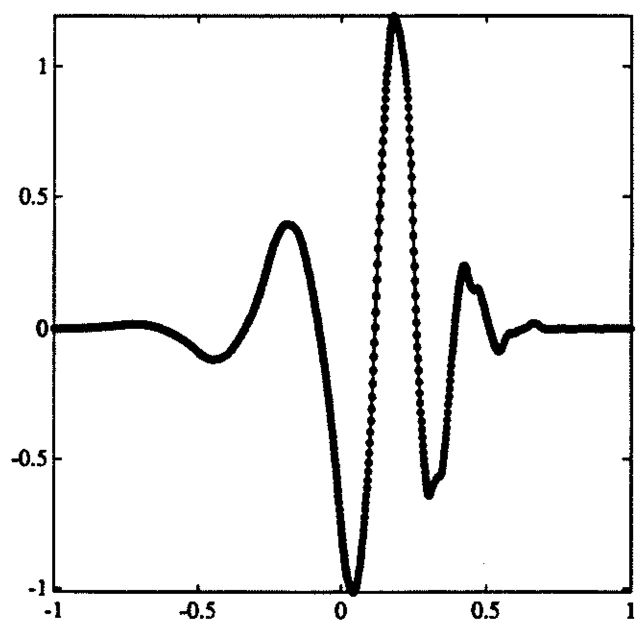

Fig. 6b. $\psi_0^{0,6}(r=5)$

Figure 6. Limiting process for Daubechies' wavelets, $r=5$.